
Nash Equilibrium Strategies in Fuzzy Games

Alireza Chakeri, Nasser Sadati and Guy A. Dumont

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1. Introduction

Modern description of game theory is generally considered to have been started with the book "*Theory of Games and Economic Behavior*" [1]. Modern game theory has grown extremely well, in particular after the influential results of Nash [2-4]. It has been widely applied to many problems in economics, engineering, politics, etc.

A game is a model of a situation where two or more groups are in dispute over some issues [5]. The participants in a game are called the players. The possible actions available to players are referred to as strategies. When each player selects a strategy, it will determine an outcome to the game and the payoffs to all players, while tries to maximize his own payoff.

Classical game theory uses the extensive form and the strategic form to explain a game. The extensive form is represented by a game tree, in which the players make sequential actions. However, the strategic form is usually used to describe games with two decision makers, in which players' choices are made simultaneously. Unlike one-player decision making, where optimality has a clear sense, in multi person decision making the optimality is in the form of NE. An NE strategy is a strategy wherein, if a player knows his opponent's strategy, he is totally satisfied and is unwilling to change his strategy.

In classical game theory it is assumed that all data of a game are known exactly by players. However, in real games, the players are often not able to evaluate exactly the game due to lack of information and precision in the available information of the situation. Harsanyi [6] treated imprecision in games with a probabilistic method and developed the theory of Bayesian games. This theory could not entirely solve the problem of imprecision in games, because it was limited to only one possible kind of imprecision. However, in reality, imprecision is of different types and can be modeled by fuzzy sets. The notion of fuzzy sets first appeared in the paper written by Zadeh [7]. This notion tries to show that to what degree an element belongs to a set. The degree, to which an element belongs to a set, is an element of the continuous interval $[0, 1]$ rather than the Boolean values. Using the notion of

fuzzy sets, each component in a game (set of players, set of strategies, set of payoffs, etc) can be fuzzified. Initially, fuzzy sets were used by Butnariu [8] in non-cooperative game theory. He used fuzzy sets to represent the belief of each player for strategies of other players. Since then, fuzzy set theory has been used in many non-cooperative [9-15] and cooperative games [16-17].

In this chapter, we will extend the NE set to fuzzy set in games with fuzzy numbers as payoffs. In this regard, using ranking fuzzy numbers, a fuzzy preference relation is constructed over payoffs and then the resultant priorities of payoffs are considered as the grades of being NE. Hence, if a player knows the opponent's strategy, he is satisfied with his own strategy by the degree that this strategy has priority for him. The more priority the players feel for each strategy, the more possible the strategy becomes the game's equilibrium. This generalization shows the distribution of Nash grades in the matrix form of the game. In other words, we can consider strategies with high grades of equilibrium which are not necessarily the equilibrium points. In the proposed approach, the effect of different viewpoints (optimism and pessimism) on the result of the game is also studied.

The remainder of this chapter is organized as follows. Section 2 reviews the backgrounds on fuzzy set theory. This section discusses the basic definition of fuzzy numbers, fuzzy extension principle and ranking fuzzy numbers. Section 3 briefly presents the fuzzy preference relation and a way to obtain priorities from fuzzy preference matrix. In this section, preference ordering of the alternatives and its transformation into fuzzy preference relation is also discussed. Section 4 introduces the proposed algorithm to find the Nash grades for pure and mixed strategies in the matrix form of the game. Two examples and their detailed results are presented in Section 5. Finally, Section 6 contains some concluding remarks.

2. Backgrounds on fuzzy set theory

There are two well-known frameworks for quantifying the lack of knowledge and precision, namely, probabilistic and possibilistic uncertainty type. The probabilistic framework deals with the uncertain events with a probability distribution function (PDF). However, there are some situations in which there is not much information about the PDF of uncertain parameters or they are inherently not repeatable. In possibilistic framework, for each uncertain event \tilde{M} , a membership function $\mu_{\tilde{M}}(x)$ is defined which describes how much each element x , of universe of discourse U (the set of all values that x can take) belongs to \tilde{M} . Different types of membership functions can be used to describe uncertain values. Fuzzy numbers have also been used in many decision making problems. The following definition of fuzzy numbers is most commonly used [18];

Definition 1: The fuzzy number \tilde{M} is a convex normalized fuzzy set of the real line R such that

1. There is exactly one x_0 with $\mu_{\tilde{M}}(x_0) = 1$.

2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

$M_\alpha = \{x \in R \mid M(x) \geq \alpha\}, \forall \alpha \in [0, 1]$ is also called α -cut of \tilde{M} . From definition 1, α -cuts of a fuzzy number \tilde{M} are closed real intervals, that is

$$M_\alpha = [a_\alpha, b_\alpha] \quad \forall \alpha \in [0, 1] \tag{1}$$

2.1. Fuzzy extension principle

Fuzzy extension principle, which was introduced by Zadeh in [19], is an essential principle in fuzzy set theory to generalize the concepts and structures of classical mathematics to fuzzy mathematics as follows [20]

Definition 2: Let U_1, \dots, U_n are UoDs and $U = U_1 \times \dots \times U_n$ be their Cartesian product. Also assume that $\tilde{M}_1, \dots, \tilde{M}_n$ are fuzzy subsets of U_1, \dots, U_n , respectively. Moreover, let $y = f(x_1, \dots, x_n)$ be a mapping from U to Y . Now if $\tilde{B} = f(\tilde{M}_1, \dots, \tilde{M}_n)$, then the membership function of \tilde{B} is defined as follows

$$\mu_{\tilde{B}}(y) = \mu_{f(\tilde{M}_1, \dots, \tilde{M}_n)}(y) = \begin{cases} \sup_{\substack{x_1, \dots, x_n \\ y=f(x_1, \dots, x_n)}} \min\{\mu_{\tilde{M}_1}(x_1), \dots, \mu_{\tilde{M}_n}(x_n)\} & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases} \tag{2}$$

Based on the above definition, one-dimensional operators and two-dimensional operators on fuzzy numbers can be represented by the following definitions, respectively.

Definition 3: Assume that \tilde{M} is a fuzzy number and $f: R \rightarrow R$ is a one-dimensional operator. According to fuzzy extension principle, $f(\tilde{M})$ is a fuzzy set with the following membership function

$$\mu_{f(\tilde{M})}(y) = \begin{cases} \sup_{x, y=f(x)} \mu_{\tilde{M}}(x) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases} \tag{3}$$

In definition 3, if $f(x) = \lambda x$, then the scalar product of the real value λ into fuzzy number \tilde{M} is a fuzzy number with the following membership function

$$\mu_{\lambda \tilde{M}}(x) = \mu_{\tilde{M}}\left(\frac{x}{\lambda}\right) \tag{4}$$

Definition 4: Assume that \tilde{M} and \tilde{N} are fuzzy numbers and $f: R \times R \rightarrow R$ is a two-dimensional operator. According to fuzzy extension principle, $\tilde{M} \otimes \tilde{N}$ is a fuzzy set with the following membership function

$$\mu_{\tilde{M} \otimes \tilde{N}}(z) = \sup_{z=x \otimes y} \min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)) \tag{5}$$

As a special case, the result of the summation operator is also a fuzzy number as follows

$$\mu_{\tilde{M} + \tilde{N}}(z) = \sup_{z=x+y} \min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)) \tag{6}$$

2.2. Ranking fuzzy numbers

Ranking fuzzy numbers seems a necessary procedure in decision making when alternatives are fuzzy numbers. Various methods for ranking fuzzy subsets have been proposed. Yager in [21] introduced a function for ranking fuzzy subsets in unit interval which is based on the integral of mean of the α -cuts. Jain in [22], Baldwin and Guild in [23] were also suggested methods for ordering fuzzy subsets in the unit interval. Ibanez and Munoz in [24] have developed a subjective approach for ranking fuzzy numbers. In this chapter, we use the subjective approach, as introduced in [24].

Ibanez and Munoz in [24] defined the following number as the average index for \tilde{M}

$$V_p(\tilde{M}) = \int_Y f_{\tilde{M}}(\alpha) dP(\alpha) \quad \forall \tilde{M} \in U \tag{7}$$

where Y is a subset of the unit interval and P is a probability distribution on Y .

The definition of $f_{\tilde{M}}$ could be subjective for decision maker. In [24] the following definition for $f_{\tilde{M}}$ has been suggested, in which the parameter λ determines the optimism-pessimism degree of the decision maker

$$f_{\tilde{M}}^\lambda : Y \rightarrow R, \quad f_{\tilde{M}}^\lambda(\alpha) = \lambda b_\alpha + (1 - \lambda) a_\alpha \tag{8}$$

where $\lambda \in [0,1]$ and $M_\alpha = [a_\alpha, b_\alpha]$.

When an optimistic decision maker ($\lambda = 1$) wants to choose the greatest value, the upper extreme of the interval (b_α) would be chosen, i.e. he prefers to choose the greatest possible value. A pessimism person ($\lambda = 0$) on the opposite prefers to decide on the lower extreme of the interval (a_α).

Using $V_p(\cdot)$ in (7), the ordering relations between fuzzy numbers \tilde{A} and \tilde{B} can be given as

$$\tilde{A} \leq \tilde{B} \Leftrightarrow V_p(\tilde{A}) \leq V_p(\tilde{B}) \quad \forall \tilde{A}, \tilde{B} \in U \tag{9}$$

The fuzzy number \tilde{A} is not preferred to \tilde{B} , if and only if their average index is the same

$$\tilde{A} \approx \tilde{B} \Leftrightarrow V_p(\tilde{A}) = V_p(\tilde{B}) \quad \forall \tilde{A}, \tilde{B} \in U \tag{10}$$

For convenience, let $Y = [0,1]$ and P is the Lebesgue measure on $[0, 1]$, as used in [24]. Then, the following $V_p(\tilde{M})$ is derived for a specific λ

$$V_p(\tilde{M}) = \int_0^1 (\lambda b_\alpha + (1-\lambda)a_\alpha) d\alpha \tag{11}$$

As a special case, the average index for triangular fuzzy numbers can be given as follows

$$V_L^\lambda = b\lambda + (a - \frac{1}{2}b) \tag{12}$$

where, a triangular fuzzy number with notation $\tilde{M} = T(a,c,b)$ is used as shown in Figure 1.

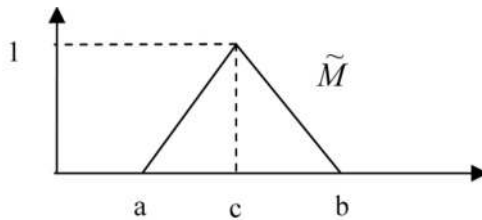


Figure 1. Triangular fuzzy number, $\tilde{M} = T(a,c,b)$.

3. Preference relations

3.1. Fuzzy preference relations

Preference relation is one of the most regular tools for stating decision maker's preferences. In the process of making decision, individuals are asked to give their preferences over an alternative, which is based on their comparison according to one's desire. Various kinds of preference relations have been developed including multiplicative preference relation [25], fuzzy preference relation [26], linguistic preference relation [27] and intuitionistic preference relation [28].

For a decision making situation, let $X = \{x_1, x_2, \dots, x_n\}$ be a discrete set of alternatives. A preference relation P on the set X is defined by a function $\mu_p = X \times X \rightarrow D$, where D is the domain of representation of preference degrees provided by the decision maker for each pair of alternatives. In many situations, due to lack of information about the problems, the goals, constraints and consequences are not precisely known. Because of these uncertainties, fuzzy set theory allows a more flexible framework to express the preferences. Fuzzy preferences show the fuzziness of the decision maker's preferences. In [26], fuzzy preference relation is defined as follows.

Definition 5: A fuzzy preference relation R on the set X is defined as a matrix $R = (r_{ij})_{n \times n}$ with some properties given as

$$\begin{cases} r_{ij} + r_{ji} = 1, \\ r_{ij} \geq 0, \\ r_{ii} = 0.5, \end{cases} \quad \forall i, j = 1, 2, \dots, n \tag{13}$$

where r_{ij} denotes the preference degree of the alternative x_i to x_j .

In particular, $r_{ij} = 0.5$ shows indifference between x_i and x_j . The case of $r_{ij} \geq 0.5$ indicates that x_i is preferred to x_j . As r_{ij} increases, the degree of preference gets larger. Also $r_{ij} = 1$ shows that x_i is absolutely preferred to x_j and vice versa.

Deriving priorities, that is the degrees of importance of alternatives, are the main aspect of preference relations. Quite a number of approaches have been developed to derive priorities from fuzzy preference relations. Lipovetsky and Michael-Conklin [29] introduced an optimization approach and an eigen-problem to produce robust priority estimation of a fuzzy preference relation. Xu [30] developed a weighted least square approach and an eigenvector method for priorities of fuzzy preference relations. Xu and Da [31] proposed the Least Deviation method to obtain the priority vector of a fuzzy preference relation and considered its properties. In this chapter, we use the Least Deviation method presented in [31] to derive priorities from fuzzy preference relation. The algorithm to derive priorities in [31] is mentioned briefly as follows.

Let $R = (r_{ij})_{n \times n}$ be a fuzzy preference relation and $W = (w_1, w_2, \dots, w_n)$ be the priority vector which is going to be calculated such that w_i shows the degree of priority of alternative x_i , with the following properties

$$\sum_{i=1}^n w_i = 1; w_i \geq 0 \tag{14}$$

Let k be the number of iterations:

Step 1. Initialize the weight $W(0) = (w_1, w_2, \dots, w_n)$, specify parameter $0 \leq \varepsilon \leq 1$ and let $k = 0$.

Step 2. Calculate the following term, where $h(r_{ij}) = 9^{(2r_{ij}-1)}$

$$\mu_i(w(k)) = \sum_{j=1}^n \left[h(r_{ij}) \left(\frac{w_j(k)}{w_i(k)} \right) - h(r_{ji}) \left(\frac{w_i(k)}{w_j(k)} \right) \right], \forall i \tag{15}$$

If $|\mu_i(w(k))| \leq \varepsilon$ for all i , then update w by $w(k)$ and go to step 5, otherwise continue with step 3.

Step 3. $|\mu_m(w(k))| = \max_{i=1}^n \{|\mu_i(w(k))|\}$, calculate $T(k)$ and then $x_i(k)$ using

$$T(k) = \sqrt{\left(\sum_{j \neq m} h(r_{mj}) \left(\frac{w_j(k)}{w_m(k)} \right) \right) / \left(\sum_{j \neq m} h(r_{jm}) \left(\frac{w_m(k)}{w_j(k)} \right) \right)} \tag{16}$$

$$x_i(k) = \begin{cases} T(k)w_m(k) \\ w_i(k) \end{cases} \tag{17}$$

Step 4. Set $k = k + 1$ and go to step 2.

Step 5. Obtain w as the priority vector.

Step 6. End.

3.2. Preference ordering of the alternatives

Individuals usually provide their preferences over the alternatives by preference ordering. Let's assume that an expert expresses his preferences on X by a preference ordering $O = \{o(x_1), o(x_2), \dots, o(x_n)\}$. It is assumed that the lower the position of the alternative in the preference ordering, the more contentment for the individual. For instance, an expert states his preference ordering on $X = \{x_1, x_2, x_3\}$ by the following ordering: $o(x_1) = 2, o(x_2) = 1, o(x_3) = 3$. This means that the alternative x_2 is the best and alternative x_3 is the worst.

3.3. Transforming preference ordering into fuzzy preference relation

An alternative satisfies the decision maker according to its position in the ordering preference relation. Various mappings to transform the ordering preference relation into fuzzy preference relation have been proposed [32-34]. In [32] a crisp relation was introduced for assessing the fuzzy preference relation, where the preference between alternatives x_i and x_j depends only on the values of $o(x_i)$ and $o(x_j)$. In [33-34], the following relations was introduced to achieve fuzzy preference relation from an ordering preference

$$v(x_i) = 1 - \frac{o(x_i) - 1}{n - 1} \tag{18}$$

$$r_{ij} = \frac{1}{2} (1 + v(x_i) - v(x_j)) \tag{19}$$

where r_{ij} denotes the preference between alternatives x_i and x_j .

4. Fuzzy games

In a strategic game, there are n players and n_i strategies for player i . Suppose X_i is the strategy set for player i defined as follows

$$X_i = \{x_{1i}, x_{2i}, \dots, x_{ns_i}\} \quad (20)$$

In pure strategy, let s_i denotes the strategy chosen by player i and s_{-i} denotes the strategies chosen by the other players. Then $\Pi_i(s_1, \dots, s_i, \dots, s_n) = \Pi_i(s_i, s_{-i})$ is the payoff achieved by player i . By definition of classical game theory, (s_1^*, \dots, s_n^*) is the pure strategy Nash equilibrium, if and only if [35]

$$\Pi_i(s_1^*, \dots, s_i^*, \dots, s_n^*) \geq \Pi_i(s_1^*, \dots, s_i, \dots, s_n^*) \quad \forall i \in \{1, \dots, n\}, \forall s_i \in X_i \quad (21)$$

In mixed strategy, each player assigns a probability distribution $\sigma_i = (\sigma_{1i}, \dots, \sigma_{ns_i}) \in \Sigma_i$ over his strategies, where Σ_i determines all possible probability distributions for player i . Then, the expected payoff for player i is the real value $E_i(\sigma_1, \dots, \sigma_n)$, where E denotes the expectation operator. In this regard, mixed strategy Nash equilibrium $(\sigma_1^*, \dots, \sigma_n^*)$ is defined as follows

$$E_i(\sigma_1^*, \dots, \sigma_i^*, \dots, \sigma_n^*) \geq E_i(\sigma_1^*, \dots, \sigma_i, \dots, \sigma_n^*) \quad \forall i \in \{1, \dots, n\}, \forall \sigma_i \in \Sigma_i \quad (22)$$

In real games, players must often make their decisions under unclear or fuzzy information. In this regard, there are several approaches for explaining games with fuzzy set theory. As discussed earlier, a game has three main components: a set of players, a set of strategies and a set of payoffs. The set of players is defined as fuzzy set when the concept of coalition in cooperative games is fuzzified. Butnariu in [17] proposed core and stable sets in fuzzy coalition games, and introduced a degree of participation for players in a coalition. Mares in [16] considered fuzzy core in fuzzy cooperative games, where possibility for each fuzzy coalition was considered as fuzzy interval, and an extension of the core in classic TU games. He discussed Shapely value in cooperative games with deterministic characteristics and fuzzy coalitions. In [9], the concept of fuzzy strategies has been introduced. It defines a strategy set consisting of fuzzy subspaces of strategy spaces and assigned a fuzzy payoff for each set of player's strategies. Hence, they have defined a fuzzy inference system (fuzzy If-Then rules). However, they have assumed some real values as strategies and have solved the games with common crisp methods. The first two steps seem rational for modeling any system according to its specifications, however the final step is not a reasonable fuzzy decision making approach to find NEs. Generally, since the final decision of a player is a number in real world problems, they can not adopt fuzzy strategies except when meaningful interpretations exist. However, we should remark that fuzzy strategies are constructive to model the games and calculate the payoffs.

Defining payoffs as fuzzy sets is reasonable in the following two main situations, and lead to fuzzy numbers as payoffs.

1. When there is not sufficient information about the payoffs and they are not inherently repeatable.
2. When calculation of payoffs is difficult or time-consuming. Hence, they are usually defined by fuzzy if-then rules.

In this chapter, we consider pure and mixed strategies in games with fuzzy numbers as payoffs. In this regard, in pure strategies, $\Pi_i(s_i, s_{-i})$ is considered as a fuzzy number. Also, in mixed strategies, the expected values are calculated based on the fuzzy extension principle and definitions 3 and 4, because the payoffs are fuzzy numbers.

5. Playing games with fuzzy numbers as payoffs

In classical game theory, a crisp payoff is either greater than or less than others. However, in fuzzy ones, there are uncertainties in comparing fuzzy payoffs. We model these uncertainties using fuzzy preference relation on the preference ordering of the expected values. In this regard, using ranking fuzzy numbers, a fuzzy preference relation reflecting the uncertainties in payoffs is constructed and then, the resultant priorities of payoffs are considered as the grades of being NE. This definition for the grade of being NE seems meaningful because if a player knows the opponent's strategy, he is satisfied with his strategy by the degree that this strategy has priority for him. The more priority the players get for each strategy, the more possible the strategy is the game's equilibrium. The proposed algorithm to find the grade of being NE for every mixed strategy is shown in Table 1.

1. $\forall i$, do steps 2-9.
2. For player i , determine the optimism-pessimism degree λ .
3. $\forall \sigma_{-i}$, do steps 4-9.
4. Fix the opponent's mixed strategy σ_{-i} .
5. $\forall \sigma_i$, calculate the average index of the resultant expected value according to (11).
6. Order the expected values according to their average index in descending order, according to (9).
7. Transform the ordering preference relation into fuzzy preference relation according to (18-19).
8. Derive priorities of expected values from the resultant fuzzy preference relation based on the least deviation method.
9. For every mixed strategy (σ_i, σ_{-i}) , set its grade of being NE equal to its corresponding priorities.
10. For every mixed strategy (σ_i, σ_{-i}) , set its overall grade of being NE equal to the minimum of its grade of being NE for each player.

Table 1. Algorithm to find the grade of being NE for each mixed strategy

6. Simulation results

6.1. Application in pure strategy

In this subsection, games with two players are considered because of their easier understanding. However, our algorithm can be implemented to more than two players. Consider a fuzzy game, as defined in Table 2. Player one and player two have three strategies, namely a_1, a_2, a_3 and b_1, b_2, b_3 , respectively. Each cell includes two fuzzy triangular numbers as payoffs; the first for player one and the second for player two.

(Π_1, Π_2)	b_1	b_2	b_3
a_1	T(4,5,6),T(1,3,5)	T(5,6,7),T(2,3,4)	T(4,5,6),T(1.5,3,4.5)
a_2	T(2,3,4),T(0,1,2)	T(1,3,5),T(3,4,5)	T(2,3,4),T(1,3,5)
a_3	T(3,4,5),T(2,4,6)	T(3,5,7),T(1,3,5)	T(6,7,8),T(4,6,8)

Table 2. A sample game with fuzzy payoffs

Implementing the algorithm presented in Table 1, there can be several grades of being NE for the game corresponding to different λ s. Tables 3 and 4 present the priority of each payoff according to possible choices of λ .

	b_1	b_2	b_3
a_1	(0.7,0.08)	(0.7,0.7)	(0.22,0.22)
a_2	(0.08,0.08)	(0.08,0.7)	(0.08,0.22)
a_3	(0.22,0.22)	(0.22,0.08)	(0.7,0.7)

Table 3. Priorities of payoffs for $\lambda \leq 1/4$

	b_1	b_2	b_3
a_1	(0.7,0.7)	(0.7,0.08)	(0.22,0.22)
a_2	(0.08,0.08)	(0.08,0.7)	(0.08,0.22)
a_3	(0.22,0.22)	(0.22,0.08)	(0.7,0.7)

Table 4. Priorities of payoffs for $\lambda > 1/4$

The minimum priority of all payoffs in a cell is interpreted now as the grade of being Nash equilibrium of that cell. The results of the mentioned game is tabulated in Tables 5 and 6 for the possible choice of λ , respectively

	b_1	b_2	b_3
a_1	0.08	0.7	0.22
a_2	0.08	0.08	0.08
a_3	0.22	0.08	0.7

Table 5. Grades of being NE for $\lambda \leq 1/4$

	b_1	b_2	b_3
a_1	0.7	0.08	0.22
a_2	0.08	0.08	0.08
a_3	0.22	0.08	0.7

Table 6. Grades of being NE for $\lambda > 1/4$

The difference between the results of Tables 5 and 6 is meaningful. As the player 2 becomes optimistic, he prefers to choose the greatest possible outcome. Hence, between three alternatives $\{T(3,2), T(3,1), T(3,1.5)\}$, $T(3,2)$ has the most priority. But a pessimistic player prefers to choose alternative $T(3,1)$.

6.2. Application in mixed strategy

In this subsection, mixed strategies in bi-matrix games are considered. Consider a fuzzy bi-matrix game as defined in Table 7.

(Π_1, Π_2)	b_1	b_2
a_1	$T(10,11,12), T(3,4,5)$	$T(3,4,5), T(0.5,2,3.5)$
a_2	$T(2,4,6), T(0,1,2)$	$T(6,7,8), T(2,4,6)$

Table 7. A sample game with fuzzy payoffs

Using the proposed approach, every mixed strategy has a grade of being NE as shown in Figures 2 and 3 for each neutral player ($\lambda = 0.5$).

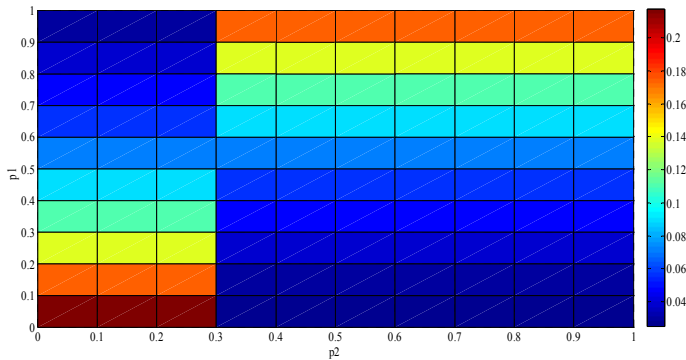


Figure 2. Grades of being NE for Player 1

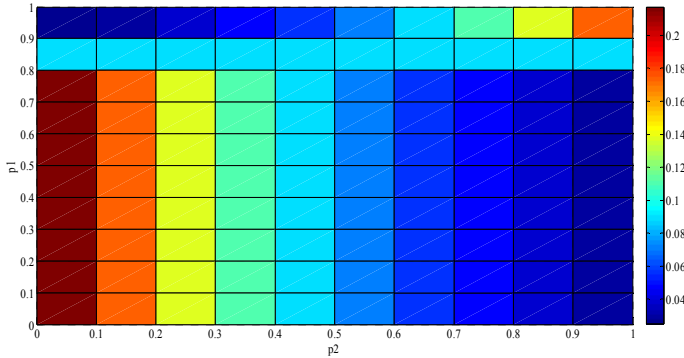


Figure 3. Grades of being NE for Player 2

Now, the minimum priorities for players can be interpreted as the overall grades of being NE. We remark that the minimum operator can be replaced by any other T-norms. The results of the game presented in Table 7, is shown in Figure 4.

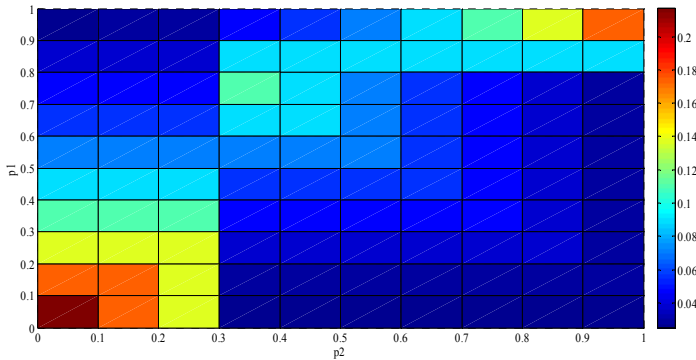


Figure 4. Grades of being NE for neutral players

Grade of being NE is high around two pure strategies, that is $(p_1 = (0,1), p_2 = (0,1))$ and $(p_1 = (1,0), p_2 = (1,0))$. The mixed strategy $(p_1 = (0.7,0.3), p_2 = (0.3,0.7))$ has also high grade of being NE.

In addition, the effect of optimism-pessimism degree of players in distribution of Nash grades is studied. For instance, if two players are optimistic ($\lambda = 1$), the Nash grades are the same as Figure 5. If two players are pessimistic ($\lambda = 0$), the Nash grades are the same as Figure 6.

The difference between the results of Figures 4 and 5 is important. As the player becomes optimistic, he prefers to choose the greatest possible outcome. Hence, payoff $T(4,1.5)$ gets more priority than payoff $T(4,1)$. In addition, Nash grades for neutral players are approximately the combination of Nash grades for optimistic and pessimistic players.

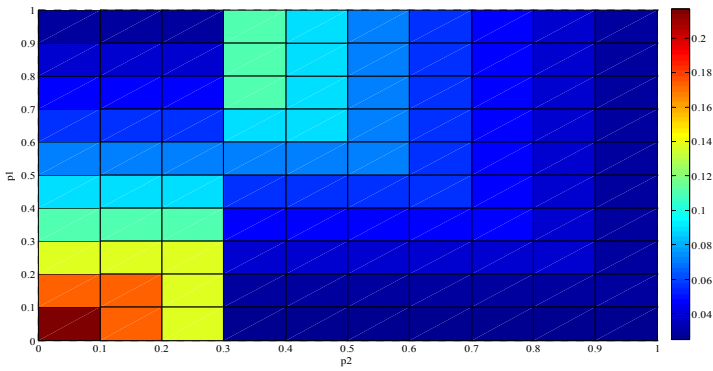


Figure 5. Grades of being NE for optimistic players

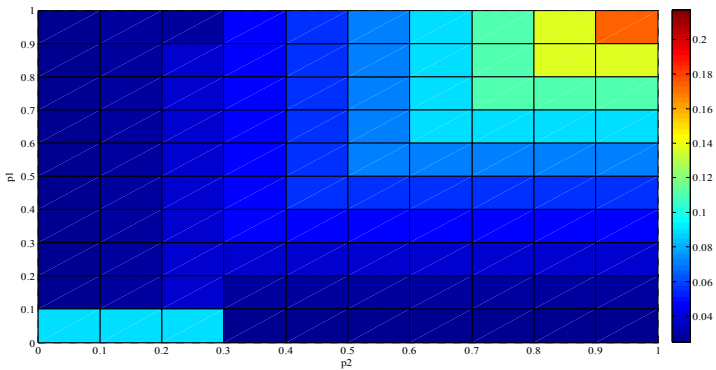


Figure 6. Grades of being NE for pessimistic players

7. Conclusion

In this chapter, some concepts in fuzzy sets including fuzzy numbers, fuzzy extension principle, ranking of fuzzy numbers and fuzzy preference relations were briefly introduced and consequently used to develop a new approach for practically analyzing the games with fuzzy numbers as payoffs. In this regard, a fuzzy preference relation was constructed on the preference ordering of payoffs using ranking fuzzy numbers. The priority of each payoff then was derived using the least deviation method. The priorities of payoffs were interpreted as the grades of being NE. We should remark that, in this chapter, we were not looking to show the pure and mixed strategy NEs, but rather we tried to assign a graded membership to each strategy to determine how much it is NE, i.e. it has the possibility for being NE.

Author details

Alireza Chakeri

Intelligent Systems Laboratory, Electrical Engineering Department, Sharif University of Technology, Tehran, Iran

Nasser Sadati

Intelligent Systems Laboratory, Electrical Engineering Department, Sharif University of Technology, Tehran, Iran

Electrical and Computer Engineering Department, The University of British Columbia, Vancouver, BC, Canada

Guy A. Dumont

Electrical and Computer Engineering Department, The University of British Columbia, Vancouver, BC, Canada

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