Chapter 5

A Game Theoretic Approach Based Adaptive Control Design for Sequentially Interconnected SISO Linear Systems

Sheng Zeng and Emmanuel Fernandez

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/50488

1. Introduction

Adaptive control has attracted a lot of research attention in control theory for many decades. In the certainty equivalence based adaptive controller design [4, 5], the unknown parameters of the uncertainty system are substituted by their online estimates, which are generated through a variety of identifiers, as long as the estimates satisfy certain properties independent of the controller. This approach leads to structurally simple adaptive controllers and has been demonstrated its effectiveness for linear systems with or without stochastic disturbance inputs [10] when long term asymptotic performance is considered. Yet, the certainty equivalence approach is unsuccessful to generalize to systems with severe nonlinearities. Also, early designs based on this approach were shown to be nonrobust [13] when the system is subject to exogenous disturbance inputs and unmodeled dynamics. Then, the stability and the performance of the closed-loop system becomes an important issue. This has motivated the study of robust adaptive control in the 1980s and 1990s, and the study of nonlinear adaptive control in the 1990s.

The topic of adaptive control design for nonlinear systems was studied intensely in the last decade after the celebrated characterization of feedback linearizable or partially feedback linearizable systems [7]. A breakthrough is achieved when the integrator backstepping methodology [8] was introduced to design adaptive controllers for parametric strict-feedback and parametric pure-feedback nonlinear systems systematically. Since then, a lot of important contributions were motivated by this approach, and a complete list of references can be found in the book [9]. Moreover, this nonlinear design approach has been applied to linear systems to compare performance with the certainty equivalence approach. However, simple designs using this approach without taking into consideration the effect of exogenous disturbance inputs have also been shown to be nonrobust when the system is subject to exogenous disturbance inputs.

The robustness of closed-loop adaptive systems has been an important research topic in late 1980s and early 1990s. Various adaptive controllers were modified to render the closed-loop systems robust [6]. Despite their successes, they still fell short of directly addressing the disturbance attenuation property of the closed-loop system.

The objectives of robust adaptive control are to improve transient response, to accommodate unmodeled dynamics, and to reject exogenous disturbance inputs, which are the same as the objectives to motivate the study of the H^∞- optimal control problem. H^∞-optimal control was proposed as a solution to the robust control problem, where these objectives are achieved by studying only the disturbance attenuation property for the closed-loop system. The game-theoretic approach to H^∞-optimal control developed for the linear quadratic problems, offers the most promising tool to generalize the results to nonlinear systems [3]. Worst-case analysis based adaptive control design was proposed in late 1990s to address the disturbance attenuation property directly, and it is motivated by the success of the game-theoretic approach to H^∞-optimal control problems [2]. In this approach, the robust adaptive control problem is formulated as a nonlinear H^∞ control problem under imperfect state measurements. By *cost-to-come function* analysis, it is converted into an H^∞ control problem with full information measurements. This full information measurements problem is then solved using nonlinear design tools for a suboptimal solution. This design scheme has been applied to worst-case parameter identification problems [11], which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems [1, 12, 14, 15, 18, 19], and the convergence properties is studied in [20]. In [14], adaptive control for a strict-feedback nonlinear systems was considered with noiseless output measurements, and more general class of nonlinear systems was studied in [1]. In [12], single-input and single output (SISO) linear systems were considered with noisy output measurements. SISO linear systems with partly measured disturbance was studied in [18], which leads to a disturbance feed-forward structure in the adaptive controller. [19] generalizes the results of [12] to the adaptive control design for SISO linear systems with zero relative degree under noisy output measurements. In [17], adaptive control for a sequentially interconnected SISO linear system was considered, and a special class of unobservable systems was also studied using the proposed approach. More recently, [16] generalized the result of [17] to adaptive control design for a linear system under simultaneous driver, plant and actuation uncertainties.

In this Chapter, we study the adaptive control design for sequentially interconnected SISO linear systems, $\mathbf{S}_1$ and $\mathbf{S}_2$(see Figure 1), under noisy output measurements and partly measured disturbance using the similar approaches as [12] and [17]. We assume that the linear systems satisfy the same assumption as [17], and the adaptive control design follows the same design method discussed above. The robust adaptive controller achieves asymptotic tracking of the reference trajectories when disturbance inputs are of finite energy. The closed-loop system is totally stable with respect to the disturbance inputs and the initial conditions. Furthermore, the closed-loop system admits a guaranteed disturbance attenuation level with respect to the exogenous disturbance inputs, where ultimate lower bound for the achievable attenuation performance level is equal to the noise intensity in the measurement channel of $\mathbf{S}_1$. The results are as same as those in [17]. In addition, the controller achieves arbitrary positive distance attenuation level with respect to the measured disturbances by proper scaling. Moreover, if the measured disturbances satisfy the assumption 2 for $\check{w}_{1,b}$ and $\check{w}_{2,b}$, the

proposed controller achieves disturbance attenuation level *zero* with respect to the measured disturbances, which further leads to a stronger asymptotic tracking property, namely, the tracking error converges to zero when the unmeasured disturbances are $\mathcal{L}_2 \cap \mathcal{L}_\infty$, and the measured disturbances are $\mathcal{L}_\infty$ only.

The balance of this Chapter is organized as follows. In Section 2, we list the notations used in the Chapter. In Section 3, we present the formulation of the adaptive control problem and discuss the general solution methodology. In Section 4, we first obtain parameter identifier and state estimator using the *cost-to-come function* analysis in Subsection 4.1, then we derive the adaptive control law in Subsection 4.2. We present the main results on the robustness of the system in Section 5, and the example in Section 6. The Chapter ends with some concluding remarks in Section 7.

2. Notations

We denote $\mathbb{R}$ to be the real line; $\mathbb{R}_e$ to be the extended real line; $\mathbf{N}$ to be the set of natural numbers; $\mathbb{C}$ to be the set of complex numbers. For a function f, we say that it belongs to $\mathcal{C}$ if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously (partial) differentiable. For any matrix A, A' denotes its transpose. For any $b \in \mathbb{R}$, $\mathrm{sgn}(b) = \begin{cases} -1 & b < 0 \\ 0 & b = 0 \\ 1 & b > 0 \end{cases}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbf{N}$, $|z|$ denotes $(z'z)^{1/2}$. For any vector $z \in \mathbb{R}^n$, and any $n \times n$-dimensional symmetric matrix M, where $n \in \mathbf{N}$, $|z|_M^2 = z'Mz$. For any matrix M, the vector $\overrightarrow{M}$ is formed by stacking up its column vectors. For any symmetric matrix M, $\overleftarrow{M}$ denotes the vector formed by stacking up the column vector of the lower triangular part of M. For $n \times n$-dimensional symmetric matrices M_1 and M_2, where $n \in \mathbf{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbf{N}$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. For $n \in \mathbf{N} \cup \{0\}$, I_n denotes the $n \times n$-dimensional identity matrix. For any matrix M, $\|M\|_p$ denotes its p-induced norm, $1 \leq p \leq \infty$. $\mathcal{L}_2$ denotes the set of square integrable functions and $\mathcal{L}_\infty$ denotes the set of bounded functions. For any n, $m \in \mathbf{N} \cup \{0\}$, $\mathbf{0}_{n\times m}$ denotes the $n \times m$-dimensional matrix whose elements are zeros. For any $n \in \mathbf{N}$ and $k \in \{1, \cdots, n\}$, $e_{n,k}$ denotes $\left[0_{1\times(k-1)}\ 1\ 0_{1\times(n-k)}\right]'$.

3. Problem Formulation

We consider the robust adaptive control problem for the system which is described by the block diagram in Figure 1.

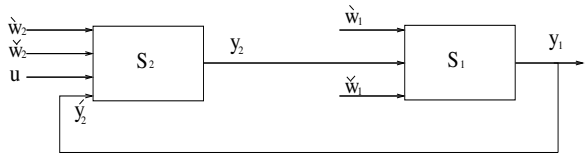

Figure 1. Diagram of two sequentially interconnected SISO linear systems.

We assume that the system dynamics for $\mathbf{S_1}$ and $\mathbf{S_2}$ are given by,

$$\dot{\grave{x}}_1 = \grave{A}_1\grave{x}_1 + \grave{B}_1 y_2 + \grave{D}_1\grave{w}_1 + \grave{\breve{D}}_1\breve{w}_1; \tag{1a}$$

$$y_1 = \grave{C}_1\grave{x}_1 + \grave{E}_1\grave{w}_1 \tag{1b}$$

$$\dot{\grave{x}}_2 = \grave{A}_2\grave{x}_2 + \grave{B}_2 u + \grave{A}_{2,y}\acute{y}_2 + \grave{D}_2\grave{w}_2 + \grave{\breve{D}}_2\breve{w}_2; \tag{1c}$$

$$y_2 = \grave{C}_2\grave{x}_2 + \grave{E}_2\grave{w}_2 \tag{1d}$$

where $\grave{x}_i$ is the n_i-dimensional state vectors with initial condition $\grave{x}_i(0) = \grave{x}_{i,0}$, $n_i \in \mathbf{N}$; u is the scalar control input; y_i is the scalar measurement output; $\grave{w}_i$ is $\grave{q}_i$-dimensional unmeasured disturbance input vector, $\grave{q}_i \in \mathbf{N}$; $\breve{w}_i$ is $\breve{q}_i$-dimensional measured disturbance input vector, $\breve{q}_i \in \mathbf{N}$; the elements of $\breve{w}_i$ are $\left[\begin{array}{ccc}\breve{w}_{i,1} & \cdots & \breve{w}_{i,\breve{q}_i}\end{array}\right]'$; $\acute{y}_2 = y_1$; the matrices $\grave{A}_i$, $\grave{A}_{i,y}$, $\grave{B}_i$, $\grave{C}_i$, $\grave{D}_i$, $\grave{\breve{D}}_i$, and $\grave{E}_i$ are of the appropriate dimensions, generally unknown or partially unknown, $i = 1, 2$. For subsystem $\mathbf{S_1}$, the transfer function from y_2 to y_1 is $H_1(s) = \grave{C}_1(sI_{n_1} - \grave{A}_1)^{-1}\grave{B}_1$, for subsystem $\mathbf{S_2}$, the transfer function from u to y_2 is $H_2(s) = \grave{C}_2(sI_{n_2} - \grave{A}_2)^{-1}\grave{B}_2$. All signals in the system are assumed to be continuous.

The subsystems $\mathbf{S_1}$ and $\mathbf{S_2}$ satisfy the following assumptions,

Assumption 1. *For $i = 1, 2$, the pair $(\grave{A}_i, \grave{C}_i)$ is observable; the transfer function $H_i(s)$ is known to have relative degree $r_i \in \mathbf{N}$, and is strictly minimum phase. The uncontrollable part of $\mathbf{S_1}$ (with respect to y_2) is stable in the sense of Lyapunov; any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}_1$ on the $j\omega$-axis is uncontrollable from $\left[\begin{array}{cc}\grave{w}_1' & \breve{w}_1'\end{array}\right]'$. The uncontrollable part of $\mathbf{S_2}$ (with respect to u) is stable in the sense of Lyapunov; any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}_2$ on the $j\omega$-axis is uncontrollable from $\left[\begin{array}{ccc}\grave{w}_2' & \acute{y}_2 & \breve{w}_2'\end{array}\right]'$.* ◇

Based on Assumption 1, for $i = 1, 2$, there exists a state diffeomorphism: $x_i = \grave{T}_i\grave{x}_i$, and a disturbance transformation: $w_i = \grave{M}_i\grave{w}_i$, such that $\mathbf{S_i}$ can be transformed into the following state space representation,

$$\dot{x}_1 = A_1x_1 + (y_1\bar{A}_{1,211} + y_2\bar{A}_{1,212} + \sum_{j=1}^{\breve{q}_1}\breve{w}_{1,j}\bar{A}_{1,213j})\theta_1 + B_1y_2 + D_1w_1 + \breve{D}_1\breve{w}_1;$$

$$y_1 = C_1x_1 + E_1w_1$$

$$\dot{x}_2 = A_2x_2 + (y_2\bar{A}_{2,211} + u\bar{A}_{2,212} + \sum_{j=1}^{\breve{q}_2}\breve{w}_{2,j}\bar{A}_{2,213j} + \acute{y}_2\bar{A}_{2,214})\theta_2 + B_2u + A_{2,y}\acute{y}_2 + D_2w_2 + \breve{D}_2\breve{w}_2;$$

$$y_2 = C_2x_2 + E_2w_2$$

where θ_i is the σ_i-dimensional vector of unknown parameters for the subsystem $\mathbf{S}_i$, $\sigma_i \in \mathbf{N}$; the matrices A_i, $\bar{A}_{i,211}$, $\bar{A}_{i,212}$, $\bar{A}_{i,2131}$, $\cdots$, $\bar{A}_{i,213\breve{q}_i}$, $\bar{A}_{2,214}$, $A_{2,y}$, B_i, D_i, $\breve{D}_i$, C_i, and E_i are known and have the following structures, $A_i = (a_{i,jk})_{n_i\times n_i}$; $a_{i,j\,(j+1)} = 1, a_{i,jk} = 0$, for $1 \le j \le r_i - 1$ and $j + 2 \le k \le n_i$; $\bar{A}_{i,212} = \left[\begin{array}{ccc}\mathbf{0}_{\sigma_i\times(r_i-1)} & \bar{A}'_{i,2120} & \bar{A}'_{i,212r_i}\end{array}\right]'$, $C_i = [\,1\ \mathbf{0}_{1\times(n_i-1)}\,]$, $A_{i,2120}$ is a row vector, $B_i = \left[\begin{array}{ccc}\mathbf{0}_{1\times(r_i-1)} & b_{i,p0} & \cdots\ b_{i,p(n_i-r_i)}\end{array}\right]'$, $b_{i,pj}$ $j = 0, 1, \cdots, n_i - r_i$ are constants. We denote the elements of x_1 and x_2 by $\left[\begin{array}{ccc}x_{1,1} & \cdots & x_{1,n_1}\end{array}\right]'$ and $\left[\begin{array}{ccc}x_{2,1} & \cdots & x_{2,n_2}\end{array}\right]'$, with initial conditions $x_{1,0}$ and $x_{2,0}$, respectively.

Assumption 2. *The measured disturbance $\breve{w}_1$ can be partitioned as: $\breve{w}_1 = \left[\begin{array}{cc}\breve{w}'_{1,a} & \breve{w}'_{1,b}\end{array}\right]'$ where $\breve{w}_{1,a}$ is $\breve{q}_{1,a}$ dimensional, $\breve{q}_{1,a} \in \mathbf{N} \cup \{0\}$, and the transfer function from each element of $\breve{w}_{1,a}$ to y_1 has relative degree less than $r_1 + r_2$; the measured disturbance $\breve{w}_2$ can be partitioned as: $\breve{w}_2 = \left[\begin{array}{cc}\breve{w}'_{2,a} & \breve{w}'_{2,b}\end{array}\right]'$ where $\breve{w}_{2,a}$ is $\breve{q}_{2,a}$ dimensional, $\breve{q}_{2,a} \in \mathbf{N} \cup \{0\}$, and the transfer function from each element of $\breve{w}_{2,a}$ to y_2 has relative degree less than r_2.* ◇

Based on Assumption 2, the matrix $\breve{D}_i$ can be partitioned into $\left[\begin{array}{cc}\breve{D}_{i,a} & \breve{D}_{i,b}\end{array}\right]$, where $\breve{D}_{i,a}$ and $\breve{D}_{i,b}$ have $n_i \times \breve{q}_{i,a}$- and $n_i \times \breve{q}_{i,b}$-dimensional, respectively; and $\breve{D}_{i,b}$, $\bar{A}_{i,213\,(\breve{q}_{i,a}+1)}, \cdots, \bar{A}_{i,213\,\breve{q}_i}$ have the following structure

$$\breve{D}_{i,b} = \left[\begin{array}{c}\mathbf{0}_{(r_i-1)\times\breve{q}_{i,b}} \\ \breve{D}_{i,b0} \\ \breve{D}_{i,br_i}\end{array}\right]; \quad \bar{A}_{i,213\,j} = \left[\begin{array}{c}\mathbf{0}_{(r_i-1)\times\sigma_i} \\ \bar{A}_{i,213\,j0} \\ \bar{A}_{i,213\,jr_i}\end{array}\right], j = \breve{q}_{i,a}+1, \cdots, \breve{q}_i$$

where $\breve{D}_{i,b0}$ and $\bar{A}_{i,213\,j0}$, $j = \breve{q}_{i,a}+1, \cdots, \breve{q}_{i,a}+\breve{q}_{i,b}$, are row vectors, $i = 1, 2$.

Since we will base our design of adaptive controllers using the model (2), we call (2) the design model, and make the following two assumptions.

Assumption 3. *For $i = 1, 2$, the matrices E_i are such that $E_i E_i' > 0$.* ◇

Define $\zeta_i := (E_i E_i')^{-\frac{1}{2}}$ and $L_i := D_i E_i'$, $i = 1, 2$.

Due to the structures of A_i, $\bar{A}_{i,212}$ and B_i, the high frequency gain of the transfer function $H_i(s)$, $b_{i,0}$, is equal to $b_{i,p0} + \bar{A}_{i,2120}\theta_i$, $i = 1, 2$.

To guarantee the stability of the identified system, we make the following assumption on the parameter vectors θ_1 and θ_2.

Assumption 4. *The sign of $b_{i,0}$ is known; there exists a known smooth nonnegative radially-unbounded strictly convex function $P_i : \mathbf{R}^{\sigma_i} \to \mathbf{R}$, such that the true value $\theta_i \in \Theta_i := \{\bar{\theta}_i \in \mathbf{R}^{\sigma_i} \mid P_i(\bar{\theta}_i) \leq 1\}$; moreover, $\forall \bar{\theta}_i \in \Theta_i$, $\mathrm{sgn}(b_{i,0})(b_{i,p0} + \bar{A}_{i,2120}\bar{\theta}_i) > 0$, $i = 1, 2$.* ◇

Assumption 4 delineates *a priori* convex compact sets where the parameter vectors θ_1 and θ_2 lie in, respectively. This will guarantee the stability of the closed-loop system and the boundedness of the estimate of θ_1 and θ_2.

We make the following assumption about the reference signal, y_d.

Assumption 5. *The reference trajectory, y_d, is $r_1 + r_2$ times continuously differentiable. Define vector $Y_d := [y_d^{(0)}, \cdots, y_d^{(r_1+r_2)}]'$, where $y_d^{(0)} = y_d$, and $y_d^{(j)}$ is the jth order time derivative of y_d, $j = 1, \cdots, r_1 + r_2$; define $Y_{d0} := [y_d^{(0)}(0), \cdots, y_d^{(r_1+r_2-1)}(0)]' \in \mathbf{R}^{r_1+r_2}$. The signal Y_d is available for feedback.* ◇

The uncertainty of subsystem $\mathbf{S_1}$ is $\dot{\omega}_1 := (x_{1,0}, \theta_1, \dot{w}_{1[0,\infty)}, \breve{w}_{1[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \dot{\mathcal{W}}_1 := \mathbf{R}^{n_1} \times \Theta_1 \times \mathcal{C} \times \mathcal{C} \times \mathbf{R}^{r_1+r_2} \times \mathcal{C}$, which comprises the initial state $x_{1,0}$, the true value of the parameters θ_1, the unmeasured disturbance waveform $\dot{w}_{1[0,\infty)}$, the measured disturbance waveform $\breve{w}_{1[0,\infty)}$, the initial conditions of the reference trajectory Y_{d0}, and the waveform

of the $(r_1 + r_2)$ th order derivative of the reference trajectory, $y_{d[0,\infty)}^{(r_1+r_2)}$. The uncertainty for subsystem $\mathbf{S_2}$ is $\grave{\omega}_2 := (x_{2,0}, \theta_2, \grave{w}_{2[0,\infty)}, \check{w}_{2[0,\infty)}) \in \grave{\mathcal{W}}_2 := \mathbb{R}^{n_2} \times \Theta_2 \times \mathcal{C} \times \mathcal{C}$, which comprises the initial state $x_{2,0}$, the true value of the parameters θ_2, the unmeasured disturbance waveform $\grave{w}_{2[0,\infty)}$, and the measured disturbance waveform $\check{w}_{2[0,\infty)}$.

Our objective is to derive a control law, which is generated by the following mapping,

$$u(t) = \mu(y_{2[0,t]}, \acute{y}_{2[0,t]}, Y_{d[0,t]}, \check{w}_1, \check{w}_2) \tag{3}$$

where $\mu : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, such that $x_{1,1}$ can asymptotically track the reference trajectory y_d, while rejecting the uncertainty $(\grave{\omega}_1, \grave{\omega}_2) \in \grave{\mathcal{W}}_1 \times \grave{\mathcal{W}}_2$, and keeping the closed-loop signals bounded. The control law μ must also satisfy that, $\forall(\grave{\omega}_1, \grave{\omega}_2) \in \grave{\mathcal{W}}_1 \times \grave{\mathcal{W}}_2$, there exists a solution $\grave{x}_{1[0,\infty)}$ and $\grave{x}_{2[0,\infty)}$ to the system (1), which yields a continuous control signal $u_{[0,\infty)}$. We denote the class of these admissible controllers by $\mathcal{M}_\mu$.

For design purposes, instead of attenuating the effect of $\left[\grave{w}_1' \ \check{w}_1' \ \grave{w}_2' \ \check{w}_2\right]'$ we design the adaptive controller to attenuate the effect of $\left[w_1' \ \check{w}_1' \ w_2' \ \check{w}_2\right]'$. This is done to allow our design paradigm to be carried out. This will result in a guaranteed attenuation level with respect to $\grave{\omega}_1$ and $\grave{\omega}_2$. To simplify the notation, we take the uncertainty $\omega_1 := (x_{1,0}, \theta_1, w_{1[0,\infty)}, \check{w}_{1[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \mathcal{W}_1 := \mathbb{R}^{n_1} \times \Theta_1 \times \mathcal{C} \times \mathcal{C} \times \mathbb{R}^{r_1+r_2} \times \mathcal{C}$, and $\omega_2 := (x_{2,0}, \theta_2, w_{2[0,\infty)}, \check{w}_{2[0,\infty)}) \in \mathcal{W}_2 := \mathbb{R}^{n_2} \times \Theta_2 \times \mathcal{C} \times \mathcal{C}$.

We state the control objective precisely as follows,

Definition 1. *A controller $\mu \in \mathcal{M}_\mu$ is said to achieve* disturbance attenuation level γ *with respect to $\left[w_1' \ \check{w}_{1,a}' \ w_2' \ \check{w}_{2,a}'\right]'$, and* disturbance attenuation level zero *with respect to $\left[\check{w}_{1,b}' \ \check{w}_{2,b}'\right]'$, if there exists functions $l_1(t, \theta_1, x_1, y_{1[0,t]}, y_{2[0,t]}, \check{w}_{1[0,t]}, \check{w}_{2[0,t]}, Y_{d[0,t]})$, $l_2(t, \theta_2, x_2, y_{1[0,t]}, y_{2[0,t]}, \check{w}_{1[0,t]}, \check{w}_{2[0,t]}, Y_{d[0,t]})$, and a known nonnegative constant $l_0(\check{x}_{1,0}, \check{x}_{2,0}, \check{\theta}_{1,0}, \check{\theta}_{2,0})$, such that*

$$\sup_{\grave{w}_1 \in \grave{\mathcal{W}}_1, \grave{w}_2 \in \grave{\mathcal{W}}_2} J_{\gamma t_f} \leq 0; \quad \forall t_f \geq 0 \tag{4}$$

and $l_1 \geq 0$ and $l_2 \geq 0$ along the closed-loop trajectory, where

$$J_{\gamma t_f} := J_{1,\gamma t_f} + J_{2,\gamma t_f} \tag{5a}$$

$$J_{1\gamma t_f} := \int_0^{t_f} \left((C_1 x_1 - y_d)^2 + l_1 - \gamma^2 |w_1|^2 - \gamma^2 |\check{w}_{1,a}|^2 \right) \mathrm{d}\tau - \gamma^2 \left| \left[\theta_1' - \check{\theta}_{1,0}' \ x_{1,0}' - \check{x}_{1,0}' \right]' \right|^2_{\bar{Q}_{1,0}} \tag{5b}$$

$$J_{2,\gamma t_f} := \int_0^{t_f} \left(l_2 - \gamma^2 |w_2|^2 - \gamma^2 |\check{w}_{2,a}|^2 \right) \mathrm{d}\tau - l_0 - \gamma^2 \left| \left[\theta_2' - \check{\theta}_{2,0}' \ x_{2,0}' - \check{x}_{2,0}' \right]' \right|^2_{\bar{Q}_{2,0}} \tag{5c}$$

$\check{\theta}_{i,0} \in \Theta_i$ is the initial guess of θ_i; $\check{x}_{i,0} \in \mathbb{R}^{n_i}$ is the initial guess of $x_{i,0}$; $\bar{Q}_{i,0} > 0$ is a $(n_i + \sigma_i) \times (n_i + \sigma_i)$-dimensional weighting matrix, quantifying the level of confidence in the estimate $\left[\check{\theta}_{i,0}' \ \check{x}_{i,0}'\right]'$; $\bar{Q}_{i,0}^{-1}$ admits the structure $\begin{bmatrix} Q_{i,0}^{-1} & Q_{i,0}^{-1}\Phi_{i,0}' \\ \Phi_{i,0}Q_{i,0}^{-1} & \Pi_{i,0} + \Phi_{i,0}Q_{i,0}^{-1}\Phi_{i,0}' \end{bmatrix}$, $Q_{i,0}$ and $\Pi_{i,0}$ are $\sigma_i \times \sigma_i$- and $n_i \times n_i$-dimensional positive definite matrices, respectively, $i = 1, 2$.

Clearly, when the inequality (4) is achieved, the squared $\mathcal{L}_2$ norm of the output tracking error $C_1x_1 - y_d$ is bounded by γ^2 times the squared $\mathcal{L}_2$ norm of the transformed disturbance input $\left[\, w_1' \;\; \breve{w}_{1,a}' \;\; w_2' \;\; \breve{w}_{2,a}' \,\right]'$, plus some constant. When the $\mathcal{L}_2$ norm of $\grave{w}_1$, $\grave{w}_2$, $\breve{w}_1$, and $\breve{w}_2$ are finite, the squared $\mathcal{L}_2$ norm of $C_1x_1 - y_d$ is also finite, which implies $\lim_{t\to\infty}(C_1x_1(t) - y_d(t)) = 0$, under additional assumptions.

Let ξ_i denote the expanded state vector $\xi_i = [\theta_i', x_i']'$, $i = 1, 2$, and note that $\dot{\theta}_i = 0$, we have the following expanded dynamics for system (2),

$$
\begin{aligned}
\dot{\xi}_1 &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ y_1\bar{A}_{1,211} + y_2\bar{A}_{1,212} + \sum_{j=1}^{\breve{q}_1} \breve{w}_{1,j}\bar{A}_{1,213j} & A_1 \end{bmatrix} \xi_1 + \begin{bmatrix} \mathbf{0} \\ B_1 \end{bmatrix} y_2 + \begin{bmatrix} \mathbf{0} \\ D_1 \end{bmatrix} w_1 + \begin{bmatrix} \mathbf{0} \\ \breve{D}_1 \end{bmatrix} \breve{w}_1 \\
&=: \bar{A}_1(y_1, y_2, \breve{w}_1)\xi_1 + \bar{B}_1 y_2 + \bar{D}_1 w_1 + \bar{\breve{D}}_1 \breve{w}_1 \\
y_1 &= \begin{bmatrix} \mathbf{0} & C_1 \end{bmatrix} \xi_1 + E_1 w_1 \\
&=: \bar{C}_1\xi_1 + E_1 w_1 \\
\dot{\xi}_2 &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ y_2\bar{A}_{2,211} + u\bar{A}_{2,212} + \sum_{j=1}^{\breve{q}_2} \breve{w}_{2,j}\bar{A}_{2,213j} + \acute{y}_2\bar{A}_{2,214} & A_2 \end{bmatrix} \xi_2 + \begin{bmatrix} \mathbf{0} \\ B_2 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ A_{2,y} \end{bmatrix} \acute{y}_2 \\
&\quad + \begin{bmatrix} \mathbf{0} \\ D_2 \end{bmatrix} w_2 + \begin{bmatrix} \mathbf{0} \\ \breve{D}_2 \end{bmatrix} \breve{w}_2 \\
&=: \bar{A}_2(y_1, y_2, , \breve{w}_2, u)\xi_2 + \bar{B}_2 u + \bar{A}_{2,y}\acute{y}_2 + \bar{D}_2 w_2 + \bar{\breve{D}}_2 \breve{w}_2 \\
y_2 &= \begin{bmatrix} \mathbf{0} & C_2 \end{bmatrix} \xi_2 + E_2 w_2 \\
&=: \bar{C}_2\xi_2 + E_2 w_2
\end{aligned}
$$

The worst-case optimization of the cost function (4) can be carried out in two steps as depicted in the following equations.

$$
\begin{aligned}
\sup_{\grave{\omega}_1\in\grave{\mathcal{W}}_1,\, \grave{\omega}_2\in\grave{\mathcal{W}}_2} J_{\gamma t_f} &= \sup_{\omega_m\in\mathcal{W}_m} \sup_{\grave{\omega}_1\in\grave{\mathcal{W}}_1,\, \grave{\omega}_2\in\grave{\mathcal{W}}_2|\omega_m\in\mathcal{W}_m} J_{\gamma t_f} \\
&\le \sup_{\omega_m\in\mathcal{W}_m} \sup_{\omega_1\in\mathcal{W}_1, \omega_2\in\mathcal{W}_2|\omega_m\in\mathcal{W}_m} J_{\gamma t_f} \\
&= \sup_{\omega_m\in\mathcal{W}_m} \Big(\sum_{i=1}^{2} \sup_{\omega_i\in\mathcal{W}_i|\omega_m\in\mathcal{W}_m} J_{i,\gamma t_f} \Big)
\end{aligned} \tag{6}
$$

where ω_m is the measured signals of the system, and defined as

$$
\omega_m := (y_{1[0,\infty)}, y_{2[0,\infty)}, \breve{w}_{1[0,\infty)}, \breve{w}_{2[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \mathcal{W}_m := \mathcal{C}\times\mathcal{C}\times\mathcal{C}\times\mathcal{C}\times\mathbb{R}^{r_1+r_2}\times\mathcal{C}.
$$

The inner supremum operators will be carried out first. We maximize over ω_i given that the measurement ω_m is available for estimator design, $i = 1, 2$. In this step, the control input, u, is a function only depended on ω_m, then u is an open-loop time function and available for the optimization. Using *cost-to-come* function analysis, we derive the dynamics of the estimators for subsystem $\mathbf{S}_1$ and $\mathbf{S}_2$ independently.

The outer supremum operator will be carried out second. In this step, we use a backstepping procedure to design the controller μ.

This completes the formulation of the robust adaptive control problem.

4. Adaptive control design

In this section, we present the adaptive control design, which involves estimation design and control design. First, we discuss estimation design.

4.1. Estimation design

In this subsection, we present the estimation design for the adaptive control problem formulated. First, we will derive the identifier of subsystem $\mathbf{S}_1$. In this step, the measurement waveform y_1, y_2 and measured disturbance $\breve{w}_1$ are assumed to be known. Then we can obtain the identifier of subsystem $\mathbf{S_1}$ from a *game-theoretic* solution methodology – *cost-to-come* function analysis.

We first set function l_1 in the definition to be $|\xi_1 - \hat{\xi}_1|^2_{\bar{Q}_1} + 2(\xi_1 - l_{1,1})' l_{1,2} + \check{l}_1$, where $\hat{\xi}_1 = [\hat{\theta}_1', \hat{x}_1']'$ is the worst-case estimate for the expanded state ξ_1, $\bar{Q}_1(y_{1[0,\tau]}, Y_{d[0,\tau]}, \breve{w}_{1[0,\tau]})$ is a matrix-valued weighting function, $l_{1,1}(y_{1[0,\tau]}, Y_{d[0,\tau]}, \breve{w}_{1[0,\tau]})$, $l_{1,2}(y_{1[0,\tau]}, Y_{d[0,\tau]}, \breve{w}_{1[0,\tau]})$ and $\check{l}_1(y_{1[0,\tau]}, Y_{d[0,\tau]}, \breve{w}_{1[0,\tau]})$ are three design functions to be introduced later, the cost function of subsystem $\mathbf{S_1}$ is then of the a linear quadratic structure.

The robust adaptive problem for $\mathbf{S_1}$ becomes an H^∞ control of affine quadratic problem, and admits a finite dimensional solution. By *cost-to-come* function analysis, we obtain the dynamics of worst-case covariance matrix $\bar{\Sigma}_1$, and state estimator $\check{\xi}_1$, which are given by

$$\begin{aligned}\dot{\bar{\Sigma}}_1 &= (\bar{A}_1 - \zeta_1^2 \bar{L}_1 \bar{C}_1)\bar{\Sigma}_1 + \bar{\Sigma}_1 (\bar{A}_1 - \zeta_1^2 \bar{L}_1 \bar{C}_1)' - \bar{\Sigma}_1 (\gamma^2 \zeta_1^2 \bar{C}_1' \bar{C}_1 - \bar{C}_1' \bar{C}_1 - \bar{Q}_1)\bar{\Sigma}_1 \\ &\quad + \gamma^{-2} \bar{D}_1 \bar{D}_1' - \gamma^{-2} \zeta_1^2 \bar{L}_1 \bar{L}_1'; \quad \bar{\Sigma}_1(0) = \gamma^{-2} \bar{Q}_{1,0}^{-1} \qquad (7a)\\ \dot{\check{\xi}}_1 &= (\bar{A}_1 + \bar{\Sigma}_1 (\bar{C}_1' \bar{C}_1 + \bar{Q}_1))\check{\xi}_1 + \bar{B}_1 y_2 + \zeta_1^2 (\gamma^2 \bar{\Sigma}_1 \bar{C}_1' + \bar{L}_1)(y_1 - \bar{C}_1 \check{\xi}_1) \\ &\quad - \bar{\Sigma}_1 (\bar{C}_1' y_d + \bar{Q}_1 \hat{\xi}_1 - l_{1,2}) + \bar{D}_1 \breve{w}_1; \quad \check{\xi}_1(0) = \left[\check{\theta}_{1,0}' \ \check{x}_{1,0}' \right]' \qquad (7b)\end{aligned}$$

where $\bar{L}_1$ is defined as $\bar{L}_1 = \left[\mathbf{0}_{1 \times \sigma_1} \ L_1' \right]'$.

We partition $\bar{\Sigma}_1$ as the same structure as

$$\bar{\Sigma}_1 = \begin{bmatrix} \Sigma_1 & \bar{\Sigma}_{1,12} \\ \bar{\Sigma}_{1,21} & \bar{\Sigma}_{1,22} \end{bmatrix} = \begin{bmatrix} \Sigma_1 & \Sigma_1 \Phi_1' \\ \Phi_1 \Sigma_1 & \frac{1}{\gamma^2} \Pi_1 + \Phi_1 \Sigma_1 \Phi_1' \end{bmatrix} \qquad (8)$$

where $\Phi_1(t) := \bar{\Sigma}_{1,21}(t)(\Sigma_1(t))^{-1}$ and $\Pi_1(t) := \gamma^2 (\bar{\Sigma}_{1,22}(t) - \bar{\Sigma}_{1,21}(t)(\Sigma_1(t))^{-1} \bar{\Sigma}_{1,12}(t))$, $\forall t \in [0, t_f]$. Then the weighting matrix $\bar{\Sigma}_1$ is positive definite if and only if Σ_1 and Π_1 are positive definite. To guarantee the boundedness of Σ_1, we choose weighing matrix $\bar{Q}_1$ as follows,

$$\bar{Q}_1 = \bar{\Sigma}_1^{-1} \begin{bmatrix} \mathbf{0}_{\sigma_1 \times \sigma_1} & \mathbf{0}_{\sigma_1 \times n_1} \\ \mathbf{0}_{n_1 \times \sigma_1} & \Delta_1(t) \end{bmatrix} \bar{\Sigma}_1^{-1} + \begin{bmatrix} \epsilon_1 \Phi_1' C_1' (\gamma^2 \zeta_1^2 - 1) C_1 \Phi_1 & \mathbf{0}_{\sigma_1 \times n_1} \\ \mathbf{0}_{n_1 \times \sigma_1} & \mathbf{0}_{n_1 \times n_1} \end{bmatrix} \qquad (9)$$

where $\Delta_1(t) = \gamma^{-2} \beta_{1,\Delta} \Pi_1(t) + \Delta_{1,1}$, with $\beta_{1,\Delta} \geq 0$ being a constant and $\Delta_{1,1}$ being an $n_1 \times n_1$-dimensional positive-definite matrix, and ϵ_1 is a scalar function defined by

$$\epsilon_1(t) := \mathrm{Tr}(\Sigma_1(t))^{-1} / K_{1,c} \quad \forall t \in [0, t_f] \qquad (10a)$$

or

$$\epsilon_1(t) := 1 \quad \forall t \in [0, t_f] \qquad (10b)$$

Then the dynamics of Σ_1, Φ_1, Π_1 are given as follows with initial conditions $\gamma^{-2}Q_{1,0}^{-1}$, $\Pi_{1,0}$, and $\Phi_{1,0}$ respectively,

$$\dot{\Sigma}_1 = (\epsilon_1 - 1)\Sigma_1\Phi_1' C_1' (\gamma^2\zeta_1^2 - 1)C_1\Phi_1\Sigma_1 \tag{11a}$$

$$\dot{\Pi\!}_1 = (A_1 - \zeta_1^2 L_1 C_1)\Pi_1 + \Pi_1 (A_1 - \zeta_1^2 L_1 C_1)' - \zeta_1^2 L_1 L_1' - \Pi_1 C_1' (\zeta_1^2 - \gamma^{-2}) C_1 \Pi_1 + D_1 D_1' + \gamma^2 \Delta_1 \tag{11b}$$

$$\dot{\Phi}_1 = A_{1,f}\Phi_1 + y_1 \bar{A}_{1,211} + y_2 \bar{A}_{1,212} + \sum_{j=1}^{\check{q}_1} \bar{A}_{1,213\,j} \check{w}_{1,j} \tag{11c}$$

where $A_{1,f} := A_1 - \zeta_1^2 L_1 C_1 - \Pi_1 C_1' C_1 (\zeta_1^2 - \gamma^{-2})$ is Hurwitz. By picking $\gamma \geq \zeta_1^{-1}$, we have the covariance matrix Σ_1 upper and lower bounded as summarized in the following Lemma [12].

Lemma 1. *Consider the dynamic equation (11a) for the covariance matrix Σ_1. Let $K_{1,c} \geq \gamma^2 \mathrm{Tr}(Q_{1,0})$, $Q_{1,0} > 0$, $\gamma \geq \zeta_1^{-1}$, and ϵ_1 be given by either (10b) or (10b). Then, the matrix Σ_1 is upper and lower bounded as follows, whenever Φ_1 is continuous on $[0, t_f]$,*

$$K_{1,c}^{-1} I_{1,\sigma_1} \leq \Sigma_1(t) \leq \Sigma_1(0) = \gamma^{-2} Q_{1,0}^{-1};$$
$$\gamma^2 \mathrm{Tr}(Q_{1,0}) \leq \mathrm{Tr}(\Sigma_1(t))^{-1} \leq K_{1,c}; \qquad \forall t \in [0, t_f]$$

To avoid the calculation of Σ_1^{-1} online, we define $s_{1,\Sigma} = \mathrm{Tr}(\Sigma_1^{-1})$. Based on the structure of $\bar{Q}_1$, we have the following assumption to guarantee the boundedness of Σ_1 and $s_{1,\Sigma}$,

Assumption 6. *If the matrix $A_1 - \zeta_1^2 L_1 C_1$ is Hurwitz, then the desired disturbance attenuation level $\gamma \geq \zeta_1^{-1}$. In case $\gamma = \zeta_1^{-1}$, choose $\beta_{1,\Delta} \geq 0$ such that $A_1 - \zeta_1^2 L_1 C_1 + \beta_{1,\Delta}/2 I_{n_1}$ is Hurwitz. If the matrix $A_1 - \zeta_1^2 L_1 C_1$ is not Hurwitz, then the desired disturbance attenuation level $\gamma > \zeta_1^{-1}$.* ◇

This assumption implies that the achievable disturbance attenuation level γ is no smaller than ζ_1^{-1}. Under this assumption, we initialize Π_1 as the unique positive definite solution of its Riccati Differential Equation (11b), which is summarized as the following assumption.

Assumption 7. *The initial weighting matrix $\Pi_{1,0}$ is chosen as the unique positive definite solutions to the following algebraic Riccati equations:*

$$(A_1 - \zeta_1^2 L_1 C_1)\Pi_1 + \Pi_1 (A_1 - \zeta_1^2 L_1 C_1)' - \Pi_1 C_1' \zeta_1^2 C_1 \Pi_1 + D_1 D_1' - \zeta_1^2 L_1 L_1' + \gamma^2 \Delta_1 = \mathbf{0}_{n_1 \times n_1} \tag{12}$$

To guarantee the estimates parameter to be bounded and the estimate of high frequency gain to be bounded away from zero, projection function scheme is applied to modify the dynamics of $\check{\xi}_1$.

Define

$$\rho_1 := \inf\{P_1(\bar{\theta}_1) \mid \bar{\theta}_1 \in \mathbb{R}^{\sigma_1}, b_{1,p0} + \bar{A}_{1,212\,0}\bar{\theta}_1 = 0\} \tag{13}$$

By Assumption 4 and Lemma 2 in [19] we have $1 < \rho_1 \leq \infty$. Fix any $\rho_{1,o} \in (1, \rho_1)$, and define the open set $\Theta_{1,o} := \{\bar{\theta}_1 \in \mathbb{R}^{\sigma_1} \mid P_1(\bar{\theta}) < \rho_{1,o}\}$. Our control design will guarantee that the

estimate $\check{\theta}_1$ lies in $\Theta_{1,o}$, which immediately implies $|b_{1,p0} + \bar{A}_{1,2120}\check{\theta}_1| > c_{1,0} > 0$, for some $c_{1,0} > 0$. Moreover, the convexity of P_1 implies the following inequality

$$\frac{\partial P_1}{\partial \theta_1}(\check{\theta}_1)\,(\theta_1 - \check{\theta}_1) < 0 \quad \forall \check{\theta}_1 \in \mathbf{R}^{\sigma_1} \backslash \Theta_1$$

We set $l_{1,1} = \check{\xi}_1$, and $l_{1,2} = \left[\, -(P_{1,r}(\check{\theta}_1))' \;\; \mathbf{0}_{1\times n_1} \,\right]'$, where

$$P_{1,r}(\check{\theta}_1) := \begin{cases} \dfrac{e^{\frac{1}{1-P_1(\check{\theta}_1)}} \left(\frac{\partial P_1}{\partial \theta_1}(\check{\theta}_1)\right)'}{\left(\rho_{1,o} - P_1(\check{\theta}_1)\right)^3} & \forall \theta_1 \in \Theta_{1,o} \backslash \Theta_1 \\ \mathbf{0}_{\sigma_1 \times 1} & \forall \theta_1 \in \Theta_1 \end{cases}$$

$$:= p_{1,r}(\check{\theta}_1) \left(\frac{\partial P_1}{\partial \theta_1}(\check{\theta}_1)\right)' \tag{14}$$

then, we obtain

$$\begin{aligned} \dot{\check{\xi}}_1 \;=\; & -\bar{\Sigma}_1 \left[\, (P_{1,r}(\check{\theta}_1))' \;\; \mathbf{0}_{1\times n_1} \,\right]' + \bar{A}_1 \check{\xi}_1 + \bar{\Sigma}_1 \bar{C}_1' (y_d - \bar{C}_1 \check{\xi}_1) - \bar{\Sigma}_1 \bar{Q}_1 (\Phi_1, s_{1,\Sigma}) \xi_{1,c} + \bar{B}_1 y_2 \\ & + \zeta_1^2 \left(\gamma^2 \bar{\Sigma}_1 \bar{C}_1' + \bar{L}_1\right)(y_1 - \bar{C}_1 \check{\xi}_1) + \bar{\check{D}}_1 \check{w}_1; \quad \check{\xi}_1(0) = \left[\, \check{\theta}_{1,0}' \;\; \check{x}_{1,0}' \,\right]' \end{aligned} \tag{15}$$

where $\xi_{1,c} = \hat{\xi}_1 - \check{\xi}_1$.

We summarize the equations for subsystem $\mathbf{S}_1$ as follows,

$$\begin{aligned} \mathbf{0} =\; & (A_1 - \zeta_1^2 L_1 C_1)\Pi_1 + \Pi_1 (A_1 - \zeta_1^2 L_1 C_1)' - \Pi_1 C_1' (\zeta_1^2 - \gamma^{-2}) C_1 \Pi_1 + D_1 D_1' - \zeta_1^2 L_1 L_1' + \gamma^2 \Delta_1 \\ \dot{\Sigma}_1 \;=\; & -(1-\epsilon_1)\Sigma_1 \Phi_1' C_1' \left(\gamma^2 \zeta_1^2 - 1\right) C_1 \Phi_1 \Sigma_1 \\ \dot{s}_{1,\Sigma} \;=\; & \left(\gamma^2 \zeta_1^2 - 1\right)(1-\epsilon_1) C_1 \Phi_1 \Phi_1' C_1' \\ \epsilon_1 \;=\; & K_{1,c}^{-1} s_{1,\Sigma} \quad \text{or} \quad 1 \\ A_{1,f} \;=\; & A_1 - \zeta_1^2 L_1 C_1 - \Pi_1 C_1' C_1 \left(\zeta_1^2 - \gamma^{-2}\right) \\ \dot{\Phi}_1 \;=\; & A_{1,f} \Phi_1 + y_1 \bar{A}_{1,211} + y_2 \bar{A}_{1,212} + \sum_{j=1}^{\check{q}_1} \bar{A}_{1,213\,j} \check{w}_{1,j} \\ \dot{\check{\theta}}_1 \;=\; & -\Sigma_1 P_{1,r}(\check{\theta}_1) - \Sigma_1 \Phi_1' C_1' (y_d - C_1 \check{x}_1) - \left[\, \Sigma_1 \;\; \Sigma_1 \Phi_1' \,\right] \bar{Q}_1 \xi_{1,c} + \gamma^2 \zeta_1^2 \Sigma_1 \Phi_1' C_1' (y_1 - C_1 \check{x}_1) \\ \dot{\check{x}}_1 \;=\; & -\Phi_1 \Sigma_1 P_{1,r}(\check{\theta}_1) + A_1 \check{x}_1 - (\gamma^{-2} \Pi_1 + \Phi_1 \Sigma_1 \Phi_1') C_1' (y_d - C_1 \check{x}_1) + B_1 y_2 + \check{D}_1 \check{w}_1 \\ & - \left[\, \Phi_1 \Sigma_1 \;\; \gamma^{-2} \Pi_1 + \Phi_1 \Sigma_1 \Phi_1' \,\right] \bar{Q}_1 \xi_{1,c} + (y_1 \bar{A}_{1,211} + y_2 \bar{A}_{1,212} + \sum_{j=1}^{\check{q}_1} \check{w}_{1,j} \bar{A}_{1,213j}) \check{\theta}_1 \\ & + \zeta_1^2 \left(\Pi_1 C_1' + \gamma^2 \Phi_1 \Sigma_1 \Phi_1' C_1' + L_1\right)(y_1 - C_1 \check{x}_1) \end{aligned}$$

This completes the estimation design of $\mathbf{S_1}$.

Next, we will derive the estimator for subsystem $\mathbf{S_2}$. In this step, the measurements waveform ω_m is assumed to be known. Since the control input, u, is a causal function of ω_m, then it is known. Again, we will apply the *cost-to-come function* methodology to derive the estimator. We briefly summarize the estimation design for $\mathbf{S}_2$ as follows.

Set function l_2 in definition to be $|\xi_2 - \hat{\xi}_2|^2_{\bar{Q}_2} + 2(\xi_2 - \check{\xi}_2)' l_{2,2} + \check{l}_2$, where $\hat{\xi}_2 = [\hat{\theta}_2', \hat{x}_2']'$ is the worst-case estimate for the expanded state ξ_2, $\check{\xi}_2$ is the estimate of ξ_2, $\bar{Q}_2$ is a matrix-valued weighting function, $l_{2,2}$ and $\check{l}_2$ are two design functions to be introduced later, the cost function of subsystem $\mathbf{S_2}$ is then of a linear quadratic structure. By *cost-to-come* function analysis, we obtain the dynamics of worst-case covariance matrix $\bar{\Sigma}_2$, and state estimator $\check{\xi}_2$. We partition $\bar{\Sigma}_2$ as $\bar{\Sigma}_2 = \begin{bmatrix} \Sigma_2 & \bar{\Sigma}_{2,12} \\ \bar{\Sigma}_{2,21} & \bar{\Sigma}_{2,22} \end{bmatrix}$ and introduce $\Phi_2 := \bar{\Sigma}_{2,21}\Sigma_2^{-1}$ and $\Pi_2 := \gamma^2(\bar{\Sigma}_{2,22} - \bar{\Sigma}_{2,21}\Sigma_2^{-1}\bar{\Sigma}_{2,12})$, then the weighting matrix $\bar{\Sigma}_2$ is positive definite if and only if Σ_2 and Π_2 are positive definite. To guarantee the boundedness of Σ_2, we choose weighing matrix $\bar{Q}_2$ as follows,

$$\bar{Q}_2 = \begin{bmatrix} -\Phi_2' \\ I_{n_2} \end{bmatrix} \gamma^4 \Pi_2^{-1} \Delta_2 \Pi_2^{-1} \begin{bmatrix} -\Phi_2' \\ I_{n_2} \end{bmatrix}' + \begin{bmatrix} \epsilon_2 \Phi_2' C_2' \gamma^2 \zeta_2^2 C_2 \Phi_2 & \mathbf{0}_{\sigma_2 \times n_2} \\ \mathbf{0}_{n_2 \times \sigma_2} & \mathbf{0}_{n_2 \times n_2} \end{bmatrix} \tag{16}$$

where $\Delta_2(t) = \gamma^{-2}\beta_{2,\Delta}\Pi_2(t) + \Delta_{2,1}$, with $\beta_{2,\Delta} \geq 0$ being a constant and $\Delta_{2,1}$ being an $n_2 \times n_2$- dimensional positive-definite matrix, and ϵ_2 is a scalar function defined by $\epsilon_2 = K_{2,c}^{-1}\mathrm{Tr}(\Sigma_2^{-1})$ or $\epsilon_2 = 1$. $K_{2,c} \geq \gamma^2 \mathrm{Tr}(Q_{2,0})$ is a design constant, $Q_{2,0}$ is an $\sigma_2 \times \sigma_2$-dimensional positive-definite matrix. Then the dynamics of Σ_2, Φ_2, Π_2 are given as follows,

$$\dot{\Sigma}_2 = (\epsilon_2 - 1)\Sigma_2 \Phi_2' C_2' \gamma^2 \zeta_2^2 C_2 \Phi_2 \Sigma_2; \quad \Sigma_2(0) = \frac{\gamma^{-2}}{Q_{2,0}} \tag{17a}$$

$$\begin{aligned} \dot{\Pi\Pi}_2 = {} & (A_2 - \zeta_2^2 L_2 C_2 + \beta_{2,\Delta}/2 I_{n_2})\Pi_2 + \Pi_2 (A_2 - \zeta_2^2 L_2 C_2 + \beta_{2,\Delta}/2 I_{n_2})' - \Pi_2 C_2' \zeta_2^2 C_2 \Pi_2 + D_2 D_2' \\ & - \zeta_2^2 L_2 L_2' + \gamma^2 \Delta_{2,1}; \quad \Pi_2(0) = \Pi_{2,0} \end{aligned} \tag{17b}$$

$$\dot{\Phi}_2 = A_{2,f}\Phi_2 + y_2 \bar{A}_{2,211} + u\bar{A}_{2,212} + \sum_{j=1}^{\check{q}_2} \bar{A}_{2,213j} \check{w}_{2,j} + \check{y}_2 \bar{A}_{2,214}; \qquad \Phi_2(0) = \Phi_{2,0} \tag{17c}$$

where $A_{2,f} := A_2 - \zeta_2^2 L_2 C_2 - \Pi_2 C_2' C_2 \zeta_2^2$ is Hurwitz. By Lemma [12], we have the covariance matrix Σ_2 upper and lower bounded as follows, $K_{2,c}^{-1} I_{\sigma_2} \leq \Sigma_2(t) \leq \Sigma_2(0) = \gamma^{-2} Q_{2,0}^{-1}$, $\gamma^2 \mathrm{Tr}(Q_{2,0}) \leq \mathrm{Tr}(\Sigma_2(t))^{-1} \leq K_{2,c}$, whenever it exists on $[0, t_f]$ and Φ_2 is continuous on $[0, t_f]$. To avoid the calculation of Σ_2^{-1} online, we define $s_{2,\Sigma} = \mathrm{Tr}(\Sigma_2^{-1})$.

To guarantee the estimates parameter to be bounded and the estimate of high frequency gain to be bounded away from zero without persistently exciting signals, we introduce the following soft projection design on the parameter estimate.

Define $\rho_2 := \inf\{P_2(\bar{\theta}_2) \mid \bar{\theta}_2 \in \mathbb{R}^{\sigma_2}, b_{2,p0} + \bar{A}_{2,2120}\bar{\theta}_2 = 0\}$, we have $1 < \rho_2 \leq \infty$. Fix any $\rho_{2,o} \in (1, \rho_2)$, we define the open set $\Theta_{2,o} := \{\bar{\theta}_2 \mid P_2(\bar{\theta}) < \rho_{2,o}\}$. Our control design will guarantee that the estimate $\check{\theta}_2$ lies in $\Theta_{2,o}$, which immediately implies $|b_{2,p0} + \bar{A}_{2,2120}\check{\theta}_2| > c_{2,0} > 0$, for some $c_{2,0} > 0$. Moreover, the convexity of P_2 implies the following inequality: $\frac{\partial P_2}{\partial \theta_2}(\check{\theta}_2)(\theta_2 - \check{\theta}_2) < 0 \quad \forall \check{\theta}_2 \in \mathbb{R}^{\sigma_2} \backslash \Theta_2$. To incorporate the modifier to the estimates dynamics,

we introduce $l_{2,2} = [-(P_{2,r}(\check{\theta}_2)' \ \mathbf{0}_{1\times n_2}]'$, where

$$P_{2,r}(\check{\theta}_2) := \begin{cases} \frac{\exp\left(\frac{1}{1-P_2(\check{\theta}_2)}\right)}{\left(\rho_{2,o}-P_2(\check{\theta}_2)\right)^3}\left(\frac{\partial P_2}{\partial\theta_2}(\check{\theta}_2)\right)' & \forall\theta_2\in\Theta_{2,o}\backslash\Theta_2 \\ 0_{\sigma_2\times 1} & \forall\theta_2\in\Theta_2 \end{cases}$$
$$:= p_{2,r}(\check{\theta}_2)\left(\frac{\partial P_2}{\partial\theta_2}(\check{\theta}_2)\right)'$$

and the dynamics of $\check{\xi}_2$ is then given as follows,

$$\dot{\check{\xi}}_2 = -\bar{\Sigma}_2\left[\,(P_{2,r}(\check{\theta}_2))'\ \mathbf{0}_{1\times n_2}\,\right]' + \bar{A}_2\check{\xi}_2 + \bar{B}_2 u + \zeta_2^2\left(\gamma^2\bar{\Sigma}_2\bar{C}_2' + \bar{A}_{2,y}\check{y}_2 + \bar{L}_2\right)(y_2 - \bar{C}_2\check{\xi}_2)$$
$$+\bar{\bar{D}}_2\check{w}_2 - \bar{\Sigma}_2\bar{Q}_2(\hat{\xi}_2 - \check{\xi}_2)$$

where $\check{\xi}_2 = [\check{\theta}_2'\ \check{x}_2]'$ with initial condition $[\check{\theta}_{2,0}'\ \check{x}_{2,0}']'$, and $\bar{L}_2$ is defined as $\bar{L}_2 = [\mathbf{0}_{1\times\sigma_2}\ L_2']'$. This completes the estimation design of $\mathbf{S_2}$.

Associated with the above identifier and estimator of subsystem $\mathbf{S}_i$, $i = 1, 2$, we introduce the value function $W_i : \mathbb{R}^{n_i+\sigma_i}\times\mathbb{R}^{n_i+\sigma_i}\times\mathcal{S}_{+(n_i+\sigma_i)} \to \mathbb{R}$ and the time derivative are as follows

$$W_i(\xi_i, \check{\xi}_i, \bar{\Sigma}_i) = |\theta_i - \check{\theta}_i|^2_{\Sigma_i^{-1}} + \gamma^2|x_i - \check{x}_i - \Phi_i(\theta_i - \check{\theta}_i)|^2_{\Pi_i^{-1}} \tag{18}$$

$$\dot{W}_1 = -|x_{1,1} - y_d|^2 - \gamma^4|x_1 - \hat{x}_1 - \Phi_1(\theta_1 - \hat{\theta}_1)|^2_{\Pi_1^{-1}\Delta_1\Pi_1^{-1}} + |C_1\check{x}_1 - y_d|^2$$
$$-\epsilon_1(\gamma^2\zeta_1^2 - 1)|\theta_1 - \hat{\theta}_1|^2_{\Phi_1'C_1'C_1\Phi_1} - \gamma^2\zeta_1^2|y_1 - C_1\check{x}_1|^2 + \gamma^2|w_1|^2 - \gamma^2|w_1 - w_{1,*}|^2$$
$$+2(\theta_1 - \check{\theta}_1)'P_{1,r}(\check{\theta}_i) + |\xi_{1,c}|^2_{\bar{Q}_1} \tag{19}$$

$$\dot{W}_2 = -\gamma^4|x_2 - \hat{x}_2 - \Phi_2(\theta_2 - \hat{\theta}_2)|^2_{\Pi_2^{-1}\Delta_2\Pi_2^{-1}} - \epsilon_2\gamma^2\zeta_2^2|\theta_2 - \hat{\theta}_2|^2_{\Phi_2'C_2'C_2\Phi_2} + |\xi_{2,c}|^2_{\bar{Q}_2}$$
$$-\gamma^2\zeta_2^2|y_2 - C_2\check{x}_2|^2 + \gamma^2|w_2|^2 - \gamma^2|w_2 - w_{2,*}|^2 + 2(\theta_2 - \check{\theta}_2)'P_{2,r}(\check{\theta}_2) \tag{20}$$

where $w_{i,*}$ is the worst-case disturbance, given by $w_{i,*} : \mathbb{R}\times\mathbb{R}^{n_i+\sigma_i}\times\mathbb{R}^{n_i+\sigma_i}\times\mathcal{S}_{+(n_i+\sigma_i)} \longrightarrow \mathbb{R}$

$$w_{i,*}(\xi_i, \check{\xi}_i, \bar{\Sigma}_i, w_i) = \zeta_i^2E_i'(y_i - \bar{C}_i\xi_i) + \gamma^{-2}(I_{q_i} - \zeta_i^2E_i'E_i)\bar{D}_i'\bar{\Sigma}_i^{-1}(\xi_i - \check{\xi}_i); \quad i = 1, 2$$

We note that (18) holds when $\Sigma_i > 0$ and $\theta_i \in \Theta_{i,0}$, and the last term in $\dot{W}_i$ is nonpositive, zero on the set Θ_i and approaches $-\infty$ as $\check{\theta}_i$ approaches the boundary of the set $\Theta_{i,o}$, which guarantees the boundedness of $\check{\theta}_i$, $i = 1, 2$.

Then (5) can be equivalently written as, $i = 1, 2$:

$$J_{1,\gamma t_f} = \int_0^{t_f}\left(|C_1\check{x}_1 - y_d|^2 + |\xi_{1,c}|^2_{\bar{Q}_1} + \check{l}_1 - \gamma^2\zeta_1^2|y_1 - C_1\check{x}_1|^2 - \gamma^2|w_1 - w_{1,*}|^2 - \gamma^2|\check{w}_{1,a}|^2\right)\mathrm{d}\tau$$
$$-l_{1,0} - |\xi_1(t_f) - \check{\xi}_1(t_f)|^2_{(\bar{\Sigma}_1(t_f))^{-1}}$$
$$J_{2,\gamma t_f} = \int_0^{t_f}\left(|\xi_{2,c}|^2_{\bar{Q}_2} + \check{l}_2 - \gamma^2\zeta_2^2|y_2 - C_2\check{x}_2|^2 - \gamma^2|w_2 - w_{2,*}|^2 - \gamma^2|\check{w}_{2,a}|^2\right)\mathrm{d}\tau$$
$$-l_{2,0} - |\xi_2(t_f) - \check{\xi}_2(t_f)|^2_{(\bar{\Sigma}_2(t_f))^{-1}}$$

This completes the identification design step.

4.2. Control design

In this section, we describe the controller design for the uncertain system under consideration. Note that, we ignored some terms in the cost function (5) in the identification step, since they are constant when y_1, y_2, $\check{w}_1$, $\check{w}_2$ and $\acute{y}_2$ are given. In the control design step, we will include such terms. Then, based on the cost function (5), the controller design is to guarantee that the following supremum is less than or equal to zero for all measurement waveforms,

$$\begin{aligned}
&\sup_{\dot{w}_1\in\dot{\mathcal{W}}_1,\dot{w}_2\in\dot{\mathcal{W}}_2} J_{\gamma t_f}\\
&\leq \sup_{\omega_m\in\mathcal{W}_m}\left(\sup_{\omega_1\in\mathcal{W}_1|\omega_m\in\mathcal{W}_m} J_{1,\gamma t_f}+\sup_{\omega_2\in\mathcal{W}_2|\omega_m\in\mathcal{W}_m} J_{2,\gamma t_f}\right)\\
&\leq \sup_{\omega_m\in\mathcal{W}_m}\left\{\int_0^{t_f}\left(|C_1\check{x}_1-y_d|^2+\sum_{i=1}^{2}\left(|\xi_{i,c}|^2_{\bar{Q}_i}+\check{l}_i-\gamma^2\zeta_i^2|y_i-C_i\check{x}_i|^2-\gamma^2|\check{w}_{i,a}|^2\right)\right)\mathrm{d}\tau\right\} \quad (21)
\end{aligned}$$

where function $\check{l}_1(\tau, y_{1[0,\tau]}, Y_{d[0,\tau]}, \check{w}_1)$ is part of the weighting function $l_1(\tau, \theta_1, x_1, y_{1[0,\tau]}, Y_{d[0,\tau]}, \check{w}_1)$, and $\check{l}_2(\tau, y_{2[0,\tau]}, Y_{d[0,\tau]}, \check{w}_2)$ is part of the weighting function $l_2(\tau, \theta_2, x_2, y_{2[0,\tau]}, Y_{d[0,\tau]}, \check{w}_2)$ to be designed, which are constants in the identifier design step and are therefore neglected.

By equation (21), we observe that the cost function is expressed in term of the states of the estimator we derived, whose dynamics are driven by the measurement y_1, y_2, $\check{w}_1$, $\check{w}_2$, $\acute{y}_2$, the reference trajectory y_d, the input u, and the worst-case estimate for the expanded state vector $\hat{\check{\xi}}_1$ and $\hat{\check{\xi}}_2$, which are signals we either measure or can construct. This is then a nonlinear H^∞-optimal control problem under full information measurements. Since $\acute{y}_2 = y_1$ in the adaptive system under consideration, we can equivalently deal with the following transformed variables instead of considering y_1, y_2, $\check{w}_1$, $\check{w}_2$, and $\acute{y}_2$ as the maximizing variable,

$$v = \begin{bmatrix} \zeta_1\left(y_1 - C_1\check{x}_1\right) \\ \check{w}_{1,a} \\ \check{w}_{1,b} \\ \hline \zeta_2\left(y_2 - C_2\check{x}_2\right) \\ \check{w}_{2,a} \\ \check{w}_{2,b} \end{bmatrix} = \begin{bmatrix} v_1 \\ \hline v_2 \end{bmatrix}$$

where $v_i = \left[\zeta_i\left(y_i - C_i\check{x}_i\right)\ \check{w}'_{i,a}\ \check{w}'_{i,b}\right]'$, $i = 1, 2$.

By the special structure of the system, we define $v_{i,a} = \left[\zeta_i\left(y_i - C_i\check{x}_i\right)\ \check{w}'_{i,a}\right]'$, $i = 1, 2$, $v_a = \left[v'_{1,a}\ v'_{2,a}\right]'$, and we will attenuate disturbance v_a, and cancel the disturbance $\check{w}_{1,b}$ and $\check{w}_{2,b}$. In view of $y_2 = \zeta_2^{-1}e'_{\check{q}_a+2,\check{q}_{1,a}+2}v_a + \check{x}_{2,1}$, we will treat $\check{x}_{2,1}$ as the virtual control input of subsystem $\mathbf{S_1}$, where $\check{q}_a = \check{q}_{1,a} + \check{q}_{2,a}$.

For $i = 1, 2$, we introduce the matrix $M_{i,f} := \left[A_{i,f}^{n_i-1}p_{i,n_i}\ \cdots\ A_{i,f}p_{i,n_i}\ p_{i,n_i}\right]$, where p_{i,n_i} is a n_i-dimensional vector such that the pair $(A_{i,f}, p_{i,n_i})$ is controllable. We note that $\acute{y}_2 = y_1$, then

the following $3n_1 + 4n_2 + \check{q}_1 + \check{q}_2$-dimensional prefiltering system for y_1, y_2, u, $\check{w}_1$, $\check{w}_2$, and $\acute{y}_2$ generates the Φ_1 and Φ_2 online:

$$
\begin{aligned}
\dot{\eta}_1 &= A_{1,f}\eta_1 + p_{1,n_1}y_1;\\
\dot{\eta}_{\check{w}_1,j} &= A_{1,f}\eta_{\check{w}_1,j} + p_{1,n_1}\check{w}_{1,j}; \quad \eta_{\check{w}_i,j}(0) = \eta_{\check{w}_i,j0}, i = 1,\cdots,\check{q}_i\\
\dot{\lambda}_1 &= A_{1,f}\lambda_1 + p_{1,n_1}y_2; \quad \lambda_1(0) = \lambda_{1,0}\\
\Phi_1 &= \left[\begin{array}{c} A_{1,f}^{n_1-1}\eta_1 \cdots A_{1,f}\eta_1\ \eta_1 \end{array}\right] M_{1,f}^{-1}\bar{A}_{1,211} + \left[\begin{array}{c} A_{1,f}^{n_1-1}\lambda_1 \cdots A_{1,f}\lambda_1\ \lambda_1 \end{array}\right] M_{1,f}^{-1}\bar{A}_{1,212}\\
&\quad + \sum_{j=1}^{\check{q}_1}\left[\begin{array}{c} A_{1,f}^{n_1-1}\eta_{\check{w}_1,j} \cdots A_{1,f}\eta_{\check{w}_1,j}\ \eta_{\check{w}_1,j} \end{array}\right] M_{1,f}^{-1}\bar{A}_{i,213j}\\
\dot{\eta}_2 &= A_{2,f}\eta_2 + p_{2,n_2}y_2;\\
\dot{\eta}_{\check{w}_2,j} &= A_{2,f}\eta_{\check{w}_2,j} + p_{2,n_2}\check{w}_{2,j}; \quad \eta_{\check{w}_2,j}(0) = \eta_{\check{w}_2,j0}, j = 1,\cdots,\check{q}_i\\
\dot{\lambda}_2 &= A_{2,f}\lambda_2 + p_{2,n_2}u; \quad \lambda_1(0) = \lambda_{1,0}\\
\dot{\eta}_{2,y} &= A_{2,f}\eta_{2,y} + p_{2,n_2}\acute{y}_2; \eta_{2,y}(0) = \eta_{2,y0}\\
\Phi_2 &= \left[\begin{array}{c} A_{2,f}^{n_2-1}\eta_1 \cdots A_{2,f}\eta_2\ \eta_2 \end{array}\right] M_{2,f}^{-1}\bar{A}_{2,211} + \left[\begin{array}{c} A_{2,f}^{n_2-1}\lambda_2 \cdots A_{2,f}\lambda_2\ \lambda_2 \end{array}\right] M_{2,f}^{-1}\bar{A}_{2,212}\\
&\quad + \sum_{j=1}^{\check{q}_2}\left[\begin{array}{c} A_{2,f}^{n_2-1}\eta_{\check{w}_2,j} \cdots A_{2,f}\eta_{\check{w}_2,j}\ \eta_{\check{w}_2,j} \end{array}\right] M_{2,f}^{-1}\bar{A}_{2,213j}\\
&\quad + \left[\begin{array}{c} A_{2,f}^{n_2-1}\eta_{2,y} \cdots A_{2,f}\eta_{2,y}\ \eta_{2,y} \end{array}\right] M_{2,f}^{-1}\bar{A}_{2,214}
\end{aligned}
$$

The variables to be designed at this stage include $\check{x}_{2,1}$, u, $\xi_{1,c}$, and $\xi_{2,c}$. Note that the structures of A_1 and A_2 in the dynamics is in strict-feedback form, we will use the backstepping methodology, see [9], to design the control input u, which will guarantee the global boundedness of the closed-loop system states and the asymptotic convergence of the tracking error. Since there are the nonnegative definite weighting on $\xi_{1,c}$ and $\xi_{2,c}$ in the cost function (21), we can not use integrator backstepping to design feedback law for $\xi_{1,c}$ and $\xi_{2,c}$. Hence, we set $\xi_{1,c} = \xi_{2,c} = 0$ in the backstepping procedure. After the completion of the backstepping procedure, we will then optimize the choice of $\xi_{1,c}$ and $\xi_{2,c}$ based on the value function obtained. Note that Σ_1, Π_1, $s_{1,\Sigma}$, $\check{\theta}_1$, Σ_2, Π_2, $s_{2,\Sigma}$, and $\check{\theta}_2$ are always bounded by the design in Section 4.1. Since Φ_1 is driven by control y_2, and Φ_2 is explicitly driven by u, they can not be stabilized in conjunction with $\check{x}_1$ and $\check{x}_2$ in the backstepping design. We will assume they are bounded and prove later they are indeed so under the derived control law.

We carry out the backstepping design for subsystem $\mathbf{S_1}$ first, and treat $\check{x}_{2,1}$ as the virtual control input of subsystem $\mathbf{S_1}$ in view of $y_2 = \zeta_2^{-1}e'_{\check{q}_a+2,\check{q}_{1,a}+2}v_a + \check{x}_{2,1}$. To stabilize η_1, we introduce variable $\eta_{1,d}$, which satisfies $\dot{\eta}_{1,d} = A_{1,f}\eta_{1,d} + p_{1,n_1}y_d$ with initial condition $\eta_{1,d}(0) = \eta_{1,d0}$, and is the reference trajectory for η_1 to track. Choosing value function $V_{1,0} := |\eta_1 - \eta_{1,d}|^2_{Z_1}$, where Z_1 is the solution to an algebraic Riccati equation. Treating $\check{x}_{1,1}$ as the virtual control input, we complete the step 0 with the virtual control law $\alpha_{1,0} = y_d$, which will guarantee the $\dot{V}_{1,0} \leq 0$ under $\check{x}_{1,1} = \alpha_{1,0}$. At step 1, we introduce $z_{1,1} := \check{x}_{1,1} - y_d$, and choose value function $V_{1,1} = V_{1,0} + \frac{1}{2}z_{1,1}^2$. Treating $\check{x}_{1,2}$ as the virtual control input, we end the

step 1 with the virtual control law $\alpha_{1,1}$, which guarantees $\dot{V}_{1,1} \leq 0$ under $\check{x}_{1,2} = \alpha_{1,1}$. Define the variable $z_{1,2} = \check{x}_{1,2} - \alpha_{1,1}$ for step 2. Repeating the backstepping procedure until step r_1, the virtual control input $\check{x}_{2,1}$ will appear in the dynamic of $\dot{z}_{1,r_1}$. Using the similar procedure as previous steps, we can derive the robust adaptive controller α_{1,r_1} such that $\dot{V}_{1,r_1} \leq 0$ under $\check{x}_{2,1} := \alpha_{1,r_1}$. This completes the control design for subsystem $\mathbf{S_1}$.

To stabilize η_2, we introduce variable $\eta_{2,d}$ as below,

$$\dot{\eta}_{2,d} = A_{2,f}\eta_{2,d} + p_{2,n_2}\alpha_{1,r_1} + p_{2,n_2}e'_{\check{q}_a+2,\check{q}_{1,a}+2}\nu_{1,r_1}; \eta_{2,d}(0) = \eta_{2,d0}$$

and is the reference trajectory for η_2 to track, where ν_{1,r_1} is a function obtained after step r_1. Choosing value function $V_{2,0} := |\eta_2 - \eta_{2,d}|^2_{Z_2} + V_{1,r_1}$, where Z_2 is the solution to an algebraic Riccati equation. We complete the step $r_1 + 1$ with the virtual control law $\alpha_{2,0} = \alpha_{1,r_1}$, which will guarantee the $\dot{V}_{2,0} \leq 0$ under $\check{x}_{2,1} = \alpha_{2,0}$. Repeating the backstepping procedure until step $r_1 + r_2 + 1$, the virtual control input u will appear in the dynamic of $\dot{z}_{2,r_2}$. Introduce $V_{2,r_2} = \sum_{j=1}^{2}(|\tilde{\eta}_j|^2_{Z_j} + \sum_{k=1}^{r_j}\frac{1}{2}z^2_{j,k})$, we then can derive the robust adaptive controller μ such that $\dot{V}_{2,r_2} \leq 0$ under $u := \mu$. Later, we will show that the control law μ will guarantee the boundedness of the closed-loop system states and the asymptotic convergence of tracking error.

For the closed-loop adaptive nonlinear system, we have the following value function, $U = W_1 + W_2 + V_{2,r_2}$, and its time derivative is given by

$$\begin{aligned}
\dot{U} = &-|x_{1,1} - y_d|^2 - \sum_{j=1}^{2}\Big(\gamma^4|x_j - \hat{x}_j - \Phi_j(\theta_j - \hat{\theta}_j)|^2_{\Pi_j^{-1}\Delta_j\Pi_j^{-1}} + \epsilon_j(\gamma^2\zeta_j^2 - 1)|\theta_j - \hat{\theta}_j|^2_{\Phi_j'C_j'C_j\Phi_j} \\
&-2(\theta_j - \check{\theta}_j)'P_{j,r}(\check{\theta}_j) + |\tilde{\eta}_j|^2_{Y_j} + \sum_{k=1}^{r_j}\beta_{j,k}z^2_{j,k} - \gamma^2|w_j|^2 + \gamma^2|w_j - w_{j,opt}|^2 - \gamma^2|\check{w}_{j,a}|^2 \\
&+\gamma^2|\check{w}_{j,a} - \check{w}_{j,opt}|^2\Big) - \epsilon_2|\theta_2 - \hat{\theta}_2|_{\Phi_2'C_2'C_2\Phi_2} - \frac{1}{4}\left|\varsigma_{1,(r_1+r_2)}\right|^2_{\bar{Q}_1} - \frac{1}{4}|\varsigma_{2,r_2}|^2_{\bar{Q}_2} \\
&+\left|\xi_{1,c} + \frac{1}{2}\varsigma_{1,(r_1+r_2)}\right|^2_{\bar{Q}_1} + \left|\xi_{2,c} + \frac{1}{2}\varsigma_{2,r_2}\right|^2_{\bar{Q}_2}
\end{aligned}$$

where ς_{1,r_1+r_2} and ς_{2,r_2} are functions obtained after step $r_1 + r_2 + 1$, $w_{1,opt}$ and $w_{2,opt}$ are the worst case disturbance with respect to the value function U, which are given by

$$\begin{aligned}
w_{1,opt} &= \zeta_1E_1'e'_{2,1}\nu_{1,r_1} + \gamma^{-2}(I_{q_1} - \zeta_1^2E_1'E_1)\bar{D}_1'\bar{\Sigma}_1^{-1}(\xi_1 - \check{\xi}_1) + \zeta_1^2E_1'C_1(\check{x}_1 - x_1) \\
w_{2,opt} &= \zeta_2E_2'e'_{2,2}\nu_{2,r_2} + \gamma^{-2}(I_{q_2} - \zeta_2^2E_2'E_2)\bar{D}_2'\bar{\Sigma}_2^{-1}(\xi_2 - \check{\xi}_2) + \zeta_2^2E_2'C_2(\check{x}_2 - x_2) \\
\check{w}_{1,opt} &= \left[\mathbf{0}'_{1\times(2+\check{q}_{1,a}+\check{q}_{2,a})}\ e_{(2+\check{q}_{1,a}+\check{q}_{2,a}),1}\ \cdots\ e_{(2+\check{q}_{1,a}+\check{q}_{2,a}),\check{q}_{1,a}}\ \mathbf{0}'_{(1+\check{q}_{2,a})\times(2+\check{q}_{1,a}+\check{q}_{2,a})}\right]'\nu_{1,r_1} \\
\check{w}_{2,opt} &= \left[\mathbf{0}'_{(2+\check{q}_{1,a})\times(2+\check{q}_{1,a}+\check{q}_{2,a})}\ e_{(2+\check{q}_{1,a}+\check{q}_{2,a}),1}\ \cdots\ e_{(2+\check{q}_{1,a}+\check{q}_{2,a}),\check{q}_{2,a}}\right]'\nu_{2,r_2}
\end{aligned}$$

where ν_{1,r_1} and ν_{2,r_2} are functions obtained after backstepping design.

Then the optimal choice for the variable $\xi_{i,c}$ and $\hat{\xi}_i$, $i = 1, 2$, are:

$$\xi_{1,c*} = -\frac{1}{2}\varsigma_{1,r_1+r_2} \Longleftrightarrow \hat{\xi}_{1,*} = \check{\xi}_1 - \frac{1}{2}\varsigma_{1,r_1+r_2};$$
$$\xi_{2,c*} = -\frac{1}{2}\varsigma_{2,r_2} \Longleftrightarrow \hat{\xi}_{2,*} = \check{\xi}_2 - \frac{1}{2}\varsigma_{2,r_2}$$

which yields that the closed-loop system is dissipative with storage function U and supply rate with optimal choice for $\hat{\xi}_i, i = 1, 2$:

$$-|x_{1,1} - y_d|^2 + \gamma^2|w_1|^2 + \gamma^2|w_2|^2 + \gamma^2|\check{w}_{1,a}|^2 + \gamma^2|\check{w}_{2,a}|^2$$

This completes the adaptive controller design step. We will discuss the robustness and tracking properties of the proposed adaptive control laws.

5. Main result

In this Section, we present the main result by stating two theorems.

For the adaptive control law, with the optimal choice of $\xi_{i,c*}$, the closed-loop system dynamics are

$$\dot{X} = F(X, y_d^{(r_1+r_2)}) + G(X)\left[\, w_1' \;\; w_2' \,\right] + G_{\check{w}}(X)\left[\, \check{w}_1' \;\; \check{w}_2' \,\right]; X(0) = X_0 \tag{22}$$

where F, G and G_M are smooth mapping of $\mathcal{D} \times \mathbb{R}$, $\mathcal{D}$ and $\mathcal{D}$, respectively; and the initial condition $X_0 \in \mathcal{D}_0 := \{X_0 \in \mathcal{D} \mid \theta_i \in \Theta_i, \check{\theta}_{i,0} \in \Theta_i, \Sigma_i(0) = \gamma^{-2}Q_{i,0}^{-1} > 0, \mathrm{Tr}\left((\Sigma_i(0))^{-1}\right) \leq K_{i,c}, s_{i,\Sigma}(0) = \gamma^2\mathrm{Tr}(Q_{i,0}); \quad i = 1, 2\}$. And the value function U satisfies an Hamilton-Jacobi-Isaacs equation, $\forall X \in \mathcal{D}, \forall y_d^{(r_1+r_2)} \in \mathbb{R}$.

$$\frac{\partial U}{\partial X}(X)F(X, y_d^{(r_1+r_2)}) + \frac{1}{4\gamma^2}\frac{\partial U}{\partial X}(X)\left[\, G(X) \;\; G_{\check{w}}(X) \,\right]\left[\, G(X)' \;\; G_{\check{w}}(X)' \,\right]'\left(\frac{\partial U}{\partial X}(X)\right)'$$
$$+Q(X, y_d^{(r_1+r_2)}) = 0;$$

where $Q : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is smooth and given by

$$Q(X, y_d^{(r_1+r_2)}) = |x_{1,1} - y_d|^2 + \sum_{j=1}^{2}\Big(\gamma^4|x_j - \hat{x}_j - \Phi_j(\theta_j - \hat{\theta})_j|^2_{\Pi_j^{-1}\Delta_j\Pi_j^{-1}}$$
$$+\epsilon_j\left(\gamma^2\varsigma_j^2 - 1\right)|\theta_j - \hat{\theta}_j|^2_{\Phi_j'C_j'C_j\Phi_j} - 2\left(\theta_j - \check{\theta}_j\right)'P_{j,r}(\check{\theta}_j) + |\tilde{\eta}_j|^2_{Y_j} + \sum_{k=1}^{r_j}\beta_{j,k}z_{j,k}^2\Big)$$
$$+\frac{1}{4}\left|\varsigma_{1,(r_1+r_2)}\right|^2_{\bar{Q}_1} + \frac{1}{4}|\varsigma_{2,r_2}|^2_{\bar{Q}_2} + \epsilon_2|\theta_2 - \hat{\theta}_2|_{\Phi_2'C_2'C_2\Phi_2}$$

The closed-loop adaptive system possesses a strong stability property, which will be stated precisely in the following theorem.

Theorem 1. *Consider the robust adaptive control problem formulated and assumptions in Section 3. The robust adaptive controller μ with the optimal choice of $\xi_{i,c}$, achieves the following strong robustness properties for the closed-loop system.*

1. *Given $c_w \geq 0$, and $c_d \geq 0$, there exists a constant $c_c \geq 0$ and compact sets $\Theta_{1,c} \subset \Theta_{1,o}$, and $\Theta_{2,c} \subset \Theta_{2,o}$ such that for any uncertainty $(x_{1,0}, \theta_1, \grave{w}_{1,[0,\infty)}, \breve{w}_{1,[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \grave{\mathcal{W}}_1$ and $(x_{2,0}, \theta_2, \grave{w}_{2,[0,\infty)}, \breve{w}_{2,[0,\infty)}) \in \grave{\mathcal{W}}_2$ with $|x_{1,0}| \leq c_w; |x_{2,0}| \leq c_w; |\grave{w}_1(t)| \leq c_w; |\grave{w}_2(t)| \leq c_w; |\breve{w}_1(t)| \leq c_w; |\breve{w}_2(t)| \leq c_w; |Y_d(t)| \leq c_d; \quad \forall t \in [0,\infty)$ all closed-loop state variables x_1, $\check{x}_1$, $\check{\theta}_1$, Σ_1, $s_{1,\Sigma}$, η_1, $\eta_{1,d}$, $\Phi_{1,u}$, x_2, $\check{x}_2$, $\check{\theta}_2$, Σ_2, $s_{2,\Sigma}$, η_2, $\eta_{2,d}$, $\Phi_{2,u}$ are bounded as follows, $\forall t \in [0,\infty)$,*

$$|x_i(t)| \leq c_c; |\check{x}_i(t)| \leq c_c; \check{\theta}_i(t) \in \Theta_{i,c}; |\eta_i(t)| \leq c_c; |\eta_{i,\breve{w},1}(t)| \leq c_c; \cdots |\eta_{i,\breve{w},\check{q}_i}| \leq c_c;$$

$$|\eta_{i,d}(t)| \leq c_c; |\Phi_{i,u}(t)| \leq c_c; K_{i,c}^{-1} I \leq \Sigma_i(t) \leq \gamma^{-2} Q_{i,0}^{-1}; \quad \gamma^2 \mathrm{Tr}(Q_{i,0}) \leq s_{i,\Sigma}(t) \leq K_{i,c}; \quad i = 1,2$$

The inputs are also bounded $|u(t)| \leq c_u$, and $\hat{\xi}_1 \leq c_u$, $\hat{\xi}_2 \leq c_u$, $\forall t \in [0,\infty)$, for some constant $c_u \geq 0$. Furthermore, there exists constant $c_\lambda \geq 0$ such that $|\lambda_{i,0}(t)| \leq c_\lambda$, $|\lambda_i(t)| \leq c_\lambda$, $i = 1,2$, and $|\eta_{2,y}(t)| \leq c_\lambda$, $\forall t \geq 0$.

2. *For any uncertainty $(x_{1,0}, \theta_1, \grave{w}_{1,[0,\infty)}, \breve{w}_{1,[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \grave{\mathcal{W}}_1$, and $(x_{2,0}, \theta_2, \grave{w}_{2,[0,\infty)}, \breve{w}_{2,[0,\infty)}) \in \grave{\mathcal{W}}_2$ the controller $\mu \in \mathcal{M}$ achieves disturbance attenuation level γ with respect to w_1 and w_2, arbitrary disturbance attenuation level $\check{\gamma}$ with respect to $\breve{w}_{1,a}$ and $\breve{w}_{2,a}$, and disturbance attenuation level zero with respect to $\breve{w}_{1,b}$ and $\breve{w}_{2,b}$, .*

3. *For any uncertainty $(x_{1,0}, \theta_1, \grave{w}_{1[0,\infty)}, \breve{w}_{1[0,\infty)}, Y_{d0}, y_{d[0,\infty)}^{(r_1+r_2)}) \in \grave{\mathcal{W}}_1$, and $(x_{2,0}, \theta_2, \grave{w}_{2[0,\infty)}, \breve{w}_{2[0,\infty)}) \in \grave{\mathcal{W}}_2$ with $\grave{w}_{1[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\grave{w}_{2[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\breve{w}_{1,a[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\breve{w}_{2,a[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ $\breve{w}_{1,b[0,\infty)} \in \mathcal{L}_\infty$, $\breve{w}_{2,b[0,\infty)} \in \mathcal{L}_\infty$, and $Y_{d[0,\infty)} \in \mathcal{L}_\infty$, the noiseless output of the system, $x_{1,1}$, asymptotically tracks the reference trajectory, y_d, i.e.,*

$$\lim_{t\to\infty} (x_{1,1}(t) - y_d(t)) = 0$$

4. *The ultimate lower bound on the achievable performance level is only relevant to the Subsystem $\mathbf{S_1}$, i.e., $\gamma \geq \zeta_1^{-1}$ or $\gamma > \zeta_1^{-1}$.*

Proof For the first statement, fix $c_w \geq 0$, and $c_d \geq 0$ consider any uncertainty $(x_{1,0}, x_{2,0}, \theta_1, \theta_2, \grave{w}_{1,[0,\infty)}, \grave{w}_{2,[0,\infty)}, \breve{w}_{1,[0,\infty)}, \breve{w}_{2,[0,\infty)}, y_{d[0,\infty)}^{(r_1+r_2)})$ that satisfies:

$$|x_{1,0}| \leq c_w; |x_{2,0}| \leq c_w; |\grave{w}_1(t)| \leq c_w; |\grave{w}_2(t)| \leq c_w; |\breve{w}_1(t)| \leq c_w; |\breve{w}_2(t)| \leq c_w; |Y_d(t)| \leq c_d; \forall t \in [0,\infty)$$

We define $[0, T_f)$ to be the maximal length interval on which the closed system (22) has a solution that lies in $\mathcal{D}$. Note that we have Σ_1, Σ_2, $s_{1,\Sigma}$ and $s_{2,\Sigma}$ are uniformly upper bounded and uniformly bounded away from 0 as desired by Section 4.

Introduce the vector of variables

$$X_e := \left[\check{\theta}_1' \ \check{\theta}_2' \ (\tilde{x}_1 - \Phi_1\tilde{\theta}_1)' \ (\tilde{x}_2 - \Phi_2\tilde{\theta}_2)' \ \tilde{\eta}_1' \ \tilde{\eta}_2' \ z_{1,1} \cdots z_{1,r_1} \ z_{2,1} \cdots z_{2,r_2} \right]'$$

and two nonnegative and continuous functions defined on $\mathbb{R}^{2n_1+2n_2+\sigma_1+\sigma_2+r_1+r_2}$

$$U_M(X_e) := \sum_{i=1}^{2} K_{i,c}|\tilde{\theta}_i|^2 + \sum_{i=1}^{2} \gamma^2 |\tilde{x}_i - \Phi_i\tilde{\theta}_i|^2_{\Pi_i^{-1}} + \sum_{i=1}^{2} |\tilde{\eta}_i|^2_{Z_i} + \sum_{j=1}^{r_1} \gamma_{1,j} z_{1,j}^2 + \sum_{j=1}^{r_2} \gamma_{2,j} z_{2,j}^2$$

$$U_m(X_e) := \sum_{i=1}^{2} \gamma^2 |\tilde{\theta}_i|^2_{Q_{i,0}} + \sum_{i=1}^{2} \gamma^2 |\tilde{x}_i - \Phi_i\tilde{\theta}_i|^2_{\Pi_i^{-1}} + \sum_{i=1}^{2} |\tilde{\eta}_i|^2_{Z_i} + \sum_{j=1}^{r_1} \gamma_{1,j} z_{1,j}^2 + \sum_{j=1}^{r_2} \gamma_{2,j} z_{2,j}^2$$

then, we have

$$U_m(X_e) \le U(t, X_e) \le U_M(X_e), \quad \forall (t, X_e) \in [0, T_f) \times \mathbb{R}^{2(n_1+n_2)+\sigma_1+\sigma_2+r_1+r_2}$$

Since $U_m(X_e)$ is continuous, nonnegative definite and radially unbounded, then $\forall \alpha \in \mathbb{R}$, the set $S_{1\alpha} := \{X_e \in \mathbb{R}^{2(n_1+n_2)+\sigma_1+\sigma_2+r_1+r_2} \mid \quad U_m(X_e) \le \alpha\}$ is compact or empty. Since $|\dot{w}_1(t)| \le c_w$, and $|\dot{w}_2(t)| \le c_w, \forall t \in [0, \infty)$, there exists a constant $c > 0$ such that we have the following inequality for the derivative of U:

$$\dot{U} \le -\sum_{i=1}^{2} \left(\frac{\gamma^4}{2} |x_i - \check{x}_i - \Phi_i (\theta_i - \check{\theta}_i)|^2_{\Pi_i^{-1}\Delta_i\Pi_i^{-1}} - 2(\theta_i - \check{\theta}_i)' P_{i,r}(\check{\theta}_i) + |\tilde{\eta}_i|^2_{\bar{Y}_i} + \sum_{j=1}^{r_i} c_{i,\beta_j} z_{i,j}^2 \right) + c$$

Since $-\sum_{i=1}^{2} (\frac{\gamma^4}{2} |x_i + \check{x}_i - \Phi_i (\theta_i - \check{\theta}_i)|^2_{\Pi_i^{-1}\Delta_i\Pi_i^{-1}} + |\tilde{\eta}_i|^2_{\bar{Y}_i} - 2(\theta_i - \check{\theta}_i)' P_{i,r}(\check{\theta}_i) + \sum_{j=1}^{r_i} c_{\beta_{i,j}} z_{i,j}^2)$ will tend to $-\infty$ when X_e approaches the boundary of $\Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2}$, then there exists a compact set $\Omega_1(c_w) \subset \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2}$, such that $\dot{U} < 0$ for $\forall X_e \in \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2} \backslash \Omega_1$.

Then we have $U(t, X_e(t)) \le c_1$, and $X_e(t)$ is in the compact set $S_{1c_1} \subseteq \mathbb{R}^{2(n_1+n_2)+\sigma_1+\sigma_2+r_1+r_2}$, $\forall t \in [0, T_f)$. It follows that the signal X_e is uniformly bounded, namely, $\tilde{\theta}_1$, $\tilde{\theta}_2$, $\tilde{x}_1 - \Phi_1\tilde{\theta}_1$, $\tilde{x}_2 - \Phi_2\tilde{\theta}_2$, $\tilde{\eta}_1$, $\tilde{\eta}_2$, $z_{1,1}, \cdots, z_{1,r_1}$ and $z_{2,1}, \cdots, z_{2,r_2}$ are uniformly bounded.

Based on the dynamics of $\eta_{1,d}$, we have $\eta_{1,d}$ is uniformly bounded. Since $\tilde{\eta}_1 = \eta_1 - \eta_{1,d}$ is uniformly bounded, then η_1 is also uniformly bounded. Furthermore, there is a particular linear combination of the components of η_1, denoted by $\eta_{1,L}$,

$$\dot{\eta}_1 = A_{1,f}\eta_1 + p_{1,n_1} y_1$$

$$\eta_{1,L} = T_{1,L}\eta_1$$

which is strictly minimum phase and has relative degree 1 with respect to y_1. Then the signal $\eta_{1,L}$ has relative degree $r_1 + 1$ with respect to the input y_2, and is uniformly bounded. The composite system of η_1 and $\dot{x}_1$ with input $\dot{w}_1$ and y_2 and output $\eta_{1,L}$ may serve as a reference system in the application of bounding Lemma [12].

Note $\Phi_1 = \Phi_{1,y} + \Phi_{1,u}$ and $\Phi_{1,y}$ is uniformly bounded. To prove Φ_1 is bounded, we need to prove $\Phi_{1,u}$ is uniformly bounded. Define the following equations to separate $\Phi_{1,u}$ into two part:

$$\Phi_{1,u} = \Phi_{1,u_s} + \lambda_{1,b}\bar{A}_{1,212\,0} \tag{23a}$$

$$\dot{\lambda}_{1,b} = A_{1,f}\lambda_{1,b} + e_{n_1,r_1} y_2; \quad \lambda_{1,b}(0) = \mathbf{0}_{n_1 \times 1} \tag{23b}$$

$$\dot{\Phi}_{1,u_s} = A_{1,f}\Phi_{1,u_s} + y_2 \begin{bmatrix} \mathbf{0}_{r_1 \times \sigma_1} \\ \bar{A}_{1,212 r_1} \end{bmatrix}; \quad \Phi_{1,u_s}(0) = \Phi_{1,u\,0} \tag{23c}$$

We observe that the relative degree for each element of $\Phi_{1,u_s 1}$ is at least $r_1 + 1$ with respect to the input y_2, and is the output of a stable linear system. Take $\eta_{1,L}$ and y_2 as output and input of

the reference system, we conclude $\Phi_{1,u_s 1}$ is uniformly bounded by bounding Lemma. Because the first row element of $\tilde{x}_1 - \Phi_1\tilde{\theta}_1$ is:

$$\tilde{x}_{1,1} - \Phi_{1,u_s 1}\tilde{\theta}_1 - \lambda_{1,b1}\bar{A}_{1,2120}\tilde{\theta}_1 - \eta_1' T_{1,1}\tilde{\theta}_1$$

we can conclude that $\tilde{x}_{1,1} - \lambda_{1,b1}\bar{A}_{1,2120}\tilde{\theta}_1$ is uniformly bounded in view of the boundedness of $\tilde{x}_1 - \Phi_1\tilde{\theta}_1$, $\tilde{\theta}_1$, $\Phi_{1,u_s 1}$, and η_1. Since $z_{1,1} = \check{x}_{1,1} - y_d$, and $z_{1,1}, y_d$ are both uniformly bounded, we have that $\check{x}_{1,1}$ is also uniformly bounded.

Notice that $A_{1,f} = A_1 - \zeta_1^2 L_1 C_1 - \Pi_1 C_1' C_1\,(\zeta_1^2 - \gamma^{-2})$, we generated the signal $x_{1,1} - b_{1,0}\lambda_{1,b1}$ by:

$$\begin{aligned}
\dot{x}_1 - b_{1,0}\dot{\lambda}_{1,b} &= A_{1,f}\,(x_1 - b_{1,0}\lambda_{1,b}) + \begin{bmatrix} \mathbf{0}_{r_1\times 1} \\ \bar{A}_{1,212r_1}\theta_1 \end{bmatrix} y_2 + \bar{A}_{1,211}\theta_1 y_1 + D_1\grave{M}_1\grave{w}_1 + (\zeta_1^2 L_1 \\
&\quad + \Pi_1 C_1'\,(\zeta_1^2 - \frac{1}{\gamma^2}))\,(y_1 - E_1\grave{M}_1\grave{w}_1) + \begin{bmatrix} \mathbf{0}_{1\times r_1} & b_{1,p1} & \cdots & b_{1,p\,n_1 - r_1} \end{bmatrix}' y_2 \\
&\quad + \sum_{j=1}^{\check{q}_1} \bar{A}_{1,213,j}\check{w}_{1,j}\theta_1 + \check{D}_1\check{w}_1 \\
x_{1,1} - b_{1,0}\lambda_{1,b1} &= C_1\,(x_1 - b_{1,0}\lambda_{1,b})
\end{aligned}$$

Now we will separate the above dynamics into y_1 dependent and y_2 dependent parts by the linearity of the system, $x_{1,1} - b_{1,0}\lambda_{1,b1} := x_{1,u\,1} + x_{1,y\,1}$, which are respectively given by,

$$\begin{aligned}
\dot{x}_{1,u} &= A_{1,f}x_{1,u} + \begin{bmatrix} \mathbf{0}_{r_1\times 1} \\ \bar{A}_{1,212r_1}\theta \end{bmatrix} y_2 + \begin{bmatrix} \mathbf{0}_{1\times r_1} & b_{1,p1} & \cdots & b_{1,p\,n_1 - r_1} \end{bmatrix}' y_2 \\
x_{1,u\,1} &= C_1 x_{1,u} \\
\dot{x}_{1,y} &= A_{1,f}x_{1,y} + (\zeta_1^2 L_1 + \Pi_1 C_1'\,(\zeta_1^2 - \frac{1}{\gamma^2}))\,(y_1 - E_1\grave{M}_1\grave{w}_1) + \bar{A}_{1,211}\theta_1 y_1 + D_1\grave{M}_1\grave{w}_1 \\
&\quad + \sum_{j=1}^{\check{q}_1} \bar{A}_{1,213,j}\check{w}_{1,j}\theta_1 + \check{D}_1\check{w}_1 \\
x_{1,y\,1} &= C_1 x_{1,y}
\end{aligned}$$

We observe that the signal $x_{1,u\,1}$ has relative degree at least $r_1 + 1$ with respect to y_2, take $\eta_{1,L}$ and y_2 as output and input of the reference system, we conclude $x_{1,u\,1}$ is uniformly bounded by bounding Lemma . Since $x_{1,y\,1}$ has relative degree at least 1 with respect to y_1, take $\eta_{1,L}$ and y_1 as output and input of the reference system, we conclude $x_{1,y\,1}$ is uniformly bounded by bounding Lemma. Then, $x_{1,1} - b_{1,0}\lambda_{1,b1}$ is uniformly bounded. It follows that $\check{x}_{1,1} - \lambda_{1,b1}\,(b_{1,p0} + A_{1,2120}\check{\theta}_1)$ is also uniformly bounded. Since $\check{x}_{1,1}$ is uniformly bounded and $\check{\theta}_1$ is uniformly bounded away from 0, we have $\lambda_{1,b1}$ is uniformly bounded. That further imply $\Phi_{1,1}$, i.e., $C_1\Phi_1$, is uniformly bounded. Furthermore, since $x_{1,1} - b_{1,0}\lambda_{1,b1}$ and $\grave{w}_1$ are bounded, we have that the signals of $x_{1,1}$ and y_1 are uniformly bounded.

Next, we need to prove the existence of a compact set $\Theta_{1,c} \subset \Theta_{1,o}$ such that $\check{\theta}_1(t) \in \Theta_{1,c}$, $\forall t \in [0, T_f)$. First introduce the function

$$\Upsilon_1 := U + (\rho_{1,o} - P_1(\check{\theta}_1))^{-1} P_1(\check{\theta}_1)$$

We notice that, when $\check{\theta}_1$ approaches the boundary of $\Theta_{1,o}$, $P_1(\check{\theta}_1)$ approaches $\rho_{1,o}$. Then Υ_1 approaches ∞ as X_e approaches the boundary of $\Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2}$. There exist some constant $c > 0$ such that the following inequalities hold.

$$\begin{aligned}\dot{\Upsilon}_1 &= \dot{U} + (\rho_{1,o} - P_1(\check{\theta}_1))^{-2}\rho_{1,o}\frac{\partial P_1}{\partial \theta_1}(\check{\theta}_1)\dot{\check{\theta}}_1 \\ &\leq -\sum_{i=1}^{2}(\frac{\gamma^4}{2}|x_{i,1} - \check{x}_{i,1} - \Phi_i\,(\theta_i - \check{\theta}_i)|^2_{\Pi_i^{-1}\Delta_i\Pi_i^{-1}} - 2\,(\theta_i - \check{\theta}_i)' P_{i,r}(\check{\theta}_i) + |\tilde{\eta}_i|^2_{Y_i} + \sum_{j=1}^{r_i} c_{i,\beta_j} z_{i,j}^2) \\ &\quad - \left|\left(\frac{\partial P_1}{\partial \theta_1}(\check{\theta}_1)\right)'\right|^2 (\rho_{1,o} - P_1(\check{\theta}_1))^{-4}\left(K_{1,c}^{-1}\rho_{1,o}p_{1,r}(\check{\theta}_1)\left(\rho_{1,o} - P_1(\check{\theta}_1)\right)^2 - c\right) + c\end{aligned}$$

Since $\dot{\Upsilon}_1$ will tend to $-\infty$ when X_e approaches the boundary of $\Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2}$, then there exists a compact set $\Omega_{1,2}(c_w) \subset \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2}$, such that $\forall X_e \in \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2} \backslash \Omega_{1,2}$, $\dot{\Upsilon}_1(X_e) < 0$.

Then there exists a compact set $\Theta_{1,c} \subset \Theta_{1,o}$, such that $\check{\theta}_1(t) \in \Theta_{1,c}$, $\forall t \in [0, T_f)$. Moreover, $\Upsilon_1(t, X_e(t)) \leq c_2$, and $X_e(t)$ is in the compact set $S_{1,2c_2} \subseteq \Theta_{1,o} \times \Theta_{2,o} \times \mathbb{R}^{2(n_1+n_2)+r_1+r_2}$, $\forall t \in [0, T_f)$.

To derive the uniformly boundedness of the closed-loop system states, we separate the relative degree, r_1, into two cases: $r_1 = 1$, and $r_1 \geq 2$. First, we consider the case 1: $r_1 = 1$.

Taking $x_{1,1}$ and y_2 as the output and input of the reference system, we note that $x_{1,1}$ is strictly minimum phase and has relative degree r_1 with respect to input y_2. Since the state x_1 can be viewed as stably filtered output signals of y_2 and y_1, it is uniformly bounded. Since λ_1 is also some stably filtered signals of y_1 and y_2, it is uniformly bounded. It further implies Φ is uniformly bounded. Then we can conclude $\check{x}_1$ is uniformly bounded from the boundedness of $\tilde{x}_1 - \Phi_1\tilde{\theta}_1$. This further implies that the inputs $\check{x}_{2,1}$ and $\check{\xi}_1$ are uniformly bounded.

Case 2: $r_1 \geq 2$. Considering the canonical form (78) in [12] for the true system (1), we denote the elements of $\bar{\xi}$ by $\left[\bar{\xi}_{11} \cdots \bar{\xi}_{1r_1}\right]'$. We will use the mathematical induction to derive the boundedness of $\Phi_{1,u_s i}$, $\tilde{x}_{1,i} - \lambda_{1,bi}\bar{A}_{1,2120}\tilde{\theta}_1$, $\check{x}_{1,i}$, $x_{1,i} - b_{1,0}\lambda_{1,bi}$, $\lambda_{1,bi}$, $\Phi_{1,u\,i}$, $x_{1,i}$, $\bar{\xi}_{1,1i}$, $\forall i = \{1, \cdots, r_1\}$. For the boundedness of $\bar{\xi}_{1i}$, we will show that $\bar{\xi}_{1i}$ is a linear combination of $x_{1,1}$, $\cdots$, $x_{1,i}$, $\bar{\xi}_3$, and $\bar{\xi}_4$, i.e.,

$$\bar{\xi}_{1i} = \tilde{a}_{1,1i}x_{1,1} + \cdots + \tilde{a}_{1,i-1\,i}x_{1,i-1} + x_{1,i} + \tilde{T}_{1,i3}\bar{\xi}_3 + \tilde{T}_{1,i4}\bar{\xi}_4; \quad 1 \leq i \leq r_1 \tag{24}$$

where $\tilde{a}_{1,1i}$, $\cdots$, $\tilde{a}_{1,i-1\,i}$ are constants, $\tilde{T}_{1,i3}$, $\tilde{T}_{1,i4}$ are constant matrices, and $\bar{\xi}_3$ and $\bar{\xi}_4$ are defined at (78) in [12].

1°: We have deduced that η_1, $\eta_{1,d}$, $\eta_{1,L}$, $\Phi_{1,u_s 1}$, $\tilde{x}_{1,1} - \lambda_{1,b1}\bar{A}_{1,2120}\tilde{\theta}_1$, $\check{x}_{1,1}$, $x_{1,1} - b_{1,0}\lambda_{1,b1}$, $\lambda_{1,b1}$, $\Phi_{1,u\,1}$, and $x_{1,1}$ are uniformly bounded in $[0, T_f)$. $\bar{\xi}_{11}$ is bounded in view of $x_{1,1} - \bar{C}_3\bar{\xi}_3 - \bar{C}_4\bar{\xi}_4$.

2°: We assume that $\Phi_{1,u_s i}$, $\tilde{x}_{1,i} - \lambda_{1,bi}\bar{A}_{1,2120}\tilde{\theta}_1$, $\check{x}_{1,i}$, $x_{1,i} - b_{1,0}\lambda_{1,bi}$, $\lambda_{1,bi}$, $\Phi_{1,u\,i}$, $x_{1,i}$, and $\bar{\xi}_{1i}$ are bounded, and

$$\bar{\xi}_{1i} = \tilde{a}_{1,1i}x_{1,1} + \cdots + \tilde{a}_{1,i-1\,i}x_{1,i-1} + x_{1,i} + \tilde{T}_{1,i3}\bar{\xi}_3 + \tilde{T}_{1,i4}\bar{\xi}_4; \quad \forall i \in \{1, \cdots k\} \tag{25}$$

where $1 \leq k < r_1$.

3°: First, we need to show that $\Phi_{1,u_s k+1}$, $\tilde{x}_{1,k+1} - \lambda_{1,b\,k+1}\bar{A}_{1,2120}\tilde{\theta}_1$, $\check{x}_{1,k+1}$, $x_{1,k+1} - b_{1,0}\lambda_{1,b\,k+1}$, $\lambda_{1,b\,k+1}$, $\Phi_{1,u\,k+1}$, $x_{1,k+1}$, and $\bar{\tilde{x}}_{1\,k+1}$ are bounded.

From equation (23c), we note that every element of $\Phi_{1,u_s k+1}$ has relative degree of at least $r_1 - k + 1$ with respect to y_2, and is the output of a stable linear system. Since the boundedness of $\bar{\tilde{x}}_{11}, \cdots, \bar{\tilde{x}}_{1k}$, we conclude $\Phi_{1,u_s k+1}$ is uniformly bounded by Lemma 11 in [12], where the reference system has input y_2 and output y_1.

Note that $k+1$st row element of $\tilde{x}_1 - \Phi_1\tilde{\theta}_1$ is

$$\tilde{x}_{1,k+1} - \Phi_{1,u_s k+1}\tilde{\theta}_1 - \lambda_{1,b\,k+1}\bar{A}_{1,2120}\tilde{\theta}_1 - \eta_1' T_{1,k+1}\tilde{\theta}_1$$

We can conclude that $\tilde{x}_{1,k+1} - \lambda_{1,b\,k+1}\bar{A}_{1,2120}\tilde{\theta}_1$ is uniformly bounded in view of the boundedness of $\tilde{x}_1 - \Phi_1\tilde{\theta}_1$, $\tilde{\theta}_1$, $\Phi_{1,u_s k+1}$, and η_1. Since the boundedness of y_d, $s_{1,\Sigma}$, η_1, $\eta_{1,d}$, Σ_1, $\check{x}_{1,1}$, $y_d^{(1)}$, $\Phi_{1,u\,1}$, $\cdots$ $\check{x}_{1,k}$, $y_d^{(k)}$, $\Phi_{1,u\,k}$, and $\check{\theta}_1(t) \in \Theta_{1,c}$, $\forall t \in [0, T_f)$, $\alpha_{1,k}$ is bounded. Since $z_{1,k+1} = \check{x}_{1,k+1} - \alpha_{1,k}$, and $z_{1,k+1}$ is uniformly bounded, we have that $\check{x}_{1,k+1}$ is also uniformly bounded.

The signal $x_{1,k+1} - b_{1,0}\lambda_{1,b\,k+1}$ is generated by:

$$\begin{aligned}
\dot{x}_1 - b_{1,0}\dot{\lambda}_{1,b} &= A_{1,f}\left(x_1 - b_{1,0}\lambda_{1,b}\right) + \begin{bmatrix} \mathbf{0}_{r_1\times 1} \\ \bar{A}_{1,212\,r_1}\theta_1 \end{bmatrix} y_2 + \bar{A}_{1,211}\theta_1 y_1 + D_1\grave{M}_1\grave{w}_1 + (\zeta_1^2 L_1 \\
&\quad + \Pi_1 C_1'\left(\zeta_1^2 - \frac{1}{\gamma^2}\right))\left(y_1 - E_1\grave{M}_1\grave{w}_1\right) + \begin{bmatrix} \mathbf{0}_{1\times r_1} & b_{1,p1} & \cdots & b_{1,p\,n_1-r_1} \end{bmatrix}' y_2 \\
&\quad + \sum_{j=1}^{\check{\eta}_1} \bar{A}_{1,213,j}\check{w}_{1,j}\theta_1 + \check{D}_1\check{w}_1 \\
x_{1,k+1} - b_{1,0}\lambda_{1,b\,k+1} &= e_{n_1,k+1}'\left(x_1 - b_{1,0}\lambda_{1,b}\right)
\end{aligned}$$

Now we will separate the above dynamics into y_1 dependent and y_2 dependent parts by the linearity of the system, $x_{1,k+1} - b_{1,0}\lambda_{1,b\,k+1} := x_{1,u\,k+1} + x_{1,y\,k+1}$, which are respectively given by,

$$\begin{aligned}
\dot{x}_{1,u} &= A_{1,f}x_{1,u} + \begin{bmatrix} \mathbf{0}_{r_1\times 1} \\ \bar{A}_{1,212\,r_1}\theta_1 \end{bmatrix} y_2 + \begin{bmatrix} \mathbf{0}_{1\times r_1} & b_{1,p1} & \cdots & b_{1,p\,n_1-r_1} \end{bmatrix}' y_2 \\
x_{1,u\,k+1} &= e_{n_1,k+1}' x_{1,u} \\
\dot{x}_{1,y} &= A_{1,f}x_{1,y} + (\zeta_1^2 L_1 + \Pi_1 C_1'\left(\zeta_1^2 - \frac{1}{\gamma^2}\right))\left(y_1 - E_1\grave{M}_1\grave{w}_1\right) + \bar{A}_{1,211}\theta_1 y_1 + D_1\grave{M}_1\grave{w}_1 \\
&\quad + \sum_{j=1}^{\check{\eta}_1} \bar{A}_{1,213,j}\check{w}_{1,j}\theta_1 + \check{D}_1\check{w}_1 \\
x_{1,y\,k+1} &= e_{n_1,k+1}' x_{1,y}
\end{aligned}$$

We observe that the signal $x_{1,u\,k+1}$ has relative degree at least $r_1 - k + 1$ with respect to y_2. Since $\bar{\tilde{x}}_{1,11}, \cdots, \bar{\tilde{x}}_{1,1k}$ are uniformly bounded, we conclude $x_{1,u\,k+1}$ is uniformly bounded by Lemma 11 in [12], where the reference system with input y_2 and output y_1. We conclude $x_{1,y\,k+1}$ is uniformly bounded since y_1 is bounded. Then, $x_{1,k+1} - b_{1,0}\lambda_{1,b\,k+1}$ is uniformly bounded. It follows that $\check{x}_{1,k+1} - \lambda_{1,b\,k+1}\left(b_{1,p0} + \bar{A}_{1,2120}\check{\theta}_1\right)$ is also uniformly bounded. Since

$\check{x}_{1,k+1}$ is uniformly bounded and $b_{1,p0} + \bar{A}_{1,2120}\check{\theta}_1$ is uniformly bounded away from 0, we have $\lambda_{1,bk+1}$ is uniformly bounded. That further imply $\Phi_{1,uk+1}$ is uniformly bounded. Furthermore, since $x_{1,k+1} - b_{1,0}\lambda_{1,bk+1}$ and $\lambda_{1,bk+1}$ are bounded, we have that the signals of $x_{1,k+1}$ is uniformly bounded.

Next, we need to show $\bar{\tilde{x}}_{1,1k+1}$ is satisfied equation (24). Comparing the design model (2) and the canonical form (78) in [12], we have $\bar{\tilde{C}}\bar{\tilde{x}} = C_1 x_1$. It further implies

$$\bar{\tilde{C}}_1 \bar{\tilde{A}}_1^k \bar{\tilde{x}}_1 = C_1 (A_1 + \bar{A}_{1,211}\theta_1 C_1)^k x_1$$

Hence, we have

$$\bar{\tilde{x}}_{1\,k+1} = \tilde{a}_{1,1\,k+1} x_1 + \cdots + \tilde{a}_{1,k\,k+1} x_{1,k} + x_{1,k+1} + \tilde{T}_{1,k+13}\bar{\tilde{x}}_3 + \tilde{T}_{1,k+14}\bar{\tilde{x}}_4 \tag{26}$$

where $\tilde{a}_{1,1\,k+1}, \cdots, \tilde{a}_{1,k\,k+1}$ are constants, and $\tilde{T}_{1,k+13}$, $\tilde{T}_{1,k+14}$ are constant matrices.

Then, we have the boundedness of $\bar{\tilde{x}}_{1\,k+1}$. Thus, we can conclude the boundedness of $\Phi_{1,u_s i}$, $\tilde{x}_{1,i} - \lambda_{1,bi}\bar{A}_{1,2120}\tilde{\theta}_1$, $\check{x}_{1,i}$, $x_{1,i} - b_{1,0}\lambda_{1,bi}$, $\lambda_{1,bi}$, $\Phi_{1,ui}$, $x_{1,i}$, and $\bar{\tilde{x}}_{1\,i}$, $\forall i \in \{1, \cdots r_1\}$.

Since the state x_1 can be viewed as stably filtered output signals of y_2 and y_1, it is uniformly bounded. Also, η_1, λ_1 are some stably filtered signals of y_2 and y_1, they are uniformly bounded. It further implies Φ_1 is uniformly bounded. Then we can conclude $\check{x}_1$ is uniformly bounded from the boundedness of $\tilde{x}_1 - \Phi_1\tilde{\theta}_1$. This further implies that the control input $\check{x}_{2,1}$ is uniformly bounded. Therefore, it follows $T_f = \infty$ and the complete system states are uniformly bounded on $[0, \infty)$.

The boundedness of closed-loop state variables of $\mathbf{S_2}$ can be proven with the similar line of reasoning above. Thus, we have established statement 1 in all cases.

We define $l_0 = l_{1,0} + l_{2,0} = V_{2,r_2}(X_1(0), X_2(0))$, and

$$\begin{aligned}
l_1 + l_2 :=& \sum_{i=1}^{2}\left(\gamma^4|x_i - \hat{x}_i - \Phi_i(\theta_i - \hat{\theta}_i)|^2_{\Pi_i^{-1}\Delta_i\Pi_i^{-1}} + |\bar{\eta}_i|^2_{\bar{Y}_i} - 2(\theta_i - \check{\theta}_i)' P_{i,r}(\check{\theta}_i) + \sum_{j=1}^{r_i}\beta_{i,j}z_{i,j}^2\right) \\
&+\frac{1}{4}|\varsigma_{1,(r_1+r_2)}|^2_{\bar{Q}_1} + \frac{1}{4}|\varsigma_{2,r_2}|^2_{\bar{Q}_2} + \epsilon_1(\gamma^2\zeta_1^2 - 1)|\theta_1 - \hat{\theta}_1|^2_{\Phi_1' C_1' C_1 \Phi_1} + \epsilon_2|\theta_2 - \hat{\theta}_2|^2_{\Phi_2' C_2' C_2 \Phi_2} \\
&\sup_{\dot{w}_1 \in \dot{\mathcal{W}}_1, \dot{w}_2 \in \dot{\mathcal{W}}_2}\left\{\int_0^{t_f}\left((x_{1,1} - y_d)^2 + l_1 + l_2 - \sum_{i=1}^{2}\gamma^2|w_i|^2 - \sum_{i=1}^{2}\gamma^2|w_{i,a}|^2\right)\mathrm{d}\tau\right. \\
&\qquad\left. - \sum_{i=1}^{2}\gamma^2\left|\left[\theta_i' - \check{\theta}_{i,0}'\ x_{i,0}' - \check{x}_{i,0}'\right]\right|^2_{\bar{Q}_{i,0}} - l_0\right\} \\
\le& \sup_{\dot{w}_1 \in \dot{\mathcal{W}}_1, \dot{w}_2 \in \dot{\mathcal{W}}_2}\left\{\int_0^{t_f}\left((x_{1,1} - y_d)^2 + l_1 + l_2 - \sum_{i=1}^{2}\gamma^2|w_i|^2 - \sum_{i=1}^{2}\check{\gamma}^2\left|\frac{\gamma}{\check{\gamma}}\check{w}_{i,a}\right|^2\right)\mathrm{d}\tau - l_0\right. \\
&\qquad\left. -\gamma^2\sum_{i=1}^{2}\left|\left[\theta_i' - \check{\theta}_{i,0}'\ x_{i,0}' - \check{x}_{i,0}'\right]\right|^2_{\bar{Q}_{i,0}} + \int_0^{t_f}\dot{U}\mathrm{d}\tau - U(t) + U(0)\right\} \\
\le& -U(t) \le 0
\end{aligned}$$

then, we establish the second statement.

For the third statement, we consider the following inequality,

$$\int_0^\infty \dot{U}\mathrm{d}\tau \leq \int_0^\infty (-|x_1 - y_d|^2 + \gamma^2|\check{w}_{1,a}|^2 + \gamma^2|\grave{M}_1\grave{w}_1|^2 + \gamma^2|\check{w}_{2,a}|^2 + \gamma^2|\grave{M}_2\grave{w}_2|^2)\mathrm{d}\tau$$

it follows that

$$\int_0^\infty |x_1 - y_d|^2\mathrm{d}\tau \leq \int_0^\infty \left(\gamma^2|\check{w}_{1,a}|^2 + \gamma^2|\grave{M}_1\grave{w}_1|^2 + \gamma^2|\check{w}_{2,a}|^2 + \gamma^2|\grave{M}_2\grave{w}_2|^2\right)\mathrm{d}\tau + U(0) < +\infty$$

By the first statement, we notice that

$$\sup_{0\leq t<\infty} |\dot{x}_1 - \dot{y}_d| < \infty.$$

Then, we have

$$\lim_{t\to\infty} |x_1(t) - y_d(t)| = 0$$

For the last statement, it's easy to establish by Section 4.

This complete the proof of the theorem. ◇

6. Example

In this section, we present one example to illustrate the main results of this Chapter. The designs were carried out using MATLAB symbolic computation tools, and the closed-loop systems were simulated using SIMULINK.

Consider the following linear systems with zeros initial conditions:

$$\dot{\hat{x}}_1 = \hat{x}_1 + \hat{x}_2 + \hat{x}_3 + 0.1\grave{w}_1; \tag{27a}$$

$$\dot{\hat{x}}_2 = (1+\theta_1)\hat{x}_3 + (1+\theta_2)\check{w}_1 \tag{27b}$$

$$\dot{\hat{x}}_3 = -\hat{x}_1 - \hat{x}_3 + \hat{x}_4 + u + \check{w}_2 \tag{27c}$$

$$\dot{\hat{x}}_4 = \hat{x}_1 + (2+\theta_3)u + 0.1\grave{w}_2 + \check{w}_2 \tag{27d}$$

$$y = \hat{x}_1 + 0.1\grave{w}_1 \tag{27e}$$

where θ_1, θ_2 and θ_3 are three unknown parameters with true value 0s. The coefficient terms, 0.1 and 1, reflect the *a priori* knowledge that the disturbances $\grave{w}_1$ and $\grave{w}_2$ are weak in power relative to that of the disturbance $\check{w}_1$ and $\check{w}_2$. We note that (27) is an unobservable system. We can decompose (27) into the following two SISO linear systems, $\mathbf{S_1}$ and $\mathbf{S_2}$, sequentially interconnected with additional output measurement,

$$\dot{x}_{11} = x_{11} + x_{12} + y_2 + w_{11}; \tag{28a}$$

$$\dot{x}_{12} = (1+\theta_1)y_2 + (1+\theta_2)\check{w}_1 + w_{12}; \tag{28b}$$

$$y_1 = x_{11} + w_{13}; \tag{28c}$$

$$\dot{x}_{21} = -x_{21} + x_{22} + u - y_1 + \breve{w}_2 + w_{21}; \tag{28d}$$

$$\dot{x}_{22} = (2 + \theta_3)u + y_1 + \breve{w}_2 + w_{22}; \tag{28e}$$

$$y_2 = x_{21} + w_{23} \tag{28f}$$

where

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}; x_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix};$$

$$w_1 = \begin{bmatrix} w_{11} \\ w_{12} \\ w_{13} \end{bmatrix} = \begin{bmatrix} 0.1\dot{w}_1 - \dot{w}_3 \\ -(1-\theta_1)\dot{w}_3 \\ 0.1\dot{w}_1 \end{bmatrix}; w_2 = \begin{bmatrix} w_{21} \\ w_{22} \\ w_{23} \end{bmatrix} = \begin{bmatrix} 0.1\dot{w}_1 \\ -0.1\dot{w}_1 + 0.1\dot{w}_2 \\ \dot{w}_3 \end{bmatrix}$$

Here $\dot{w}_3$ is the measurement disturbance of the state $\dot{x}_3$. It is easy to check that $\mathbf{S_1}$ and $\mathbf{S_2}$ in (28) satisfied the assumptions 1–5.

For the adaptive control design, we set the desired disturbance attenuation level $\gamma = 10$. We select the true value of the parameters in subsystem $\mathbf{S_1}$ and subsystem $\mathbf{S_2}$ are zeros, and belong to the interval $[-1, 1]$. The projection function $P_1(\theta_1)$ and $P_2(\theta_2)$ are chosen as $P_1(\theta_1) = 0.5(\theta_1^2 + \theta_2^2)$, $P_2(\theta_2) = \theta_3^2$. The reference trajectory, y_d, is generated by the following linear system $\dot{x}_{d,1} = -x_{d,2}$, $\dot{x}_{d,2} = x_{d,1} - x_{d,2} + d$, $y_d = x_{d,1}$ with zeros initial condition, where d is the command input signal. The objective is to achieve asymptotic tracking of $\dot{x}_1$ to the reference trajectory y_d.

For design and simulation parameters of $\mathbf{S}_i$, $(i = 1, 2)$, we select

$$\breve{x}_{1,0} = \begin{bmatrix} 0.2 & 0 \end{bmatrix}'; \ \breve{x}_{2,0} = \begin{bmatrix} 0.1 & 0 \end{bmatrix}'; \breve{\theta}_{1,0} = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}'; \ \breve{\theta}_{2,0} = -1/2; Q_{i,0} = 0.001 I_2;$$

$$K_{i,c} = 0.2; \ \Delta_i = I_2; \ p_{i,n_i} = e_{2,2}; \ \Phi_{i,0} = \mathbf{0}_{2\times 1}; \ \rho_{i,o} = 2; \ \beta_{i,\Delta} = 0; \ \epsilon_i = K_{i,c}^{-1} s_{i,\Sigma}; \ \lambda_{i,0} = \mathbf{0}_{2\times 1};$$

$$\beta_{i,1} = 0.5; \quad \eta_{i,0} = \mathbf{0}_{2\times 1}; \ Z_1 = \begin{bmatrix} 0.0893 & -0.0081 \\ -0.0081 & 0.0097 \end{bmatrix}; \ Z_2 = \begin{bmatrix} 0.1094 & -0.0099 \\ -0.0099 & 0.0099 \end{bmatrix}$$

We present one set of simulation results in this example to illustrate the regulatory behavior of the adaptive controller. We set $d(t) = 0.4\sin(0.1t) + \sin(0.6t)$, $\dot{w}_1(t) = 0$, $\dot{w}_2(t) = 0$, $\dot{w}_3(t) = 0$, $\breve{w}_1(t) = \sin(12t + \frac{\pi}{9}) + 0.8\sin(3t)$, and $\breve{w}_2(t) = 3\sin(3t + \frac{\pi}{3})$. The results are shown in Figure 2(a)–(f). To illustrate that the proposed controller can improve the system performance by incorporating the measurements and/or the estimation of the significant external disturbances into the control design, the simulation results based on [17] are presented in Figure 2(c)(d), where the measured disturbances $\breve{w}_1$ and $\breve{w}_2$ are treated as arbitrary disturbances and θ_3 is treated as constant in control design. We observe that the output tracking error asymptotically converges to zero and the parameter estimates asymptotically converge to its true value 0 in (a) and (b) even if there is a non-zero measured disturbance in the system. But the parameter estimates doesn't converge to the true value,

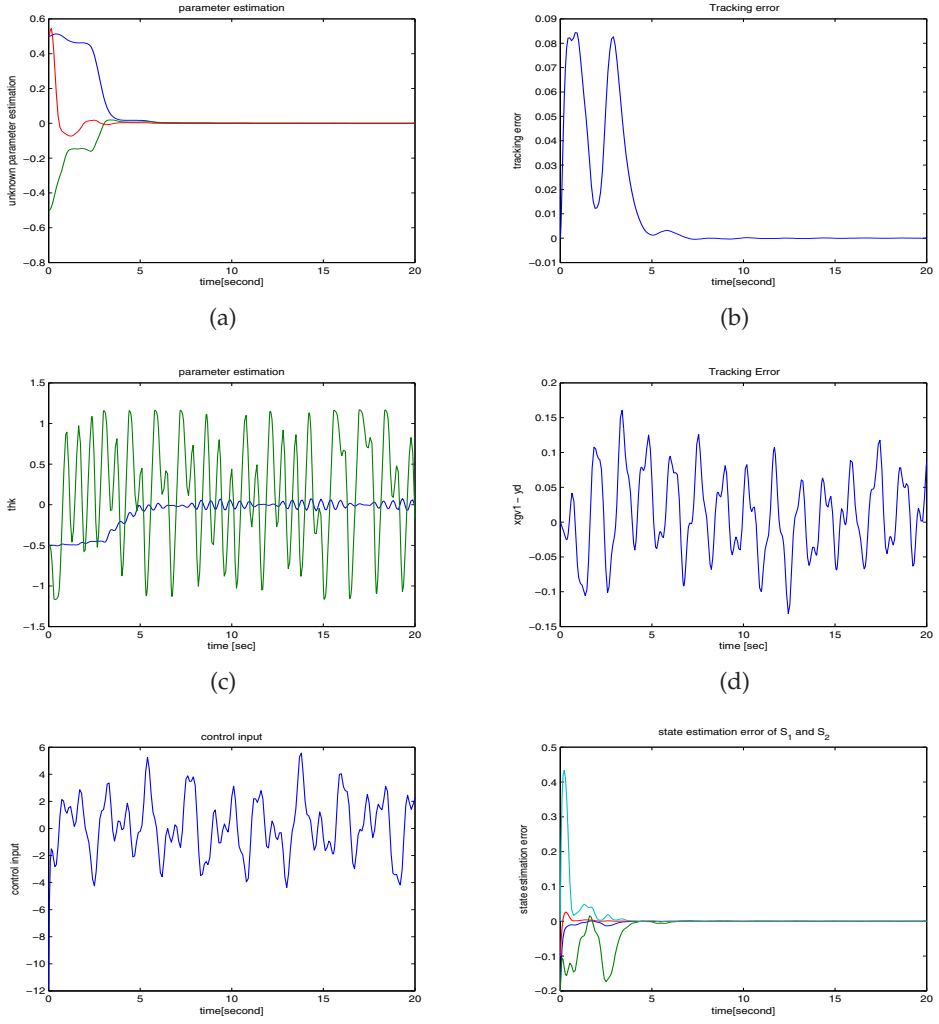

Figure 2. System response for Example under command input $d(t) = 0.4\sin(0.1t) + \sin(0.6t)$, $\dot{w}_1 = 0$, $\dot{w}_2 = 0$, $\dot{w}_3 = 0$, $\ddot{w}_1(t) = \sin(12t + \frac{\pi}{9}) + 0.8\sin(3t)$, and $\ddot{w}_2(t) = 3\sin(3t + \frac{\pi}{3})$. (a) Parameter estimate; (b) Tracking error; (c) Parameter estimate(based on [17]); (d) Tracking error(based on [17]); (e) control input; (f) State estimation error;

and the tracking error doesn't converge to zero in (c) and (d). State estimation error, $x_1 - \check{x}_1$ and $x_2 - \check{x}_2$, converge to zero in (f), and the transient performance behaves well as in (e).

7. Conclusions

In this Chapter, we present the game-theoretical approach based adaptive control design for a special class of MIMO linear systems, which is composed of two sequentially interconnected SISO linear systems, $\mathbf{S}_1$ and $\mathbf{S}_2$. We assume the subsystem under studied subject to noisy output measurements, unknown initial state conditions, linear unknown parametric uncertainties, measured and unmeasured additive exogenous disturbance input uncertainties. Our design objective is to address the asymptotical tracking, the transient response and robustness of the closed-loop system, which are the same as the objectives to motivate the study of the H^∞- optimal control problem. In view of the similar solution between H^∞ optimal control design and *zero sum differential game*, we convert the original adaptive control design problem into a zero-sum game with soft constraints on the disturbance input uncertainties and the unknown initial state uncertainties, which incorporates the measures of transient response, disturbance attenuation, and asymptotic tracking into a single game-theoretic cost function and formulates the design problem as a nonlinear H^∞ control problem under imperfect state measurements. A game-theoretical approach, *cost-to-come function* analysis, is then applied to obtain the finite dimensional estimators of $\mathbf{S_1}$ and $\mathbf{S_2}$ independently, which is also converted the control design as an H^∞ control problem with full information measurements. The integrator backstepping methodology is finally applied on this full information measurements problem to obtain a suboptimal solution. The controller achieves the same result as [17], namely the total stability of the closed-loop system, the desired disturbance attenuation level, and asymptotic tracking of the reference trajectory when the disturbance is of finite energy and uniformly bounded. In addition, the proposed controller may achieve arbitrary positive disturbance attenuation level with respect to the measured disturbances by proper scaling. The contribution of the measurements of part of the disturbance inputs is that we can design an adaptive controller with disturbance feedforward structure with respect to $\check{w}_{1,b}$ and $\check{w}_{2,b}$ to eliminate their effect on the squared $\mathcal{L}_2$ norm of the tracking error. Moreover, the asymptotic tracking is achieved even if the measured disturbances are only uniformly bounded without requiring them to be of finite energy.

Author details

Sheng Zeng
Critical Care R&D Engineering, Carefusion Corporation, CA, 92887, USA

Emmanuel Fernandez
School of Electronics & Computing Systems, University of Cincinnati, OH, 45221-0030, USA

8. References

[1] Arslan, G. & Başar, T. [2001]. Disturbance attenuating controller design for strict-feedback systems with structurally unknown dynamics, *Automatica* 37(8): 1175–1188.

[2] Başar, T. & Bernhard, P. [1995]. *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd edn, Birkhäuser, Boston, MA.
[3] Didinsky, G. [1994]. *Design of minimax controllers for nonlinear systems using cost-to-come methods*, PhD thesis, University of Illinois, Urbana, IL.
[4] Goodwin, G. C. & Mayne, D. Q. [1987]. A parameter estimation perspective of continuous time adaptive control, *Automatica* 23: 57–70.
[5] Goodwin, G. C. & Sin, K. S. [1984]. *Adaptive Filtering, Prediction and Control*, Prentice-Hall, Englewood Cliffs.
[6] Ioannou, P. A. & Sun, J. [1996]. *Robust Adaptive Control*, Prentice Hall, Upper Saddle River, NJ.
[7] Isidori, A. [1995]. *Nonlinear Control Systems*, 3rd edn, Springer-Verlag, London.
[8] Kanellakopoulos, I., Kokotović, P. V. & Morse, A. S. [1991]. Systematic design of adaptive controllers for feedback linearizable systems, *IEEE Transactions on Automatic Control* 36: 1241–1253.
[9] Krstić, M., Kanellakopoulos, I. & Kokotović, P. V. [1995]. *Nonlinear and Adaptive Control Design*, Wiley, New York, NY.
[10] Kumar, P. R. [1985]. A survey of some results in stochastic adaptive control, *SIAM Journal on Control and Optimization* 23(3): 329–380.
[11] Pan, Z. & Başar, T. [1996]. Parameter identification for uncertain linear systems with partial state measurements under an H^∞ criterion, *IEEE Transactions on Automatic Control* 41: 1295–1311.
[12] Pan, Z. & Başar, T. [2000]. Adaptive controller design and disturbance attenuation for SISO linear systems with noisy output measurements, *CSL report*, University of Illinois at Urbana-Champaign, Urbana, IL.
[13] Rohrs, C. E., Valavani, L., Athans, M. & Stein, G. [1985]. Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics, *IEEE Transactions on Automatic Control* 30: 881–889.
[14] Tezcan, I. E. & Başar, T. [1999]. Disturbance attenuating adaptive controllers for parametric strict feedback nonlinear systems with output measurements, *Journal of Dynamic Systems, Measurement and Control, Transactions of the ASME* 121(1): 48–57.
[15] Zeng, S. [2010]. Adaptive controller design and disturbance attenuation for a general class of sequentially interconnected siso linear systems with noisy output measurements, *Proceedings of the 49th IEEE Conference on Decision and Control(CDC)*, Atlanta, GA, pp. 2608–2613.
[16] Zeng, S. [2011]. Worst-case analysis based adaptive control design for siso linear systems with plant and actuation uncertainties, *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, Orlando, FL, pp. 6349–6354.
[17] Zeng, S. & Fernandez, E. [2010]. Adaptive controller design and disturbance attenuation for sequentially interconnected siso linear systems under noisy output measurements, *IEEE Transactions on Automatic Control* 55: 2123–2129.
[18] Zeng, S. & Pan, Z. [2009]. Adaptive controller design and disturbance attenuation for SISO linear systems with noisy output measurements and partly measured disturbances, *International Journal of Control* 82(2): 310–334.
[19] Zeng, S., Pan, Z. & Fernandez, E. [2010]. Adaptive controller design and disturbance attenuation for SISO linear systems with zero relative degree under noisy output measurements, *International Journal of Adaptive Control and Signal Processing* 24: 287–310.

[20] Zhao, Q., Pan, Z. & Fernandez, E. [2009]. Convergence analysis for reduced-order adaptive controller design of uncertain siso linear systems with noisy output measurements, *International Journal of Control* 82(11): 1971–1990.