Chapter 4

Spectral Theory of Operators on Manifolds

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Additional information is available at the end of the chapter

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Abstract

Differential operators that are defined on a differentiable manifold can be used to study various properties of manifolds. The spectrum and eigenfunctions play a very significant role in this process. The objective of this chapter is to develop the heat equation method and to describe how it can be used to prove the Hodge Theorem. The Minakshisundaram-Pleijel parametrix and asymptotic expansion are then derived. The heat equation asymptotics can be used to give a development of the Gauss-Bonnet theorem for two-dimensional manifolds.

Keywords: manifold, operator, differential form, Hodge theory, eigenvalue, partial differential operator, Gauss-Bonnet

1. Introduction

Topological and geometric properties of a manifold can be characterized and further studied by means of differential operators, which can be introduced on the manifold. The only natural differential operator on a manifold is the exterior derivative operator which takes k-forms to $k+1$ forms. This operation is defined purely in terms of the smooth structure of the manifold, used to define de Rham cohomology groups. These groups can be related to other topological quantities such as the Euler characteristic. When a Riemannian metric is defined on the manifold, a set of differential operators can be introduced. The Laplacian on k-forms is perhaps the most well known, as well as other elliptic operators.

On a compact manifold, the spectrum of the Laplacian on k-forms contains topological as well as geometric information about the manifold. The Hodge theorem relates the dimension of the kernel of the Laplacian to the k-th Betti number requiring them to be equal. The Laplacian determines the Euler characteristic of the manifold. A sophisticated approach to obtaining information related to the manifold is to consider the heat equation on k-forms with its solution given by the heat semigroup [1–3].

The heat kernel is one of the more important objects in such diverse areas as global analysis, spectral geometry, differential geometry, as well as in mathematical physics in general. As an example from physics, the main objects that are investigated in quantum field theory are described by Green functions of self-adjoint, elliptic partial differential operators on manifolds as well as their spectral invariants, such as functional determinants. In spectral geometry, there is interest in the relation of the spectrum of natural elliptic partial differential operators with respect to the geometry of the manifold [4–6].

Currently, there is great interest in the study of nontrivial links between the spectral invariants and nonlinear, completely integrable evolutionary systems, such as the Korteweg-de Vries hierarchy. In many interesting situations, these systems are actually infinite-dimensional Hamiltonian systems. The spectral invariants of a linear elliptic partial differential operator are nothing but the integrals of motion of the system. There are many other applications to physics such as to gauge theories and gravity [7].

In general, the existence of nonisometric isospectral manifolds implies that the spectrum alone does not determine the geometry entirely. It is also important to study more general invariants of partial differential operators that are not spectral invariants. This means that they depend not only on the eigenvalues but also on the eigenfunctions of the operator. Therefore, they contain much more information with respect to the underlying geometry of the manifold.

The spectrum of a differential operator is not only studied directly, but the related spectral functions such as the spectral traces of functions of the operator, such as the zeta function and the heat trace, are relevant as well [8, 9]. Often the spectrum is not known exactly, which is why different asymptotic regimes are investigated [10, 11]. The small parameter asymptotic expansion of the heat trace yields information concerning the asymptotic properties of the spectrum. The trace of the heat semigroup as the parameter approaches zero is controlled by an infinite sequence of geometric quantities, such as the volume of the manifold and the integral of the scalar curvature of the manifold. The large parameter behavior of the traces of the heat kernels is parameter independent and in fact equals the Euler characteristic of the manifold. The small parameter behavior is given by an integral of a complicated curvature-dependent expression. It is quite remarkable that when the dimension of the manifold equals two, the equality of the short- and long-term behaviors of the heat flow implies the classic Gauss-Bonnet theorem. The main objectives of the chapter are to develop the heat equation approach with Schrödinger operator on a vector bundle and outline how it leads to the Hodge theorem [12, 13]. The heat equation asymptotics will be developed [14, 15] andit is seen that the Gauss-Bonnet theorem can be proved for a two-dimensional manifold based on it. Moreover, this kind of approach implies that there is a generalization of the Gauss-Bonnet theorem as well in higher dimensions greater than two [16, 17].

2. Geometrical preliminaries

For an n-dimensional Riemannian manifold M, an orthonormal moving frame $\{e_1,\dots,e_n\}$ can be chosen with $\{\omega_1,\dots,\omega_n\}$ the accompanying dual coframe which satisfy

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$$\omega_i(e_j) = \delta_{ij}, \qquad i,j = 1,\ldots,n \tag{1}$$

It is then possible to define a system of one-forms ω_{ij} and two-forms Ω_{ij} by solving the equations,

$$\nabla_X e_i = \sum_j \omega_{ji}(X)\, e_j, \qquad R(X,Y)e_i = \sum_j \Omega_{ji}(X,Y)e_j \tag{2}$$

It then follows that the Christoffel coefficients and components of the Riemann tensor for M are

$$\omega_{ji}(e_k) = \sum_a \langle \omega_{aj}(e_k)e_a, e_i\rangle_g = \langle \nabla_{e_k} e_j, e_i\rangle_g = \Gamma^i_{kj} \tag{3}$$

$$\Omega_{ij}(e_k,e_s) = \sum_a \langle \Omega_{aj}(e_k,e_s)e_a, e_i\rangle_g = \langle R(e_k,e_s)e_j, e_i\rangle_g = R_{ksji} \tag{4}$$

The inner product induced by the Riemannian metric on M is denoted here by $\langle \cdot , \cdot \rangle : \Gamma(TM) \times \Gamma(TM) \to \mathcal{F}(M)$ and it induces a metric on $\Lambda^k(M)$ as well. Using the Riemannian metric and the measure on M, an inner product denoted $\langle\langle \cdot , \cdot \rangle\rangle : \Lambda^k(M) \times \Lambda^k(M) \to \mathbb{R}$ can be defined on $\Lambda^k(M)$ so that for $\alpha, \beta \in \Lambda^k(M)$,

$$\langle\langle \alpha,\beta \rangle\rangle = \int_M \langle \alpha,\beta\rangle_g \, dv_M \tag{5}$$

where if $(x^1,\ldots,x^m)$ is a system of local coordinates,

$$dv_M = \det(g_{ij})\, dx^1 \wedge \ldots \wedge dx^m$$

is the Riemannian measure on M. Clearly, $\langle\langle \alpha,\beta\rangle\rangle$ is linear with respect to α, β and $\langle\langle \alpha,\alpha\rangle\rangle \geq 0$ with equality if and only if $\alpha = 0$. Hodge introduced a star homomorphism $* : \Lambda^k(M) \to \Lambda^{n-k}(M)$, which is defined next.

Definition 2.1. (i) For $\omega = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k}\, \omega_{i_1} \wedge \cdots \omega_{i_k}$, define

$$^*\omega = \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_{n-k}}} f_{i_1\cdots i_k} \epsilon(i_1,\ldots, i_k, j_1,\ldots, j_{n-k}) \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}},$$

where ϵ is 1, −1, or 0 depending on whether $(i_1,\ldots,i_k,j_1,\ldots,j_{n-k})$ is an even or odd permutation of $(1,\ldots,n)$, respectively.

(ii) If M is an oriented Riemannian manifold with dimension n, define the operator

$$\delta = (-1)^{nk+n+1*} d^* : \Lambda^k(M) \to \Lambda^{k-1}(M) \tag{6}$$

In terms of the two operators d and δ, the Laplacian acting on k-forms can be defined on the two subspaces

$$\Lambda^{\text{even}}(M) = \oplus_{\text{even}} \Lambda^k(M), \qquad \Lambda^{\text{odd}}(M) = \oplus_{\text{odd}} \Lambda^k(M) \tag{7}$$

The operator $d + \delta$ can be regarded as the operators on these subspaces,

$$D_0 = d + \delta : \Lambda^{\text{even}}(M) \to \Lambda^{\text{odd}}(M), \quad D_1 = d + \delta : \Lambda^{\text{odd}}(M) \to \Lambda^{\text{even}}(M) \tag{8}$$

Definition 2.2. Let M be a Riemannian manifold, then the operator

$$D_0 = d + \delta : \Lambda^{\text{even}}(M) \to \Lambda^{\text{odd}}(M) \tag{9}$$

is called the Hodge-de Rham operator. It has the property that it is a self-conjugate operator, $D_0^* = D_1$ and $D_1^* = D_0$. It is useful in studying the Laplacian to have a formula for the operator $\Delta = (d + \delta)^2$ and hence for $D_0^* D_0$ and $D_1^* D_1$ as well.

Let $\{e_1,\dots,e_n\}$ be an orthonormal moving frame defined on an open set U. Define as well the pair of operators

$$E_j^+ = \omega_j \wedge \cdot + i(e_j) : \Lambda^*(U) \to \Lambda^*(U), \qquad E_j^- = \omega_j \wedge \cdot - i(e_j) : \Lambda^*(U) \to \Lambda^*(U) \tag{10}$$

Lemma 2.1. The operators $E_j^{\pm}$ satisfy the following relations

$$E_i^+ E_j^+ + E_j^+ E_i^+ = 2\delta_{ij}, \quad E_i^+ E_j^- + E_j^- E_i^+ = 0, \quad E_i^- E_j^- + E_j^- E_i^- = -2\delta_{ij} \tag{11}$$

If M is a Riemannian manifold and $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is a Levi-Civita connection, then a connection on the space $\Lambda^*(M)$, namely $(X,\omega) \to \nabla_X \omega$, can also be defined such that

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y), \quad Y \in \Gamma(TM)$$

The connection may be regarded as a first-order derivative operator $(X,Y,\omega) \to D(X,Y)\omega$.

Definition 2.3. The second-order derivative operator $(X,Y,\omega) \to D(X,Y)\omega$ is defined to be

$$D(X,Y)\omega = \nabla_X \nabla_Y \omega - \nabla_{\nabla_X Y} \omega \tag{12}$$

In terms of the operator (Eq. (12)), define a second-order differential operator $\Delta_0 : \Lambda^*(M) \to \Lambda^*(M)$ by

$$\Delta_0 = \sum_i D(e_i,e_i), \tag{13}$$

where $\{e_i\}_1^n$ is an orthonormal moving frame. The operator Δ_0 in Eq. (13) is referred to as the *Laplace-Beltrami* operator.

Theorem 2.1. (Weitzenböck) Let M be a Riemannian manifold M with an associated orthonormal moving frame $\{e_i\}_1^n$. The Laplace operator can be expressed as

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$$\Delta = (d+\delta)^2 = -\Delta_0 - \frac{1}{8}\sum_{i,j,k,s} R_{ijks}\, E_i^+ E_j^+ E_k^- E_s^- + \frac{1}{4}R \tag{14}$$

In Eq. (14), R is the scalar curvature, $R = -\sum_{i,j} R_{ijij}$ and Δ_0 is the Laplace-Beltrami operator (13).

The operator defined by Eq. (14) does not contain first-order covariant derivatives and is of a type called a Schrödinger operator. Thus, Weitzenböck formula (14) implies the that Laplacian can be expressed in the form $\Delta = -\Delta_0 - F$ and is an elliptic operator. The Schrödinger operator (14) can be used to define an operator that plays an important role in mathematical physics. The heat operator is defined to be

$$\mathcal{H} = \frac{\partial}{\partial t} + \Delta \tag{15}$$

The crucial point for the theory of the heat operator is the existence of a fundamental solution. In fact, the Hodge theorem can be proved by making use of the fundamental solution.

Definition 2.4. Let M be a Riemannian manifold, $\pi : E \to M$ is a vector bundle with connection. Let $\Delta_0 : \Gamma(E) \to \Gamma(E)$ be the Laplace-Beltrami operator, which is defined by means of the Levi-Civita connection on M and the connection on the vector bundle E. Let $F : \Gamma(E) \to \Gamma(E)$ be a $\mathcal{F}(M)$-linear map. Then, $\Delta = -\Delta_0 - F$ is a Schrödinger operator. If a family of R-linear maps

$$G(t,q,p) : E_p \to E_q$$

with parameter $t > 0$ and $q,p \in M$ satisfies the following three conditions, the family is called a fundamental solution of the heat operator (15) where $E_p = \pi^{-1}(p)$. First, $G(t,q,p) : E_p \to E_q$ is an R-linear map of vector spaces and continuous in all variables t,q,p. Second, for a fixed $w \in E_p$, let $\theta(t,q) = G(t,q,p)w$, for all $t > 0$, then θ has first and second continuous derivatives in t and q, respectively andsatisfies the heat equation, which for $t > 0$ is given by $\mathcal{H}\theta(t,q) = 0$, which can be written as

$$\left(\frac{\partial}{\partial t} + \Delta_q\right) G(t,q,p) = 0 \tag{16}$$

where Δ_q acts on the variable q. Finally, if φ is a continuous section of the vector bundle E, then

$$\lim_{t \to 0^+} \int_M G(t,q,p)\varphi(p)\, dv_p = \varphi(q)$$

for all φ, where dv_p is the volume measure with respect to the coordinates of p given in terms of the Riemannian metric.

Definition 2.5. Suppose a $G_0(t,q,p)$ is given. The following procedure taking $G_0(t,q,p)$ to $G(t,q,p)$ is called the Levi algorithm:

$$K_0(t,q,p) = \left(\frac{\partial}{\partial t} + \Delta_q\right) G(t,q,p),$$
$$K_{m+1}(t,q,p) = \int_0^t d\tau \int_M K_0(t-\tau,q,z) K_m(\tau,z,p)\, dv_z$$
$$\overline{K}(t,q,p) = \sum_{m=0}^{\infty} (-1)^{m+1} K_m(t,q,p), \tag{17}$$
$$G(t,q,p) = G_0(t,q,p) + \int_0^t d\tau \int_M G_0(t-\tau,q,z)\overline{K}(\tau,z,p)\, dv_z$$

The Cauchy problem can be formulated for the heat equation such that existence, regularity and uniqueness of solution can be established. The Hilbert-Schmidt theorem can be invoked to develop a Fourier expansion theorem applicable to this Schrödinger operator.

Suppose $\Delta : \Gamma(E) \rightarrow \Gamma(E)$ is a self-adjoint nonnegative Schrödinger operator, then there exists a set of C^∞ sections $\{\psi_i\} \subset \Gamma(E)$ such that

$$\langle\langle \psi_i, \psi_j \rangle\rangle = \int_M \langle \psi_i(x), \psi_j(x) \rangle\, dv_x = \delta_{ij}$$

Moreover, denoting the completion of the inner product space $\Gamma(E)$ by $\overline{\Gamma(E)}$, the set $\{\psi_i\}$ is a complete set in $\overline{\Gamma(E)}$, so for any $\psi \in \overline{\Gamma(E)}$,

$$\psi = \sum_{i=1}^{\infty} \langle\langle \psi, \psi_i \rangle\rangle\, \psi_i$$

Finally, the set $\{\psi_i\}$ satisfies the equation

$$\Delta \psi_i = \lambda_i \psi_i, \qquad T_t \psi_i = e^{-t\lambda_i} \psi_i$$

where λ_i are the eigenvalues of Δ andform an increasing sequence: $0 \le \lambda_1 \le \lambda_2 \le \cdots$ where $\lim_{k\rightarrow\infty} \lambda_k = \infty$.

Denote $U(t,q)$ by $(T_t\psi)(q)$ when $U(0,q) = \psi(q)$ and T_t satisfies the semigroup property andT_t is a self-adjoint, compact operator.

Theorem 2.2. Let $G(t,q,p)$ be the fundamental solution of the heat operator (15), then

$$G(t,q,p)w = \sum_{i=1}^{\infty} e^{\lambda_i t} \langle\, \psi_i(p), w \rangle \psi_i(q) \tag{18}$$

with $w \in E_p$ holds in $\overline{\Gamma(E)}$.

Proof: For fixed $t > 0$ and $w \in E_p$, expand $G(t,q,p)w$ in terms of eigenfunctions $\psi_i(q)$,

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$$G(t,q,p)w = \sum_{i=1}^{\infty} \sigma_i(t,p,w)\psi_i(q), \quad \sigma_i(t,p,w) = \int_M \langle \psi_i(q), G(t,q,p)w \rangle \, dv_q$$

Differentiating with respect to t and using $\Delta\psi_i = \lambda_i\psi_i$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_i(t,p,w) &= \int_M \langle \psi_i(q), \frac{\partial}{\partial t} G(t,q,p)w \rangle \, dv_q = \int_M \langle \psi_i(q), -\Delta_q G(t,q,p)w \rangle dv_q \\ &= -\int_M \langle \Delta_q \psi_i(q), G(t,q,p)w \rangle \, dv_q = -\lambda_i \int_M \langle \psi_i(q), G(t,q,p)w \rangle \, dv_q \\ &= -\lambda_i \sigma_i(t,p,w)\end{aligned}$$

It follows from this that

$$\sigma_i(t,p,w) = c_i(p,w)e^{-\lambda_i t}$$

and since σ_i depend linearly on w, so $c_i(p,w) = c_i(p)w$, where $c_i(p) : E_p \to \mathbb{R}$ is a linear function. There exists $\tilde{c}_i(p)$ independent of w such that $c_i(p)w = \langle \tilde{c}_i(p), w \rangle$ so that

$$G(t,q,p)w = \sum_{i=1}^{\infty} e^{\lambda_i t} \, \psi_i(q) \langle \tilde{c}_i(p), w \rangle$$

Consequently, for any $\beta \in \Gamma(E)$, we have

$$\beta(q) = \lim_{t \to 0} \int_M G(t,q,p)\beta(p) \, dv_p = \sum_{k=1}^{\infty} \psi_k(q) \int_M \langle \tilde{c}_k(p), \beta(p) \rangle \, dv_p$$

Moreover, $\beta(q)$ can also be expanded in terms of the ψ_k basis set,

$$\beta(q) = \sum_{k=1}^{\infty} \psi_k(q) \int_M \langle \psi_k(p), \beta(p) \rangle \, dv_p$$

Upon comparing these last two expressions, it is clear that $\tilde{c}_k(p) = \psi_k(p)$ for all k andwe are done.

One application of the heat equation method developed so far is to develop and give a proof of the Hodge theorem.

Theorem 2.3. Let M, E, Δ be defined as done already, then

1. $H = \{\varphi \in \Gamma(E) | \Delta\varphi = 0\}$ is a finite-dimensional vector space.
2. For any $\psi \in \Gamma(E)$, there is a unique decomposition of ψ as $\psi = \psi_1 \oplus \psi_2$, where $\psi_1 \in H$ and $\psi_2 \in \Delta(\Gamma(E))$.

The first part is a direct consequence of the expansion theorem and due to the fact $H \perp \Delta(\Gamma(E))$, the decomposition is unique.

The Hodge theorem has many applications, but one in particular fits here. It is used in conjunction with the de Rham cohomology group $H^*_{dR}(M)$. Define

$$Z^k(M) = \ker\{d : \Lambda^k(M) \to \Lambda^{k+1}(M)\} \equiv \{\alpha \in \Lambda^k(M) |\ d\alpha = 0\} \tag{19}$$

$$B^k(M) = \ \mathrm{Im}\ \{d : \Lambda^{k-1}(M) \to \Lambda^k(M)\} \equiv d(\Lambda^{k-1}(M)) \tag{20}$$

Since $d^2 = 0$, it follows that $B^k(M) \subset Z^k(M)$ andthe k-th de Rham cohomology group of M is defined to be

$$H^k_{dR}(M) = Z^k(M)/B^k(M) \tag{21}$$

From Eq. (21), construct

$$H^*_{dR}(M) = \oplus_k H^k_{dR}(M) \tag{22}$$

In 1935, Hodge claimed a theorem, which stated every element in $H^k_{dR}(M)$ can be represented by a unique harmonic form α, one which satisfies both $d\alpha = 0$ and $\delta\alpha = 0$. Denote the set of harmonic forms as $H^k(M)$.

Theorem 2.4. Let M be a Riemannian manifold of dimension n, then

$$H^k(M) = \ker\ \{d + \delta : \Lambda^k(M) \to \Lambda^*(M)\} = \ker\ \{\Delta : \Lambda^k(M) \to \Lambda^k(M)\} \tag{23}$$

where $\Delta = (d + \delta)^2$.

Proof: Since $\Delta = d\delta + \delta d$, this implies that $\Delta(\Lambda^k(M)) \subset \Lambda^k(M)$ andit is clear that

$H^k(M) \subset \ker\{d + \delta : \Lambda^k(M) \to \Lambda^*(M)\} \subset \ker\{\Delta : \Lambda^k(M) \to \Lambda^*(M)\} = \ker\ \{\Delta : \Lambda^k(M) \to \Lambda^k(M)\}$. To finish the proof, it suffices to show that $\ker\{\Delta : \Lambda^k(M) \to \Lambda^k(M)\} \subset H^k(M)$. If $\alpha \in \ker\{\Delta : \Lambda^k(M) \to \Lambda^k(M)\}$, that is $\Delta\alpha = 0$, then

$$\begin{aligned}\langle\langle \Delta\alpha,\alpha,\rangle\rangle = \langle\langle (d+\delta)^2\alpha,\alpha\rangle\rangle = \langle\langle (d+\delta)\alpha,(d+\delta)\alpha\rangle\rangle = \langle\langle d\alpha,d\alpha\rangle\rangle + \langle\langle \delta\alpha,\delta\alpha\rangle\rangle + 2\langle\langle d\alpha,\delta\alpha\rangle\rangle \\ = \langle\langle d\alpha,d\alpha\rangle\rangle + \langle\langle \delta\alpha,\delta\alpha\rangle\rangle = 0\end{aligned}$$

This implies that $d\alpha = 0$ and $\delta\alpha = 0$, hence $\alpha \in H^k(M)$.

Theorem 2.5. Let M be a Riemannian manifold of dimension n, then

1. $H^k(M)$ is a finite dimensional vector space for $k = 0, 1, 2, \ldots, n$.
2. There is an orthogonal decomposition of $\Lambda^k(M)$ as

$$\Lambda^k(M) = H^k(M) + d(\Lambda^{k-1}(M)) + \delta(\Lambda^{k+1}(M)) \tag{24}$$

Proof: By Theorem 2.1, $\Delta : \Lambda^k(M) \to \Lambda^k(M)$ is a Schrödinger operator, so the Hodge theorem applies. Thus $H^k(M)$ is of finite dimension, so the first holds. The second part of the Hodge theorem is $\Lambda^k(M) = H^k(M) + \Delta(\Lambda^k(M))$. Since $\Delta(\Lambda^k(M)) \subset d(\Lambda^{k-1}(M)) + \delta(\Lambda^{k+1}(M))$, we have $\Lambda^k(M) = H^k(M) + d(\Lambda^{k-1}(M)) + \delta(\Lambda^{k+1}(M))$. The three spaces in this decomposition are orthogonal to each other, so (ii) holds as well.

Theorem 2.6. (Duality theorem) For an oriented Riemannian manifold M of dimension n, the star isomorphism $* : H^k(M) \to H^{n-k}(M)$ induces an isomorphism

$$H^k_{dR}(M) \simeq H^{n-k}_{dR}(M) \tag{25}$$

The k-th Betti number defined as $b_k(M) = \dim H^k(M, \mathbb{R})$ also satisfies $b_k(M) = b_{n-k}(M)$ for $0 \leq k \leq n$.

3. The Minakshisundaran-Pleijel paramatrix

Let M be a Riemannian manifold with dimension n and E a vector bundle over M with an inner product and a metric connection. Here, the following formal power series is considered with a special transcendental multiplier $e^{-\rho^2/4t}$ and parameters $(t,p,q) \in (0,\infty) \times M \times M$, defined by

$$H_\infty(t,q,p) = \frac{1}{(4\pi t)^{n/2}} e^{-\rho^2/4t} \sum_{k=0}^{\infty} t^k \, u_k(p,q) : E_p \to E_q \tag{26}$$

In Eq. (26), the function $\rho = \rho(p,q)$ is the metric distance between p and q in M, $E_p = \pi^{-1}(p)$ is the fiber of E over p and $u_k(p,q) : E_p \to E_q$ are R-linear map.

It is the objective to find conditions for which Eq. (26) satisfies the heat equation or the following equality:

$$\left(\frac{\partial}{\partial t} + \Delta_q\right) H_\infty(t,q,p) w = 0 \tag{27}$$

To carry out this, a normal coordinate system denoted by $\{x_1,\ldots,x_n\}$ is chosen in a neighborhood of point p and is centered at p. This means that if q is in this neighborhood about p, which has coordinates $(x_1,\ldots,x_n)$, then the function $\rho(p,q)$ is

$$\rho(p, q) = \sqrt{x_1^2 + \cdots + x_n^2} \tag{28}$$

In terms of these coordinates, we calculate the components of g,

$$g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle, \qquad G = \det(g_{ij}) \tag{29}$$

and define the differential operator

$$\hat{\partial} = \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}$$

The notion of the heat operator (15) on Eq. (26) is worked out one term at a time. First, the derivative with respect to t is calculated

$$\begin{aligned} \frac{\partial}{\partial t} H_\infty(t,p,q)w &= \frac{1}{(4\pi t)^{n/2}} e^{-\rho^2/4t} \left\{ \left(\frac{\rho^2}{4t^2} - \frac{n}{2t} \right) \sum_{k=0}^{\infty} t^k \, u_k(p,\, q)w + \sum_{k=0}^{\infty} k t^{k-1} u_k(p,q)w \right\} \\ &= \frac{1}{(4\pi t)^{n/2}} \; e^{-\rho^2/4t} \sum_{k=0}^{\infty} \left\{ \frac{\rho^2}{4t^2} - \frac{n}{2t} + \frac{k}{t} \right\} t^k u_k(p,q)w \end{aligned} \tag{30}$$

It is very convenient to abbreviate the function appearing in front of the sum in Eq. (30) as follows:

$$\Phi(\rho) = \frac{e^{-\rho^2/4t}}{(4\pi t)^{n/2}} \tag{31}$$

Let $\{e_1,\ldots,e_n\}$ be a frame that is parallel along geodesics passing through p and satisfies

$$e_i(p) = \frac{\partial}{\partial x_i}|_p$$

In terms of the function in Eq. (31), the operator Δ_0 acting on Eq. (26) is given as

$$\begin{aligned} \Delta_0 H_\infty(t,p,q)w = (\Delta_0 \; \Phi) \cdot \left(\sum_{k=0}^{\infty} t^k u_k(p,q)w \right) \\ +2 \sum_{a=1}^{n} (e_a \Phi) \cdot \nabla_{e_a} \left(\sum_{k=0}^{\infty} t^k u_k(p,q)w \right) + \Phi \cdot \Delta_0 \left(\sum_{k=0}^{\infty} t^k u_k(p,q)w \right) \end{aligned} \tag{32}$$

The individual components of (32) can be calculated as follows; since Φ is a function $\nabla_{e_a} \Phi = e_a \Phi$ and so

$$\begin{aligned} e_a \Phi(\rho) &= \Phi'(\rho) e_a(\rho), \\ \Delta_0 \Phi = \sum_a \{ e_a \, e_a \Phi(\rho) - (\nabla_{e_a} e_a) \Phi(\rho) \} &= \Phi''(\rho) \cdot \sum_a (e_a \rho)^2 + \Phi'(\rho) \cdot \Delta_0 \rho, \\ \Phi'(\rho) &= -\frac{\rho}{2t} \Phi(\rho), \\ \Phi''(\rho) &= \left(\frac{\rho^2}{4t^2} - \frac{1}{2t} \right) \Phi(\rho) \end{aligned} \tag{33}$$

Consequently,

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$$e_a\rho = \frac{x_a}{\rho}, \qquad \sum_a (e_a\rho)^2 = 1, \qquad \Delta_0\rho = \frac{n-1}{\rho} + \frac{1}{\rho}\hat{\partial}\ \log\sqrt{G}$$

and the Laplace-Beltrami operator on the function Φ is given by

$$\Delta_0\ \Phi = \Phi(\rho)\left(\left(\frac{\rho^2}{4t^2} - \frac{1}{2t}\right) - \frac{1}{2t}(n-1-\hat{\partial}\log\sqrt{G})\right) \tag{34}$$

Expression (34) goes into the first term on the right side of Eq. (32). The second term on the right-hand side of (32) takes the form,

$$\begin{aligned} 2\sum_{a=1}^{n}\ (e_a\Phi)\cdot\nabla_{e_a}\left(\sum_{k=0}^{\infty} t^k u_k(p,q)w\right) &= 2\Phi'(\rho)\ \sum_{a=1}^{n}\ \frac{x_a}{\rho}\cdot\nabla_{e_a}\left(\sum_{k=0}^{\infty} t^k u_k(p,q)w\right) \\ &= -\frac{\rho}{t}\Phi(\rho)\nabla_{\hat{\partial}/\rho}\left(\sum_{k=0}^{\infty} t^k u_k(p,q)w\right) \end{aligned} \tag{35}$$

Substituting these results into (32), it follows that

$$\Delta_0\ H_\infty(t,q,p) = \Phi(\rho)\ \left[\frac{\rho^2}{4t^2} - \frac{1}{2t} - \frac{1}{2t}(n-1-\hat{\partial}\log\sqrt{G}) - \frac{\rho}{t}\nabla_{\hat{\partial}/\rho} + \Delta_0\right]\sum_{m=0}^{\infty} t^m u_m(p,q)w \tag{36}$$

Combining Eq. (36) with the derivative of H_∞ with respect to t in Eq. (35), the following version of the heat equation results:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_0 - F\right)H_\infty(t,q,p)w = \Phi\Bigg[\left(\nabla_{\hat{\partial}} + \frac{1}{4G}\hat{\partial}G\right)\cdot\frac{1}{t}u_0(p,q)w + \sum_{k=1}^{\infty}\Bigg[\left(\nabla_{\hat{\partial}} + k + \frac{1}{4G}\hat{\partial}G\right)u_k(p,q)w \\ -(\Delta_0 + F)u_{k-1}(p,q)w\Bigg]t^{k-1}\Bigg] \end{aligned} \tag{37}$$

This is summarized in the following Lemma.

Lemma 3.1. Heat equation (27) for $H_\infty(t,p,q)$ is equivalent to

$$\left(\nabla_{\hat{\partial}} + k + \frac{1}{4G}\hat{\partial}G\right)\ u_k(p,q)w = (\Delta_0 + F)\ u_{k-1}(p,q)w \tag{38}$$

for all $k = 0, 1, 2, \ldots$ and Eq. (38) is initialized with $u_{-1}(p,q) = 0$.

In fact, for fixed $p \in M$ and $w \in E_p$, there always exists a unique solution to problem (Eq. (38)) over a small coordinate neighborhood about p.

Definition 3.1. Denote the solution of Eq. (38) by $u(p,q)w$, which depends linearly on w. Then, $u_m(p,q) : E_p \to E_q$ and the Minakshisundaram-Pleijel parametrix for heat operator (Eq. 15) is defined by

$$H_{\infty}(t,p,q) = \frac{1}{(4\pi t)^{n/2}} e^{-\rho^2/4t} \sum_{m=0}^{\infty} t^m u_m(p,q) : E_p \to E_q \tag{39}$$

Based on Eq. (39), the N-truncated parametrix is defined based on Eq. (39) to be

$$H_N(t,q,p) = \frac{1}{(4\pi t)^{n/2}} e^{-\rho^2/4t} \sum_{m=0}^{N} t^m u_m(p,q) : E_p \to E_q \tag{40}$$

Theorem 3.1. Choose a smooth function $\phi : M \times M \to M$ and let $G_0(t,q,p) = \phi(q,p)H_N(t,q,p)$. Then $G_0(t,q,p)$ is a k-th initial solution of the heat operator (15), where $k = \lfloor \frac{N}{2} - \frac{n}{4} \rfloor$ and $\lfloor z \rfloor$ is the greatest integer less than or equal to z.

Proof: Clearly, G_0 is a linear map of vector spaces andis continuous and C^{∞} in all parameters. From the previous calculation, it holds that

$$\left(\frac{\partial}{\partial t} - \Delta_0 - F\right) H_N(t,q,p)w = -\frac{1}{(4\pi t)^{n/2}} e^{-\rho^2/4t} t^{N-\frac{n}{2}} (\Delta_0 + F) u_N(p,q)w \tag{41}$$

and $u_N(p,q)$ is C^{∞} with respect to p and q. Since $t^{N-\frac{n}{2}} e^{-\rho^2/4t}$ is $C^k([0,\infty) \times M \times M)$, hence $\mathcal{H}(\varphi(p,q)H_N(t,q,p)) \in C^k([0,\infty) \times M \times M)$. Consider integrating G_0 against $\psi(s,\beta)$,

$$\int_M G_0(t,q,s)\psi(s,\beta)\, dv_s = \sum_{m=0}^{N} t^m \int_M \frac{1}{(4\pi t)^{n/2}} e^{-\rho^2/4t} \psi(q,s) u_m(s,q) \psi(s,\beta)\, dv_s \tag{42}$$

The integral of Eq. (42) over M can be broken up into an integral over $Q_q(\frac{\epsilon}{2}) = \{s \in M | \rho(q,s) < \epsilon/2\}$ anda second integral over the set $M - M_q(\frac{\epsilon}{2})$. On the latter set, the limit converges uniformly hence

$$\lim_{t \to \infty} \frac{e^{-\rho^2/4t}}{(4\pi t)^{n/2}} = 0$$

To estimate the remaining integral, choose a normal coordinate system at q and denote the integration coordinates as $(s_1, \ldots, s_n)$, then the integrand of Eq. (42) is given as

$$\frac{1}{(4\pi t)^{n/2}} e^{-|s|^2/4t}\, \varphi(q,s) u_m(s,q) \psi(s,\beta) \sqrt{\det\langle \frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j} \rangle}\, ds_1 \cdots ds_n$$

Therefore, in the limit using Definition 2.4,

$$\lim_{t \to 0} \int_{M(\epsilon/2)} \frac{1}{(4\pi t)^{n/2}} e^{-\rho^2/4t}\, \varphi(q,s) u_m(s,q) \psi(s,\beta)\, dv_s = u_m(q,q)\psi(q,\beta)$$

This result implies that

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$$\lim_{t\to 0}\int_M G_0(t,q,s)\psi(s,\beta)\,dv_s = \sum_{m=0}^{N}\lim_{t\to 0} t^m u_m(q,q)\psi(q,\beta) = \psi(q,\beta)u_0(q,q) = \psi(q,\beta) \tag{43}$$

The convergence here is uniform.

There exists an asymptotic expansion for the heat kernel which is extremely useful and has several applications. It is one of the main intentions here to present this. An application of its use appears later.

Theorem 3.2. (Asymptotic expansion) Let M be a Riemannian manifold with dimension n and$E$ a vector bundle over M with inner product and metric Riemannian connection. Let $G(t,q,p)$ be the heat kernel or fundamental solution for heat operator (Eq. (15)) and (Eq. (39)) the MP parametrix. Then as $t \to 0$, $G(t,p,p)$ has the asymptotic expansion $G(t,p,p)\sim H\infty(t,p,p)$,

that is, for any $N > 0$, it is the case that

$$G(t,p,p) - \frac{1}{(4\pi t)^{n/2}}\sum_{m=0}^{N} t^m u_m(p,p) = O\left(t^{N-\frac{n}{2}}\right) \tag{44}$$

and the symbol on the right-hand side of Eq. (44) signifies a quantity ξ with the property that

$$\lim_{t\to 0}\frac{\xi}{t^{N-\frac{n}{2}}} = 0$$

Proof: It suffices to prove the theorem for any large N. Let $G_0(t,q,p) = \varphi(q,p)H_N(t,q,p)$ as in Theorem 3.2. The conclusion of the theorem is equivalent to the statement

$$G(t,p,p) - G_0(t,p,p) = O\left(t^{N-\frac{n}{2}}\right)$$

From the previous theorem and existence and regularity of the fundamental solution, the result G of Levi iteration initialized by G_0 is exactly the fundamental solution. Equality (Eq. (41)) means that there exists a constant A such that for any $t\in(0,T)$,

$$|K_0(t,q,p)| = \left|\left(\frac{\partial}{\partial t} + \Delta\right)G_0(t,q,p)\right| \le A t^{N-\frac{n}{2}}$$

Let $v(M)$ be the volume of the manifold M. Using this result, the following upper bound is obtained

$$|K_1(t,q,p)| \le \int_0^t d\tau \int_M |K_0(t-\tau,q,s)K_0(\tau,s,p)|\ dv_s$$

$$\le \int_0^t [A^2(t-\tau)^{N-\frac{n}{2}}\tau^{N-\frac{n}{2}}v(M)]\,d\tau \le \int_0^t A^2 T^{N-\frac{n}{2}}\tau^{N-\frac{n}{2}}v(M)\,d\tau \le AB\frac{t^{N-\frac{n}{2}+1}}{N-\frac{n}{2}+1}$$

We have set $B = A\cdot T^{N-\frac{n}{2}}v(M)$. Exactly the same procedure applies to $|K_2(t,q,p)|$. Based on the pattern established this way, induction implies that the following bound results

$$|K_m(t,q,p)| \le A \cdot B^m \frac{t^{N-\frac{n}{2}+m}}{\left(N-\frac{n}{2}+1\right)\left(N-\frac{n}{2}+2\right)\cdots\left(N-\frac{n}{2}+m\right)} \le A \cdot B^m \frac{t^m}{m!} t^{N-\frac{n}{2}}$$

The formula for Levi iteration yields upon summing this over m the following upper bound

$$|\tilde{K}(t,q,p)| \le \sum_{m=0}^{\infty} |K_m(t,q,p)| \le A \cdot e^{Bt} t^{N-\frac{n}{2}}$$

Using this bound, the required estimate is obtained,

$$|G(t,q,p) - G_0(t,q,p)| \le \left| \int_0^t d\tau \int_M dv_z G_0(t-\tau,q,z) \tilde{K}(\tau,z,p) \right|$$

$$\le \int_0^t d\tau \int_M \frac{e^{-\rho^2/4(t-\tau)}}{(4\pi(t-\tau))^{n/2}} A \cdot e^{B\tau} \cdot \tau^{N-\frac{n}{2}} dv_s$$

$$\le M_n A e^{Bt} \int_0^t \tau^{N-\frac{n}{2}} d\tau \, v(M) = \frac{1}{N-\frac{n}{2}+1} M_n A \cdot e^{Bt} v(M) t^{N-\frac{n}{2}+1}$$

This finishes the proof.

Now if all the Hodge theorem is used, formal expressions for the index can be obtained. Suppose $D : \Gamma(E) \to \Gamma(F)$ is an operator such that D^*D and DD^* are Schrödinger operators andD^* is the adjoint of D. Suppose the operators $D^*D : \Gamma(E) \to \Gamma(E)$ and $DD^* : \Gamma(F) \to \Gamma(E)$ are defined, so they are self-adjoint and have nonnegative real eigenvalues. Then the spaces $\Gamma_\mu(E)$ and $\Gamma_\mu(F)$ can be defined this way

$$\Gamma_\mu(E) = \{\varphi \in \Gamma(E) | D^* D\varphi = \mu\varphi\}, \qquad \Gamma_\mu(F) = \{\varphi \in \Gamma(F) | DD^*\varphi = \mu\varphi\} \tag{45}$$

For any $m > 0$, the dimensions of the spaces in (44) are finite and moreover,

$$\Gamma_0(E) = \ker\{D : \Gamma(E) \to \Gamma(F)\}, \qquad \Gamma_0(F) = \ker\{D^* : \Gamma(F) \to \Gamma(E)\}$$

Consequently, an expression for the index Ind (D) can be obtained from Eq. (45) as follows

$$\text{Ind } D = \dim \ker D - \dim \ker D^* = \dim \Gamma_0(E) - \dim \Gamma_0(F)$$

Definition 3.2. For the Schrödinger operator Δ, let $e^{-t\Delta} : \Gamma(E) \to \Gamma(E)$, for $t > 0$ be defined as

$$(e^{-t\Delta}\varphi)(q) = \int_M G(t,q,p)\varphi(p) dv_p \tag{46}$$

where $G(t,q,p)$ is the fundamental solution of heat operator (Eq. (15)).

Let $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$ be the eigenvalues of the operator Δ and $\{\psi_1, \psi_2, \ldots\}$ the corresponding eigenfunctions. Intuitively, the trace of $e^{-t\Delta}$ is defined as

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$$tr\ e^{-t\Delta} = \sum_{k=1}^{\infty} \langle\langle e^{-t\Delta}\ \psi_k, \psi_k \rangle\rangle \tag{47}$$

This is clearly $\sum_k e^{-\lambda_k t}$ or $\sum_\mu e^{-t\mu}$ dim $\Gamma_\mu(E)$, so the definition of tr is well-defined if and only if

$$\sum_k e^{-\lambda_k t} < \infty \tag{48}$$

Theorem 3.3. For any $p,q \in M$, let $\{e_1(p),\ldots,e_N(p)\}$ and $\{f_1(q),\ldots,f_N(q)\}$ be orthonormal bases on E_p and E_q, respectively, then the following two results hold for $t > 0$,

$$\begin{aligned} &(a) && \int_M \int_M \sum_{a,b=1}^{N} \langle G(t,q,p) e_a(p), f_b(q) \rangle^2\ dv_q dv_p < \infty, \\ &(b) && \sum_{k=1}^{\infty} e^{2\lambda_k t} < \int_M \int_M \sum_{a,b=1}^{N} \langle G(t,q,p) e_a(p), f_b(q) \rangle^2\ dv_q dv_p < \infty \end{aligned} \tag{49}$$

Proof: When $t > 0$, $G(t,q,p)$ is continuous and hence satisfies (a). For and $w \in \Gamma(E)$, Theorem 2.5 yields the following expansion for $G(t,q,p) \in \overline{\Gamma(E)}$, hence the Parseval equality yields

$$\int_M |G(t,q,p)w|^2\ dv_q = \sum_{k=1}^{\infty} e^{-2\lambda_k t} \langle \psi_k(p), w \rangle^2$$

Replacing w by the basis element $e_a(p)$, this implies that

$$\begin{aligned} &\sum_{a=1}^{N} \int_M |G(t,q,p) e_a(p)|^2\ dv_q \\ &= \sum_{a=1}^{N} \sum_{k=1}^{\infty} e^{-2\lambda_k t} \langle \psi_k(p), e_a(p) \rangle^2 = \sum_{k=1}^{\infty} \sum_{a=1}^{N} e^{-2\lambda_k t}\ \langle \psi_k(p), e_a(p) \rangle^2 = \sum_{k=1}^{\infty} e^{-2\lambda_k t} \langle \psi_k(p), \psi_k(p) \rangle \end{aligned}$$

Then for any m, it follows that

$$\begin{aligned} &\sum_{k=1}^{m} e^{-2\lambda_k t} = \sum_{k=1}^{m} \int_M e^{-2\lambda_k t} \langle \psi_k(p), \psi_k(p) \rangle\ dv_p \le \int_M \sum_{k=1}^{\infty} e^{-2\lambda_k t} \langle \psi_k(p), \psi_k(p) \rangle\ dv_p \\ &= \int_M dv_p \int_M \sum_{a=1}^{N} |G(t,q,p) e_a(p)|^2\ dv_q = \int_M \int_M \sum_{a,b=1}^{N} \langle G(t,q,p) e_a(p), f_b(q) \rangle^2\ dv_q dv_p < \infty \end{aligned}$$

Theorem 3.4. For any $t > 0$,

$$\text{tr}\ (e^{-t\Delta}) = \int_M tr\ G(t,p,p)\ dv_p \tag{50}$$

Proof: From Theorem 2.2, it follows that

$$tr\ G(t,p,p) = \sum_{a=1}^{N} \langle G(t,p,p)e_a(p),e_a(p)\rangle = \sum_{a=1}^{N} \left\langle \sum_{k=1}^{\infty} e^{-t\lambda_k}\langle\psi_k(p)e_a(p)\rangle\psi_k(p),e_a(p)\right\rangle$$
$$= \sum_{a=1}^{N}\sum_{k=1}^{\infty} e^{-t\lambda_k}\langle\psi_k(p),e_a(p)\rangle^2 = \sum_{k=1}^{\infty} e^{-t\lambda_k}\langle\psi_k(p),\psi_k(p)\rangle^2$$

Integrating this on both sides, it is found that

$$\int_M \text{tr}\ G(t,p,p)\ dv_p = \int_M \sum_{k=1}^{\infty} e^{-t\lambda_k}\langle\psi_k(p),\psi_k(p)\rangle^2\ dv_p = \sum_{k=1}^{\infty} e^{-t\lambda_k} = \text{tr}\ (e^{-t\Delta})$$

Note that Eq. (48) is a series with positive terms which converges uniformly as $t \to \infty$. Therefore,

$$\lim_{t\to\infty} \text{tr}\ e^{-t\Delta} = \sum_{k=1}^{\infty} \lim_{t\to\infty} e^{-t\lambda_k} = \dim\ \Gamma_0(E) \tag{51}$$

In fact, as $t \to 0$, the equality

$$G(t,p,p) = \frac{1}{(4\pi t)^{n/2}} + O\left(\frac{1}{t^{n/2}}\right)$$

and the previous theorem imply that $\lim_{t\to 0}$ tr $e^{-t\Delta} = \infty$.

4. An application of the expansions: the Gauss Bonnet theorem

As far as Ind (D) is concerned, it is the case for all $t > 0$ that,

$$\text{Ind}\ (D) = \text{tr}\ e^{-tD^*D} - \text{tr}\ e^{-tDD^*} = \int_M \text{tr}\ G_+(t,p,p)\ dv_p - \int_M \text{tr}\ G_-(t,p,p)\ dv_p$$

by Theorem 3.5, where $G_\pm(t,p,p)$ are the fundamental solutions of $\partial_t + D^*D$ and $\partial_t + DD^*$. As $t \to 0$, Theorem 3.2 assumes the form

$$G_\pm(t,p,p) \sim H_\infty^\pm(t,p,p) = \frac{1}{(4\pi t)^{n/2}} \sum_{m=0}^{\infty} t^m u_{\pm m}(p,p)$$

Lemma 4.1. Let $\{\lambda_i\}$ be the spectrum of the Laplacian on zero-forms, or functions, on M. Then,

$$\sum_k e^{-\lambda_k t} = \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} \int_M u_k(x,x)\ dv_x \tag{52}$$

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Proof:

$$\sum_k e^{-\lambda_k t} = \int_M \text{ tr } G(t,x,x)\, dv_x = \frac{1}{(4\pi t)^{n/2}} \sum_k \left(\int_M u_k(x,x)\, dv_x \right) t^k$$

The spectrum of the Laplacian on functions characterizes a lot of interesting geometric information. Note that Eq. (52) can be written as

$$\sum_i e^{\lambda_i t} \sim \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} a_k\, t^k, \qquad a_k = \int_M u_k(x,x)\, dv_x$$

and the trace does not appear in the case of functions. The superscript on the Laplacian Δ^p denotes the form degree acted upon andsimilarly on other objects throughout this section.

Two Riemannian manifolds are said to be isospectral if the eigenvalues of their Laplacians on functions counted with multiplicities coincide.

Corollary 4.1. Let M and N be compact isospectral Riemannian manifolds. Then M and N have the same dimension and the same volume.

Proof: Let $\{\lambda_i\}$ denote the spectrum of both M and N with $\dim M = m$ and $\dim N = n$. Then it follows that

$$\frac{1}{(4\pi t)^{m/2}} \sum_{k=0}^{\infty} \left(\int_M u_k^M(p,p)\, dv_p \right) t^k = \sum_{i=0}^{\infty} e^{-\lambda_i t} = \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} \left(\int_N u_k^N(q,q)\, dv_q \right) t^k$$

This implies that $m = n$, which in turn implies that

$$\frac{1}{(4\pi t)^{m/2}} \left[\int_M u_0^M(p,p)\, dv_p - \int_N u^N(q,q)\, dv_q \right] = \frac{1}{(4\pi t)^{m/2}} \sum_{k=1}^{\infty} \left(\int_M u_k^M(p,p)\, dv_p - \int_N u^N(q,q)\, dv_q \right) t^k$$

Since the right-hand side of the equation depends on t, but the left-hand side does not, this result implies that

$$\int_M u_0^M(p,\, p)\, dv_p = \int_N u_0^N(q,\, q)\, dv_q \tag{53}$$

Iterating this argument leads to the set of equations

$$\int_M u_k^M(p,\, p)\, dv_p = \int_N u_k^N(q,\, q)\, dv_q \tag{54}$$

for all $k > 0$. In particular, since $u_0 = 1$, Eq. (53) leads to the conclusion vol $(M) =$ vol (N).

The proof illustrates that in fact there exist an infinite sequence of obstructions to claiming that two manifolds are isospectral, namely the set of integrals $\int_M u_k \, dv_p$. The first integral contains basic geometric information. It is then natural to investigate the other integrals in sequence as well. Recall that $R_p, \nabla R_p, \cdots$ denote the covariant derivatives of the curvature tensor at p. A polynomial P in the curvature and its covariant derivatives is called universal if its coefficients depend only on the dimension of M. The notation $P(R_p, \nabla R_p, \ldots, \nabla^k R_p)$ is used to denote a polynomial in the components of the curvature tensor and its covariant derivatives calculated in a normal Riemannian coordinate chart at p. The following theorem will not be proved, but it will be used shortly.

Theorem 4.2. On a manifold of dimension n,

$$u_1(p,p) = P_1^n(R_p), \qquad u_k(p,p) = P_k^n(R_p, \nabla R_p, \ldots, \nabla^{2k-2} R_p), \qquad k \geq 2 \tag{55}$$

for some universal polynomials P_k^n.

Thus, P_1^n is a linear function with no constant term and $u_1(p,p)$ is a linear function of the components of the curvature tensor at p, with no covariant derivative terms. The only linear combination of curvature components that produces a well-defined function $u_1(p,p)$ on a manifold is the scalar curvature $R(p) = R_{ij}^{ij}$ andso there exists a constant C such that $u_1(p,p) = C \cdot R(p)$.

Theorem 4.3.

$$u_1(p,p) = \frac{1}{6} R(p) \tag{56}$$

Proof: The proof amounts to noticing that P_1^n is a universal polynomial, so it suffices to compute C over one kind of manifold. A good choice is to integrate over S^n with the standard metric and work it out explicitly in normal coordinates. It is found that $u_1(p,p) = n(n-1)/6$ andit is known that $R(p) = n(n-1)$ for all $p \in S^n$ andthis implies Eq. (56).

The large t or long-time behavior of the heat operator for the Laplacian on differential forms is then controlled by the topology of the manifold through the means of the de Rham cohomology. The small t or short-time behavior is controlled by the geometry of the asymptotic expansion. The combination of topological information has a geometric interpretation. This is made explicit by means of the Chern-Gauss-Bonnet theorem. The two-dimensional version of this theorem will be developed here.

These results can be summarized by the elegant formula

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} = \frac{1}{(4\pi t)^{n/2}} \left\{ v(M) + \frac{1}{6} \int_M R(x) \, dv_x \cdot t + O(t^2) \right\}$$

where $v(M)$ is the volume of M.

Suppose that λ is positive and here we let E_λ^p denote the possibly trivial eigenspace of Δ on p-forms. If $\omega \in E_\lambda^p$ then it follows that $\Delta^{p+1} d\omega = d\Delta^p \omega = \lambda\, d\omega$, hence $d\omega \in E_\lambda^{p+1}$. Thus, a well-defined sequential ordering of the spaces can be established. If $\omega \in E_\lambda^p$ has the property that $d\omega = 0$, then $\lambda\omega = \Delta^p\omega = (\delta d + d\delta)\omega = d\,\delta\omega$. Therefore, since $\lambda \neq 0$, it is found that $\omega = d\left(\frac{1}{\lambda}\delta\omega\right)$. Thus, the sequence $0 \to E_\lambda^0 \to^d \cdots \to^d E_\lambda^n \to 0$ is exact. Since the operator $d + \delta$ is an isomorphism on $\oplus_k E_\lambda^{2k}$, it follows that

$$\sum_s (-1)^s \dim E_\lambda^s = 0 \tag{57}$$

Theorem 4.4. Let $\{\lambda_i^s\}$ be the spectrum of the operator Δ, then

$$\sum_s (-1)^s \sum_i e^{-\lambda_i^s t} = \sum_s (-1)^s \dim \ker \Delta^s. \tag{58}$$

Proof: By (57),

$$\sum_s (-1)^s \sum_k e^{-\lambda_k^s t} = \sum_s (-1)^s {\sum}' e^{-\lambda_i t}$$

The sum on the right $\sum'$ is only over eigenvalues such that $\lambda_i^p = 0$ and so

$${\sum}' e^{-\lambda_i^p t} = \dim \ker \Delta^p.$$

This has the consequence that

$$\sum_p (-1)^p \ \operatorname{tr} e^{-t\Delta} = \sum_p (-1)^p \sum_k e^{-\lambda_k^p t} \tag{59}$$

is independent of the parameter t. This means that its large or long t behavior is the same as its short or small t behavior. To put it another way, the long-time behavior of $\operatorname{tr} e^{-t\Delta}$ is given by the de Rham cohomology, while the short-time behavior is dictated by the geometry of the manifold. Using the definition of the Euler characteristic, it follows that

$$\begin{aligned} \chi(M) = \sum\nolimits_p (-1)^p \dim H_{dH}^p(M) &= \sum\nolimits_p (-1)^p \dim \ker \Delta^p = \sum\nolimits_p (-1)^p \ \operatorname{tr} e^{-t\Delta^p} \\ &= \sum\nolimits_p (-1)^p \int_M \operatorname{tr}\ G(t,x,x)\, dv_x \end{aligned} \tag{60}$$

From the asymptotic expansion theorem, the following expression for $\chi(M)$ results

$$\chi(M) = \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} \left(\int_M \sum_{s=0}^{n} \operatorname{tr}\ u_k^s(x,x)\, dv_x \right) t^k \tag{61}$$

The u_k^s in Eq. (61) are the coefficients in the asymptotic expansion for $\operatorname{tr}(e^{-t\Delta^s})$. Since $\chi(M)$ is independent of t, only the constant or t-independent term on the right-hand side of Eq. (61) can be nonzero. This implies the following important theorem.

Theorem 4.5. If the dimension of M is even, then

$$\frac{1}{(4\pi)^{n/2}} \int_M \sum_{s=0}^{n} (-1)^s \text{ tr } u_k^s(x,x)\, dv_x = \begin{cases} 0, & k \neq \frac{n}{2}; \\ \chi(M), & k = \frac{n}{2}. \end{cases} \tag{62}$$

Theorem 4.6. (Gauss-Bonnet) Let M be a closed oriented manifold with Gaussian curvature K and area measure da_M, then

$$\chi(M) = \frac{1}{2\pi} \int_M K\, da_M \tag{63}$$

Proof: By the last theorem and the fact that tr $u_k^p(x,x) =$ tr $u_k^{p-1}(x,x)$, it follows that

$$\begin{aligned} \chi(M) &= \frac{1}{4\pi} \int_{M} \sum_{p=0}^{2} (^{-1)p} \text{ tr } u_1^p\, da_M = \frac{1}{4\pi} \int_M (\text{ tr } u_1^0 - \text{ tr } u_1^1 + \text{ tr } u_1^2)\, da_M \\ &= \frac{1}{4\pi} \int_M (2 \text{ tr } u_1^0 - \text{ tr } u_1^1)\, da_M = \frac{1}{4\pi} \int_M (\frac{2}{3} K - \text{ tr } u_1^1)\, da_M \end{aligned} \tag{64}$$

since the scalar curvature is two times the Gaussian. Now it must be that tr $u_1^1(x,x) = CR(x) = 2CK(x)$, for some constant C. The standard sphere S^2 has Gaussian curvature one andso C can be calculated from Eq. (64),

$$2 = \frac{1}{2\pi} \int_{S^2} (\frac{1}{3} - C)\, da_M = \frac{1}{2\pi} (\frac{1}{3} - C) \cdot (4\pi)$$

Therefore, $C = -2/3$ and putting all of these results into Eq. (64), Eq. (62) results.

As an application of this theorem, note that the calculation of u_1 gives another topological obstruction to manifolds having the same spectrum.

Theorem 4.7. Let (M,g) and (N,h) be compact isospectral surfaces, then M and N are diffeomorphic.

Proof: As noted in Corollary 4.1,

$$\int_M u_1^M(x,x)\, dv_x = \int_N u_1^N(y,y)\, dv_y$$

On a surface, the scalar curvature is twice the Gaussian curvature, so by the Gauss-Bonnet theorem,

$$6\pi\chi(M) = \int_M u_1^M(x,x)\, dv_x = \int_N u_1^N(y,y)\, dv_y = 6\pi\chi(N) \tag{65}$$

However, oriented surfaces with the same Euler characteristic are diffeomorphic.

5. Summary and outlook

The heat equation approach has been seen to be quite deep, leading both to the Hodge theorem and also to a proof of the Gauss-Bonnet theorem. Moreover, it is clear from the asymptotic development that there is a generalization of this theorem to higher dimensions. The four-dimensional Chern-Gauss-Bonnet integrand is given by the invariant $\frac{1}{32\pi^2}\{K^2-4|\rho_r|^2+|R|^2\}$, where K is the scalar curvature, $|\rho_r|^2$ is the norm of the Ricci tensor, $|R|^2$ is the norm of the total curvature tensor andthe signature is Riemannian. This comes up in physics especially in the study of Einstein-Gauss-Bonnet gravity where this invariant is used to get the associated Euler-Lagrange equations.

Let R_{ijkl} be the components of the Riemann curvature tensor relative to an arbitrary local frame field $\{e_i\}$ for the tangent bundle TM and adopt the Einstein summation convention. Let $m = 2s$ be even, then the Pfaffian $E_m(g)$ is defined to be

$$E_m(g) = \frac{1}{(8\pi)^s s!} R_{i_1 i_2 j_2 j_1} \cdots R_{i_{2s-1} i_{2s} j_{2s} j_{2s-1}} \, g(e^{i_1}\wedge\cdots\wedge e^{i_{2s}}, e^{j_1}\wedge\cdots\wedge e^{j_{2s}}) \tag{66}$$

The Euler characteristic $\chi(M)$ of any compact manifold of odd dimension without boundary vanishes. Only the even dimensional case is of interest.

Theorem 5.1. Let (M,g) be a compact Riemannian manifold without boundary of even dimension m. Then

$$\chi(M) = \int_M E_m(g)\, dv_M \tag{67}$$

This was proved first by Chern, but of greater significance here, this can be deduced from the heat equation approach that has been introduced here. There is a proof by Patodi [18], but there is no room for it now. It should be hoped that more interesting results will come out in this area as well in the future.

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