Chapter 1

# Fixed Point Theory Approach to Existence of Solutions with Differential Equations

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Additional information is available at the end of the chapter

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**Abstract**

In this chapter, we introduce a generalized contractions and prove some fixed point theorems in generalized metric spaces by using the generalized contractions. Moreover, we will apply the fixed point theorems to show the existence and uniqueness of solution to the ordinary difference equation (ODE), Partial difference equation (PDEs) and fractional boundary value problem.

**Keywords:** fixed point, contraction, generalized contraction, differential equation, partial differential equation, fractional difference equation

## 1. Introduction

The study of differential equations is a wide field in pure and applied mathematics, chemistry, physics, engineering and biological science. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics investigated the existence and uniqueness of solutions, but applied mathematics focuses on the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods.

Following the ordinary differential equations with boundary value condition

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

where $y(x_0) = 0,\ y'(x_1) = c_1, \ldots, y^{(n-1)}(x_{n-1}) = c_{n-1}$ the positive integer $n$ (the order of the highest derivative). This will be discussed. Existence and uniqueness of solution for initial value problem (IVP).

$$u'(t) = f(t, u(t))$$
$$u(t_0) = u_0.$$

Differential equations contains derivatives with respect to two or more variables is called a *partial differential equation* (PDEs). For example,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where $u$ is dependent variable and $A, B, C, D, E, F$ and $G$ are function of $x, y$ above equation is classified according to discriminant $\left(B^2 - 4AC\right)$ as follows,

1. Elliptic equation if $\left(B^2 - 4AC\right) < 0$,
2. Hyperbolic equation if $\left(B^2 - 4AC\right) > 0$,
3. Parabolic equation if $\left(B^2 - 4AC\right) = 0$.

This will be discussed. Existence of solution for semilinear elliptic equation. Consider a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ that solves,

$$-\Delta u = f(u) \quad \text{in} \quad \Omega$$
$$u = u_0 \quad \text{on} \quad \partial\Omega$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a typically nonlinear function. And fractional differential equations. This will be discussed. Fractional differential equations are of two kinds, they are Riemann-Liouville fractional differential equations and Caputo fractional differential equations with boundary value.

$${}^cD_t^\alpha u(t) = Bu(t);\ t > 0$$
$$u(0) = u_0 \in X$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, and $t \in [0, \tau]$, for all $\tau > 0$.

The following fractional differential equation will boundary value condition.

$$D_{0+}^\alpha u(t) + f(t, u(t)) = 0,\ \ 0 < t < 1,\ \ 1 < \alpha \leq 2$$
$$u(0) = 0,\ \ u(1) = \int_0^1 u(s)ds,$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function and $D^{\alpha}_{0+}$ is the standard Riemann-Liouville fractional derivative.

One method for existence and uniqueness of solution of difference equation due to fixed point theory. The primary result in fixed point theory which is known as *Banach's contraction principle* was introduced by Banach [1] in 1922.

**Theorem 1.1.** *Let $(X, d)$ be a complete metric spaces and $T : X \to X$ be a contraction mapping (that is, there exists $0 \leq \alpha < 1$) such that*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$, then $T$ has a unique fixed point.

Since Banach contraction is a very popular and important tool for solving many kinds of mathematics problems, many authors have improved, extended and generalized it (see in [2–4]) and references therein.

In this chapter, we discuss on the existence and uniqueness of the differential equations by using fixed point theory to approach the solution.

## 2. Basic results

Throughout the rest of the chapter unless otherwise stated $(X, d)$ stands for a complete metric space.

### 2.1. Fixed point

**Definition 2.1.** Let $X$ be a nonempty set and $T : X \to X$ be a mapping. A point $x^* \in X$ is said to be a *fixed point* of $T$ if $T(x^*) = x^*$.

**Definition 2.2.** Let $(X, d)$ be a metric space. The mapping $T : X \to X$ is said to be Lipschitzian if there exists a constant $\alpha > 0$ (called Lipschitz constant) such that

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all } x, y \in X.$$

A mapping $T$ with a Lipschitz constant $\alpha < 1$ is called contraction.

**Definition 2.3.** Let $F$ and $X$ be normed spaces over the field $\mathbb{K}$, $T : F \to X$ an operator and $c \in F$. We say that $T$ is continuous at $c$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If $T$ is continuous at each $x \in F$, then $T$ is said to be continuous on $T$.

**Definition 2.4.** Let $X$ and $Y$ be normed spaces. The mapping $T : X \to Y$ is said to be completely continuous if $T(C)$ is a compact subset of $Y$ for every bounded subset $C$ of $X$.

**Definition 2.5.** Compact operator is a linear operator $L$ form a Banach space $X$ to another Banach space $Y$, such that the image under $L$ of any bounded subset of $X$ is a relatively compact subset (has compact closure) of $Y$ such an operator is necessarily a bounded operator, and so continuous.

**Definition 2.6.** A subset $C$ of a normed linear space $X$ is said to be convex subset in $X$ if $\lambda x + (1-\lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

**Definition 2.7.** $v$ is called the $\alpha^{th}$ weak derivative of $u$

$$D^\alpha u = v$$

if

$$\int_\Omega u D^\alpha \psi dx = (-1)^{|\alpha|} \int_\Omega v \psi dx$$

for all test function $\psi \in C_c^\infty(\Omega)$.

**Theorem 2.8.** *(Schauder's Fixed Point Theorem) Let X be a Banach space, $M \subset X$ be nonempty, convex, bounded, closed and $T : M \subset X \to M$ be a compact operator. Then T has a fixed point.*

**Lemma 2.9.** ref. [5] Given $f \in C(\mathbb{R})$ such that $|f(t)| \leq a = b|t|^r$ where $a > 0, b > 0$ and $r > 0$ are positive constants. Then the map $u \mapsto f(u)$ is continuous for $L^p(\Omega)$ to $L^{\frac{p}{r}}(\Omega)$ for $p \geq \max(1, r)$ and maps bounded subset of $L^p(\Omega)$ to bounded subset of $L^{\frac{p}{r}}(\Omega)$.

*Proof.* Form to Jensen's inequality

$$(a + b|t|^r)^{\left(\frac{p}{r}\right)} \leq 2^{\frac{p}{r}-1} a^{\frac{p}{r}} + 2^{\frac{p}{r}-1} b^{\frac{p}{r}} |t|^p \leq C(1 + |t|^p)$$

where $C$ is a positive constant depending on $a, b, p$ and $r$ only, since $u \in L^p(\Omega)$, we have

$$\int_\Omega |f(u)|^{\frac{p}{r}} dx \leq C(a, b, p, r)\left(|\Omega| + \int_\Omega u^p dx\right) < \infty$$

therefore $f(u) \in L^{\frac{p}{r}}(\Omega)$. Let $u_n$ be a sequence converging to $u$ in $L^p(\Omega)$. There exists a subsequence $u_n$, and a function $g \in L^p(\Omega)$ such that set, $u_{n'} \to u(x)$, and $|u_{n'}(x)| \leq g(x)$, almost everywhere. This is sometimes called the generalized DCT, or the partial converse of the DCT, or the Riesz-Fisher Theorem. From the continuity of $f$, $|f(u(x)) - f(u_{n'})| \to 0$ on $\Omega \setminus \mathbb{N}$, and

$$|f(u(x)) - f(u_{n'})|^{\frac{p}{r}} \leq C(1 + g(x)^p + |f(u)|^p)$$

where $C$ is another positive constant depending on $a, b, p$ and $r$ only, the left-hand-side is independent of $n'$ and is in $L^1(\Omega)$. We can apply the Dominated Convergence Theorem to conclude the

$$\int_\Omega |f(u(x)) - f(u_{n'})|^{\frac{p}{r}} dx \to 0$$

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or in other words, $\|f(u(x)) - f(u_{n'})\|_{L^p(\Omega)} \to 0$. Since the limit does not depend on the subsequence this convergence $u$ holds for $u_n$. □

**Corollary 2.10.** ref. [5] Let $\mu \geq 0$. Then the map $g \mapsto (-\Delta + \mu I_d)^{-1} g$ is

**i.** continuous as map from $L^2(\Omega)$ to $H_0^1(\Omega)$ in other words

$$\|v\|_{H_0^1(\Omega)} \leq C(\Omega)\|g\|_{L^2(\Omega)}.$$

**ii.** compact as map form $L^2(\Omega)$ to $L^2(\Omega)$.

*Proof.* The first part is due to the fact that $L^2(\Omega)$ is continuously in $H^{-1}(\Omega)$. The second part follows as $(-\Delta + \mu I_d)^{-1} : L^2(\Omega) \to L^2(\Omega)$ can be viewed as composition of the continuous map $(-\Delta + \mu I_d)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ and the compact embedding $H_0^1(\Omega) \rightarrowtail L^2(\Omega)$ and as the composition of a compact linear operator a continuous linear operator is again compact.

**Theorem 2.11.** *(Poincare) For $p \in [1, \infty)$, there exists a constant $C = C(\Omega, p)$ such that $\forall \in W_0^{1,p}(\Omega)$; $\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega:\mathbb{R}^n)}$. A key tool to obtain the compactness of the fixed point maps.*

### 2.2. Fuzzy

A fuzzy set in $X$ is a function with domain X and values in $[0, 1]$. If $A$ is a fuzzy set on $X$ and $x \in X$, then the functional value $Ax$ is called the grade of membership of $x$ in $A$. The $\alpha-$ level set of A, denoted by $A_\alpha$ is defined by

$$A_\alpha = \{x : Ax \geq \alpha\} \quad \text{if} \quad \alpha \in (0, 1], \quad A_0 = \overline{\{x : Ax > 0\}},$$

where denotes by $\overline{A}$ the closure of the set $A$. For any $A$ and $B$ are subset of $X$ we denote by $H(A, B)$ the Huasdorff distance.

**Definition 2.12.** A fuzzy set $A$ in a metric linear space is called an approximate quantity if and only if $A_\alpha$ is convex and compact in $X$ for each $\alpha \in [0, 1]$ and $\sup_{x \in X} Ax = 1$.

Let $I = [0, 1]$ and $W(X) \subset I^X$ be the collection of all approximate in X. For $\alpha \in [0, 1]$, the family $W_\alpha(X)$ is given by $\{A \in I^X : A_\alpha$ is nonempty and compact$\}$.

For a metric space $(X, d)$ we denoted by $V(X)$ the collection of fuzzy sets $A$ in $X$ for which $A_\alpha$ is compact and $\sup Ax = 1$ for all $\alpha \in [0, 1]$. Clearly, when $X$ is a metric linear space $W(X) \subset V(X)$.

**Definition 2.13.** Let $A, B \in V(X)$, $\alpha \in [0, 1]$. Then

$$p_\alpha(A, B) = \inf_{x \in A_\alpha,\, y \in B_\alpha} d(x, y), \quad D_\alpha(A, B) = H(A_\alpha, B_\alpha)$$

where $H$ is the Hausdorff distance.

**Definition 2.14.** Let $A, B \in V(X)$. Then $A$ is said to be more accurate than $B$ (or $B$ includes $A$), denoted by $A \subset B$, if and only if $Ax \leq Bx$ for each $x \in X$.

Denote with $\Phi$, the family of nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

**Theorem 2.15.** ref. [6] Let $(X, d, \preccurlyeq)$ *be a complete ordered metric space and* $T_1, T_2 : X \to W_\alpha(X)$ *be two fuzzy mapping satisfying*

$$D_\alpha(T_1x, T_2y) \leq \phi(M(x, y)) + L \min\{p_\alpha(x, T_1x), p_\alpha(y, T_2y), p_\alpha(x, T_2y), p_\alpha(y, T_1x)\}$$

for all comparable element $x, y \in X$, where $L \geq 0$ and

$$M(x, y) = \max\left\{d(x, y), p_\alpha(x, T_1x), p_\alpha(y, T_2y), \frac{1}{2}\left[p_\alpha(x, T_2y) + p_\alpha(y, T_1x)\right]\right\}.$$

*Also suppose that*

**i.** if $y \in (T_1x_0)_\alpha$, then $y, x_0 \in X$ are comparable,

**ii.** if $x, y \in X$ are comparable, then every $u \in (T_1x)_\alpha$ and every $v \in (T_2y)_\alpha$ are comparable,

**iii.** if a sequence $\{x_n\}$ in $X$ converges to $x \in X$ and its consecutive terms are comparable, then $x_n$ and $x$ are comparable for all $n$.

*Then there exists a point* $x \in X$ *such that* $x_\alpha \subset T_1x$ *and* $x_\alpha \subset T_2x$.

*Proof.* See in [6].

**Corollary 2.16.** ref. [6] Let $(X, d, \preccurlyeq)$ *be a complete ordered metric space and* $T_1, T_2 : X \to W_\alpha(X)$ *be two fuzzy mappings satisfying*

$$D_\alpha(T_1x, T_2y) \leq q \max\left\{d(x, y), p_\alpha(x, T_1x), p_\alpha(y, T_2y), \frac{1}{2}\left[p_\alpha(x, T_2y) + p_\alpha(y, T_1x)\right]\right\}$$

for all comparable elements $x, y \in X$. Also suppose that

**i.** if $y \in (T_1x_0)_\alpha$, then $y, x_0 \in X$ are comparable,

**ii.** if $x, y \in X$ are comparable, then every $u \in (T_1x)_\alpha$ and every $v \in (T_2y)_\alpha$ are comparable,

**iii.** if a sequence $\{x_n\}$ in $X$ converges to $x \in X$ and its consecutive terms are comparable, then $x_n$ and $x$ are comparable for all $n$.

*Then there exists a point* $x \in X$ *such that* $x_\alpha \subset T_1x$ *and* $x_\alpha \subset T_2x$.

### 2.3. Metric-like space

**Definition 2.17.** [7] Let $X$ be nonempty set and function $p : X \times X \to \mathbb{R}^+$ be a function satisfying the following condition: for all $x, y, z \in X$,

$(p_1)$ $p(x,x) = p(x,y) = p(y,y)$ if and only if $x = y$,

$(p_2)$ $p(x,x) \leq p(x,y)$,

$(p_3)$ $p(x,x) = p(y,x)$,

$(p_4)$ $p(x,y) = p(x,z) + p(z,y) - p(z,z)$.

Then $p$ is called a partial metric on $X$, so a pair $(X,p)$ is said to be a partial metric space.

**Definition 2.18.** [8] A metric-like on nonempty set $X$ is a function $\sigma : X \times X \to \mathbb{R}^+$. If for all $x, y, z \in X$, the following conditions hold:

$$(\sigma_1)\ \sigma(x,y) = 0 \Rightarrow x = y;$$

$$(\sigma_2)\ \sigma(x,y) = \sigma(y,x);$$

$$(\sigma_3)\ \sigma(x,y) = \sigma(x,z) + \sigma(z,y).$$

Then a pair $(X,\sigma)$ is called a metric-like space.

It is easy to see that a metric space is a partial metric space and each partial metric space is a metric-like space, but the converse is not true but the converse is not true as in the following examples:

**Example 2.19.** [8] Let $X = \{0,1\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2, & \text{if } x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then $(X,\sigma)$ is a metric-like space, but it is not a partial metric space, cause $\sigma(0,0) \nleq \sigma(0,1)$.

**Lemma 2.20.** ref. [9] Let $(X,p)$ be a partial metric space. Then

**i.** $\{x_n\}$ is a Cauchy sequence in $(X,p)$ if and only if it is a Cauchy sequence in the metric space $(X,d_p)$,

**ii.** $X$ is complete if and only if the metric space $(X,d_p)$ is complete.

**Definition 2.21.** [8, 10] Let $(X,\sigma)$ be a metric-like space. Then:

**i.** A sequence $\{x_n\}$ in $X$ converges to a point $x \in X$ if $\lim_{n\to\infty}\sigma(x_n,x) = \sigma(x,x)$. The sequence $\{x_n\}$ is said to be $\sigma-$ Cauchy if $\lim_{n,m\to\infty}\sigma(x_n,x_m)$ exists and is finite. The space $(X,\sigma)$ is called complete if for every $\sigma-$ Cauchy sequence in $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n\to\infty}\sigma(x_n,x) = \sigma(x,x) = \lim_{n,m\to\infty}\sigma(x_n,x_m).$$

**ii.** A sequence $\{x_n\}$ in $(X,\sigma)$ is said to be a $0-\sigma-$ Cauchy sequence if $\lim_{n,m\to\infty}\sigma(x_n,x_m) = 0$. The space $(X,\sigma)$ is said to be $0-\sigma-$ complete if every $0-\sigma-$ Cauchy sequence in $X$ converges (in $\tau_\sigma$) to a point $x \in X$ such that $\sigma(x,x) = 0$.

**iii.** A mapping $T : X \to X$ is continuous, if the following limits exist (finite) and

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(Tx, x).$$

Following Wardowski [11], we denote by $\mathcal{F}$ the family of all function, $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:
($F1$) $F$ is strictly increasing on $\mathbb{R}^+$,

($F2$) for every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$,

($F3$) there exists a number $k \in (0, 1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

**Example 2.22.** The following function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$:
**i.** $F(s) = \ln s,$ with $s > 0$,

**ii.** $F(s) = \ln s + s,$ with $s > 0$.

**Definition 2.23.** [11] Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an $F$-contraction mapping if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\forall x, y \in X, \quad [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{2.1}$$

**Definition 2.24.** [12] Let $(X, \sigma)$ be a metric-like space. A mapping $T : X \to X$ is called a generalized Roger Hardy type $F-$ contraction mapping, if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\begin{aligned} \sigma(Tx, Ty) > 0 \Rightarrow \tau + F(\sigma(Tx, Ty)) \quad \leq \quad & F(\alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) \\ & + \eta\sigma(x, Ty) + \delta\sigma(y, Tx)) \end{aligned} \tag{2.2}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \eta, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\eta + 2\delta < 1$.

**Theorem 2.25.** ref. [12] Let $(X, \sigma)$ be $0 - \sigma-$ complete metric-like spaces and $T : X \to X$ be a generalized Roger Hardy type $F-$ contraction. Then $T$ has a unique fixed point in $X$, either $T$ or $F$ is continuous.

*Proof.* See in [12]. □

### 2.4. Modular metric space

Let $X$ be a nonempty set. Throughout this paper, for a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, we write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y)$$

for all $\lambda > 0$ and $x, y \in X$.

**Definition 2.26** [13, 14] Let $X$ be a nonempty set. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is called a metric modular on $X$ if satisfying, for all $x, y, z \in X$ the following conditions hold:

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**i.** $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$,

**ii.** $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$,

**iii.** $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda$, $\mu > 0$.

If instead of (i) we have only the condition (i')

$$\omega_\lambda(x, x) = 0 \text{ for all } \lambda > 0, x \in X,$$

then $\omega$ is said to be a pseudomodular (metric) on $X$. A modular metric $\omega$ on $X$ is said to be regular if the following weaker version of (i) is satisfied:

$x = y$ if and only if $\omega_\lambda(x, y) = 0$ for some $\lambda > 0$.

Note that for a metric (pseudo)modular $\omega$ on a set $X$, and any $x, y \in X$, the function $\lambda \mapsto \omega_\lambda(x, y)$ is nonincreasing on $(0, \infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Note that every modular metric is regular but converse may not necessarily be true.

**Example 2.27.** Let $X = \mathbb{R}$ and $\omega$ is defined by $\omega_\lambda(x, y) = \infty$ if $\lambda < 1$, and $\omega_\lambda(x, y) = |x - y|$ if $\lambda \geq 1$, it is easy to verify that $\omega$ is regular modular metric but not modular metric.

**Definition 2.28.** [13, 14] Let $X_\omega$ be a (pseudo)modular on $X$. Fix $x_0 \in X$. The two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

are said to be modular spaces (around $x_0$).

Throughout this section we assume that $(X, \omega)$ is a modular metric space, $D$ be a nonempty subset of $X_\omega$ and $\mathcal{G}$:=$\{G_\omega$ is a directed graph with $V(G_\omega) = D$ and $\Delta \subseteq E(G_\omega)\}$.

**Definition 2.29.** [15, 16] The pair $(D, G_\omega)$ has Property (A) if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $D$, with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G_\omega)$, then $(x_n, x) \in E(G_\omega)$, for all $n$.

**Definition 2.30.** ref. [17] Let $F \in \mathcal{F}$ and $G_\omega \in \mathcal{G}$. A mapping $T : D \to D$ is said to be $F$-$G_\omega$-contraction with respect to $R : D \to D$ if

**i.** $(Rx, Ry) \in E(G_\omega) \Rightarrow (Tx, Ty) \in E(G_\omega)$ for all $x, y \in D$, i.e. $T$ preserves edges w.r.t. $R$,

**ii.** there exists a number $\tau > 0$ such that

$$\omega_1(Tx, Ty) > 0 \Rightarrow \tau + F(\omega_1(Tx, Ty)) \leq F(\omega_1(Rx, Ry))$$

for all $x, y \in D$ with $(Rx, Ry) \in E(G_\omega)$.

**Example 2.31.** ref. [17] Let $F \in \mathcal{F}$ be arbitrary. Then every $F$-contractive mapping w.r.t. $R$ is an $F$-$G_\omega$-contraction w.r.t. $R$ for the graph $G_\omega$ given by $V(G_\omega) = D$ and $E(G_\omega) = D \times D$.

We denote $C(T, R) := \{x \in D : Tx = Rx\}$ the set of all coincidence points of two self-mappings $T$ and $R$, defined on $D$.

**Theorem 2.32.** ref. [17] Let $(X, \omega)$ be a regular modular metric space with a graph $G_\omega$. Assume that $D = V(G_\omega)$ is a nonempty $\omega$-bounded subset of $X_\omega$ and the pair $(D, G_\omega)$ has property (A) and also satisfy $\Delta_M$-condition. Let $R, T : D \to D$ be two self mappings satisfying the following conditions:

**i.** there exists $x_0 \in D$ such that $(Rx_0, Tx_0) \in E(G_\omega)$,

**ii.** $T$ is an $F$-$G_\omega$-contraction w.r.t $R$,

**iii.** $T(D) \subseteq R(D)$,

**iv.** $R(D)$ is $\omega$ complete.

Then $C(R, T) \neq \emptyset$.

*Proof.* See in [17]. □

## 3. Fixed point approach to the solution of differential equations

Next, we will show a differential equation which solving by fixed point theorem in suitable spaces.

### 3.1. Ordinary differential equation

**Lemma 3.1.** ref. [18] $u$ is a solution of $u'(t) = f(t, u(t))$ satisfying the initial condition $u(t_0) = u_0$ if and only if $u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds$.

*Proof.* Suppose that $u$ is a solution of $u'(t) = f(t, u(t))$ defined on an interval $I$ and satisfying $u(t_0) = u_0$. We integrate both sides of the equation $u'(t) = f(t, u(t))$ from $t_0$ to $t$, where $t$ is in $I$

$$\int_{t_0}^{t} u'(s)ds = \int_{t_0}^{t} f(s, u(s))ds$$
$$u(t) - u(t_0) = \int_{t_0}^{t} f(s, u(s))ds.$$

Since $u(t_0) = u_0$, we have

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \quad t \in I. \tag{3.1}$$

We will show that, conversely, any function which satisfies this integral equation satisfies both the differential equation and the initial condition. Suppose that $u$ is a function defined on an interval $I$ and satisfies (3.1). Setting $t = t_0$ yields $u(t_0) = u_0$, so that $u$ satisfies the initial

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condition. Next, we note that an integral is always a continuous function, so that a solution of (3.1) is automatically continuous. Since both $u$ and $f$ are continuous, it follows that the integrand $f(s, u(s))$ is continuous. We may therefore apply the fundamental theorem of calculus to (3.1) and conclude that $u$ is differentiable, and that is $u'(t) = f(t, u(t))$. □

The contraction mapping theorem may by used to prove the existence and uniqueness of the initial problem for ordinary differential equations. We consider a first-order of ODEs for a function $u(t)$ that take value in $\mathbb{R}^n$

$$u'(t) = f(t, u(t)) \tag{3.2}$$

$$u(t_0) = u_0. \tag{3.3}$$

The function $f(t, u(t))$ also take value in $\mathbb{R}^n$ and is assumed to be a continuous function of $t$ and a Lipschitz continuous function of $u$ on suitable domain.

**Definition 3.2.** Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ where $I$ is on interval in $\mathbb{R}$. We say that $f(t, u(t))$ is a globally Lipschitz continuous function of $u$ uniformly in t if there is a constant $C > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq C\|u - v\| \tag{3.4}$$

for all $x, y \in \mathbb{R}^n$ and all $t \in I$.

The initial value problem can be reformulated as an integral equation.

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds. \tag{3.5}$$

By the fundamental theorem of calculus, a continuous solution of (3.5) is a continuously differentiable solution of (3.2). Eq. (3.5) may by written as fixed point equation.

$$u = Tu$$

for the map $T$ defined by

$$Tu(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds.$$

**Theorem 3.3.** ref. [19] Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ where $I$ is on interval in $\mathbb{R}$ and $t_0$ is a point in the interior of $I$. If $f(t, u)$, is a continuous function of $(t, u)$ and a globally Lipschitz continuous function of $u$ uniformly in $t$ on $I \times \mathbb{R}^n$, then there is a unique continuously differentiable function $u : I \to \mathbb{R}^n$ that satisfies (3.2).

*Proof.* We will show that $T$ is a contraction on the space of continuous function defined on a time interval $t_0 \leqslant t \leqslant t_0 + \delta$, for sufficiently small $\delta$.

Suppose that $u, v : [t_0, t_0 + \delta] \to \mathbb{R}^n$ are two continuous function. Then, form (3.4), (3.5) we estimate,

$$\begin{aligned}
|Tu - Tv|_\infty &= \sup_{t_0 \leqslant t \leqslant t_0+\delta} |Tu(t) - Tv(t)| \\
&= \sup_{t_0 \leqslant t \leqslant t_0+\delta} \left| \int_{t_0}^{t} f(s,u(s)) - f(s,v(s))ds \right| \\
&\leq \sup_{t_0 \leqslant t \leqslant t_0+\delta} \int_{t_0}^{t} |f(s,u(s)) - f(s,v(s))|ds \\
&\leq \sup_{t_0 \leqslant t \leqslant t_0+\delta} C|u(s) - v(s)| \int_{t_0}^{t} ds \\
&\leq C\delta|u - v|_\infty.
\end{aligned}$$

If follow that if $\delta \leq \frac{1}{c}$ then $T$ is contraction on $C([t_0, t_0 + \delta])$. Therefore, there is a unique solution $u : [t_0, t_0 + \delta] \rightarrow \mathbb{R}^n$.

Let $f(x,y)$ be a continuous real-valued function on $[a,b] \times [c,d]$. The Cauchy initial value problem is to find a continuous differentiable function $y$ on $[a,b]$ satisfying the differential equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{3.6}$$

Consider the Banach space $C[a,b]$ of continuous real-valued functions with supremum norm defined by $\|y\| = \sup\{y(x)| : x \in [a,b]\}$.

Integrating (3.6), that yield an integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{3.7}$$

The problem (3.6) is equivalent the problem solving the integral Eq. (3.7).

We define an integral operator $T : C[a,b] \rightarrow C[a,b]$ by

$$Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Therefore, a solution of Cauchy initial value problem (3.6) corresponds with a fixed point of $T$. One may easily check that if $T$ is contraction, then the problem (3.6) has a unique solution.

**Theorem 3.4.** ref. [20] Let $f(x,y)$ be a continuous function of $Dom(f) = [a,b] \times [c,d]$ such that $f$ is Lipschitzian with respect to $y$, i.e., there exists $k > 0$ such that

$$|f(x,u) - f(x,v)| \leq k|u - v| \text{ for all } u,v \in [c,d] \text{ and for } x \in [a,b].$$

Suppose $(x_0, y_0) \in \text{int}(Dom(f))$. Then for sufficiently small $h > 0$, there exists a unique solution of the problem (3.6).

*Proof.* Let $M = \sup\{|f(x,y)| : x,y \in Dom(f)\}$ and choose $h > 0$ such that

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$$C:=\{y\in C[x_0-h,x_0+h]:|y(x)-y_0|\le Mh\}.$$

Then $C$ is a closed subset of the complete metric space $C[x_0-h,x_0+h]$ and hence $C$ is complete. Note $T:C\to C$ is a contraction mapping. Indeed, for $x\in[x_0-h,x_0+h]$ and two continuous functions $y_1,y_2\in C$, we have

$$\begin{aligned}\|Ty_1-Ty_2\| &= \|\textstyle\int_{x_0}^{x} f(x,y_1)-f(x,y_2)dt\|\\ &\le |x-x_0|\sup_{s\in[x_0-h,x_0+h]} k|y_1(s)-y_2(s)|\\ &\le kh\|y_1-y_2\|.\end{aligned}$$

Therefore, $T$ has a unique fixed point implying that the problem (3.6) has a unique fixed point.

### 3.2. Ordinary fuzzy differential equation

Now, we consider the existence of solution for the second order nonlinear boundary value problem:

$$\begin{cases} x''(t)=k(t,x(t),x'(t)), & t\in[0,\Lambda],\ \ \Lambda>0,\\ x(t_1)=x_1, & \\ x(t_2)=x_2, & t_1,t_2\in[0,\Lambda,]\end{cases} \tag{3.8}$$

where $k:[0,\Lambda]\times W(X)\times W(X)\to W(X)$ is a continuous function. This problem is equivalent to the integral equation

$$x(t)=\int_{t_1}^{t_2} G(t,s)k(s,x(s),x'(s))ds+\beta(t),\quad t\in[0,\Lambda], \tag{3.9}$$

where the Green's function $G$ is given by

$$G(t,s)=\begin{cases}\dfrac{(t_2-t)(s-t_1)}{t_2-t_1} & \text{if } t_1\le s\le t\le t_2,\\ \dfrac{(t_2-s)(t-t_1)}{t_2-t_1} & \text{if } t_1\le t\le s\le t_2,\end{cases}$$

and $\beta(t)$ satisfies $\beta''=0,\beta(t_1)=x_1,\beta(t_2)=x_2$. Let us recall some properties of $G(t,s)$, precisely we have

$$\int_{t_1}^{t_2}|G(t,s)|ds\le\frac{(t_2-t_1)^2}{8}$$

and

$$\int_{t_1}^{t_2}|G_t(t,s)|ds\le\frac{(t_2-t_1)}{2}.$$

If necessary, for a more detailed explanation of the background of the problem, the reader can refer to the reference [21, 22]. Here, we will prove our results, by establishing the existence of a common fixed point for pair of integral operators defined as

$$T_i(x)(t) = \int_{t_1}^{t_2} G(t,s)k_i(s,x(s),x'(s))ds + \beta(t), \quad t\in[0,\Lambda], \ \ i\in\{1,2\} \tag{3.10}$$

where $k_1, k_2 \in C([0,\Lambda]\times W(X)\times W(X), W(X)),\ \ x\in C^1([0,\Lambda], W(X)),$ and $\beta\in C([0,\Lambda], W(X))$.

**Theorem 3.5** ref. [6] *Assume that the following conditions are satisfied:*

**i.** $k_1, k_2 : [0,\Lambda]\times W(X)\times W(X)\to W(X)$ *are increasing in its second and third variables,*

**ii.** *there exists* $x_0\in C^1([0,\Lambda], W(X))$ *such that, for all* $t\in[0,\Lambda]$, *we have*

$$x_0(t) \le \int_{t_1}^{t_2} G(t,s)k_1\big(t, x_0(s), x_0'(s)\big)ds + \beta(t),$$

where $t_1, t_2 \in [0,\Lambda]$,

**iii.** *there exist constants* $\gamma, \delta > 0$ *such that, for all* $t\in[0,\Lambda]$, *we have*

$$|k_1(t,x(t),x'(t)) - k_2(t,y(t),y'(t))| \le \gamma|x(t)-y(t)| + \delta|x'(t)-y'(t)|$$

for all comparable $x, y\in C^1([0,\Lambda], W(X))$,

**iv.** *for* $\gamma,\delta>0$ *and* $t_1, t_2\in[0,\Lambda]$ *we have*

$$\gamma\frac{(t_2-t_1)^2}{8} + \delta\frac{(t_2-t_1)}{2} < 1,$$

**v.** *if* $x, y\in C^1([0,\Lambda], W(X))$ *are comparable, then every* $u\in(T_1x)_1$ *and every* $v\in(T_2y)_1$ *are comparable.*

*Then the pair of nonlinear integral equations*

$$x(t) = \int_{t_1}^{t_2} G(t,s)k_i(s,x(s),x'(s))ds + \beta(t) \quad t\in[0,\Lambda], \ \ i\in\{1,2\} \tag{3.11}$$

has a common solution in $C^1([t_1,t_2], W(X))$.

*Proof.* Consider $\mathcal{C} = C^1([t_1,t_2], W(X))$ with the metric

$$D(x,y) = \max_{t_1\le t\le t_2}(\gamma|x(t)-y(t)| + \delta|x'(t)-y'(t)|).$$

The $(\mathcal{C}, D)$ is a complete metric space, which can also be equipped with the partial ordering given by

$$x, y\in\mathcal{C}, \quad \Leftrightarrow x(t)\le y(t) \ \text{ for all } \ t\in[0,\Lambda].$$

In [23], it is proved that $(\mathcal{C}, \preccurlyeq)$ satisfies the following condition:

(r) for every nondecreasing sequence $\{x_n\}$ in $\mathcal{C}$ convergent to some $x \in \mathcal{C}$, we have $x_n \preccurlyeq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Let $T_1, T_2 : \mathcal{C} \to \mathcal{C}$ be two integral operators defined by (3.10); clearly, $T_1, T - 2$ are well defined since $k_1, k_2,$ and $\beta$ are continuous functions. Now, $x^*$ is a solution of (3.9) if and only if $x^*$ is a common fixed point of $T_1$ and $T_2$.

By hypothesis (a), $T_1, T_2$ are increasing and, by hypothesis (b), $x_0 \preccurlyeq T_1(x_0)$. Consequently, in view of condition (r), hypothesis (i)-(iii) of Corollary 2.16 hold true.

Next, for all comparable $x, y \in \mathcal{C}$, From hypothesis (c) we obtain successively

$$\begin{aligned} |T_1(x)(t) - T_2(y)(t)| &\leq \int_{t_1} t_2 |G(t,s)| |k_1(s, x(s), x'(s)) - k_1(s, y(s), y'(s))| ds \\ &\leq D(x,y) \int_{t_1}^{t_2} |G(t,s)| ds \\ &\leq \frac{(t_2 - t_1)^2}{8} D(x,y) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} |(T_1(x))'(t) - (T_2(y))'(t)| &\leq \int_{t_1} t_2 |G_t(t,s)| \left|k_1\left(s, x(s), x'(s)\right) - k_1\left(s, y(s), y'(s)\right)\right| ds \\ &\leq D(x,y) \int_{t_1}^{t_2} |G_t(t,s)| ds \\ &\leq \frac{(t_2 - t_1)}{2} D(x,y). \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we obtain easily

$$D(T_1 x, T_2 y) \leq \left( \gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} \right) D(x,y).$$

Consequently, in view of hypothesis (d), the contractive condition (5) is satisfied with

$$q = \gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} < 1.$$

Therefore, Corollary 2.16 applied to $T_1$ and $T_2$, which have common fixed point $x^* \in \mathcal{C}$, that is, $x^*$ is a common solution of (3.9). □

### 3.3. Second-order differential equation

Now, we consider the boundary value problem for second order differential equation

$$\begin{cases} x''(t) = -f(t, x(t)), & t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{3.14}$$

where $I = [0, 1]$ and $f : I \times \mathbb{R} \to \mathbb{R}$. is a continuous function.

It is known, and easy to check, that the problem (3.14) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t, s) f(s, x(s)) ds, \quad \text{for} \quad t \in I, \tag{3.15}$$

where $G$ is the Green's function define by

$$G(t, s) = \begin{cases} t(1 - s) & \text{if} \quad 0 \le t \le s \le 1 \\ s(1 - t) & \text{if} \quad 0 \le s \le t \le 1. \end{cases}$$

That is, if $x \in C^2(I, \mathbb{R})$, then $x$ is a solution of problem (3.14) iff $x$ is a solution of the integral Eq. (3.15).

Let $X = C(I)$ be the space of all continuous functions defined on $I$. Consider the metric-like $\sigma$ on $X$ define by

$$\sigma(x, y) = \|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty \quad \text{for all} \quad x, y \in X,$$

where $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$ for all $u \in X$.

Note that $\sigma$ is also a partial metric on $X$ and since

$$d_\sigma(x, y) := 2\sigma(x, y) - \sigma(x, x) - \sigma(y, y) = 2\|x - y\|_\infty.$$

By Lemma 2.20, hence $(X, \sigma)$ is complete since the metric space $(X, \| \cdot \|)$ is complete.

**Theorem 3.6.** ref. [12] Suppose the following conditions:

**i.** there exist continuous functions $p : I \to \mathbb{R}^+$ such that

$$|f(s, a) - f(s, b)| \le 8p(s)|a - b|$$

for all $s \in I$ and $a, b \in \mathbb{R}$;

**ii.** there exist continuous functions $q : I \to \mathbb{R}^+$ such that

$$|f(s, a)| \le 8q(s)|a|$$

for all $s \in I$ and $a \in \mathbb{R}$;

**iii.** $\max_{s \in I} p(s) = \alpha\lambda_1 < \frac{1}{49}$, which is $0 \le \alpha < \frac{1}{7}$;

**iv.** $\max_{s \in I} q(s) = \alpha\lambda_2 < \frac{1}{49}$ which is $0 \le \alpha < \frac{1}{7}$.

Then problem (3.14) has a unique solution $u \in X = C(I, \mathbb{R})$.

*Proof.* Define the mapping $T : X \to X$ by

$$Tx(t) = \int_0^1 G(t,s)f(s,x(s))ds,$$

for all $x \in X$ and $t \in T$. Then the problem (3.14) is equivalent to finding a fixed point $u$ of $T$ in $X$. Let $x, y \in X$, we obtain

$$\begin{aligned} |Tx(t) - Ty(t)| &= |\int_0^1 G(t,s)f(s,x(s))ds - \int_0^1 G(t,s)f(s,y(s))ds| \\ &\leq \int_0^1 G(t,s)|f(s,x(s)) - f(s,y(s)|ds \\ &\leq 8\int_0^1 G(t,s)p(s)|x(s) - y(s)|ds \\ &\leq 8\alpha\lambda_1\|x-y\|_\infty \int_0^1 G(t,s)ds \\ &= \alpha\lambda_1\|x-y\|_\infty. \end{aligned}$$

In the above equality, we used that for each $t \in I$, we have $\int_0^1 G(t,s)ds = \frac{t}{2}(1-t)$ and so $\sup_{t\in I}\int_0^1 G(t,s)ds = \frac{1}{8}$. Therefore,

$$\|Tx - Ty\|_\infty \leq \alpha\lambda_1\|x-y\|_\infty. \tag{3.16}$$

Moreover, we have

$$\begin{aligned} Tx(t) = |&\int_0^1 G(t,s)f(s,x(s))ds| \\ &\leq 8\int_0^1 G(t,s)q(s)|x(s)|ds \\ &\leq 8\alpha\lambda_2\|x\|_\infty. \end{aligned}$$

Hence,

$$\|Tx\|_\infty \leq \alpha\lambda_2\|x\|_\infty. \tag{3.17}$$

Similar method, we obtain

$$\|Ty\|_\infty \leq \alpha\lambda_2\|y\|_\infty. \tag{3.18}$$

Let $e^{-\tau} = \lambda_1 + 2\lambda_2 < 1$ where $\tau > 0$. Form (3.16), (3.17) and (3.18), we obtain

$$\begin{aligned} \sigma(Tx,Ty) &= |Tx - Ty|_\infty + |Tx|_\infty + |Ty|_\infty \\ &\leq \alpha\lambda_1|x-y|_\infty + \alpha\lambda_2|x|_\infty + \alpha\lambda_2|y|_\infty \\ &\leq (\lambda_1 + 2\lambda_2)[(\alpha)(|Tx - Ty|_\infty + |Tx|_\infty + |Ty|_\infty)] \\ &= (e^{-\tau})\alpha\sigma(x,y). \end{aligned} \tag{3.19}$$

Let $\beta, \gamma, \eta, \delta > 0$ where $\beta < \frac{1}{7}, \gamma < \frac{1}{7}, \eta < \frac{1}{7}, \delta < \frac{1}{7}$. It following (3.19), we get

$$\sigma(Tx, Ty) \leq (e^{-\tau})\left[\alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \eta\sigma(x,Ty) + \delta\sigma(y,Tx)\right], \tag{3.20}$$

where $\alpha + \beta + \gamma + 2\eta + 2\delta < 1$. Taking the function $F : \mathbb{R}^+ \to \mathbb{R}$ in (3.20), where $F(t) = \ln(t)$, which is $F \in \mathcal{F}$, we get

$$\tau + F(\sigma(Tx, Ty)) \leq F\left(\alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \eta\sigma(x,Ty) + \delta\sigma(y,Tx)\right).$$

Therefore all hypothesis of Theorem (2.25) are satisfied, and so $T$ has a unique fixed point $u \in X$, that is, the problem (3.14) has a unique solution $u \in C^2(I)$. □

### 3.4. Partial differential equation

Consider the Laplace operator is a second order differential operator in the n-dimensional Euclidean space, defined as the divergence $(\nabla\cdot)$ of the gradient $(\nabla f)$. Thus if $f$ is a twice-differentiable real-valued function, then the Laplacian of $f$ is defined by

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f \tag{3.21}$$

where the latter notations derive from formally writing $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$. Equivalently, the Laplacian of $f$ the sum of all the unmixed

$$\Delta f = \sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i^2}. \tag{3.22}$$

As a second-order differential operator, the Laplace operator maps $C^k$ functions to $C^{k-2}$ functions for $k \geq 2$. the expression (3.21)(or equivalently(3.22)) defines an operator $\Delta : C^{(k)}(\mathbb{R}^n) \to C^{(k-2)}(\mathbb{R}^n)$ or more generator $\Delta : C^{(k)}(\Omega) \to C^{(k-2)}(\Omega)$ for any open set $\Omega$ Consider semilinear elliptic equation. Look for a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ that solves

$$-\Delta u = f(u) \quad \text{in} \quad \Omega \tag{3.23}$$

$$u = u_0 \quad \text{on} \quad \partial\Omega \tag{3.24}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a typically nonlinear function. Equivalently look for a fixed point of $Tu := (-\Delta u_0)^{-1}(f(u))$.

**Theorem 3.7.** ref. [5] Let $f \in C(\mathbb{R})$ and $\sup_{x \in \mathbb{R}} |f(x)| = a < \infty$. then (3.23) has a weak solution $u \in H_0^1(\Omega)$, i.e.

$$\int_\Omega \nabla u \cdot \nabla \Phi dx = \int_\Omega f(u)\Phi dx, \qquad \forall \Phi \in C_0^\infty(\Omega).$$

*Proof.* Our strategy is to apply Schauder's Fixed Point Theorem to the map

$$T : L^2(\Omega) \to L^2(\Omega)$$
$$u \mapsto (-\Delta)^{-1}(f(u)),$$

where $T$ is continuous. Lemma (2.9) show that $u \to f(u)$ is continuous form $L^2(\Omega)$ into itself. Corollary (2.10) shows that $(-\Delta)^{-1}$ is continuous form $L^2(\Omega)$ into $H_0^1(\Omega)$, which is continuously embedded in $L^2(\Omega)$. Find a closed, non-empty bounded convex set such that $T : M \to M$. Given $u \in L^2(\Omega)$, $Tu$ satisfies

$$\int_\Omega \nabla Tu \cdot \nabla Tudx = \int_\Omega f(u)Tudx \le a|\Omega| \|Tu\|_{L^2(\Omega)} \tag{3.25}$$

Cauchy-Schwarz. $T$ here fore, using Ponincare's inequality

$$\|Tu\|^2_{L^2(\Omega)} \le C(\Omega)\|Tu\|^2_{L^2(\Omega)} \le a|\Omega| \|Tu\|^2_{L^2(\Omega)}.$$

Thus if we set $R = a|\Omega|C(\Omega)$ and choose $M = \left\{u : \quad \|u\|^2_{L^2(\Omega)} \le R\right\}$. We have established that $T : M \to M$, $T$ is compact. Using Poincare's inequality on the right-hand-side in (3.25), we obtain. $\|\nabla Tu\|^2_{L^2(\Omega)} \le R\|\nabla Tu\|_{L^2(\Omega)}$. Thus $T(M) \subset \left\{u : \quad \|u\|_{H^1(\Omega)} \le R\right\}$, and since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, $T$ is compact. □

### 3.5. A non-homogeneous linear parabolic partial differential equation

We consider the following initial value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t)), & -\infty < x < \infty, 0 < t \le T, \\ u(x,0) = \varphi(x) \ge 0, & -\infty < x < \infty, \end{cases} \tag{3.26}$$

where $H$ is continuous and $\varphi$ assume to be continuously differentiable such that $\varphi$ and $\varphi'$ are bounded.

By a *solution* of the problem (3.26), we mean a function $u \equiv u(x,t)$ defined on $\mathbb{R} \times I$, where $I{:=}[0,T]$, satisfying the following conditions:

**i.** $u, u_t, u_x, u_{xx} \in C(\mathbb{R} \times I)$. $\{C(\mathbb{R} \times I)$ denote the space of all continuous real valued functions$\}$,

**ii.** $u$ and $u_x$ are bounded in $\mathbb{R} \times I$,

**iii.** $u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t))$ for all $(x,t) \in \mathbb{R} \times I$,

**iv.** $u(x,0) = \varphi(x)$ for all $x \in \mathbb{R}$.

It is important to note that the differential equation problem (3.26) is equivalent to the following integral equation problem

$$u(x,t) = \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi d\tau \qquad (3.27)$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

The problem (3.26) admits a solution if and only if the corresponding problem (3.27) has a solution.

Let

$$\Omega := \{u(x,t); u, u_x \in C(\mathbb{R} \times I) \text{ and } \|u\| < \infty\},$$

where

$$\|u\| := \sup_{x \in \mathbb{R},\, t \in I} |u(x,t)| + \sup_{x \in \mathbb{R},\, t \in I} |u_x(x,t)|.$$

Obviously, the function $\omega : \mathbb{R}^+ \times \Omega \times \Omega \to \mathbb{R}_+$ given by

$$\omega_\lambda(u,v) := \frac{1}{\lambda} \|u - v\| = \frac{1}{\lambda} d(u,v)$$

is a metric modular on $\Omega$. Clearly, the set $\Omega_\omega$ is a complete modular metric space independent of generators.

**Theorem 3.8.** ref. [17] Consider the problem (3.26) and assume the following:

**i.** for $c > 0$ with $|s| < c$ and $|p| < c$, the function $F(x,t,s,p)$ is uniformly Hölder continuous in $x$ and $t$ for each compact subset of $\mathbb{R} \times I$,

**ii.** there exists a constant $c_H \leq \left(T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}}\right)^{-1} \leq q$, where $q \in (0,1)$ such that

$$0 \leq \frac{1}{\lambda}\left[H\big(x,t,s_2,p_2\big) - H\big(x,t,s_1,p_1\big)\right]$$

$$\leq c_H \left[\frac{s_2 - s_1 + p_2 - p_1}{\lambda}\right]$$

for all $(s_1,p_1), (s_2,p_2) \in \mathbb{R} \times \mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$,

**iii.** $H$ is bounded for bounded $s$ and $p$.

Then the problem (3.26) admits a solution.

*Proof.* It is well known that $u \in \Omega_\omega$ is a solution (3.26) iff $u \in \Omega_\omega$ is a solution integral Eq. (3.27).

Consider the graph $G$ with $V(G) = D = \Omega_\omega$ and $E(G) = \{(u,v) \in D \times D : u(x,t) \leq v(x,t)$ and $u_x(x,t) \leq v_x(x,t)$ at each $(x,t) \in \mathbb{R} \times I\}$. Clearly $E(G)$ is partial ordered and $(D, E(G))$ satisfy property (A).

Also, define a mapping $\Lambda : \Omega_\omega \to \Omega_\omega$ by

$$(\Lambda u)(x,t) := \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi d\tau$$

for all $(x,t) \in \mathbb{R} \times I$. Then, finding solution of problem (3.27) is equivalent to the ensuring the existence of fixed point of $\Lambda$.

Since $(u,v) \in E(G)$, $(u_x, v_x) \in E(G)$ and hence $(\Lambda u, \Lambda v) \in E(G)$, $(\Lambda u_x, \Lambda v_x) \in E(G)$.

Thus, from the definition of $\Lambda$ and by (ii) we have

$$\begin{aligned} &\frac{1}{\lambda}|(\Lambda v)(x,t) - (\Lambda u)(x,t)| \\ &\leq \frac{1}{\lambda}\int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)|H(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau)) - H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))|d\xi d\tau \\ &\leq \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)c_H\left[\frac{1}{\lambda}|(v(\xi,\tau) - u(\xi,\tau) + v_x(\xi,\tau) - u_x(\xi,\tau))|\right]d\xi d\tau \\ &\leq c_H\omega_\lambda(u,v)T. \end{aligned} \tag{3.28}$$

Similarly, we have

$$\begin{aligned} \frac{1}{\lambda}|(\Lambda v)_x(x,t) - (\Lambda u)_x(x,t)| &\leq c_H\omega_\lambda(u,v)\int_0^t \int_{-\infty}^{\infty} |k_x(x-\xi,t-\tau)|d\xi d\tau \\ &\leq 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}c_H\omega_\lambda(u,v). \end{aligned} \tag{3.29}$$

Therefore, from (3.28) and (3.29) we have

$$\omega_\lambda(\Lambda u, \Lambda v) \leq \left(T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)c_H\omega_\lambda(u,v)$$

i.e.

$$\omega_\lambda(\Lambda u, \Lambda v) \leq q\omega_\lambda(u,v), \quad q \in (0,1)$$

i.e.

$$d(\Lambda u, \Lambda v) \leq e^{-\tau}d(u,v), \tau > 0.$$

Now, by passing to logarithms, we can write this as

$$\ln(d(\Lambda u, \Lambda v)) \leq \ln(e^{-\tau}d(u,v))$$

$$\tau + \ln(d(\Lambda u, \Lambda v)) \leq \ln(d(u,v)).$$

Now, from example 2.22 (i) and taking $T = \Lambda$ and $R = \mathcal{I}$ (Identity map), we deduce that the operator T satisfies all the hypothesis of theorem 2.32.

Therefore, as an application of theorem 2.32 we conclude the existence of $u^* \in \Omega_\omega$ such that $u^* = \Lambda u^*$ and so $u^*$ is a solution of the problem 3.26.

### 3.6. Fractional differential equation

Before we will discuss the source of fractional differential equation.

*Cauchy's formula for repeated integration.* Let $f$ be a continuous function on the real line. Then the $n_{th}$ repeated integral of $f$ based at $a$,

$$f^{(-n)}(x) = \int_a^x \int_a^{\sigma_1} \int_a^{\sigma_2} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n) d\sigma_n \ldots d\sigma_3 d\sigma_2 d\sigma_1$$

is given by single integration

$$f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt.$$

A proof is given by mathematical induction. Since $f$ is continuous, the base case follows from the fundamental theorem of calculus.

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

where

$$f^{-1}(a) = \int_a^a f(t) dt = 0.$$

Now, suppose this is true for n, and let us prove it for $n+1$.

Firstly, using the Leibniz integral rule. Then applying the induction hypothesis

$$\begin{aligned} f^{(-n+1)}(x) &= \int_a^x \int_a^{\sigma_1} \int_a^{\sigma_2} \cdots \int_a^{\sigma_n} f(\sigma_{n+1}) d\sigma_n \ldots d\sigma_3 d\sigma_2 d\sigma_1 \\ &= \int_a^x \frac{1}{(n-1)!} \int_a^{\sigma_1} (\sigma_1 - t)^{n-1} f(t) dt d\sigma_1 \\ &= \int_a^x \frac{d}{d\sigma_1} \left[ \frac{1}{n!} \int_a^{\sigma_1} (\sigma_1 - t)^n f(t) dt \right] d\sigma_1 \\ &= \frac{1}{n!} \int_a^x (x-t)^n f(t) dt. \end{aligned}$$

This completes the proof. In fractional calculus, this formula can be used to construct a notion of differintegral, allowing one to differentiate or integrate a fractional number of time.

Integrating a fractional number of time with this formula is straightforward, one can use fractional $n$ by interpreting $(n-1)!$ as $\Gamma(n)$, that is the Riemann-Liouville integral which is defined by

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}dt.$$

This also makes sense if $a = -\infty$, with suitable restriction on $f$. The fundamental relation hold

$$\frac{d}{dx}I^{\alpha+1}f(x) = I^{\alpha}f(x)$$
$$I^{\alpha}\left(I^{\beta}f\right) = I^{\alpha+\beta}f(x)$$

the latter of which is semigroup properties. These properties make possible not only the definition of fractional differentiation by taking enough derivative of $I^{\alpha}f$. One can define fractional-order derivative of as well by

$$\frac{d^{\alpha}}{dx^{\alpha}}f = \frac{d^{[\alpha]}}{dx^{[\alpha]}}I^{[\alpha]-\alpha}f$$

where $[\cdot]$ denote the ceilling function. One also obtains a differintegral interpolation between differential and integration by defining

$$D_x^{\alpha}f(x) = \begin{cases} \frac{d^{[\alpha]}}{dx^{[\alpha]}}I^{[\alpha]-\alpha}f(x) & \text{if } \alpha > 0 \\ f(x) & \text{if } \alpha = 0 \\ I^{-\alpha}f(x) & \text{if } \alpha < 0. \end{cases}$$

An alternative fractional derivative was introduced by Caputo in 1967, and produce a derivative that has different properties it produces zero from constant function and more importantly the initial value terms of the Laplace Transform are expressed by means of the value of that function and of its derivative of integer order rather than the derivative of fractional order as in the Riemann-Liouville derivative. The Caputo fractional derivative with base point $x$ is then

$$^{c}D_x^{\alpha}f(x) = I^{[\alpha]-\alpha}\frac{d^{[\alpha]}}{dx^{[\alpha]}}f(x).$$

**Lemma 3.9.** ref. [24] Let $u : [0,\infty] \to X$ be continuous function such that $u \in C([0,\tau],X)$ for all $\tau > 0$. Then $u$ is a global solution of

$$^{c}D_t^{\alpha}u(t) = Bu(t); t > 0 \tag{3.30}$$

$$u(0) = u_0 \in X \tag{3.31}$$

if and only if $u$ the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s)ds, \, t \geq 0.$$

*Proof.* ($\Rightarrow$) Let $\tau > 0$. Since $u$ is a global solution of (3.30), then $u \in C([0,\tau], X)$, ${}^cD_t^\alpha u \in C([0,\tau], X)$ andt

$${}^cD_t^\alpha u(t) = Bu(t), \quad t \in (0,\tau].$$

Thus, by applying $I_t^\alpha$ in both sides of the equality (since ${}^cD_t^\alpha u \in L^1(0,\tau; X)$) we obtain

$$u(t) = u(0) + I_t^\alpha Bu(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s)ds, \, t \geq 0.$$

Since $\tau > 0$ was an arbitrary choice, $u$ satisfies the integral equation for all $t \geq 0$, as we wish.

($\Leftarrow$) On the other hand, choose $\tau > 0$ (but arbitrary). By hypothesis, $u \in C([0,\tau], X)$, and satisfies the integral equation,

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s)ds, \quad t \in [0,\tau].$$

Observing also $u(0) = u_0$ and rewriting the equality above, we obtain

$$u(t) = u(0) + I_t^\alpha Bu(s), \quad t \in [0,\tau].$$

Since $Bu(s) \in C([0,\tau], X)$, we conclude, by ${}^cD_t^\alpha I_t^\alpha f(t) = f(t)$ of the fractional integral and derivative property that we can apply ${}^cD_t^\alpha$ in both sides of the integral equation, obtaining

$${}^cD_t^\alpha u(t) = Bu(t), \quad t \in [0,\tau]$$

what lead us to verify that ${}^cD_t^\alpha u \in C([0,\tau], X)$. Since $\tau > 0$ was an arbitrary choice, we conclude that the function $u$ is a global solution of (3.30). □

**Theorem 3.10.** ref. [24] Let $\alpha \in (0,1)$, $B \in L(X)$ and $u_0 \in X$ then the problem (3.30).

have a unique global solution.

*Proof.* Choose $\tau > 0$. then consider $K_\tau = u \in C([0,\tau], X); u(0) = u_0$ and operator.

$T : K_\tau \to K_\tau$ given by

$$T(u(t)) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t Bu(t)dt.$$

We will show that a power (with respect to be composition) of this operator is a contraction and therefore by Banach's Fixed Point Theorem, $T$ have a unique fixed point in $K_\tau$ to this end, observe that for any $u, v \in K_\tau$

$$\begin{aligned}
\|T(u(t)) - T(v(t))\| &= \|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} B(u(s)-v(s))ds\| \\
&\leq \|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} \|B\|_{L(X)}\|u(s)-v(s))\|ds \\
&\leq \frac{\|B\|_{L(X)}}{\Gamma(\alpha)}\|u(s)-v(s))\|\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{t^\alpha\|B\|_{L(X)}}{\alpha\Gamma(\alpha)}\|u(s)-v(s))\| \\
&\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\|u(s)-v(s))\| \\
&\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\sup_{0\leqslant s\leqslant\tau}\|u(s)-v(s))\|.
\end{aligned}$$

By iterating this relation, we find that

$$\begin{aligned}
\|T^2(u(t)) - T^2(v(t))\| &\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\sup_{0\leqslant s\leqslant\tau}\|Tu(s)-Tv(s))\| \\
&\leq \frac{t^2\alpha}{\Gamma^2(\alpha+1)}\|B\|^2_{L(X)}\sup_{0\leqslant s\leqslant\tau}\|u(s)-v(s))\| \\
\|T^3(u(t)) - T^3(v(t))\| &\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\sup_{0\leqslant s\leqslant\tau}\|T^2u(s)-T^2v(s))\| \\
&\leq \frac{t^3\alpha}{\Gamma^3(\alpha+1)}\|B\|^3_{L(X)}\sup_{0\leqslant s\leqslant\tau}\|u(s)-v(s))\| \\
&\leq \cdots \\
\|T^n(u(t)) - T^n(v(t))\| &\leq \frac{t^n\alpha}{\Gamma^n(\alpha+1)}\|B\|^n_{L(X)}\sup_{0\leqslant s\leqslant\tau}\|u(s)-v(s))\|
\end{aligned}$$

and for an sufficiently large $n$,the constant in question is less than 1, i.e., there exists a fixed point $u \in K_\tau$. Observe now that $\tau > 0$ was an arbitrary choice, so we conclude that the fixed point $u \in C([0,\tau], X)$ for all $\tau > 0$ and Lemma (3.9), we obtain the existence and uniqueness of a global solution to the problem (3.30). □

**Corollary 3.11.** ref. [24] Consider the same hypothesis of theorem (3.10).

**i.** Let $\{U_n(t)\}|_{n=0}^\infty$ be a sequence of continuous functions $U_n : [0,\infty) \to X$ given by $U_0(t) = u_0$, $U_n = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} BU_{n-1}(s)ds$, $n \in \{1, 2, \ldots\}$.

Then there exists a continuous function $U : [0,\infty) \to X$, such that for any $\tau > 0$, we conclude that $U_n \to U$ in $C([0,\tau], X)$. Moreover, $U(t)$ is the unique global solution of (3.30).

**ii.** It holds that

$$U(t) = \sum_{k=0}^\infty \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)}.$$

*Proof.* (i) It follows directly from proof of Theorem (3.10).

(ii) It is trivial that $U_0(t) = u_0$. So we compute, using the gamma function properties, that

$$U_1(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B u_0(s) ds = u_0 + \frac{t^\alpha B u_0}{\alpha\Gamma(\alpha)} = u_0 + \frac{t^\alpha B u_0}{\Gamma(\alpha+1)}.$$

By a simple induction process, we conclude that

$$U_n(t) = \sum_{k=0}^{n} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)}$$

and therefore

$$U(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)} := E_\alpha(t^\alpha B) u_0.$$

□

From the above works, we can see a fact, although the fractional boundary value problems have been studied, to the best of our knowledge, there have been a few works using the lower and upper solution method. However, only positive solution are useful for many application, motivated by the above works, we study the existence and uniqueness of positive solution of the following integral boundary value problem.

$$D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \le 2 \tag{3.32}$$

$$u(0) = 0, \quad u(1) = \int_0^1 u(s) ds, \tag{3.33}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function and $D_{0+}^\alpha$ is the standard Riemann-Liouville fractional derivative.

We need the following lemmas that will be used to prove our main results.

**Lemma 3.12.** ref. [25] Let $\alpha > 0$ and $u \in C(0,1) \cap L(0,1)$. Then fractional differential equation

$$D_{0+}^\alpha u(t) = 0$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{3.34}$$

$C_i \in \mathbb{R}, \ \ i = 1, 2, \cdots, N, \ \ N = [\alpha] + 1$ as unique solution.

**Lemma 3.13.** ref. [25] Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2} - \cdots - C_N t^{\alpha-N} \tag{3.35}$$

for some $C_i \in \mathbb{R}, i = 1, 2, \cdots, N, \quad N = [\alpha] + 1$.

In the following, we present the Green function of fractional differential equation with integral boundary value condition.

**Theorem 3.14.** ref. [26] Let $1 < \alpha < 2$, Assume $y(t) \in C[0, 1]$, then the following equation

$$D^{\alpha}_{0+}u(t) + y(t) = 0, \quad 0 < t < 1 \tag{3.36}$$

$$u(0) = 0, \quad u(1) = \int_0^1 u(s)ds, \tag{3.37}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds \tag{3.38}$$

where

$$G(t, s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1}(\alpha - 1 + s) - [t - s]^{\alpha-1}(\alpha - 1)}{(\alpha - 1)\Gamma(\alpha)} & \text{if } 0 \le s \le t \le 1 \\ \dfrac{[t(1-s)]^{\alpha-1}(\alpha - 1 + s)}{(\alpha - 1)\Gamma(\alpha)} & \text{if } 0 \le t \le s \le 1. \end{cases}$$

*Proof.* We may apply Lemma (3.13) to reduce Eq. (3.36) to an equivalent integral equation

$$u(t) = -I^{\alpha}_{0+}y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$$

for some $C_1, C_2 \in \mathbb{R}$. Therefore, the general solution of (3.36) is

$$u(t) = -\int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \tag{3.39}$$

By $u(0) = 0$, we can get $C_2 = 0$. In addition, $u(1) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + C_1$, it follows

$$C_1 = \int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \int_0^1 u(s)ds. \tag{3.40}$$

Take (3.40) into (3.39), we have

$$u(t) = -\int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + t^{\alpha-1}\int_0^1 u(s)ds. \tag{3.41}$$

Let $\int_0^1 u(s)ds = A$,by (3.41), we can get

$$\begin{aligned}\int_0^1 u(t)dt &= -\int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)dsdt + \int_0^1 t^{\alpha-1}\int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)dsdt + A\int_0^1 t^{\alpha-1}dt \\ &= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}y(s)ds + \frac{A}{\alpha} \\ &= \int_0^1 \frac{s(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}y(s)ds + \frac{A}{\alpha}.\end{aligned}$$

So,

$$A = \frac{\alpha}{\alpha-1}\int_0^1 \frac{s(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}y(s)ds = \int_0^1 \frac{s(1-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)}y(s)ds.$$

Combine with (3.41), we have

$$\begin{aligned}u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + t^{\alpha-1}\int_0^1 \frac{s(1-s)^{\alpha-1}}{(\alpha-1)\Gamma()\alpha}y(s)ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \int_0^1 \frac{\left[t(1-s)^{\alpha-1}(\alpha-1+s)\right]}{(\alpha-1)\Gamma(\alpha)}y(s)ds \\ &= \int_0^1 \frac{\left[t(1-s)^{\alpha-1}(\alpha-1+s) - (t-s)^{\alpha-1}(\alpha-1)\right]}{(\alpha-1)\Gamma(\alpha)}y(s)ds + \int_t^1 \frac{[t(1-s)^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds.\end{aligned}$$

This complete the proof.

**Remark 3.15.** Obviously, the Green function $G(t,s)$ satisfies the following properties:

**i.** $G(t,s) > 0, \quad t, s \in (0,1);$

**ii.** $G(t,s) \leq \frac{2}{(\alpha-1)\Gamma(\alpha)}; \quad 0 \leq t, s \leq 1.$

**Theorem 3.16.** ref. [26] Assume that function $f$ satisfies

$$|f(t,u) - f(t,v)| \leq a(t)|u-v| \tag{3.42}$$

where $t \in [0,1], \quad u, v \in [0,\infty), \quad a:[0,1] \to [0,\infty)$ is a continuous function. If

$$\int_0^1 s^{\alpha-1}(\alpha-1+s)a(s)ds < (\alpha-1)\Gamma(\alpha) \tag{3.43}$$

then the Eq. (3.32) has a unique positive solution.

*Proof.* If $T^n$ is a contraction operator for $n$ sufficiently large, then the Eq. (3.32) has a unique positive solution.

In fact, by the definition of Green function $G(t,s)$, for $u, v \in P$, we have the estimate

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \int_0^1 G(t,s)|f(s,u(s)) - f(s,v(s))|ds \\
&\leq \int_0^1 G(t,s)a(s)|u(s) - v(s)|ds \\
&\leq \int_0^1 \frac{[t(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)} a(s)\|u-v\|ds \\
&\leq \frac{\|u-v\|t^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha-1+s)a(s)ds.
\end{aligned}$$

Denote $K = \int_0^1 (1-s)^{\alpha-1}(\alpha-1+s)a(s)ds$, then

$$|Tu(t) - Tv(t)| \leq \frac{Kt^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \|u-v\|.$$

Similarly,

$$\begin{aligned}
|T^2u(t) - T^2v(t)| &= \int_0^1 G(t,s)|f(s,Tu(s)) - f(s,Tv(s))|ds \\
&\leq \int_0^1 G(t,s)a(s)|Tu(s) - Tv(s)|ds \\
&\leq \int_0^1 G(t,s)a(s) \frac{Ks^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \|u-v\|ds \\
&\leq \int_0^1 \frac{K[t(1-s)]^{\alpha-1}(\alpha-1+s)}{(\alpha-1)^2\Gamma^2(\alpha)} a(s)s^{\alpha-1}\|u-v\|ds \\
&\leq \frac{K\|u-v\|t^{\alpha-1}}{(\alpha-1)^2\Gamma^2(\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}(\alpha-1+s)a(s)ds \\
&= \frac{KHt^{\alpha-1}}{(\alpha-1)^2\Gamma^2(\alpha)} \|u-v\|
\end{aligned}$$

where $H = \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}(\alpha-1+s)a(s)ds$. By mathematical induction, it follows

$$|T^n u(t) - T^n v(t)| \leq \frac{KH^{n-1}t^{\alpha-1}}{(\alpha-1)^n\Gamma^n(\alpha)} \|u-v\|$$

by (3.43), for $n$ large enough, we have

$$\frac{KH^{n-1}t^{\alpha-1}}{(\alpha-1)^n\Gamma^n(\alpha)} = \frac{K}{(\alpha-1)\Gamma(\alpha)} \left(\frac{H}{(\alpha-1)\Gamma(\alpha)}\right)^{n-1} < 1.$$

Hence, it holds

$$\|T^n u - T^n v\| < \|u - v\|,$$

which implies $T^n$ is a contraction operator for $n$ sufficiently large, then the Eq. (3.32) has a unique positive solution. □

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