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# New Stabilization of Complex Networks with Non-delayed and Delayed Couplings over Random Exchanges

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Additional information is available at the end of the chapter

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## Abstract

In this chapter, the stabilization problem of complex dynamical network with non-delayed and delayed couplings is realized by a new kind of stochastic pinning controller being partially delay dependent, where the topologies related to couplings may be exchanged. The designed pinning controller is different from the traditional ones, whose non-delay and delay state terms occur asynchronously with a certain probability, respectively. Sufficient conditions for the stabilization of complex dynamical network over topology exchange are derived by the robust method and are presented with liner matrix inequities (LMIs). The switching between the non-delayed and delayed couplings is modeled by the related coupling matrices containing uncertainties. It has shown that the bound of such uncertainties play very important roles in the controller design. Moreover, when the bound is inaccessible, a kind of adaptive partially delay-dependent controller is proposed to deal with this general case, where another adaptive control problem in terms of unknown probability is considered too. Finally, some numerical simulations are used to demonstrate the correctness and effectiveness of our theoretical analysis.

**Keywords:** complex dynamical network, partially delay-dependent pinning controller, non-delayed and delayed couplings, robust method, adaptive control

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## 1. Introduction

With the rapid development of science and technology, human beings have marched into the network era, and complex network has become a hot topic. Complex network is an important

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method to describe and study complex systems, and all complex systems can be abstracted from practical background by different perspectives and become a complex network of interacting individuals, such as ecological network, food network, gene regulation network, social network, and distributed sensor network. Research on complex network has become a frontier subject with many subjects and challenges. Over the past few years, studies on complex network have received more and more attention from various fields of scientific research See [1–5]. The popularization of complex network has also caused a series of important problems about the network structures and studies of the network dynamic behaviors. Particularly, special attention has been paid to the studies of synchronization control problems of complex dynamical networks. As one of the significant dynamic behaviors of complex dynamical network, synchronization is widely used in neural network, public transit scheduling, laser system, secure communication system, information science, etc. [6–11]. So it is concerned by more and more scholars. In real networks, because of the complex dynamical network having a great many nodes, and every node has its dynamical behavior, it is hard for the complex dynamical network itself to make the states of the network to desired trajectory. Thus, the studies on the control strategy of complex dynamical network will be meaningful. So far, many control methods for complex dynamical network have been reported in refs. [12–17]. Pinning control such as in refs. [18–20] is widely welcomed for its advantages. It is easy to be realized and can save the cost effectively. The main idea of pinning control is to control a part of nodes in the complex networks to realize the whole network to the expected states and to reduce the number of the controllers effectively. When there exist some unknown parameters, the adaptive control method could be exploited, some of which was mentioned in refs. [21–23].

On the other hand, there are many factors that affect the stability of complex network, where time delay and network topology are two important factors. First, time delay is an objective phenomenon in nature and human society. In the process of transmission and response of complex network, it is inevitable to produce time delay, which is because of the physical limitations of the speed of transmission and the existence of network congestion, such as the existence of time delay in communication network and virus transmission. There are some typical time delay network systems such as circuit system [24], satellite communication system [25], and laser array system [26]. It is noticed that the majority of the studies on complex network have been performed on some absolute assumptions. For example, the stabilization referred to state feedback control is realized only by a non-delay or delay controller, which is relied on some absolute assumptions [18, 19, 27]. However, in many practical applications, these assumptions do not accord with the peculiarities of the real networks. Based on these facts, we may design a kind of controller that contains non-delay and delay states simultaneously. Second, the topology of the network plays an important role in determining the network characteristics and the synchronization control. The research of coupling delay also plays a significant role in complex networks. In most of the above papers, it is seen that the topologies are fixed. But in practical applications, the topological structure of the complex network is not constant and may be changed randomly. That is because of the influence of various stochastic factors. In this case, how to ensure the stabilization of networks by the proposed controller when the topologies related to couplings change is worth discussing.

Motivated by the above discussions, in this chapter, the stabilization problem of complex networks with non-delayed and delayed couplings over random exchanges is studied by exploiting the robust method to describe the topologies exchanging randomly. A kind of stochastic pinning controller being partially delay-dependent is developed, which contains non-delay and delay terms simultaneously but occur asynchronously. Here, the probability distributions are taken into account in the proposed controller design. The rest of this chapter is organized as follows: In Section 2, the model of complex dynamical networks with non-delayed and delayed couplings over random exchanges is established. In Section 3, the stabilization of the underlying complex networks is considered, which is realized by partially delay-dependent controller and adaptive controller respectively. A numerical example is demonstrated in Section 4; the conclusion of this chapter is given in Section 5.

**Notation:**  $\mathbb{R}^n$  denotes the  $n$  dimensional Euclidean space,  $\mathbb{R}^{m \times n}$  is the set of all  $m \times n$  real matrices.  $E\{\cdot\}$  is the expectation operator with respect to some probability measure.  $\text{diag}\{\cdot\cdot\cdot\}$  stands for a block-diagonal matrix.  $I_N$  is an identity matrix being of  $N$  dimensions.  $S=S_\ell \cup \bar{S}_\ell$ , where  $S_\ell=\{1, 2, \dots, l\}$ ,  $\bar{S}_\ell=\{l+1, l+2, \dots, N\}$ .  $\lambda_{\max}(M)$  is the maximum eigenvalue of  $M$ , while  $\sigma_{\max}(M)$  is the maximum singular value of  $M$ .  $\|G\|$  denotes the 2-norm of matrix  $G$ . \* stands for an ellipsis for the term induced by symmetry.

## 2. Model of complex networks with non-delayed and delayed couplings over random exchanges

As is known, time delay is ubiquitous in many network systems. When time delay exists in the interaction, it may affect the dynamic behavior and even destabilize the network system. Thus, time delay should be taken into consideration, which could accurately reflect some characteristics of networks. By investing the existing literatures, it is easy to find that most of the results on complex networks have been carried out under some implicit assumptions. That is the communication information of nodes is only related to  $x(t)$  or  $x(t - \tau)$ . However, in many cases, this simplification is not satisfactory for the special nature of the networks. In fact, the information communication of nodes is not only related to  $x(t)$  but also to  $x(t - \tau)$ . Unfortunately, this property has been ignored in many literatures that are about the complex systems with non-delayed and delayed couplings simultaneously. In this section, we will consider a general stabilization problem of complex systems with non-delayed and delayed couplings exchanging randomly.

Considering a kind of complex dynamical network consisting of  $N$  nodes and every node is a  $n$ -dimensional dynamical system, which is described as

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^N a_{ij} x_j(t) + c \sum_{j=1}^N b_{ij} x_j(t - \tau), \quad i \in S \tag{1}$$

where  $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$  is the state vector of the  $i$ th node.  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuously differentiable function that describes the activity of an individual system.

$c > 0$  is the coupling strength among the nodes.  $\tau > 0$  is the coupling delay.  $A = (a_{ij}) \in \mathbb{R}^{N \times N}$  and  $B = (b_{ij}) \in \mathbb{R}^{N \times N}$  stand for the configuration matrices of the complex dynamical network with the non-delayed and delayed couplings, respectively.  $A$  and  $B$  can be defined as follows: for  $i \neq j$ , if there exist non-delayed and delayed couplings between nodes  $i$  and  $j$ , then  $a_{ij} > 0$  and  $b_{ij} > 0$ ; Otherwise,  $a_{ij} = 0$  and  $b_{ij} = 0$ , respectively. Assuming both  $A$  and  $B$  are symmetric and also satisfy

$$a_{ii} = - \sum_{j=1, j \neq i}^N a_{ij}, b_{ii} = - \sum_{j=1, j \neq i}^N b_{ij}, i \in S$$

Here, the topologies of the complex network are more general, whose related coupling matrices exchange each other randomly. That is,  $A$  changes into  $B$ , while  $B$  changes into  $A$  simultaneously. In other words, matrices  $A$  and  $B$  exchange. In this case, we have the following complex network:

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^N b_{ij} x_j(t) + c \sum_{j=1}^N a_{ij} x_j(t - \tau), \quad i \in S \quad (2)$$

From these demonstrations, it is seen that the above two complex networks occur separately and randomly. To describe the above random switching between coupling matrices  $A$  and  $B$ , a robust method will be exploited. That is

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) + c \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau), \quad i \in S \quad (3)$$

when  $\Delta A = (\Delta a_{ij}) \in \mathbb{R}^{N \times N}$  and  $\Delta B = (\Delta b_{ij}) \in \mathbb{R}^{N \times N}$ . Especially, such uncertainties are selected to be  $\Delta A = B - A$  and  $\Delta B = A - B$ , which is assumed to be

$$B - A \leq \delta^* \quad (4)$$

where  $\delta^*$  is a given positive scalar.

Before giving the main results, a definition is needed.

**Definition 1.** The complex network (1) is asymptotically stable over topologies exchanging randomly, if the complex network (3) with condition (4) is asymptotically stable.

### 3. Stabilization of complex networks with couplings exchanging randomly

Based on the proposed model, this section focuses on the design of stochastic pinning controller. By investigating the existing references, it is found that most of the stabilization results of complex networks are achieved by either non-delay or delay controllers. However, from the above explanations, it is said that two such controllers may not describe the actual systems very well. Here, a kind of partially delay-dependent pinning controller containing both non-delay and delay states that take place with a certain probability is proposed to deal with the general case. Without loss of generality, it is assumed that the first  $l$  nodes are selected to be added the desired pinning controller  $u_i(t)$ , which are described as

$$\begin{cases} u_i(t) = -c\alpha(t)k_i x_i(t) - c(1-\alpha(t))k_{di} x_i(t-\tau), & i \in \mathbb{S}_\ell \\ u_i(t) = 0, & i \in \bar{\mathbb{S}}_\ell \end{cases} \quad (5)$$

where  $k_i$  and  $k_{di}$  are the non-delayed and delayed coupling control gains, respectively.  $\alpha(t)$  is the Bernoulli stochastic variable and is described as follows:

$$\alpha(t) = \begin{cases} 1, & \text{if } x(t) \text{ is valid} \\ 0, & \text{if } x(t-\tau) \text{ is valid} \end{cases} \quad (6)$$

whose probabilities are expressed by

$$P_r\{\alpha(t) = 1\} = E\{\alpha(t)\} = \alpha^*, P_r\{\alpha(t) = 0\} = 1 - \alpha^* \quad (7)$$

where  $\alpha^* \in [0, 1]$ . In addition, it is obtained that

$$E\{\alpha(t) - \alpha^*\} = 0 \quad (8)$$

Substituting  $u_i(t)$  into complex network (3), one has

$$\left\{ \begin{array}{l} \dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) \\ \quad + c \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau) \\ \quad - c\alpha(t) k_i x_i(t) - c(1 - \alpha(t)) k_{di} x_i(t - \tau), i \in S_\ell \\ \dot{x}_i(t) = f(x_i(t)) \\ \quad + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) \\ \quad + c \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau), i \in \bar{S}_\ell \end{array} \right. \quad (9)$$

which is equivalent to

$$\left\{ \begin{array}{l} \dot{x}_i(t) = f(x_i(t)) \\ \quad + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) \\ \quad + c \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau) \\ \quad - c(\alpha(t) - \alpha^*) k_i x_i(t) \\ \quad \quad + c(\alpha(t) - \alpha^*) k_{di} x_i(t - \tau) \\ \quad - c\alpha^* k_i x_i(t) - c(1 - \alpha^*) k_{di} x_i(t - \tau), i \in S_\ell \\ \dot{x}_i(t) = f(x_i(t)) \\ \quad + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) \\ \quad + c \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau), i \in \bar{S}_\ell \end{array} \right. \quad (10)$$

**Assumption 1.** Supposing that there exists a positive definite diagonal matrix  $P = \text{diag}\{p_1, p_2, \dots, p_n\}$  and  $\eta > 0$ , such that

$$x_i^T(t) P f(x(t)) \leq \eta x_i^T(t) x_i(t), \forall x_i(t) \in \mathbb{R}^n, t \geq 0 \quad (11)$$

### 3.1 Stabilization realized by a partially delay-dependent pinning controller

**THEOREM 1.** Let Assumption 1 hold, for given scalars  $\alpha^*$  and  $\delta^*$ , there exists a pinning controller (5) such that the complex network (9) is asymptotically stable over topology exchange (4), if there exist  $Q > 0$ ,  $k_i > 0$ , and  $k_{d_i} > 0$ ,  $\forall i \in S_p$ , such that the following condition

$$\begin{bmatrix} 2\varphi I_N + 2c\tilde{A} + 2c\delta^* I_N + Q & c(\tilde{B} + \delta^* I_N) \\ * & -Q \end{bmatrix} < 0 \tag{12}$$

is satisfied, where

$$\varphi = \frac{\eta}{\min_{1 \leq i \leq n}(p_i)}$$

$$\tilde{A} = A - \text{diag}\{\underbrace{\alpha^* k_1, \alpha^* k_2, \dots, \alpha^* k_l}_l, \underbrace{0, \dots, 0}_{N-l}\},$$

$$\tilde{B} = B - \text{diag}\{\underbrace{(1 - \alpha^*) k_{d_1}, (1 - \alpha^*) k_{d_2}, \dots, (1 - \alpha^*) k_{d_l}}_l, \underbrace{0, \dots, 0}_{N-l}\}.$$

**Proof.** For complex network (9), we choose a Lyapunov function as follows:

$$V(x(t)) = \frac{1}{2} \sum_{i=1}^N x_i^T(t) P x_i(t) + \frac{1}{2} \sum_{j=1}^n p_j \int_{t-\tau}^t \tilde{x}_j^T(s) Q \tilde{x}_j(s) ds \tag{13}$$

where  $\tilde{x}_j(t) = (x_{1j}(t), x_{2j}(t), \dots, x_{N_j}(t))^T \in \mathbb{R}^N, j = 1, 2, \dots, n$ , and  $Q$  is a positive definite of suitable dimensions matrix. Let  $L$  be the weak infinitesimal generator of stochastic process, it is defined as

$$LV(x(t)) = \lim_{\Delta \rightarrow 0^+} \frac{E\{V(x(t + \Delta))\} - V(x(t))}{\Delta} \tag{14}$$

Then, one has

$$\begin{aligned}
LV(x(t)) &= \sum_{i=1}^N x_i^T(t) P [f(x_i(t)) + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) + c \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau)] \\
&\quad - c \alpha^* \sum_{i=1}^l k_i x_i^T(t) P x_i(t) - c(1 - \alpha^*) \sum_{i=1}^l k_{di} x_i^T(t) P x_i(t - \tau) \\
&\quad + \frac{1}{2} \sum_{j=1}^n p_j [\tilde{x}_j^T(t) Q \tilde{x}_j(t) - \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau)] \\
&\leq \eta \sum_{i=1}^N x_i^T(t) x_i(t) + c \sum_{i=1}^N x_i^T(t) P [\sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) + \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau)] \\
&\quad - c \alpha^* \sum_{i=1}^l k_i x_i^T(t) P x_i(t) - c(1 - \alpha^*) \sum_{i=1}^l k_{di} x_i^T(t) P x_i(t - \tau) \\
&\quad + \frac{1}{2} \sum_{j=1}^n p_j [\tilde{x}_j^T(t) Q \tilde{x}_j(t) - \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau)] \\
&\leq \frac{\eta}{\min_{1 \leq i \leq n} (p_i)} \sum_{i=1}^N x_i^T(t) P x_i(t) + c \sum_{i=1}^N x_i^T(t) P \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) \\
&\quad + c \sum_{i=1}^N x_i^T(t) P \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau) \\
&\quad - c \alpha^* \sum_{i=1}^l k_i x_i^T(t) P x_i(t) - c(1 - \alpha^*) \sum_{i=1}^l k_{di} x_i^T(t) P x_i(t - \tau) \\
&\quad + \frac{1}{2} \sum_{j=1}^n p_j [\tilde{x}_j^T(t) Q \tilde{x}_j(t) - \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau)] \tag{15} \\
&= \varphi \sum_{i=1}^N x_i^T(t) P x_i(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{A} \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{B} \tilde{x}_j(t - \tau) \\
&\quad + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \Delta A \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \Delta B \tilde{x}_j(t - \tau) \\
&\quad + \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t) Q \tilde{x}_j(t) - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau) \\
&\leq \varphi \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{A} \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{B} \tilde{x}_j(t - \tau) \\
&\quad + c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t - \tau) \\
&\quad + \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t) Q \tilde{x}_j(t) - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau) \\
&\leq \sum_{j=1}^n p_j \tilde{x}_j^T(t) (\varphi I_N + c \tilde{A} + c \delta^* I_N + \frac{1}{2} Q) \tilde{x}_j(t) \\
&\quad + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) (\tilde{B} + \delta^* I_N) \tilde{x}_j(t - \tau) - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau)
\end{aligned}$$



$$= \frac{1}{2} \sum_{j=1}^n p_j \begin{bmatrix} \tilde{x}_j^T(t) & \tilde{x}_j^T(t-\tau) \end{bmatrix} \Pi_1 \begin{bmatrix} \tilde{x}_j(t) \\ \tilde{x}_j(t-\tau) \end{bmatrix} < 0$$

where

$$\Pi_1 = \begin{bmatrix} 2\varphi I_N + 2c\tilde{A} + 2c\delta^* I_N + Q & c(\tilde{B} + \delta^* I_N) \\ * & -Q \end{bmatrix}$$

It is guaranteed by  $\Pi_1 < 0$ . By condition (12), it is known that  $LV(x(t)) < 0$ . This completes the proof.

**REMARK 1.** It is worth mentioning that for any given function  $f(x_i(t))$ , it is necessary to find suitable parameters  $P$  and  $\eta$ . There,  $P$  is related to  $f(x_i(t))$ , where  $\eta$  can be obtained by the given matrix  $P$ . Moreover, Theorem 1 is also extended to other general cases that the coupling matrices  $A$  and  $B$  change to the other ones independently. Here, we only consider the special case that  $A$  and  $B$  exchanges each other.

Based on Theorem 1, it is claimed that  $Q$  is selected with a general case. However, it may be selected to be some special cases. When  $Q$  is chosen as the special case that  $Q = c\sigma_{\max}(\tilde{B} + \delta^* I_N)I_N$ , we will have the following corollary.

**COROLLARY 1.** Let Assumption 1 hold, for given scalars  $\alpha^*$  and  $\delta^* > 0$ , there exists a pinning controller (5) such that the complex network (9) is asymptotically stable over topology exchange (4), if there exist  $k_i > 0$ , and  $k_{di} > 0, \forall i \in S_{\ell}$ , such that the following condition

$$\varphi I_N + c\tilde{A} + c\delta^* I_N + c\sigma_{\max}(\tilde{B} + \delta^* I_N)I_N < 0 \tag{16}$$

is satisfied, where the other symbols are defined in Theorem 1.

**Proof.** Based on Theorem 1 and using the Schur complement lemma, one has

$$2\varphi I_N + 2c\tilde{A} + 2c\delta^* I_N + Q + c^2(\tilde{B} + \delta^* I_N)Q^{-1}(\tilde{B} + \delta^* I_N)^T < 0 \tag{17}$$

implying  $\Pi_1 < 0$ . By choosing  $Q = c\sigma_{\max}(\tilde{B} + \delta^* I_N)I_N$ , it is concluded that (17) is guaranteed by

$$2\varphi I_N + 2c\tilde{A} + 2c\delta^* I_N + 2c\sigma_{\max}(\tilde{B} + \delta^* I_N)I_N < 0 \tag{18}$$

This completes the proof.

When there is no topology exchange, we will have the following corollary directly.

**COROLLARY 2.** Let Assumption 1 hold, for given scalar  $\alpha^*$ , there exists a pinning controller (5) such that the complex network (9) is asymptotically stable over topology exchange (4), if there exist  $Q > 0$ ,  $k_i > 0$ , and  $k_{di} > 0$ ,  $\forall i \in S_{\rho}$ , such that the following condition holds:

$$\begin{bmatrix} 2\varphi I_N + 2c\tilde{A} + Q & c\tilde{B} \\ * & -Q \end{bmatrix} < 0 \quad (19)$$

where  $\varphi$ ,  $\tilde{A}$ , and  $\tilde{B}$  are defined as those in (12).

It is seen that the expectation of  $\alpha(t)$  in Theorem 1 plays a vital role in the control of the complex network, which needs to be given exactly. However, in practice, it may be very hard to get  $\alpha^*$  exactly, and only its estimation  $\tilde{\alpha}$  is available. For an uncertain  $\alpha^*$  with its estimation  $\tilde{\alpha}$ , its admissible uncertainty  $\Delta\alpha$  is defined as

$$\Delta\alpha = \alpha^* - \tilde{\alpha}, \tilde{\alpha} \in [0, 1] \quad (20)$$

where  $\Delta\alpha \in [-\mu, \mu]$  with  $\mu \in [0, 1]$ . Then, we have the following theorem.

**THEOREM 2.** Let Assumption 1 hold, for given scalars  $\tilde{\alpha}$  and  $\delta^* > 0$ , there exists a pinning controller (5) satisfying condition (20) such that the complex network (9) is asymptotically stable over topology exchange (4), if there exist  $Q > 0$ ,  $W > 0$ ,  $k_i > 0$ , and  $k_{di} > 0$ ,  $\forall i \in S_{\rho}$ , such that the following conditions

$$\begin{bmatrix} 2\varphi I_N + 2c\tilde{A} + 2c\mu K_1 + 2c\mu W_{11} + 2c\delta^* I_N + Q & c(\tilde{B} - \mu K_2 + 2\mu W_{12} + \delta^* I_N) \\ * & 2c\mu W_{22} - Q \end{bmatrix} < 0 \quad (21)$$

$$\begin{bmatrix} -2K_1 - W_{11} & K_2 - W_{12} \\ * & -W_{22} \end{bmatrix} < 0 \quad (22)$$

hold, where

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

$$K_1 = \text{diag}\{\underbrace{k_1, k_2, \dots, k_l}_l, \underbrace{0, \dots, 0}_{N-l}\},$$

$$K_2 = \text{diag}\{\underbrace{k_{d1}, k_{d2}, \dots, k_{dl}}_l, \underbrace{0, \dots, 0}_{N-l}\},$$

$$\bar{A} = A - \text{diag}\{\underbrace{\tilde{\alpha}k_1, \tilde{\alpha}k_2, \dots, \tilde{\alpha}k_l}_l, \underbrace{0, \dots, 0}_{N-l}\},$$

$$\bar{B} = B - \text{diag}\{\underbrace{(1 - \tilde{\alpha})k_{d1}, (1 - \tilde{\alpha})k_{d2}, \dots, (1 - \tilde{\alpha})k_{dl}}_l, \underbrace{0, \dots, 0}_{N-l}\}$$

**Proof.** Based on the proof of Theorem 1, it is known that the stabilization of complex network (9) over random exchanges with (20) is guaranteed by (12), which is equivalent to

$$\begin{bmatrix} 2\varphi I_N + 2c\bar{A} - 2c\Delta\alpha K_1 + 2c\delta^* I_N + Q & c(\bar{B} + \Delta\alpha K_2 + \delta^* I_N) \\ * & -Q \end{bmatrix} < 0 \tag{23}$$

It could be rewritten as

$$\begin{bmatrix} 2\varphi I_N + 2c\bar{A} + 2c\delta^* I_N + Q & c(\bar{B} + \delta^* I_N) \\ * & -Q \end{bmatrix} + c\Delta\alpha \begin{bmatrix} -2K_1 & K_2 \\ * & 0 \end{bmatrix} < 0 \tag{24}$$

That is

$$\begin{bmatrix} 2\varphi I_N + 2c\bar{A} + 2c\delta^* I_N + Q & c(\bar{B} + \delta^* I_N) \\ * & -Q \end{bmatrix} + c(\Delta\alpha + \mu) \begin{bmatrix} -2K_1 & K_2 \\ * & 0 \end{bmatrix} - c(\Delta\alpha + \mu)W - c\mu \begin{bmatrix} -2K_1 & K_2 \\ * & 0 \end{bmatrix} + c(\Delta\alpha + \mu)W < 0 \tag{25}$$

which is implied by

$$\begin{aligned}
& \begin{bmatrix} 2\varphi I_N + 2c\bar{A} + 2c\delta^* I_N + Q & c(\bar{B} + \delta^* I_N) \\ * & -Q \end{bmatrix} \\
& + c(\Delta\alpha + \mu) \begin{bmatrix} -2K_1 - W_{11} & K_2 - W_{12} \\ * & -W_{22} \end{bmatrix} - c\mu \begin{bmatrix} -2K_1 & K_2 \\ * & 0 \end{bmatrix} \\
& + 2c\mu \begin{bmatrix} W_{11} & W_{12} \\ * & W_{22} \end{bmatrix} < 0
\end{aligned} \tag{26}$$

Taking into account condition (22), it is further guaranteed by

$$\begin{aligned}
& \begin{bmatrix} 2\varphi I_N + 2c\bar{A} + 2c\delta^* I_N + Q & c(\bar{B} + \delta^* I_N) \\ * & -Q \end{bmatrix} - c\mu \begin{bmatrix} -2K_1 & K_2 \\ * & 0 \end{bmatrix} \\
& + 2c\mu \begin{bmatrix} W_{11} & W_{12} \\ * & W_{22} \end{bmatrix} < 0
\end{aligned} \tag{27}$$

which is (21) actually. This completes the proof.

### 3.2 Stabilization realized by adaptive pinning controller

When  $a^*$  is unknown, how to stabilize a complex network through a pinning controller should also be taken into consideration. In this section, we will exploit the adaptive pinning control method to deal with this general case.

**THEOREM 3.** Let Assumption 1 hold, for given scalar  $\delta^*$ , if there exist  $Q > 0$ ,  $k_i > 0$ , and  $k_{di} > 0$ ,  $\forall i \in S_\rho$ , such that the following condition

$$\begin{bmatrix} 2\varphi I_N + 2c\hat{A} + 2c\delta^* I_N + Q & c(\hat{B} + \delta^* I_N) \\ * & -Q \end{bmatrix} < 0 \tag{28}$$

holds with  $\hat{A} = A - K_1$  and  $\hat{B} = B - K_2$ , then the complex network (9) is asymptotically stable over topology exchange (4) under the adaptive pinning controller

$$\begin{cases} u_i(t) = -ck_i x_i(t) - ck_{di} x_i(t - \tau) + v_i(t), i \in S_\ell \\ u_i(t) = 0, i \in \bar{S}_\ell \end{cases} \tag{29}$$

where

$$v_i(t) = -c\hat{\alpha}(t)x_i(t)$$

and the updating law

$$\dot{\hat{\alpha}}(t) = \begin{cases} 0, & \text{if } \hat{\alpha}(t) = 1 \\ \hat{\alpha}c \sum_{i=1}^l x_i^T(t)Px_i(t), & \text{others} \end{cases} \quad (30)$$

where  $\forall \delta > 0$  and  $\hat{\alpha}_0 \in [0, 1]$ .

**Proof.** Here, the Lyapunov function is defined as

$$\begin{aligned} V(x(t)) &= \frac{1}{2} \sum_{i=1}^N x_i^T(t)Px_i(t) \\ &\quad + \frac{1}{2} \sum_{j=1}^n p_j \int_{t-\tau}^t \tilde{x}_j^T(s)Q\tilde{x}_j(s)ds \\ &\quad + \frac{1}{2\delta} \tilde{\alpha}(t)\tilde{\alpha}(t) \end{aligned} \quad (31)$$

where  $\tilde{\alpha}(t) = \hat{\alpha}(t) - \alpha^*$ ,  $\tilde{x}_j(t)$ , and  $Q$  are same as the ones in (13). Then, it is obtained

$$\begin{aligned} LV(x(t)) &= \sum_{i=1}^N x_i^T(t)P[f(x_i(t)) + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij})x_j(t) + c \sum_{j=1}^N (b_{ij} + \Delta b_{ij})x_j(t - \tau)] \\ &\quad - c \sum_{i=1}^l k_i x_i^T(t)Px_i(t) - c \sum_{i=1}^l k_{di} x_i^T(t)Px_i(t - \tau) + \sum_{i=1}^l x_i^T(t)Pv_i(t) \\ &\quad + \frac{1}{2} \sum_{j=1}^n p_j [\tilde{x}_j^T(t)Q\tilde{x}_j(t) - \tilde{x}_j^T(t - \tau)Q\tilde{x}_j(t - \tau)] + \frac{1}{\delta} (\hat{\alpha}(t) - \alpha^*)\dot{\hat{\alpha}}(t) \\ &\leq \eta \sum_{i=1}^N x_i^T(t)x_i(t) + c \sum_{i=1}^N x_i^T(t)P[\sum_{j=1}^N (a_{ij} + \Delta a_{ij})x_j(t) + \sum_{j=1}^N (b_{ij} + \Delta b_{ij})x_j(t - \tau)] \\ &\quad - c \sum_{i=1}^l k_i x_i^T(t)Px_i(t) - c \sum_{i=1}^l k_{di} x_i^T(t)Px_i(t - \tau) + \sum_{i=1}^l x_i^T(t)Pv_i(t) \\ &\quad + \frac{1}{2} \sum_{j=1}^n p_j [\tilde{x}_j^T(t)Q\tilde{x}_j(t) - \tilde{x}_j^T(t - \tau)Q\tilde{x}_j(t - \tau)] + \frac{1}{\delta} (\hat{\alpha}(t) - \alpha^*)\dot{\hat{\alpha}}(t) \\ &\leq \frac{\eta}{\min_{1 \leq i \leq n} (p_i)} \sum_{j=1}^n p_j \tilde{x}_j^T(t)\tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t)A\tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t)B\tilde{x}_j(t - \tau) \\ &\quad - c \sum_{j=1}^n p_j \tilde{x}_j^T(t)K_1\tilde{x}_j(t) - c \sum_{j=1}^n p_j \tilde{x}_j^T(t)K_2\tilde{x}_j(t - \tau) \\ &\quad + c \sum_{j=1}^n p_j \tilde{x}_j^T(t)\Delta A\tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t)\Delta B\tilde{x}_j(t - \tau) + \sum_{i=1}^l x_i^T(t)Pv_i(t) \end{aligned} \quad (32)$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t) Q \tilde{x}_j(t) - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t-\tau) Q \tilde{x}_j(t-\tau) \\
& + \frac{1}{\delta} (\hat{\alpha}(t) - \alpha^*) \dot{\hat{\alpha}}(t) \\
& \leq \varphi \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \hat{A} \tilde{x}_j(t) \\
& + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \hat{B} \tilde{x}_j(t-\tau) \\
& + c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t-\tau) \\
& + \sum_{i=1}^l x_i^T(t) P v_i(t) \\
& + \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t) Q \tilde{x}_j(t) - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t-\tau) Q \tilde{x}_j(t-\tau) \\
& + \frac{1}{\delta} (\hat{\alpha}(t) - \alpha^*) \dot{\hat{\alpha}}(t) \\
& \leq \varphi \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \hat{A} \tilde{x}_j(t) \\
& + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \hat{B} \tilde{x}_j(t-\tau) \\
& + c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) \\
& + c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t-\tau) \\
& + \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t) Q \tilde{x}_j(t) \\
& - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t-\tau) Q \tilde{x}_j(t-\tau) \\
& = \frac{1}{2} \sum_{j=1}^n p_j \begin{bmatrix} \tilde{x}_j^T(t) & \tilde{x}_j^T(t-\tau) \end{bmatrix} \Pi_2 \begin{bmatrix} \tilde{x}_j(t) \\ \tilde{x}_j(t-\tau) \end{bmatrix} < 0
\end{aligned}$$

where

$$\Pi_2 = \begin{bmatrix} 2\varphi I_N + 2c\hat{A} + 2c\delta^* I_N + Q & c(\hat{B} + \delta^* I_N) \\ * & -Q \end{bmatrix}$$

This completes the proof.

On the other hand, it is obtained that  $\delta^*$  is also important to the control of the complex network. When it is unavailable, how to get the sufficient condition for the stabilization of complex network is an interesting problem to be discussed. In the next, such a problem will be solved by the following theorem.

**THEOREM 4.** Let Assumption 1 hold, for given scalar  $\alpha^*$ , if there exist  $Q > 0$ ,  $k_i > 0$ , and  $k_{di} > 0$ ,  $\forall i \in S_p$ , such that the following condition

$$\begin{bmatrix} 2\varphi I_N + 2c\tilde{A} + Q & c\tilde{B} \\ * & -Q \end{bmatrix} < 0 \tag{33}$$

holds, then the complex network (9) is asymptotically stable over topology exchange (4) under the adaptive pinning controller

$$\begin{cases} u_i(t) = -c\alpha(t)k_i x_i(t) - c(1 - \alpha(t))k_{di} x_i(t - \tau) + \varpi_i(t), & i \in S_\ell \\ u_i(t) = 0, & i \in \bar{S}_\ell \end{cases} \tag{34}$$

where

$$\varpi_i(t) = \begin{cases} 0, & \text{if } \sum_{i=1}^l x_i^T(t) P x_i(t) = 0 \\ \frac{-c\hat{\delta} x_i(t) [2 \sum_{i=1}^N x_i^T(t) P x_i(t) + \sum_{i=1}^N x_i^T(t - \tau) P x_i(t - \tau)]}{\sum_{i=1}^l x_i^T(t) P x_i(t)}, & \text{others} \end{cases}$$

and the updating law

$$\dot{\hat{\delta}} = 2\xi c \sum_{i=1}^N x_i^T(t) P x_i(t) + \xi c \sum_{i=1}^N x_i^T(t - \tau) P x_i(t - \tau) \tag{35}$$

where  $\xi$  is a positive constant and  $\delta_0 \geq 0$ .

**Proof.** For this case, we choose the Lyapunov function as

$$V(t) = \frac{1}{2} \sum_{i=1}^N x_i^T(t) P x_i(t) + \frac{1}{2} \sum_{j=1}^n p_j \int_{t-\tau}^t \tilde{x}_j^T(s) Q \tilde{x}_j(s) ds + \frac{1}{2\xi} \hat{\delta}^2 \tag{36}$$

where  $\delta = \hat{\delta} - \delta^*$ . Then, it is obtained

$$\begin{aligned}
& LV(x(t)) \\
&= \sum_{i=1}^N x_i^T(t) P [f(x_i(t)) + c \sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) \\
&+ c \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau)] \\
&- c \alpha^* \sum_{i=1}^l k_i x_i^T(t) P x_i(t) - c(1 - \alpha^*) \sum_{i=1}^l k_{di} x_i^T(t) P x_i(t - \tau) \\
&+ \sum_{i=1}^l x_i^T(t) P \varpi_i(t) \\
&+ \frac{1}{2} \sum_{j=1}^n p_j [\tilde{x}_j^T(t) Q \tilde{x}_j(t) - \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau)] \\
&+ \frac{1}{\xi} (\hat{\delta} - \delta^*) \hat{\delta} \\
&\leq \eta \sum_{i=1}^N x_i^T(t) x_i(t) + c \sum_{i=1}^N x_i^T(t) P [\sum_{j=1}^N (a_{ij} + \Delta a_{ij}) x_j(t) \\
&+ \sum_{j=1}^N (b_{ij} + \Delta b_{ij}) x_j(t - \tau)] \\
&- c \alpha^* \sum_{i=1}^l k_i x_i^T(t) P x_i(t) - c(1 - \alpha^*) \sum_{i=1}^l k_{di} x_i^T(t) P x_i(t - \tau) \\
&+ \sum_{i=1}^l x_i^T(t) P \varpi_i(t) \\
&+ \frac{1}{2} \sum_{j=1}^n p_j [\tilde{x}_j^T(t) Q \tilde{x}_j(t) - \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau)] \\
&+ \frac{1}{\xi} (\hat{\delta} - \delta^*) \hat{\delta} \\
&\leq \varphi \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{A} \tilde{x}_j(t) \\
&+ c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{B} \tilde{x}_j(t - \tau) \\
&+ c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t - \tau) \\
&+ \sum_{i=1}^l x_i^T(t) P \varpi_i(t) \\
&+ \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t) Q \tilde{x}_j(t) - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau) \\
&+ \frac{1}{\xi} (\hat{\delta} - \delta^*) \hat{\delta}
\end{aligned} \tag{37}$$



$$\begin{aligned}
 &\leq \varphi \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) + c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{A} \tilde{x}_j(t) \\
 &+ c \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{B} \tilde{x}_j(t - \tau) \\
 &- c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t) \\
 &+ c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t) \tilde{x}_j(t - \tau) \\
 &- c \delta^* \sum_{j=1}^n p_j \tilde{x}_j^T(t - \tau) \tilde{x}_j(t - \tau) \\
 &+ \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t) Q \tilde{x}_j(t) - \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau) \\
 &\leq \sum_{j=1}^n p_j \tilde{x}_j^T(t) (\varphi I_N + c \tilde{A} + \frac{1}{2} Q) \tilde{x}_j(t) \\
 &+ \sum_{j=1}^n p_j \tilde{x}_j^T(t) c \tilde{B} \tilde{x}_j(t - \tau) \\
 &- \frac{1}{2} \sum_{j=1}^n p_j \tilde{x}_j^T(t - \tau) Q \tilde{x}_j(t - \tau) \\
 &+ c \delta^* \sum_{j=1}^n p_j [-\tilde{x}_j^T(t) \tilde{x}_j(t) + \tilde{x}_j^T(t) \tilde{x}_j(t - \tau) \\
 &- \tilde{x}_j^T(t - \tau) \tilde{x}_j(t - \tau)] \\
 &= \frac{1}{2} \sum_{j=1}^n p_j \begin{bmatrix} \tilde{x}_j^T(t) & \tilde{x}_j^T(t - \tau) \end{bmatrix} \Pi_2 \begin{bmatrix} \tilde{x}_j(t) \\ \tilde{x}_j(t - \tau) \end{bmatrix} \\
 &+ c \delta^* \sum_{j=1}^n p_j \begin{bmatrix} \tilde{x}_j^T(t) & \tilde{x}_j^T(t - \tau) \end{bmatrix} \begin{bmatrix} -I_N & \frac{1}{2} I_N \\ \frac{1}{2} I_N & -I_N \end{bmatrix} \begin{bmatrix} \tilde{x}_j(t) \\ \tilde{x}_j(t - \tau) \end{bmatrix} \\
 &\leq \frac{1}{2} \sum_{j=1}^n p_j \begin{bmatrix} \tilde{x}_j^T(t) & \tilde{x}_j^T(t - \tau) \end{bmatrix} \Pi_3 \begin{bmatrix} \tilde{x}_j(t) \\ \tilde{x}_j(t - \tau) \end{bmatrix} < 0
 \end{aligned}$$

where

$$\Pi_3 = \begin{bmatrix} 2\varphi I_N + 2c\tilde{A} + Q & c\tilde{B} \\ c\tilde{B}^T & -Q \end{bmatrix}$$

It is guaranteed by  $\Pi_3 < 0$  which is equivalent to (33). This completes the proof.

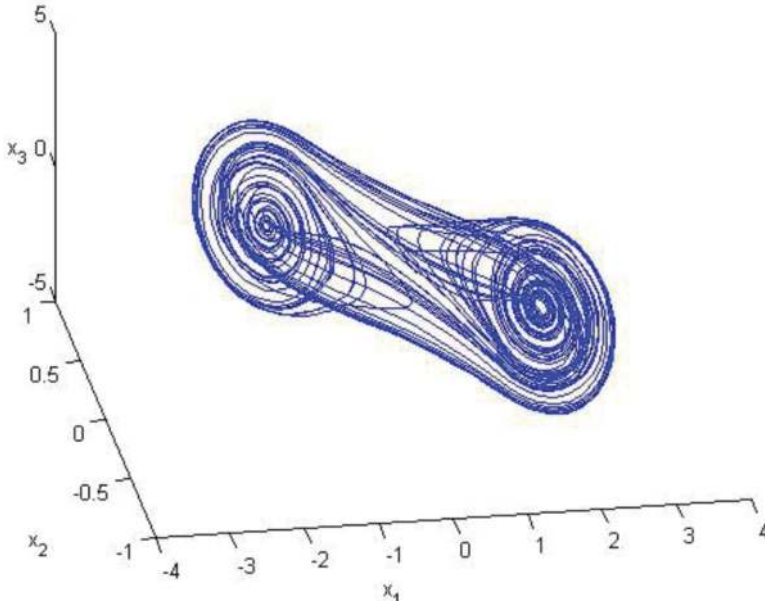
## 4. Numerical example

In this section, a numerical example is used to verify the effectiveness of the proposed methods.

**Example 1.** Consider a dynamical network consisting of 10 nodes that are identical Chua's circuits. A single Chua's circuit is described by

$$\begin{cases} \dot{x}_1 = \vartheta(-x_1 + x_2 - \zeta(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\omega x_2 \end{cases} \quad (38)$$

where  $\vartheta = 10$ ,  $\omega = 14.87$ ,  $\zeta(x_1) = bx_1 + \frac{a-b}{2}(|x_1 + 1| - |x_1 - 1|)$ ,  $a = -1.27$ , and  $b = -0.68$ . It is known that the Chua's system has a chaotic attractor which is shown in **Figure 1**.



**Figure 1.** The chaotic attractor of Chua's circuit.

It is obvious that system (38) is also be rewritten as

$$\dot{x} = Hx + g(x) \quad (39)$$

where

$$x = [x_1 \quad x_2 \quad x_3]^T$$

$$H = \begin{bmatrix} -g & g & 0 \\ 1 & -1 & 1 \\ 0 & -\omega & 0 \end{bmatrix}$$

$$g(x) = [-g\zeta(x_1) \quad 0 \quad 0]^T$$

Without loss of generality, matrix  $P$  here is selected as  $P = \text{diag}\{1, \omega, 1\}$ . Next, we will check whether there is a suitable  $\eta$  satisfying condition (11) in Assumption 1. It is obtained that

$$\begin{aligned} x^T P(Hx + g(x)) &\leq \frac{1}{2} x^T (PH + H^T P)x - gax_1^2 \\ &= \frac{1}{2} x^T (\tilde{H} + \tilde{H}^T)x \\ &\leq \frac{1}{2} \lambda_{\max}(\tilde{H} + \tilde{H}^T) x^T x \\ &= \eta x^T x \end{aligned} \tag{40}$$

where  $\tilde{H} = PH + \text{diag}\{-ga, 0, 0\}$  and  $\eta = \frac{1}{2} \lambda_{\max}(\tilde{H} + \tilde{H}^T) = 9.0620$ . Thus, condition (11) is satisfied. Then, the resulting network closed by controller (4) is described as

$$\begin{cases} \dot{x}_{i1} = 10(-x_{i1} + x_{i2} - \zeta(x_{i1})) + c \sum_{j=1}^N a_{ij} x_{j1} + c \sum_{j=1}^N b_{ij} x_{j1}(t - \tau) + u_{i1} \\ \dot{x}_{i2} = x_{i1} - x_{i2} + x_{i3} + c \sum_{j=1}^N a_{ij} x_{j2} + c \sum_{j=1}^N b_{ij} x_{j2}(t - \tau) + u_{i2} \\ \dot{x}_{i3} = -14.87x_{i2} + c \sum_{j=1}^N a_{ij} x_{j3} + c \sum_{j=1}^N b_{ij} x_{j3}(t - \tau) + u_{i3} \end{cases} \quad i \in S_\ell \tag{41}$$

Without loss of generality, the coupling matrices  $A$  and  $B$  are expressed by small-world and scale-free networks, which are depicted in **Figures 2** and **3**, respectively.

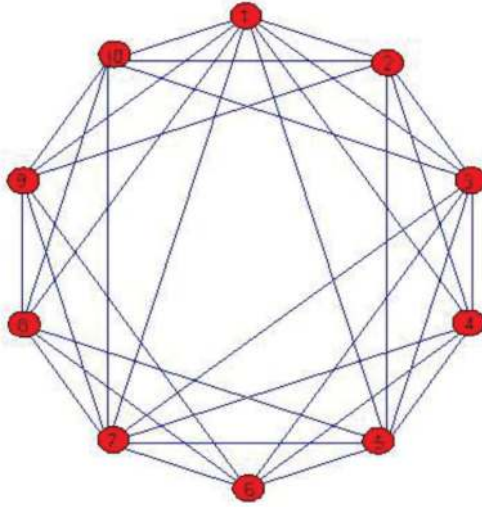


Figure 2. The simulation of coupling matrix  $A$ .

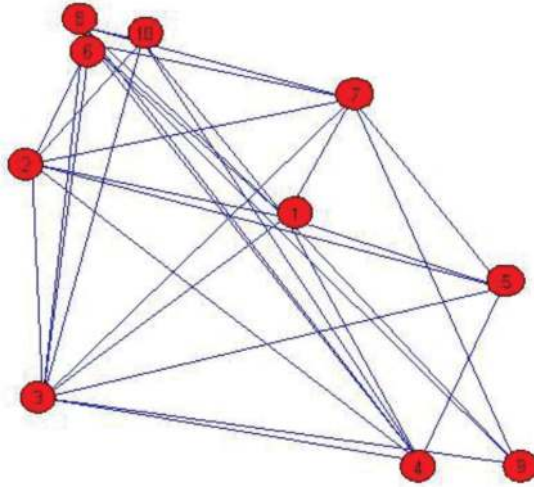


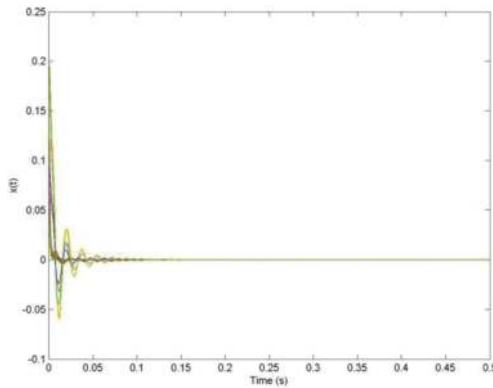
Figure 3. The simulation of coupling matrix  $B$ .

When such coupling matrices exchange randomly, under conditions such that  $c = 50$ ,  $\alpha^* = 0.85$ ,  $\delta^* = 3.6$ , and pinning fraction = 0.8, based on Theorem 1, we have the corresponding parameters computed as follows:

$$k_i = 22.8791, k_{di} = 2.3840, i \in S_{\theta}, \text{ and}$$

$$Q = \begin{bmatrix} 1137.4 & -35.5 & -28.5 & -34.6 & -43.2 & -2.5 & -36.5 & -34.6 & -47.9 & -51.3 \\ * & 1074.4 & -34.8 & -35.9 & -41.1 & -9.1 & -9.7 & -4.9 & -54.9 & -53.3 \\ * & * & 1.1085 & -33.8 & -42.2 & -32.2 & -32.1 & -6.1 & -2.7 & -50.3 \\ * & * & * & 1075.2 & -45.4 & -40.3 & -47.0 & -4.8 & -3.5 & -3.2 \\ * & * & * & * & 1097.1 & -50.2 & -37.0 & -38.8 & -2.6 & -6.6 \\ * & * & * & * & * & 1075.5 & -36.3 & -39.2 & -54.0 & -0.8 \\ * & * & * & * & * & * & 1134.6 & -41.1 & -48.4 & -53.5 \\ * & * & * & * & * & * & * & 1074.8 & -54.7 & -51.9 \\ * & * & * & * & * & * & * & * & 156.6 & -78.4 \\ * & * & * & * & * & * & * & * & * & 148.1 \end{bmatrix}$$

Under the initial condition  $x_i(t)=[0.1 \ 0.1 \ 0.2]^T$ , where  $i = 1, 2, \dots, 10$  and  $\tau = 0.005$ , we have the state response of the closed-loop network by the stochastic pinning controller (5) shown in **Figure 4** and is stable.



**Figure 4.** The state response of the complex network by controller (5).

Based on the results in this chapter, it is known that probability  $\alpha^*$  plays important roles in the stabilization of complex networks, where non-delay and delay control gains  $k_i$  and  $k_{di}$  are very close to  $\alpha^*$ . Let  $k_a = \sqrt{k_i^2 + k_{di}^2}$ , we have the relationship between parameters  $\alpha^*$  and  $k_i$ ,  $k_{di}$  and  $k_a$  given in **Table 1**, where the more detailed correlation between  $\alpha^*$  and  $k_i$ ,  $k_{di}$  and  $k_a$  is simulated in **Figure 5**. From **Table 1** and **Figure 5**, it is seen that both gains of  $k_i$  and  $k_{di}$  have effects in the stabilization of the underlying complex network. It is also found that there is not a phenomenon that larger  $\alpha^*$  results in larger  $k_{di}$  or smaller  $k_i$ . This property further demonstrates the necessity of considering the probability distribution of non-delay and delay states while the stabilization problem of delayed systems is considered. Particularly, it is seen that when  $\alpha^* = 0$ , there are no solutions to  $k_i$  and  $k_{di}$ . This is determined by condition (10), which is



$$W_{11} = \begin{bmatrix} 813.9 & -10.19 & -10.86 & -10.07 & -9.55 & -2.78 & -9.14 & -10.63 & -7.00 & -6.26 \\ * & 805.26 & -11.63 & -11.87 & -9.71 & -0.94 & -2.32 & -1.89 & -7.14 & -6.40 \\ * & * & 810.49 & -11.34 & -10.21 & -11.38 & -10.06 & -2.69 & 0.19 & -6.32 \\ * & * & * & 805.28 & -10.36 & -9.80 & -9.56 & -2.88 & 0.04 & 0.67 \\ * & * & * & * & 813.78 & -9.40 & -9.63 & -8.92 & 0.04 & -0.15 \\ * & * & * & * & * & 804.55 & -11.81 & -9.13 & -7.08 & -0.08 \\ * & * & * & * & * & * & 820.49 & -9.54 & -7.09 & -7.30 \\ * & * & * & * & * & * & * & 805.14 & -7.17 & -6.52 \\ * & * & * & * & * & * & * & * & 7.17 & -2.97 \\ * & * & * & * & * & * & * & * & * & 7.36 \end{bmatrix}$$

$$W_{12} = \begin{bmatrix} 101.76 & -4.67 & -4.64 & -4.71 & -4.72 & -4.70 & -4.69 & -4.69 & -4.18 & -4.18 \\ * & 92.71 & -4.70 & -4.72 & -4.70 & -4.77 & -4.71 & -0.11 & -0.06 & -4.17 \\ * & * & 101.91 & -4.68 & -4.73 & -4.74 & -4.76 & -4.71 & -4.06 & -4.18 \\ * & * & * & 92.62 & -4.75 & -4.74 & -0.08 & -4.77 & 0.02 & -4.10 \\ * & * & * & * & 83.40 & -0.11 & -4.74 & -0.09 & 0.01 & 0.02 \\ * & * & * & * & * & 88.09 & -4.74 & -0.09 & -4.18 & 0.02 \\ * & * & * & * & * & * & 92.76 & -4.68 & -4.16 & -0.05 \\ * & * & * & * & * & * & * & 83.62 & -0.02 & -4.15 \\ * & * & * & * & * & * & * & * & 1.39 & 0.01 \\ * & * & * & * & * & * & * & * & * & 4.82 \end{bmatrix}$$

$$W_{22} = \begin{bmatrix} 209.59 & 0.33 & 1.05 & 0.23 & -0.36 & 2.42 & -0.62 & 0.64 & -1.96 & -2.52 \\ * & 200.26 & 1.49 & 1.64 & -0.35 & 0.76 & 1.93 & 1.91 & -1.71 & -2.38 \\ * & * & 204.41 & 1.37 & 0.23 & 1.28 & 0.12 & 2.36 & -0.23 & -2.48 \\ * & * & * & 200.41 & 0.38 & -0.08 & -0.21 & 2.53 & -0.04 & -0.62 \\ * & * & * & * & 201.24 & -0.20 & -0.28 & -0.69 & -0.02 & 0.14 \\ * & * & * & * & * & 200.74 & 1.57 & -0.56 & -1.81 & 0.07 \\ * & * & * & * & * & * & 204.31 & -0.48 & -1.83 & -1.55 \\ * & * & * & * & * & * & * & 199.66 & -1.69 & -2.27 \\ * & * & * & * & * & * & * & * & 8.06 & -3.82 \\ * & * & * & * & * & * & * & * & * & 7.90 \end{bmatrix}$$

When probability  $\alpha^*$  is inaccessible, a kind of adaptive pinning control method may be exploited. Let the corresponding parameters  $P, \eta$ , and  $\delta^*$  same to the above values, by Theorem 3, one could get the related parameters computed as follows:  $k_i = 102.2258, k_{d_i} = 22.3035, i \in S_{\psi}$ , and

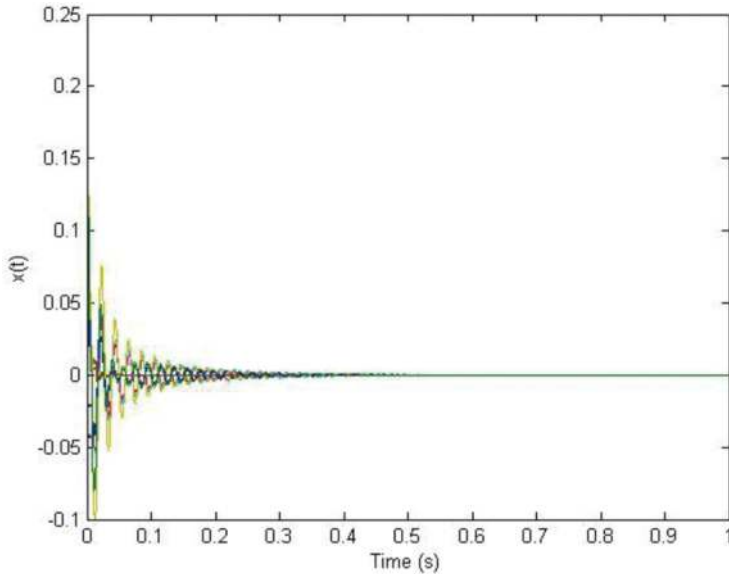


Figure 6. The state response of the complex network by controller (29).

$$Q = \begin{bmatrix} 5134.0 & -28.2 & 82.8 & 40.8 & -112.6 & -153.4 & -36.7 & 62.2 & -50.0 & -54.5 \\ * & 5437.0 & -82.4 & -61.7 & -100.9 & 2.1 & -84.6 & -87.9 & -49.7 & -51.8 \\ * & * & 5214.8 & 35.4 & -232.0 & -93.1 & 119.2 & 20.6 & -4.9 & -52.3 \\ * & * & * & 5516.0 & -173.6 & -12.1 & -303.0 & 24.9 & -5.2 & -2.3 \\ * & * & * & * & 5644.9 & -59.9 & -107.9 & -206.4 & -0.4 & 7.0 \\ * & * & * & * & * & 5431.7 & -195.1 & -196.0 & -47.2 & -0.6 \\ * & * & * & * & * & * & 5589.6 & -123.3 & -46.6 & -50.7 \\ * & * & * & * & * & * & * & 5101.7 & -50.8 & -50.5 \\ * & * & * & * & * & * & * & * & 143.9 & -66.4 \\ * & * & * & * & * & * & * & * & * & 133.0 \end{bmatrix}$$

where  $\delta$  is selected to be  $\delta = 5$ . Under the same initial condition and topologies having couplings exchanges, the simulations of the resulting complex network are given in **Figures 6** and **7**, where **Figure 6** is state response of the closed-loop system through the desired adaptive pinning controller with form (29) and updating law with form (30), and **Figure 7** is the curve of estimation  $\alpha(t)$  with  $\alpha_0=0.2$ .



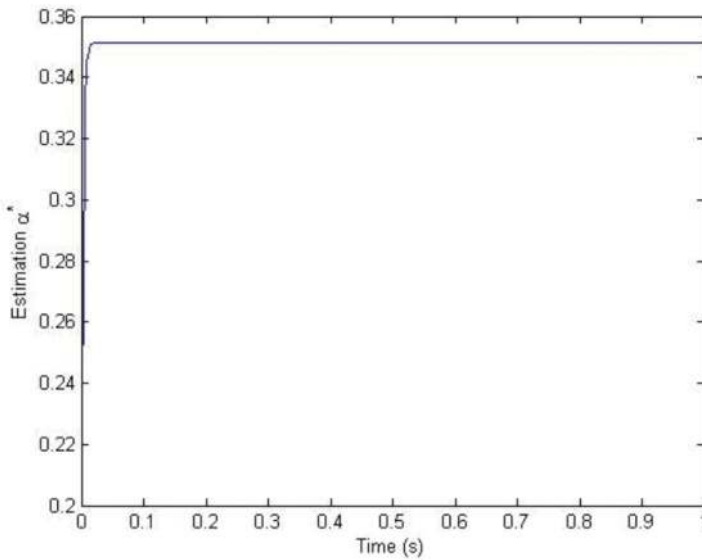


Figure 7. The curve of estimation of  $\alpha^*$ .

From these simulations, it is said that the desired partially delay-dependent controllers in terms of stochastic pinning controller (5) and adaptive controller (29) are both effective, where the resulting complex network is stable even if the coupling matrices experience random exchanges. On the other hand, when  $\alpha$  is obtained exactly but  $\delta^*$  is unavailable, using Theorem 4, we have the corresponding parameters obtained as follows:  $k_i = 30.6104$ ,  $k_{di} = 16.7135$ ,  $i \in S_{\rho}$ , and

$$Q = \begin{bmatrix} 1648.4 & -36.0 & -31.3 & -34.5 & -39.7 & 2.5 & -36.2 & -37.5 & -48.8 & -52.0 \\ * & 1573.4 & -35.6 & -33.0 & -38.4 & -5.7 & -6.0 & -1.5 & -53.0 & -52.1 \\ * & * & 1613.6 & -33.0 & -38.9 & -35.4 & -34.2 & 4.1 & -1.7 & -50.7 \\ * & * & * & 1574.9 & -41.5 & -40.3 & -42.6 & -0.5 & -2.0 & 5.0 \\ * & * & * & * & 1603.5 & -45.1 & -37.1 & -36.6 & 1.5 & 5.8 \\ * & * & * & * & * & 1576.6 & -34.3 & -44.8 & -43.6 & 0.3 \\ * & * & * & * & * & * & 1643.7 & -39.6 & -45.3 & -52.0 \\ * & * & * & * & * & * & * & 1578.1 & -52.8 & -48.6 \\ * & * & * & * & * & * & * & * & 334.8 & -75.6 \\ * & * & * & * & * & * & * & * & * & 308.0 \end{bmatrix}$$

where  $\xi$  is selected to be  $\xi = 1$ .

## 5. Conclusion

In this chapter, the stabilization problem of complex dynamical network with non-delayed and delayed couplings exchanging randomly has been realized by a new kind of stochastic pinning controller being partially delay-dependent, where the switching between the non-delayed and delayed couplings is modeled by the related coupling matrices containing uncertainties. Different from the traditional pinning methods, the designed pinning controller contains non-delay and delay state terms simultaneously but occurs asynchronously with a certain probability, respectively. Sufficient conditions for the stabilization of complex dynamical network over topology exchange are derived by the robust method and presented with linear matrix inequalities (LMIs). It has been shown that the probability distributions of non-delay and delay states in addition to the bound of such uncertainties play very important roles in the controller design. Moreover, when the probability is inaccessible, a kind of adaptive partially delay-dependent controller is proposed to deal with this general case, where another adaptive control problem in terms of unknown bound is also considered. Finally, the correctness and feasibility of the proposed method are verified by a numerical simulation.

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