12

Frictional Property of Flexible Element

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1. Introduction

In the calculation of frictional force of a flexible element such as a belt, rope or cable wrapped around the cylinder, the famous Euler's belt formula (Hashimoto, 2006) or simply known as the belt friction equation (Joseph F. Shelley, 1990) is used. The formula is useful for designing a belt drive or band brake (J. A. Williams, 1994). On the other hand, a belt or rope is conveniently used to tighten a luggage to a carrier or lift up the luggage from the carrier. In that case, for the sake of adjusting the belt length and keeping an appropriate tension during transportation, various kinds of belt buckles are used. These belt buckles have been devised empirically and there was no theory about why it can fix the belt. The first purpose of this chapter is to present the theory of belt buckle clearly by considering the self-locking mechanism generated by wrapping the belt on the belt. Making use of the belt tension for a locking mechanism, a belt buckle with no locking mechanism can be made. The principle and some basic property of this new belt buckle are also shown.

The self-locking of belt may occur even in the case where a belt is wrapped on an axis two or more times. The second purpose of this chapter is to present the frictional property of belt wrapped on an axis two and three times through deriving the formulas corresponding to an each condition. Making use of this self-locking property of belt, a belt-type one-way clutch can be made (Imado, 2010). The principle and fundamental property of this new clutch are described.

As the last part of this chapter, the frictional property of flexible element wrapped on a hard body with any contour is discussed. The frictional force can be calculated by the curvilinear integral of the curvature with respect to line element along the contact curve.

2. Theory of belt buckle

Notation

C	Magnification factor of belt tension
F	Frictional force, N
$F_{ij} = F_{ji}$	Frictional force between point P_i and P_j, N
L	Distance between two cylinder centers, m
N	Normal force of belt to surface, N
$N_{ij} = N_{ji}$	Normal force of belt between point P_i and P_j, N
P_i	Boundary of contact angle
R	Radius of main cylinder, m

T_i	Tension of belt in i'th interval, N
r	Radius of accompanied cylinder, m
μ	Coefficient of friction for belt-cylinder contact
μ_b	Coefficient of friction for belt-belt contact
θ	Angle
θ_i	Angle of point P_i
$\theta_{ij} = \theta_{ji}$	Contact angle between P_i and P_j

2.1 Friction of belt in belt buckle

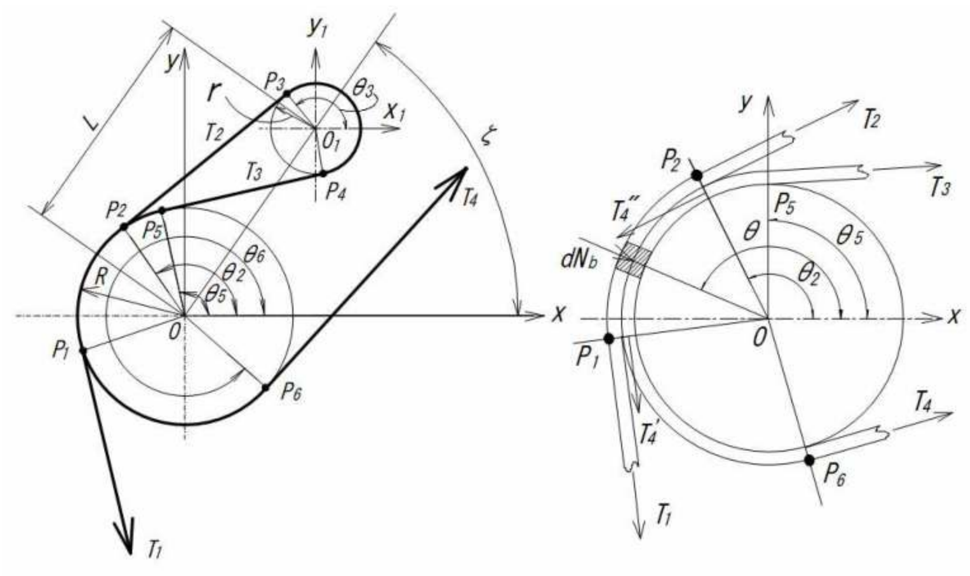
Figure 1 (a) shows a cross sectional view of a belt buckle and a belt wrapped around the two cylindrical surfaces. T_1 and T_4 ($T_1 > T_4$) are tensions of the belt at both ends. There is a double-layered part where the belt is wrapped over the belt. Figure 1 (b) shows the enlarged view around the main axis. For simplicity, the thickness of the belt was neglected. According to the theory of belt friction, following equations are known for belt tensions of T_1, T_2 and T_3 (Joseph F. Shelley, 1990).

$$T_1 = e^{\mu_b \theta_{12}} T_2, \quad T_2 = e^{\mu \theta_{34}} T_3 \tag{1}$$

T_4' and T_4'' are of inner belt tension at P_1 and P_2 respectively. The normal force to a small element of the inner belt at angle θ is denoted as dN_b, which can be written as

$$dN_b = e^{\mu_b (\theta - \theta_2)} T_2 d\theta \tag{2}$$

Making use of T_4' and T_4'', the normal forces of inner belt for an each section are expressed as



(a) Belt buckle (b) Enlarged view

Fig. 1. Mechanical model of belt buckle and enlarged veiw around main axis

$$\left.\begin{aligned} dN_{25} &= e^{\mu(\theta_2-\theta)}T_4''d\theta \\ dN_{12} &= e^{\mu(\theta_1-\theta)}T_4'd\theta \\ dN_{16} &= e^{\mu(\theta_6-\theta)}T_4\ d\theta \end{aligned}\right\} \tag{3}$$

The frictional force between P_1 and P_6 is

$$F_{16} = \int_{\theta_1}^{\theta_6} \mu dN_{16} = (e^{\mu\theta_{16}} - 1)T_4 \tag{4}$$

The inner belt tension T_4' is the sum of the frictional force F_{16} and the belt tension T_4.

$$T_4' = T_4 + F_{16} = e^{\mu\theta_{16}}T_4 \tag{5}$$

The frictional force F_{12} acting on the inner belt is composed of two forces denoted as F_{12in} and F_{12out}. The frictional force F_{12in} is acting on the cylindrical surface, which is generated by the normal forces dN_b and dN_{12}. The normal force dN_b is exerted from the outer belt. The other normal force dN_{12} is generated by the inner belt tension. So, F_{12in} is given by

$$F_{12in} = \int_{\theta_2}^{\theta_1} \mu\, dN_b + \int_{\theta_2}^{\theta_1} \mu\, dN_{12} = (e^{\mu_b\theta_{12}} - 1)\frac{\mu T_2}{\mu_b} + (e^{\mu\theta_{12}} - 1)T_4' \tag{6}$$

Making use of Eq. (2), the frictional force F_{12out} acting on the belt-belt boundary can be written as

$$F_{12out} = \int_{\theta_2}^{\theta_1} \mu_b\, dN_b = (e^{\mu_b\theta_{12}} - 1)T_2 \tag{7}$$

The frictional force F_{12} is the sum of Eqs. (6) and (7).

$$F_{12} = (e^{\mu_b\theta_{12}} - 1)\left(1 + \frac{\mu}{\mu_b}\right)T_2 + (e^{\mu\theta_{12}} - 1)T_4' \tag{8}$$

As the belt tension T_4'' is the sum of F_{12} and T_4', making use of Eq. (5) and (8), T_4'' can be written as

$$T_4'' = F_{12} + T_4' = e^{\mu\theta_{26}}T_4 + (e^{\mu_b\theta_{12}} - 1)\left(1 + \frac{\mu}{\mu_b}\right)T_2 \tag{9}$$

Making use of Eq. (3), the frictional force F_{25} can be written as

$$F_{25} = \int_{\theta_5}^{\theta_2} \mu\, dN_{25} = (e^{\mu\theta_{25}} - 1)T_4'' \tag{10}$$

As the belt tension T_3 is the sum of F_{25} and T_4'', making use of Eqs. (9) and (10), T_3 can be expressed as

$$T_3 = F_{25} + T_4'' = e^{\mu\theta_{25}}\left\{e^{\mu\theta_{26}}T_4 + (e^{\mu_b\theta_{12}} - 1)\left(1 + \frac{\mu}{\mu_b}\right)T_2\right\} \tag{11}$$

Substituting Eq. (1) into Eq. (11) to eliminate T_2 gives

$$T_3 = \frac{e^{\mu\theta_{56}}}{1 - e^{\mu(\theta_{34}+\theta_{25})}(e^{\mu_b\theta_{12}} - 1)(1 + \mu/\mu_b)} T_4 \tag{12}$$

Substituting Eq. (1) into Eq. (12) to get the relation between T_1 and T_4 gives

$$T_1 = \frac{e^{\mu_b\theta_{12}} e^{\mu(\theta_{34}+\theta_{56})}}{1 - e^{\mu(\theta_{34}+\theta_{25})}(e^{\mu_b\theta_{12}} - 1)(1 + \mu/\mu_b)} T_4 \tag{13}$$

In the same manner from Eq. (1) to Eq. (13), in the case of $T_1 < T_4$, corresponding relation of Eq. (13) yields as

$$T_4 = \left\{ e^{\mu(\theta_{34}+\theta_{56})} e^{\mu_b\theta_{12}} + e^{\mu\theta_{16}}(e^{\mu_b\theta_{12}} - 1)\left(1 + \frac{\mu}{\mu_b}\right) \right\} T_1 \tag{14}$$

2.2 Property of formulas of belt buckle

The validity of Eqs. (13) and (14) might be checked by supposing an extreme case of either μ=0 or μ_b=0. Substituting μ=0 into Eq. (13) gives

$$T_1 = \frac{e^{\mu_b\theta_{12}}}{2 - e^{\mu_b\theta_{12}}} T_4 \tag{15}$$

Next, substituting μ_b=0 into Eq. (13) gives

$$T_1 = \frac{e^{\mu(\theta_{34}+\theta_{56})}}{1 - \theta_{12}\,\mu\, e^{\mu(\theta_{34}+\theta_{25})}} T_4 = C\, e^{\mu(\theta_{34}+\theta_{56})} T_4 \tag{16}$$

Substituting μ_b=0 into Eq. (15) or substituting μ=0 into Eq. (16) gives T_1=T_4. Substituting $\theta_{12} = 0$ into Eq. (13) to remove the double-layered segment on the ratio of belt tension yields the conventional equation of belt friction.

$$T_1 = e^{\mu(\theta_{34}+\theta_{56})} T_4 \tag{17}$$

Equation (17) is also obtained by substituting $\theta_{12} = 0$ into Eq. (16). This means that the ratio of belt tension is magnified by the factor C

$$C = \frac{1}{1 - \theta_{12}\,\mu\, e^{\mu(\theta_{34}+\theta_{25})}} \tag{18}$$

due to the double-layered segment even in the case of μ_b=0. As far as these inspections are concerned, there is no contradiction in Eq. (13). As Eqs. (13), (15) and (16) are of fractions, the factor of T_4 might become infinity meaning T_4/T_1=0. This fact virtually implies the occurrence of self-locking. Figure 2 shows the relation of μ_b and θ_{12} satisfying $e^{\mu_b\theta_{12}} = 2$ in Eq. (15). Self-locking occurs in the region above this curve where $e^{\mu_b\theta_{12}} > 2$. On the other hand, in the region below this curve, self-locking does not occur. In the case of μ=0, the equilibrium of moment of belt tension about O in Fig. 1 gives

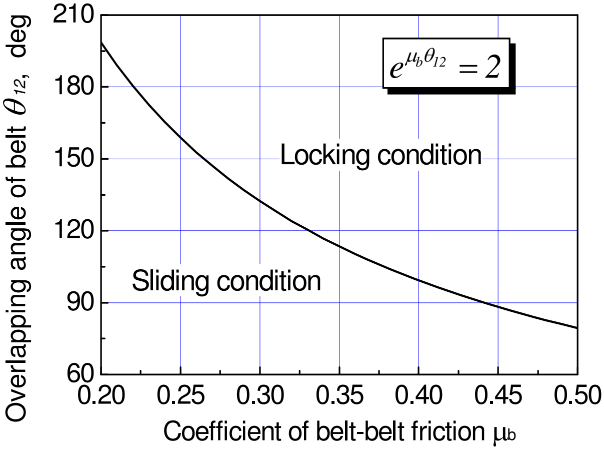


Fig. 2. Boundary curve between self-locking condition and sliding condition

$$T_4 = 2T_2 - T_1 \tag{19}$$

In the locking state with μ=0, T_4=0 so that T_1=2T_2=2T_3. It means that belt tension T_1 is halved to T_2 by the belt-belt friction.

As the angle of double-layered segment θ_{12} is determined by the geometry of the buckle, some calculations were carried out to know the properties of Eq. (13) and Eq. (15) providing r/L=R/L=1/4. The direction of belt tension T_1 and T_4 were assumed to be the same direction for simplicity. Results are shown in Figs. 3 and 4. Figure 3 corresponds to the Eq. (15) where

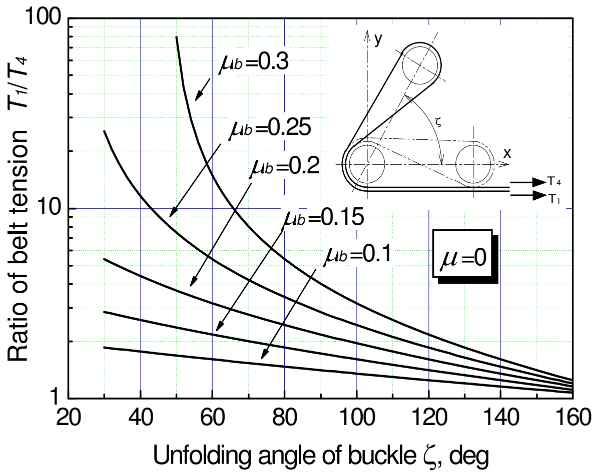


Fig. 3. Change of belt tension ratio with unfolding angle ζ in the case of μ=0. Belt tension ratio increases greatly with an increment of the coefficient of friction μ_b especially in the vicinity of locking condition. It is very sensitive to angle ζ.

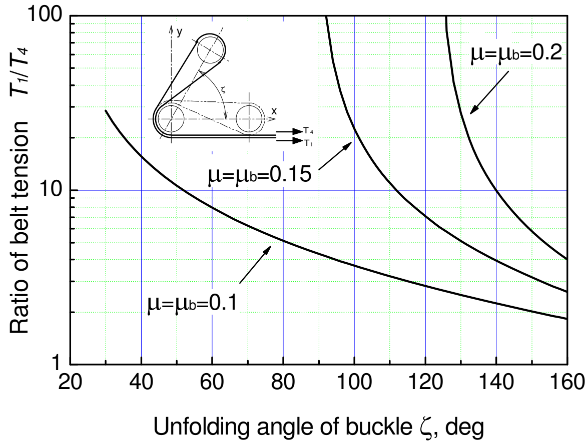


Fig. 4. Change of belt tension ratio with unfolding angle ζ in the case of $\mu=\mu_b$. Belt tension ratio increases greatly with an increment of the coefficient of friction.

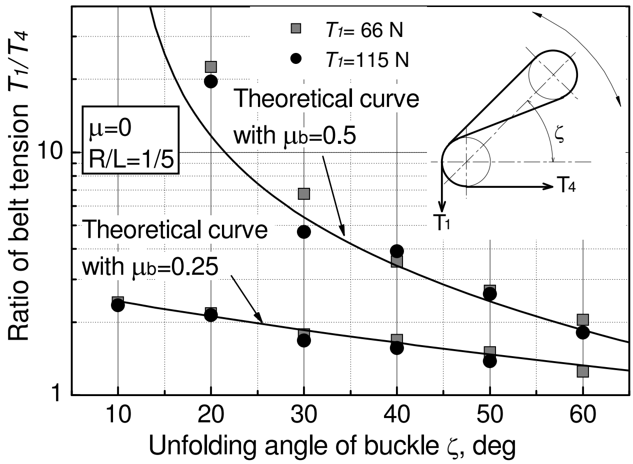


Fig. 5. Change of belt tension ratio with unfolding angle ζ in the case of $\mu=0$. The ratio of belt tension changes according to Eqs. (13) or (15).

the coefficient of friction is $\mu=0$. The ratio of belt tension increases with an increment of the coefficient of friction μ_b. It increases greatly when it approaches the locking condition. Figure 4 shows some results obtained by Eq. (13) providing $\mu=\mu_b$. The ratio of belt tension becomes far bigger than the that of Fig. 3.

Some experiments were carried out to verify the validity of Eq. (15) by wrapping a belt around the outer rings of rolling bearings to realize the condition of $\mu=0$. Belt tension T_1 was applied by the weight. Belt tension T_4 was measured by the force gauge. Figure 5 shows the results. Experimental data are almost on the theoretical curves. As predicted by the Eq. (15), self-locking was confirmed for the belt with $\mu_b=0.5$ in the region of $\zeta<10^\circ$ where $e^{\mu_b \theta_{12}} > 2$.

2.3 Calculation of arm torque

Figure 6 (a) shows the mechanical model of belt buckle (Imado, 2008 a). Figure 6 (b) shows a three-dimensional model of the buckle. The arm of the buckle rotates around the point O_2. The angle of arm is denoted by α. The intersection angle of the line O-O_2 and O_2-O_1 is denoted by β. From geometrical consideration, the angle β is given by

$$\beta = \pi + \alpha - \phi \tag{20}$$

Applying the cosine theorem to the triangle OO_1O_2, length L is given by

$$L = L_2\sqrt{1+\kappa^2+2\kappa\cos(\alpha-\phi)}\ ,\ \text{ where } \kappa = L_1 / L_2 \tag{21}$$

The symbol ζ denotes the angle of line O-O_1.

$$\zeta = \phi - \beta_1 \tag{22}$$

Applying the cosine theorem and sine theorem to the triangle OO_1O_2 gives

$$\cos\beta_1 = \frac{L^2+L_1^2-L_2^2}{2LL_1},\quad \sin\beta_1 = \frac{L_2}{L}\sin(\phi-\alpha) \tag{23}$$

Substituting Eq. (21) into Eq. (23) and substituting Eq. (23) into Eq. (22) gives

$$\zeta = \phi - \tan^{-1}\left\{\frac{1}{\kappa+\cos(\alpha-\phi)}\sin(\phi-\alpha)\right\} \tag{24}$$

ζ, the angle of center line O-O_1, can be calculated from the arm angle α by Eq. (24). Note the angle ζ is equal to α when L_1 becomes 0.
The moment of the arm about point O_2 due to belt tensions T_2 and T_3 is expressed by

$$M = c_2T_2 + c_3T_3 \tag{25}$$

where c_2 and c_3 are geometrical variables that can be calculated from the position of contact boundaries P_2, P_3, P_4 and P_5. Dividing the arm torque M with RT_1, torque due to belt tension T_1 about point O, gives non-dimensional moment N.

$$N = \frac{1}{R}\left(c_2\frac{T_2}{T_1} + c_3\frac{T_3}{T_1}\right) \tag{26}$$

Making use of Eq. (1), the fractions of belt tension in Eq. (26) can be calculated by

$$\frac{T_2}{T_1} = \frac{1}{e^{\mu_b\theta_{12}}},\quad \frac{T_3}{T_1} = \frac{1}{e^{\mu_b\theta_{12}}e^{\mu\theta_{34}}} \tag{27}$$

Figure 7 shows some examples of non-dimensional torque N. For simplicity, the coefficients of friction were taken to be $\mu=\mu_b$. The non-dimensional torque N decreases to be negative value with decrement of arm angle α. It means an occurrence of directional change in arm torque. This negative torque acts so as to hold the arm angle in a locking state without any locking mechanism. The angle where arm torque N becomes 0 is denoted by α_C. It depends on the geometry of buckle and the coefficients of friction μ and μ_b. Making use of Eqs. (13) and (24), the fraction of belt tension can be calculated. Figure 8 shows some results. The fraction of belt tension, T_4/T_1, decreases with arm angle α. It becomes 0 at $\alpha = \alpha_L$. According to Eq. (13), the fraction of belt tension T_4/T_1 becomes negative when arm angle α becomes less than α_L, $\alpha < \alpha_L$. The physical meaning of negative value in the fraction of belt tension is that the belt tension T_4 should be compressive so as to satisfy the equilibrium condition of the force. But a belt cannot bear compressive force so that negative value in the fraction of belt tension is actually unrealistic. It means the belt was locked with the buckle. The angle α_L becomes larger with an increment of the coefficients of friction. As the coefficient of friction is generally greater than 0.15, the locking condition is easily satisfied. Once the locking condition is satisfied, the belt is dragged into the buckle with a decrement of arm angle α. Then the belt tension becomes greater.

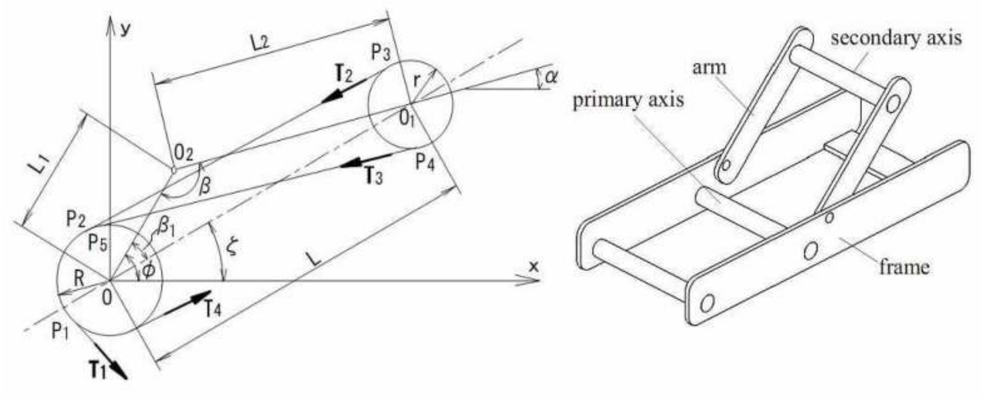


Fig. 6. Mechanical model of belt buckle to calculate arm torque and 3D model

3. Theory of belt friction in over-wrapped condition

3.1 Friction of belt wrapped two times around an axis

Figure 9 shows a mechanical model (Imado, 2008 b). The point P_i (i=1, 2, 3) is a boundary of contact and T_i (i=1, 2, 3, 4) is tension of the belt. Symbol θ_i denotes the angle of point P_i. The belt is over-wrapped around the belt in the range from P_1 to P_2 denoted by θ_1. The axis x is taken so as to pass through the point P_2, which is an end of the belt. T_1 is bigger than T_4. T_4 is an imaginary belt tension. There is no contact from P_2 to P_3 due to the thickness of the belt-end. According to the theory of belt friction (Joseph F. Shelley, 1990), analysis starts with the conventional equation.

$$T_1 = e^{\mu_b \theta_1} T_2 = e^{\mu_b \theta_1} T_3 \tag{28}$$

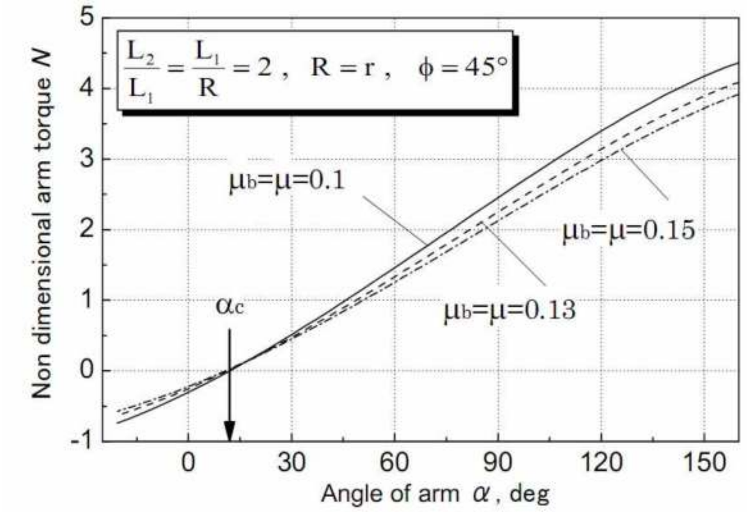


Fig. 7. Non-dimensional arm toruque N decreases with arm angle α

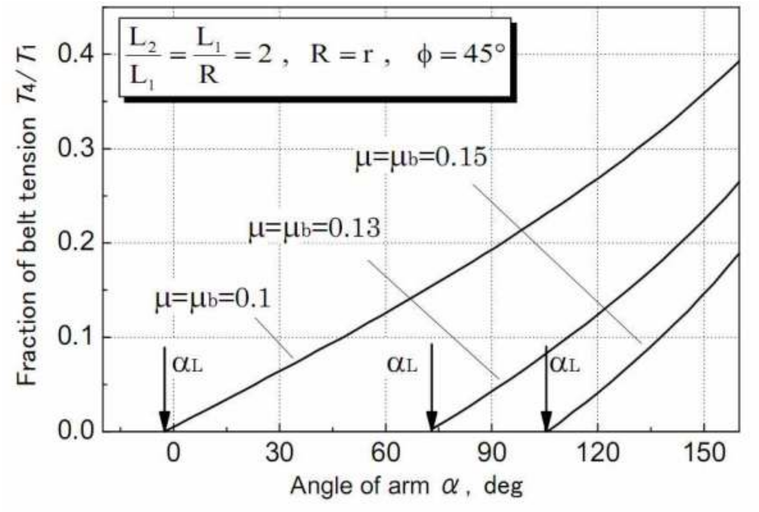


Fig. 8. Fraction of belt tension T_4/T_1 decreases with arm angle α

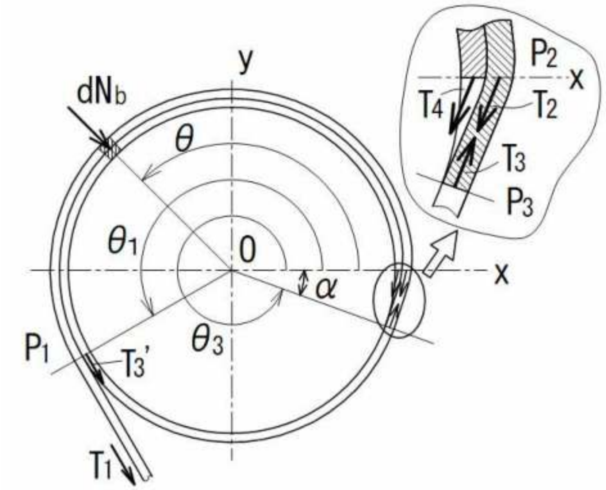


Fig. 9. Mechanical model of belt wrapped two times around an axis

The belt tension T_2 or T_3 can be expressed by the belt tension T_3', where T_3' is inner belt tension at the point P_1 as shown in Fig. 9.

$$T_2 = T_3 = e^{\mu(\theta_3 - \theta_1)} T_3' \tag{29}$$

Making use of Eqs. (28) and (29), T_1 can be expressed as

$$T_1 = e^{\{\mu_b \theta_1 + \mu(\theta_3 - \theta_1)\}} T_3' \tag{30}$$

The inner belt is normally pressed onto the cylinder by the outer belt. The normal force to a small segment of the inner belt at angle θ denoted by dN_b is

$$dN_b = e^{\mu_b \theta} T_2 d\theta \tag{31}$$

On the other hand, the normal force is also generated by inner belt tension itself. The normal force exerted on the cylinder between P_i and P_j is denoted by N_{ij}. Normal force acting to a small segment of the cylinder at angle θ is given by

$$dN_{21} = e^{\mu\theta} T_4 d\theta \tag{32}$$

Then, making use of Eqs. (31) and (32), the frictional force between the inner belt and cylinder denoted by F_{12in} is given by

$$F_{12in} = \int_0^{\theta_1} \mu dN_b + \int_0^{\theta_1} \mu dN_{21} = (e^{\mu_b \theta_1} - 1)\frac{\mu T_2}{\mu_b} + (e^{\mu\theta_1} - 1)T_4 \tag{33}$$

Denoting the radius of cylinder by r and neglecting the thickness of the belt, the equilibrium equation of moment of the cylinder is

$$T_1 r = (F_{12in} + F_{13} + T_4) r \tag{34}$$

Here, the frictional force F_{13} exerted on the surface between P_1 and P_3 is given by

$$F_{13} = \mu \int_{\theta_1}^{\theta_3} e^{\mu(\theta-\theta_1)} T_3' d\theta = \{e^{\mu(\theta_3-\theta_1)} - 1\} T_3' \tag{35}$$

Substituting Eqs. (33) and (35) into Eq. (34) gives

$$T_1 = (e^{\mu_b \theta_1} - 1)\frac{\mu T_2}{\mu_b} + \{e^{\mu(\theta_3-\theta_1)} - 1\} T_3' + e^{\mu\theta_1} T_4 \tag{36}$$

Substituting T_2 and T_3' in Eq. (36) as functions of T_1 by making use of Eqs. (28) and (30) gives

$$T_1 = \frac{e^{\theta_1(\mu+\mu_b)}}{(1-e^{\mu_b\theta_1})\left(\frac{\mu}{\mu_b} - 1\right) + e^{-\mu(\theta_3-\theta_1)}} T_4 \tag{37}$$

This is the targeted equation that expresses the relation between T_1 and T_4.
Equation (37) can be checked by supposing an extreme case of either μ=0 or μ_b=0. Substituting μ=0 into Eq. (37) gives T_1=T_4 as a matter of course. Substituting of μ_b=0 into Eq. (37) requires limiting operation.

$$\lim_{\mu_b \to 0} (1-e^{\mu_b\theta_1})\frac{\mu}{\mu_b} = -\mu\theta_1 \tag{38}$$

Making use of Eq. (38), Eq. (37) becomes Eq. (39) for the case of μ_b=0.

$$T_1 = \frac{e^{\mu\theta_1}}{-\mu\theta_1 + e^{-\mu(\theta_3-\theta_1)}} T_4 \tag{39}$$

Equation (39) implies the belt may be locked firmly around an axis when the denominator of the fraction in Eq. (39) becomes 0. Substituting μ=0 into Eq. (39) gives T_1=T_4 again as a matter of course.
Substituting μ=μ_b into Eq. (37) gives

$$T_1 = e^{\mu(\theta_1+\theta_3)} T_4 \tag{40}$$

Equation (40) is exactly the same form as the Euler's belt formula though was derived from the expression that took an effect of over-wrapping of belt into account. Equation (40) implies that the belt cannot be locked on the cylinder as far as the wrapping angle is finite.
Letting θ_1=0 in Eq. (37) to eliminate the over-wrapping part gives

$$T_1 = e^{\mu\theta_3} T_4 \tag{41}$$

This is the well-known Euler's belt formula. So the Euler's belt formula was proved to be included as a special case in Eq. (37). Equation (41) can also be obtained from Eqs. (39) and (40).
Next, let's consider some locking conditions. According to Eq. (37), the belt tension ratio T_4 / T_1 can be expressed as

$$\frac{T_4}{T_1} = \frac{(1-e^{\mu_b \theta_1})\left(\dfrac{\mu}{\mu_b}-1\right)+e^{-\mu(\theta_3-\theta_1)}}{e^{\theta_1(\mu+\mu_b)}} = \frac{\Gamma}{e^{\theta_1(\mu+\mu_b)}} \tag{42}$$

The locking condition is satisfied when the numerator of Eq. (42) becomes 0 meaning T_4 =0. So, the discriminant of locking condition can be expressed as

$$\Gamma = (1-e^{\kappa\mu\theta_1})\left(\frac{1}{\kappa}-1\right)+e^{-\mu(\theta_3-\theta_1)} \tag{43}$$

Locking condition is satisfied in the case of $\Gamma \le 0$. Critical point is $\Gamma=0$. Here, κ denotes a ratio of the coefficient of friction.

$$\kappa = \mu_b / \mu \tag{44}$$

As $e^{\kappa\mu\theta_1} \ge 1$ and $e^{-\mu(\theta_3-\theta_1)} > 0$, κ should be less than unity to make the value of locking discriminant of Eq. (43) be $\Gamma<0$. As can be seen in Fig. 9, the angle θ_3 is smaller than 2π due to the thickness of the belt. From geometrical consideration in Fig. 9, following equation is obtained.

$$\cos\alpha = \frac{r}{r+t} \approx 1-\frac{t}{r} \tag{45}$$

Here, t is thickness of the belt and r is a radius of the cylinder. When angle α is small, the angle α can be roughly estimated by

$$\alpha \approx \sqrt{2t/r} \tag{46}$$

Supposing the angle of non-contact is α =15˚, the corresponding critical locking condition can be evaluated by solving Eq. (43). Figure 10 shows some solutions. The critical angle of belt locking θ_1 decreases with an increment of the coefficient of friction μ. Provided the coefficient of friction is constant, the critical angle of belt locking θ_1 increases with an increment of κ. This fact means that the belt is likely to lock with a decrement of κ. So the smaller coefficient of friction μ_b is preferable for self-locking. The limiting condition for the belt locking is κ =0 or μ_b=0.

Figure 11 illustrates the effect of κ on the fraction of belt tension T_4/T_1 for the case of μ=0.3 and θ_3=345°. Making use of Eq. (41), the convergence point is calculated. It is $T_4/T_1 = \exp(-\mu\theta_3) \approx 0.164$. It is clear that the fraction of belt tension T_4/T_1 is greatly influenced by the magnitude of κ, μ_b/μ. The belt tension ratio T_4/T_1 decreases with an increment of over-wrapping angle θ_1 except for the case of κ=1.4. When $\kappa \ge 1$, the fraction of belt tension is always positive, so that the self-locking never occurs. Provided θ_1=360°, θ_3=345° and μ=0.3, the critical ratio of the coefficient of friction κ_c for the self-locking with two times over-wrapping condition was calculated by using the discriminant Eq. (43). It was κ_c=0.735. The corresponding line was plotted with a dashed line in Fig. 11. The magnitude of κ should be smaller than κ_c to cause the self-locking.

Figure 12 shows a method by which the coefficient of friction between the belt and belt can be reduced so as to satisfy the self-locking condition. When a polyethylene film was wrapped together with belt, an occurrence of self-locking was confirmed. But self-locking never occurred without polyethylene film.

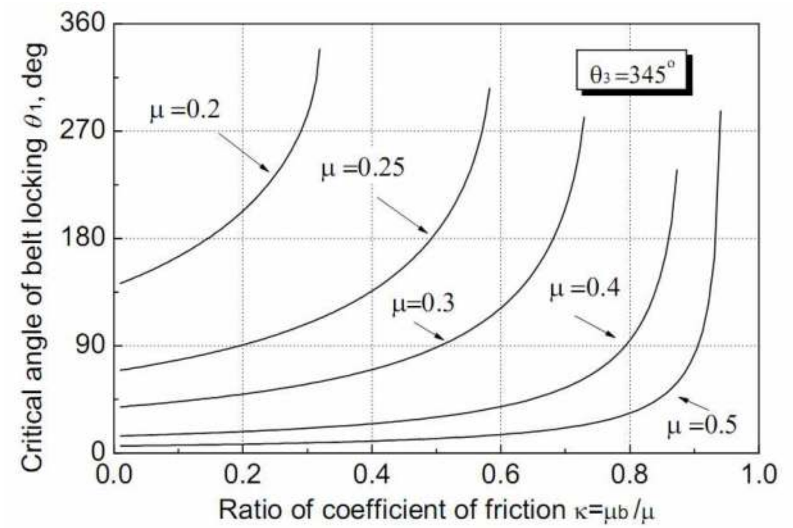


Fig. 10. Change of critical over-wrapping angle θ_1 for self-locking with ratio of the coefficients of friction κ.

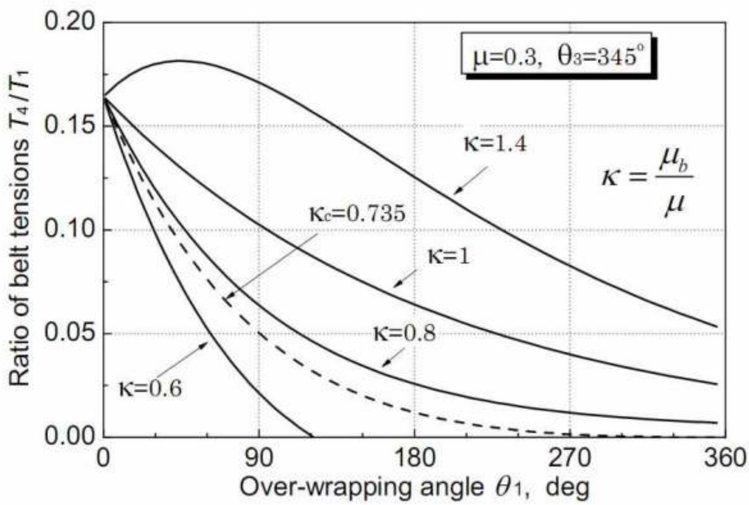


Fig. 11. Fraction of belt tension T_4 / T_1 decreases rapidly with increment of over-wrapping angle θ_1 for the case of smaller κ.

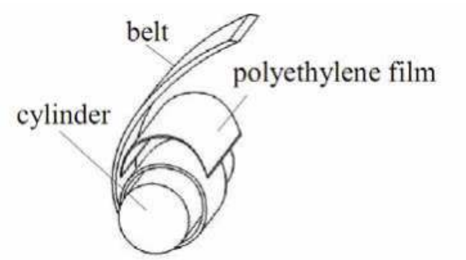


Fig. 12. Polyethylene film was wrapped together with belt to reduce the coefficient of friction μ_b. Self-locking was recognized in experiment with polyethylene film. But it never occurred without polyethylene film.

3.2 Friction of belt wrapped three times around axis

A belt can be wrapped more than two times around an axis. Let us consider the case where a belt is wrapped three times around an axis as shown in Fig. 13. The point P_i (i=1, 2, 3) is a boundary of contact. Tension of belt is denoted by T_i (i=1, 2, 3, 4) or T_i' and T_1>T_4. There are two kinds of the coefficients of friction μ and μ_b. μ_b is the coefficient of friction between belt and belt. The belt does not in contact with the axis from the point P_2 to P_3 due to the thickness of belt-end. In order to consider the equation of belt friction, the belt is divided into 5 sections from outside to inside as a, b, c, d and e in terms of frictional force as shown in Fig. 14. The frictional force working on an each section is expressed by either F_{si} or F_{so} , where the first subscript s means the name of section and the second subscript i means inside and o means outside respectively. Note that F_{si} works clockwisely and F_{so} works in a counter-clockwise direction. Considering the equilibrium of the force in an each section, following equations are obtained.

$$T_1 = F_{ai} + T_2 \tag{47}$$

$$T_3 = T_2 = F_{bi} + T_1' \tag{48}$$

$$T_1' = F_{ci} - F_{co} + T_2' \tag{49}$$

$$T_3' = T_2' = F_{di} - F_{do} + T_1'' \tag{50}$$

$$T_1'' = F_{ei} - F_{eo} + T_4 \tag{51}$$

Denothing the normal force from the section a to c by N_{ac}, the normal force acting to a small segment at angle θ is given by

$$dN_{ac} = e^{\mu_b \theta} T_2 d\theta \tag{52}$$

Frictional force F_{ai} is calculated by integrating Eq. (52).

$$F_{ai} = \int_{\theta=0}^{\theta_1} \mu_b dN_{ac} = \left(e^{\mu_b \theta_1} - 1\right) T_2 \tag{53}$$

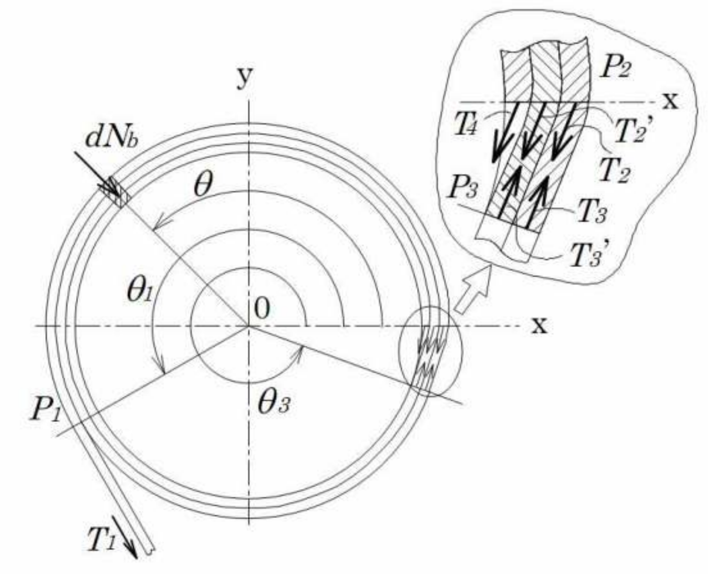


Fig. 13. Mechanical model of belt wrapped three times around an axis

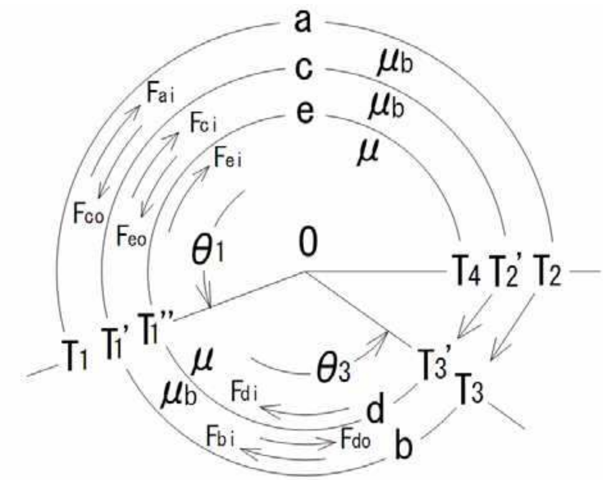


Fig. 14. Mechanical model with frictional force direction and the coefficients of friction corresponding to an each section.

In the same manner, infinitesimal normal force from the section b belt to d belt is given by

$$dN_{bd} = e^{\mu_b(\theta-\theta_1)} T_1' d\theta \tag{54}$$

Frictional force F_{bi} is calculated by integrating Eq. (54).

$$F_{bi}=\int_{\theta_1}^{\theta_3}\mu_b dN_{bd}=\int_{\theta_1}^{\theta_3}\mu_b e^{\mu_b(\theta-\theta_1)}T_1'd\theta=\left(e^{\mu_b(\theta_3-\theta_1)}-1\right)T_1' \tag{55}$$

Making use of Eq. (52), infinitesimal normal force from section c belt to e belt is given by

$$dN_{ce}=e^{\mu_b\theta}T_2'd\theta+dN_{ac}=e^{\mu_b\theta}T_2'd\theta+e^{\mu_b\theta}T_2 d\theta \tag{56}$$

Frictional force F_{ci} is calculated by integrating Eq. (56).

$$F_{ci}=\int_0^{\theta_1}\mu_b dN_{ce}=\int_0^{\theta_1}\mu_b e^{\mu_b\theta}\left(T_2'+T_2\right)d\theta=\left(e^{\mu_b\theta_1}-1\right)\left(T_2'+T_2\right) \tag{57}$$

Making use of Eq. (54), infinitesimal normal force from section d belt to the axis is given by

$$dN_d=e^{\mu(\theta-\theta_1)}T_1''d\theta+dN_{bd}=e^{\mu(\theta-\theta_1)}T_1''d\theta+e^{\mu_b(\theta-\theta_1)}T_1'd\theta \tag{58}$$

Frictional force F_{di} is calculated by integrating Eq. (58).

$$F_{di}=\int_{\theta_1}^{\theta_3}\mu dN_d=\int_{\theta_1}^{\theta_3}\mu\left(e^{\mu(\theta-\theta_1)}T_1''+e^{\mu_b(\theta-\theta_1)}T_1'\right)d\theta=\left(e^{\mu(\theta_3-\theta_1)}-1\right)T_1''+\left(e^{\mu_b(\theta_3-\theta_1)}-1\right)\frac{\mu}{\mu_b}T_1' \tag{59}$$

Making use of Eq. (56), infinitesimal normal force from section e belt to the axis is given by

$$dN_e=e^{\mu\theta}T_4 d\theta+dN_{ce}=e^{\mu\theta}T_4 d\theta+e^{\mu_b\theta}T_2'd\theta+e^{\mu_b\theta}T_2 d\theta \tag{60}$$

Then, the frictional force F_{ei} is given by

$$F_{ei}=\int_0^{\theta_1}\mu dN_e=\int_0^{\theta_1}\mu\left(e^{\mu\theta}T_4+e^{\mu_b\theta}T_2'+e^{\mu_b\theta}T_2\right)d\theta=\left(e^{\mu\theta_1}-1\right)T_4+\left(e^{\mu_b\theta_1}-1\right)\left(T_2+T_2'\right)\frac{\mu}{\mu_b} \tag{61}$$

Neglecting the thickness of the belt, the equilibrium requirement of the moment gives

$$T_1=F_{di}+F_{ei}+T_4 \tag{62}$$

Substituting Eqs. (59) and (61) into Eq. (62) gives

$$T_1=\left(e^{\mu(\theta_3-\theta_1)}-1\right)T_1''+\left(e^{\mu_b(\theta_3-\theta_1)}-1\right)\frac{\mu}{\mu_b}T_1'+\left(e^{\mu_b\theta_1}-1\right)\left(T_2+T_2'\right)\frac{\mu}{\mu_b}+e^{\mu\theta_1}T_4 \tag{63}$$

The belt tensions T_1', T_1'', T_2 and T_2' in Eq. (63) should be expressed by the function of T_4. From the law of action and reaction,

$$F_{co}=F_{ai},\quad F_{eo}=F_{ci},\quad F_{do}=F_{bi} \tag{64}$$

Substituting Eqs. (53), (55), (57), (59) and (61) into Eqs. (47) to (51) give

$$T_1=F_{ai}+T_2=e^{\mu_b\theta_1}T_2 \tag{65}$$

$$T_2=T_3=F_{bi}+T_1'=e^{\mu_b(\theta_3-\theta_1)}T_1' \tag{66}$$

$$T_1'=F_{ci}-F_{co}+T_2'=\left(e^{\mu_b\theta_1}-1\right)\left(T_2'+T_2\right)-\left(e^{\mu_b\theta_1}-1\right)T_2+T_2'=e^{\mu_b\theta_1}T_2' \tag{67}$$

$$T_3' = T_2' = F_{di} - F_{do} + T_1'' = F_{di} - F_{bi} + T_1'' = e^{\mu(\theta_3-\theta_1)}T_1'' + \left(e^{\mu_b(\theta_3-\theta_1)} - 1\right)\left(\frac{\mu}{\mu_b} - 1\right)T_1' \quad (68)$$

$$T_1'' = F_{ei} - F_{eo} + T_4 = \left(e^{\mu_b\theta_1} - 1\right)\left(\frac{\mu}{\mu_b} - 1\right)(T_2 + T_2') + e^{\mu\theta_1}T_4 \quad (69)$$

Making use of Eqs. (65), (66) and (67) gives,

$$T_1 = e^{\mu_b(\theta_1+\theta_3)}T_2' = e^{\mu_b(\theta_1+\theta_3)}T_3' \quad (70)$$

Substituting Eq. (68) into Eq. (70) and making use of Eq. (67) gives

$$T_1 = e^{\mu_b(\theta_1+\theta_3)}\left\{e^{\mu(\theta_3-\theta_1)}T_1'' + \left(e^{\mu_b(\theta_3-\theta_1)} - 1\right)\left(\frac{\mu}{\mu_b} - 1\right)e^{\mu_b\theta_1}T_2'\right\} \quad (71)$$

Making use of Eqs. (65), (66) and (67) gives

$$T_2' = \frac{T_1}{e^{\mu_b(\theta_1+\theta_3)}} \quad (72)$$

Substituting Eq. (72) into Eq. (71) gives

$$T_1 = e^{\mu_b(\theta_1+\theta_3)+\mu(\theta_3-\theta_1)}T_1'' + e^{\mu_b\theta_1}\left(e^{\mu_b(\theta_3-\theta_1)} - 1\right)\left(\frac{\mu}{\mu_b} - 1\right)T_1 \quad (73)$$

Rearranging Eq. (73) gives,

$$T_1'' = \frac{1 - \left(e^{\mu_b\theta_3} - e^{\mu_b\theta_1}\right)\left(\frac{\mu}{\mu_b} - 1\right)}{e^{\mu_b(\theta_1+\theta_3)+\mu(\theta_3-\theta_1)}}T_1 = AT_1 \quad (74)$$

Making use of Eqs. (65) and (72) gives

$$T_2 + T_2' = \frac{e^{\mu_b\theta_3} + 1}{e^{\mu_b(\theta_1+\theta_3)}}T_1 \quad (75)$$

Substituting Eq. (75) into Eq. (69) gives

$$T_1'' = \left(\frac{\mu}{\mu_b} - 1\right)\frac{\left(e^{\mu_b\theta_3} + 1\right)\left(e^{\mu_b\theta_1} - 1\right)}{e^{\mu_b(\theta_1+\theta_3)}}T_1 + e^{\mu\theta_1}T_4 = BT_1 + e^{\mu\theta_1}T_4 \quad (76)$$

Substituting Eq. (76) into the left hand side of Eq. (74) gives,

$$T_1" = \left(\frac{\mu}{\mu_b} - 1\right)\frac{\left(e^{\mu_b\theta_3}+1\right)\left(e^{\mu_b\theta_1}-1\right)}{e^{\mu_b(\theta_1+\theta_3)}}T_1 + e^{\mu\theta_1}T_4 = BT_1 + e^{\mu\theta_1}T_4$$
$$= \frac{1-\left(e^{\mu_b\theta_3}-e^{\mu_b\theta_1}\right)\left(\frac{\mu}{\mu_b}-1\right)}{e^{\mu_b(\theta_1+\theta_3)+\mu(\theta_3-\theta_1)}}T_1 = AT_1 \tag{77}$$

Equation (77) can be written in the form of

$$T_1 = \frac{e^{\mu\theta_1}}{A-B}T_4 \tag{78}$$

where

$$A = \frac{1-\left(e^{\mu_b\theta_3}-e^{\mu_b\theta_1}\right)\left(\frac{\mu}{\mu_b}-1\right)}{e^{\mu_b(\theta_1+\theta_3)+\mu(\theta_3-\theta_1)}}, \quad B = \left(\frac{\mu}{\mu_b}-1\right)\frac{\left(e^{\mu_b\theta_3}+1\right)\left(e^{\mu_b\theta_1}-1\right)}{e^{\mu_b(\theta_1+\theta_3)}} \tag{79}$$

Eqs. (78) and (79) are the targeted equations that express the relation between T_1 and T_4 in the case of a belt wrapped three times around an axis.

3.3 Characteristics of belt friction equation with three times wrapping around axis

The equation derived in the previous section seems complex. It can be checked by assuming some extreme cases such as μ=0, μ_b=0 and $\mu=\mu_b$. In the case of μ=0, Eq. (79) becomes,

$$A = \frac{1+\left(e^{\mu_b\theta_3}-e^{\mu_b\theta_1}\right)}{e^{\mu_b(\theta_1+\theta_3)}}, \quad B = -\frac{\left(e^{\mu_b\theta_3}+1\right)\left(e^{\mu_b\theta_1}-1\right)}{e^{\mu_b(\theta_1+\theta_3)}} \tag{80}$$

then

$$A - B = \frac{e^{\mu_b(\theta_1+\theta_3)}}{e^{\mu_b(\theta_1+\theta_3)}} = 1 \tag{81}$$

Substituting Eq. (81) and μ=0 into Eq. (78) gives T_1=T_4.
In the case of μ_b=0, limiting operations are required. For the term A in Eq. (79),

$$\lim_{\mu_b\to 0}\frac{\mu}{\mu_b}\left(e^{\mu_b\theta_3}-e^{\mu_b\theta_1}\right) = \mu\left(\theta_3-\theta_1\right) \tag{82}$$

For the term B in Eq. (79),

$$\lim_{\mu_b\to 0}\frac{\mu}{\mu_b}\left(e^{\mu_b\theta_3}+1\right)\left(e^{\mu_b\theta_1}-1\right) = 2\mu\theta_1 \tag{83}$$

Then Eq. (79) becomes,

$$A = \frac{1-\mu\left(\theta_3-\theta_1\right)}{e^{\mu(\theta_3-\theta_1)}}, \quad B = 2\mu\theta_1 \tag{84}$$

Substituting Eq. (84) into (78) gives

$$T_1 = \frac{e^{\mu\theta_3}}{1-\mu\left(\theta_3-\theta_1+2\theta_1 e^{\mu(\theta_3-\theta_1)}\right)}T_4 \tag{85}$$

In order to consider the smallest wrapping angle of three times wrapping, substituting θ_1=0 into Eq. (85) gives,

$$T_1 = \frac{e^{\mu\theta_3}}{1-\mu\theta_3}T_4 \tag{86}$$

On the other hand, substituting $\theta_1=\theta_3$ into Eq. (85) gives,

$$T_1 = \frac{e^{\mu\theta_3}}{1-2\mu\theta_3}T_4 \tag{87}$$

Equation (87) shows the relation of belt tension with the largest wrapping angle of three times wrapping. The locking condition is satisfied when the denominator of Eqs. (86) and (87) become 0, so that in the case of $\theta_1=\theta_3$, only 1/2 of the coefficient of friction is required for self locking in compared with the case of θ_1=0.

In the case of $\mu_b=\mu$, Eq. (79) becomes,

$$A = \frac{1}{e^{2\mu\theta_3}}, \quad B = 0 \tag{88}$$

so that Eq. (78) becomes,

$$T_1 = \frac{e^{\mu\theta_1}}{A-B}T_4 = e^{\mu(\theta_1+2\theta_3)}T_4 \tag{89}$$

Substituting θ_1=0 into Eq. (89) gives,

$$T_1 = e^{2\mu\theta_3}T_4 \tag{90}$$

Substituting $\theta_1=\theta_3$ into Eq. (89) gives,

$$T_1 = e^{3\mu\theta_3}T_4 \tag{91}$$

Note the magnitude of the wrapping angle of Eqs. (90) and (91). They are exactly the same form as the Euler's belt formula though they were derived considering the effect of over-wrapping of belt on belt friction.

Next, Substituting θ_1=0 into Eq. (79) provided the boundary of two and three times over-wrapping of belt gives,

$$A = \frac{1+\left(1-e^{\mu_b\theta_3}\right)\left(\dfrac{\mu}{\mu_b}-1\right)}{e^{\theta_3(\mu+\mu_b)}}, \quad B = 0 \tag{92}$$

then Eq. (78) becomes

$$T_1 = \frac{1}{A-B}T_4 = \frac{e^{\theta_3(\mu+\mu_b)}}{(1-e^{\mu_b\theta_3})\left(\dfrac{\mu}{\mu_b}-1\right)+1}T_4 \quad (93)$$

On the other hand, substituting θ_1=θ_3 into Eq. (37) in the section 3.1 that was the equation for two times over-wrapping conditions gives,

$$T_1 = \frac{e^{\theta_3(\mu+\mu_b)}}{(1-e^{\mu_b\theta_3})\left(\dfrac{\mu}{\mu_b}-1\right)+1}T_4 \quad (94)$$

Equation (93) is completely corresponding to Eq. (94) so that both equations are continuous. Figures 15 and 16 show some calculated results by using Eqs. (37), (78) and (79). Figure 15 is of μ=0.25 and θ_3=350°. With an increment of κ, namely with an increment of μ_b, the ratio of belt tension T_4 / T_1 increases. Self-locking occurs with wrap angle less than 720° in the case of κ=0.5 and 0.6, so that they were calculated by Eq. (37). On the other hand, in the case of κ=0.7, 0.8 and 0.9, the wrap angle less than 720° is not enough for self-locking to occur. They requires wrap angle greater than 720° so that they were calculated by Eqs. (78) and (79). Figure 16 is of μ=0.2 and θ_3=350°. All of them require wrap angle greater than 720° to enter the self-locking condition.
The threshold of self-locking for three times wrapped belt is obtained by equating A to B in Eq. (79).

$$\left(\frac{\mu}{\mu_b}-1\right)\left\{\left(e^{\mu_b\theta_3}+1\right)\left(e^{\mu_b\theta_1}-1\right)e^{\mu(\theta_3-\theta_1)}+e^{\mu_b\theta_3}-e^{\mu_b\theta_1}\right\}=1 \quad (95)$$

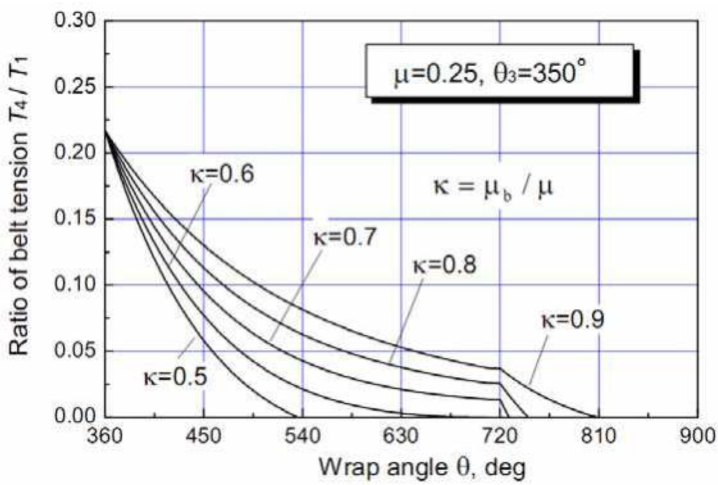


Fig. 15. Change of belt tension ratio with wrap angle

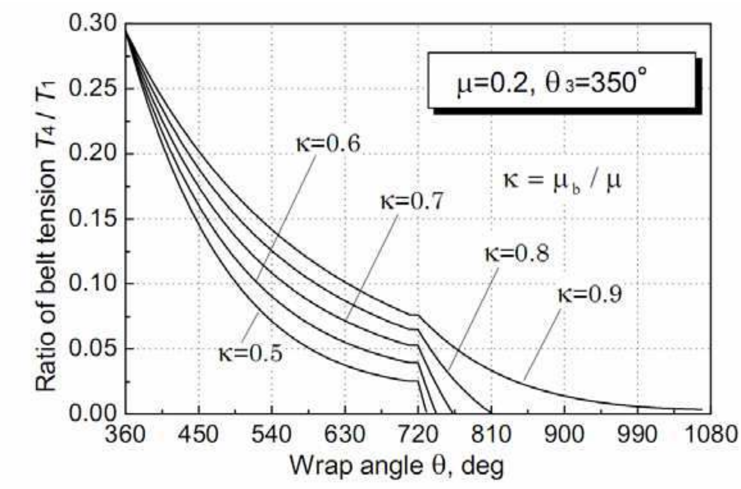


Fig. 16. Change of belt tension ratio with wrap angle

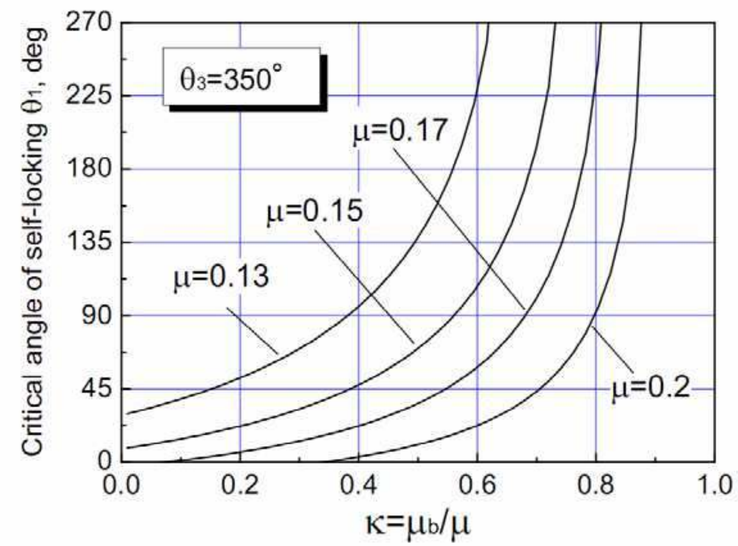


Fig. 17. Change of critical angle for self-locking with ratio of the coefficients of friction

Equation (95) is the discriminant of the self-locking condition for three times wrapped belt. When the coefficients of friction μ, μ_b and the angle θ_3 are given, the magnitude of critical angle θ_1 necessary for self-locking is calculated by solving Eq. (95). Figure 17 shows some solutions of Eq. (95) with angle θ_3=350°. If an angle θ_3 and the coefficient of frictions μ and μ_b are given, self-locking occurs with the wrap angle θ_1 over the corresponding curve. But it does not occur with wrap angle θ_1 under the corresponding curve. According to Fig. 17, it is clearly seen that wrap angle θ_1 becomes larger with an increment of κ. It also becomes larger with a decrement of the coefficient of friction μ. Provided κ is small enough, it is noticeable that the self-locking occurs theoretically even with these small coefficients of friction.

4. Novel clutch utilizing self-locking property of belt

Paying attention to the self-locking property of belt as described in the previous section, a novel clutch mechanism can be developed (Imado et al., 2010). Figure 18 shows a simplified three-dimensional image of the novel clutch. Figure 19 shows a cross sectional view of the clutch. Rotational torque is transmitted from the power ring to the inner axis by the belt. In declutching condition, a belt is only rotating with the power ring. Due to the centrifugal force or some restitutive property of belt, the belt is pressed against the internal face of the power ring. To transmit the rotation of power ring to the internal axis, the sleeve on the inner axis is slid along the axis to push the end face of the trigger pin that is attached at the end of the belt and rotating with the power ring. As the sleeve is rotating with the same angular speed of the inner axis, the frictional force to the trigger pin drags the belt so as to coil around the inner axis. The trigger pin works as a synchronizer. As soon as the belt comes in contact with the axis, the belt coils automatically around the axis by the frictional force between the belt and axis. Then due to the self-locking property of belt, the rotation of the power ring is transmitted to the inner axis without any slip as far as self-locking

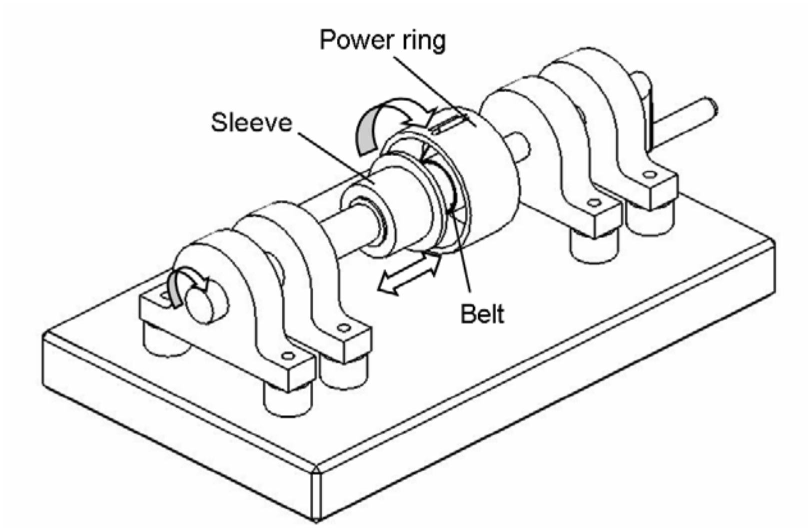


Fig. 18. Three-dimensional image of novel clutch. Rotational torque is transmitted from power ring to inner axis by self-locking belt.

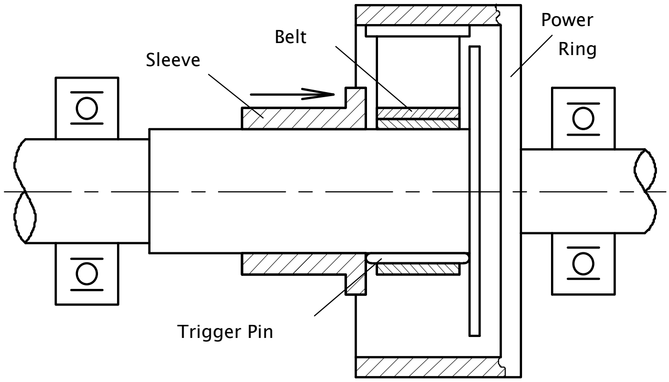


Fig. 19. Cross section of novel clutch

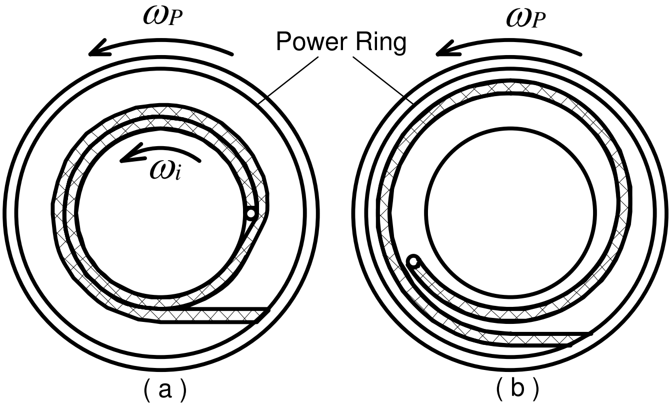


Fig. 20. Frontal views of main part of belt-type clutch in (a) locked-up condition and (b) unlocked condition

condition is satisfied. As long as driving torque is applied, the self-locking state is maintained. Semi-locking state can be realized by adjusting the over-wrapping angle of the belt. When the rotational speed of the power ring becomes smaller than that of the inner axis, the rotation of the inner axis uncoils the belt so that declutching occurs automatically.

Figure 20 shows frontal views of the main part of the clutch in a state of locked-up condition and unlocked condition respectively. From the mechanical point of view, an accurate centering operation is required in assembling individual rotational machine components. Because the torque is transmitted through a flexible belt, this delicate centering operation is not so strictly required for this novel clutch. The belt-type clutch works even in the case where a power ring and an inner axes are either slightly off-centered or inclined with each other.

Figure 21 shows prototype clutch. Brake torque can be applied by the belt brake. It was confirmed experimentally that rotational torque could be transmitted without any slip

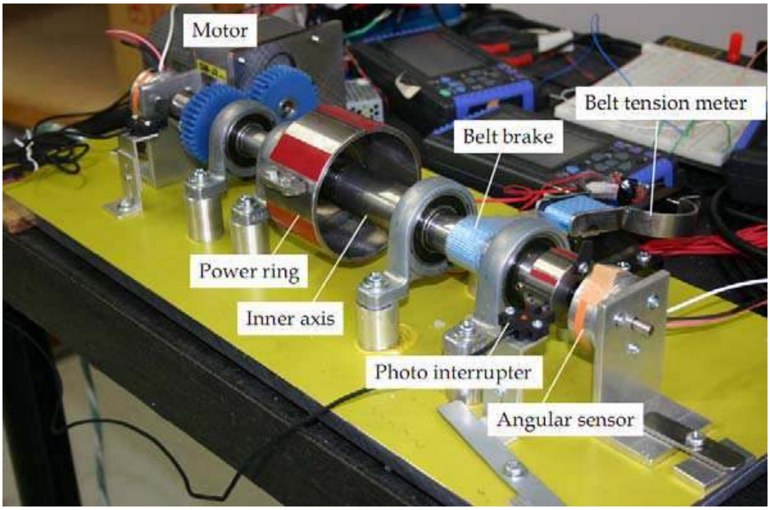


Fig. 21. Photograph of belt-type clutch

where there was an eccentricity. A steel belt with 12 mm wide and 0.12 mm in thickness was used in the prototype clutch. In order to reduce the coefficient of friction between belt and belt μ_b, a small amount of grease of molybdenum disulfide, MoS_2, was spread between the belt and belt. Test condition was summarized in Table 1. According to Eq. (43), the critical wrap angle θ_1 of the clutch in Fig. 9 was 105° as shown in Table 1. Considering unsteadiness of the coefficients of friction, two kinds of experiments were carried out. One was of θ_1=90°, the other was of θ_1=120°. Then, self-locking occurred in the case of wrap angle θ_1=120°. On the other hand, self-locking never occurred in the case of θ_1=90°. As far as this experimental result was concerned, the validity of Eq. (43) was verified.

Item	
Rotational speed	60 rpm
Diameter of power ring	89 mm
Diameter of inner axis	30 mm
Width of steel belt	12 mm
Thickness of steel belt	0.12 mm
Coefficient of friction μ	0.25
Coefficient of friction with MoS_2 grease μ_b	0.074
Maximum center offset	8.2 mm
Critical over-wrap angle θ_1	105°

Table 1. Dimensions of clutch and the coefficients of friction

5. Generalization of belt/rope friction formula

The belt formula written in a text, it is usually explained by a figure illustrating a flexible element partially wrapped on a cylindrical surface. But actually there are many kinds of surface. So far, frictional force calculation of a flexible element to these surfaces has not been clearly explained in a text. In this section, the friction of flexible element in the generalized condition is studied. Fige 22 shows a belt wrapped around an arbitrary surface. The equilibrium equation of the force acting to an infinitesimal line element ds is (Hashimoto, 2006)

$$\mu d\theta = \frac{dT}{T} \tag{96}$$

Let denote the curvature and the radius of curvature by κ and ρ respectively. The small wrap angle dθ can be written as

$$d\theta = \frac{1}{\rho} ds = \kappa ds \tag{97}$$

Substituting Eq. (97) into Eq. (96) gives

$$\mu \kappa ds = \frac{dT}{T} \tag{98}$$

Equation (98) means that the friction of a flexible element on a generalized curve can be evaluated by line integral of the curvature with respect to curvilinear length s.

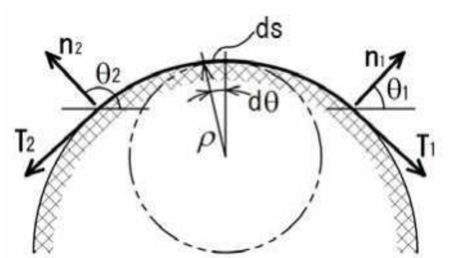


Fig. 22. Flexible element wrapped around body of arbitrary profile

Now, a position vector **r** of a curve C in parametric expression with t is

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{99}$$

Differential coefficient with respect to the parameter t is expressed as $\dot{\mathbf{r}}$ or $\dot{x}$. On the other hand, the differential coefficient with respect to curvilinear length s is expressed as $\mathbf{r}'$ or x'. The unit tangential vector $\mathbf{u}$, curvature κ and a line element ds of the curve C are (Yano & Ishihara, 1964)

$$\mathbf{u} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \tag{100}$$

$$\kappa = \frac{\sqrt{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{3/2}} , \quad ds = |\dot{\mathbf{r}}| dt \tag{101}$$

For a plane curve of z=0 in Eq. (99), substituting Eq. (99) into Eq. (101) gives

$$\kappa = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \quad ds = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt \tag{102}$$

Substituting Eq. (102) into the left side of Eq. (98) gives

$$\mu\kappa ds = \mu \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} dt \tag{103}$$

The unit principal normal vector **m** of the curve C is given by the formula (Yano & Ishihara, 1964)

$$\mathbf{m} = \frac{\mathbf{u}'}{|\mathbf{u}'|} = \frac{d\mathbf{u}}{dt}\frac{dt}{ds} / |\mathbf{u}'| \tag{104}$$

Making use of Eq. (100) gives

$$\mathbf{u}' = \frac{1}{(\dot{x}^2 + \dot{y}^2)^2}\{\dot{y}(\ddot{x}\dot{y} - \dot{x}\ddot{y})\mathbf{i} + \dot{x}(\dot{x}\ddot{y} - \ddot{x}\dot{y})\mathbf{j}\} \tag{105}$$

$$|\mathbf{u}'| = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^2}\sqrt{\dot{x}^2 + \dot{y}^2} = \kappa \tag{106}$$

Substituting Eqs. (105) and (106) into Eq. (104) gives

$$\mathbf{m} = \frac{-\dot{y}\,\mathbf{i} + \dot{x}\,\mathbf{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \tag{107}$$

Here, the direction of the vector **m** is toward the center of curvature. Then, an outward normal vector **n** can be defined as

$$\mathbf{n} = -\mathbf{m} = \frac{\dot{y}\,\mathbf{i} - \dot{x}\,\mathbf{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \tag{108}$$

The direction of the normal vector **n** is denoted by θ

$$\theta = \tan^{-1}\left(\frac{-\dot{x}}{\dot{y}}\right) \tag{109}$$

Differentiating Eq. (109) with respect to t gives

$$\dot{\theta} = \frac{\dot{x}\,\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \tag{110}$$

Comparing Eq. (110) with Eq. (103) gives

$$\mu\kappa ds = \mu \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} dt = \mu\dot{\theta}dt \tag{111}$$

Hence, making use of Eq. (111), integration of Eq. (98) becomes

$$\int \mu\kappa ds = \int \mu\dot{\theta}dt = \mu(\theta_2 - \theta_1) = \log(T_2 / T_1) \tag{112}$$

Equation (112) means that fraction of belt tension is determined by angular difference of the outward normal vectors at the contact boundaries and is unrelated to the intermediate profile. Equation (98) might be applied to the three dimensional problems.
As an example, let's consider a rope spirally wrapped around a cylinder with radius a. The parametric expression of a spiral with parameter t is (Yano & Ishihara, 1964)

$$x = a\cos t, \quad y = a\sin t, \quad z = bt \tag{113}$$

Substituting Eq. (113) into Eqs. (99) and (101) gives

$$\kappa = a / \left(a^2 + b^2\right), \; ds = \sqrt{a^2 + b^2}\, dt \tag{114}$$

Substituting Eq. (114) into Eq. (98) and integrating with respect to the parameter t from t_1 to t_2 gives

$$\frac{T_2}{T_1} = \exp\left\{\mu \frac{1}{\sqrt{1 + (b/a)^2}}(t_2 - t_1)\right\} \tag{115}$$

The member t_2-t_1 in Eq. (115) is usually a wrap angle for a plane problem. But it is not wrap angle in the three dimensional problem. In the case of b=0, Eq. (115) becomes well known Euler's belt formula. On the other hand, when b becomes infinity, Eq. (115) yields T_1=T_2. Hence, for a three-dimensional problem, the frictional force of a rope is influenced on a way of wrapping. Figure 23 (a) shows some results of calculation of Eq. (115) provided t_1=0 and t_2=$2n\pi$. Tension ratio T_2/T_1 decreases with an increment of the fraction of b/a.
Let's consider another example of a modified spiral defined by Eq. (116).

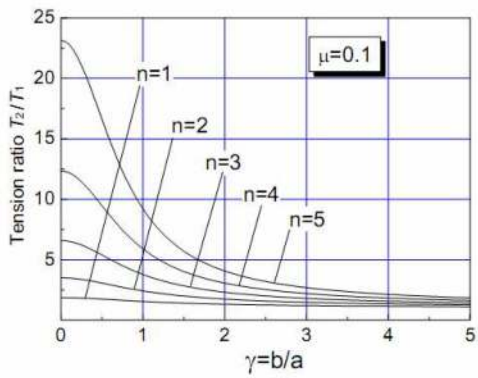
$$x = a\cos t, \quad y = a\sin t, \quad z = \left(1 - \frac{t}{4n\pi}\right)bt \tag{116}$$

According to Eq. (116), it can be seen that the velocity component in z direction decreases linearly with parameter t and becomes 0 at t=$2n\pi$. The components of Eq. (101) for the curve of Eq. (116) are

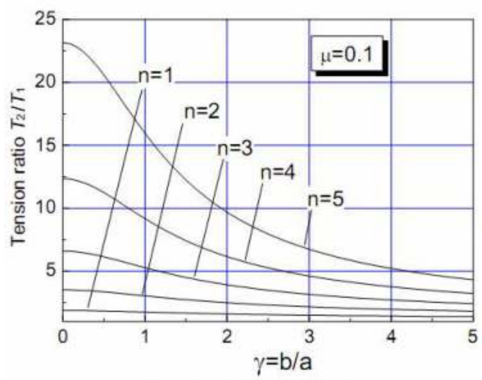
$$\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = a^2 + \left(1 - \frac{t}{2n\pi}\right)^2 b^2, \quad \ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = a^2 + \left(\frac{b}{2n\pi}\right)^2, \quad \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \frac{b^2(t - 2n\pi)}{4n^2\pi^2} \tag{117}$$

Substituting Eq. (117) into Eq. (101) gives

$$\kappa = \frac{a\sqrt{4a^2n^2\pi^2 + b^2\left(1 + 4n^2\pi^2 - 4n\pi t + t^2\right)}}{2n\pi\left\{a^2 + \left(1 - \frac{t}{2n\pi}\right)^2 b^2\right\}^{\frac{3}{2}}}, \quad ds = \sqrt{a^2 + \left(1 - \frac{t}{2n\pi}\right)^2 b^2}\, dt \tag{118}$$



(a) Spiral



(b) Modified spiral

Fig. 23. Change of tension ratio T_2/T_1 with coefficient ratio b/a of spiral

Substituting Eq. (118) into Eq. (98) and integrating with respect to parameter t from t_1=0 to t_2=$2n\pi$ gives

$$\log\left(\frac{T_2}{T_1}\right)=\mu\left\{\frac{1}{\beta\gamma}\log\left(\frac{\sqrt{1+\beta^2\gamma^2}}{\sqrt{1+\beta^2\gamma^2+\gamma^2}-\gamma}\right)+\tan^{-1}\left(\frac{\beta\gamma^2}{\sqrt{1+\beta^2\gamma^2+\gamma^2}}\right)\right\} \tag{119}$$

where $\gamma=b/a,\quad \beta=1/(2n\pi)$

Considering the case of γ=0, namely b=0 of Eq. (119) requires limiting operation.

$$\lim_{\gamma\to 0}\frac{\mu}{\beta\gamma}\log\left(\frac{\sqrt{1+\beta^2\gamma^2}}{\sqrt{1+\beta^2\gamma^2+\gamma^2}-\gamma}\right)=\frac{\mu}{\beta}=\mu 2n\pi \tag{120}$$

Hence, the result of plane problem is included as a special case of γ=0 in Eq. (119). Figure 23 (b) shows some results of calculation of Eq. (119). The tension ratio of T_2/T_1 in Fig. 23 (b) becomes larger than that of the corresponding value of Fig. 23 (a).

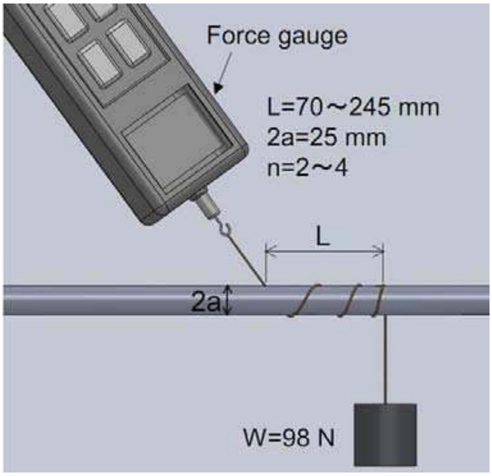


Fig. 24. Experimental method to evaluate the coefficient of friction of string wrapped spirally around cylinder

n	L, mm	b/a	μ
2	70	0.8913	0.143
2	140	1.7825	0.144
3	110	0.9337	0.142
3	250	2.1221	0.134
4	245	1.5597	0.151

Table 2. Summary of experimental result to evaluate the coefficient of friction of string wrapped spirally around cylinder

In order to confirm the validity of Eq. (119), simple experiments were carried out. Figure 24 shows experimental method. The diameter of the pipe was 25 mm. The string of 1.8 mm in diameter was wrapped around the pipe in a way according to Eq. (116) by using a steel scale. The weight of 98 N was hung at the end of the string. The other end of the string was connected to the force gauge that was fixed firmly to the stay. The weight was lifted up by hand at first. Then the weight was released quietly and string tension was measured by the force gauge. As the parameter t in Eq. (116) was taken from t=0 at z=0 to $t=2n\pi$ at $z=L$, the constant b in Eq. (116) can be calculated by $b=L/(n\pi)$. The coefficient of friction was calculated by Eq. (119). Experimental results are summarized in Table 2. Because almost same values were obtained regardless of the test condition, the validity of Eq. (119) was confirmed.

6. Closure

Frictional property of a flexible element was considered in this chapter. The theory of belt buckle has been clarified by considering an effect of over-wrapping of belt on belt friction. Frictional fixation of the belt buckle is caused by self-locking property of belt friction. Self-locking occurs even in the case where a belt is wrapped around an axis two or more times. Two conditions are required to bring about self-locking. One is smaller coefficient of belt-belt friction than that of belt-axis friction. The other is larger wrap angle than the critical wrap angle. Utilizing the self-locking property of belt, a novel one-way clutch was developed. The problem of this clutch is how to get the smaller and stable coefficient of belt-belt friction for long time use. Friction of a flexible element wrapped around a generalized profile was studied. However, the friction of twisted flexible element in a thread, rope and wire has not been clarified yet. Further research is required.

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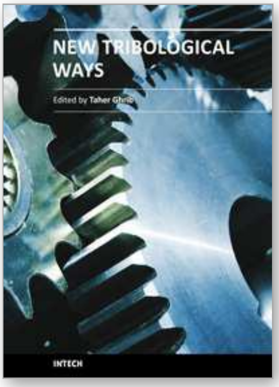

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This book aims to recapitulate old information's available and brings new information's that are with the fashion research on an atomic and nanometric scale in various fields by introducing several mathematical models to measure some parameters characterizing metals like the hydrodynamic elasticity coefficient, hardness, lubricant viscosity, viscosity coefficient, tensile strength .... It uses new measurement techniques very developed and nondestructive. Its principal distinctions of the other books, that it brings practical manners to model and to optimize the cutting process using various parameters and different techniques, namely, using water of high-velocity stream, tool with different form and radius, the cutting temperature effect, that can be measured with sufficient accuracy not only at a research lab and also with a theoretical forecast. This book aspire to minimize and eliminate the losses resulting from surfaces friction and wear which leads to a greater machining efficiency and to a better execution, fewer breakdowns and a significant saving. A great part is devoted to lubrication, of which the goal is to find the famous techniques using solid and liquid lubricant films applied for giving super low friction coefficients and improving the lubricant properties on surfaces.

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