Transverse Vibration Analysis of Euler-Bernoulli Beams Using Analytical Approximate Techniques

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1. Introduction

The vibration problems of uniform and nonuniform Euler-Bernoulli beams have been solved analytically or approximately [1-5] for various end conditions. In order to calculate fundamental natural frequencies and related mode shapes, well known variational techniques such as Rayleigh_Ritz and Galerkin methods have been applied in the past. Besides these techniques, some discretized numerical methods were also applied to beam vibration analysis successfully.

Recently, by the emergence of new and innovative semi analytical approximation methods, research on this subject has gained momentum. Among these studies, Liu and Gurram [6] used He's Variational Iteration Method to analyze the free vibration of an Euler-Bernoulli beam under various supporting conditions. Similarly, Lai et al [7] used Adomian Decomposition Method (ADM) as an innovative eigenvalue solver for free vibration of Euler-Bernoulli beam again under various supporting conditions. By doing some mathematical elaborations on the method, the authors obtained i^{th} natural frequencies and modes shapes one at a time. Hsu et al. [8] again used Modified Adomian Decomposition Method to solve free vibration of non-uniform Euler-Bernoulli beams with general elastically end conditions. Ozgumus and Kaya [9] used a new analytical approximation method namely Differential Transforms Method to analyze flapwise bending vibration analysis of double tapered rotating Euler-Bernoulli beam. Hsu et al. [10] also used Modified Adomian Decomposition Method, a new analytical approximation method, to solve eigenvalue problem for free vibration of uniform Timoshenko beams. Ho and Chen [11] studied the problem of free transverse vibration of an axially loaded non-uniform spinning twisted Timoshenko beam using Differential Transform Method. Another researcher, Register [12] found a general expression for the modal frequencies of a beam with symmetric spring boundary conditions. In addition, Wang [13] studied the dynamic analysis of generally supported beam. Yieh [14] determined the natural frequencies and natural

modes of the Euler_Bernoulli beam using the singular value decomposition method. Also, Kim [15] studied the vibration of uniform beams with generally restrained boundary conditions. Naguleswaran [16] derived an approximate solution to the transverse vibration of the uniform Euler-Bernoulli beam under linearly varying axial force. Chen and Ho [17] studied the problem of transverse vibration of rotating twisted Timoshenko beams under axial loading using differential transform method to obtain natural frequencies and mode shapes.

In this study, transverse vibration analysis of uniform and nonuniform Euler-Bernoulli beams will be briefly explained and demonstrated with some examples by using some of these novel approaches. To this aim, the theory and analytical techniques about lateral vibration of Euler-Bernoulli beams will be explained first, and then the methods used in the analysis will be described. Finally, some case studies will be presented by using the proposed techniques and the advantages of those methods will be discussed.

2. Transverse vibration of the beams

2.1 Formulation of the problem

Lateral vibration of beams is governed by well-known Bernoulli-Euler equation. To develop the governing equation, consider the free body diagram of a beam element in bending shown in Fig.1. In this figure, $M(x,t)$ is the bending moment, $Q(x,t)$ is the shear force, and $f(x,t)$ is the external force per unit length acting on the beam.
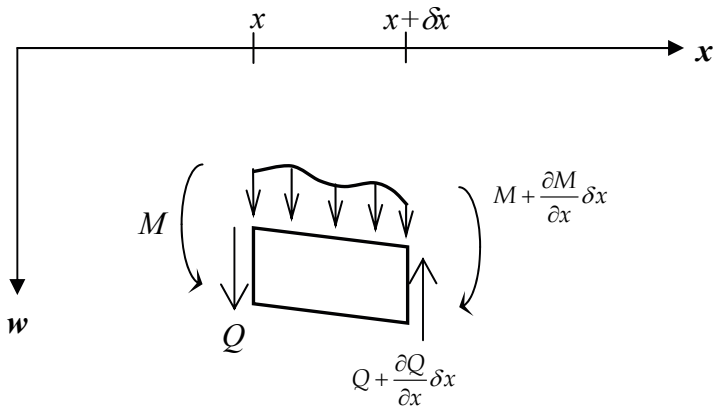

Fig. 1. Free-body diagram of a beam element in bending

Equilibrium condition of moments leads to the following equation:

$$M + Q\delta x - \left(M + \frac{\partial M}{\partial x} \delta x \right) = 0 \tag{1}$$

or

$$Q = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) \tag{2}$$

Since a uniform beam is not assumed in the formulation, $I(x)$ will be variable along beam length.

The equation of motion in the tranverse direction for the beam element is:

$$(\rho A \delta x)\frac{\partial^2 w}{\partial t^2} = f(x,t)\delta x + Q - \left(Q + \frac{\partial Q}{\partial x}\right) \tag{3}$$

In Eq.(3), ρ is mass density of the material of the beam. After simplifications, Eq.(3) can be rewritten as follows:

$$\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial Q}{\partial x} = f(x,t) \tag{4}$$

In view of Eq.(2), governing equation for forced transverse vibration is obtained as below which is the well known Euler-Bernoulli equation.

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) \tag{5}$$

For a uniform beam Eq.(5) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) \tag{6}$$

For the free vibration case, *i.e. f(x,t)=0*, the equation of motion becomes

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \tag{7}$$

If the beam is uniform, i.e. EI is constant, the equation of motion in Eq.(7) reduces to

$$c^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0 \tag{8}$$

where

$$c = \sqrt{\frac{EI}{\rho A}} \, . \tag{9}$$

Transverse vibration of beams is an initial-boundary value problem. Hence, both initial and boundary conditions are required to obtain a unique solution $w(x,t)$. Since the equation involves a second order derivative with respect to time and a fourth order derivative with respect to a space coordinate, two initial conditions and four boundary conditions are needed.

2.2 Modal analysis
The solution to problem given by Eq.(5) can be produced by, first obtaining the natural frequencies and mode shapes and then expressing the general solution as a summation of

modal responses. In each mode, the system will vibrate in a fixed shape ratio which leads to providing a separable displacement function into two separate time and space functions. This approach is the same for both free and forced vibration problems. Hence, the displacement function $w(x,t)$ can be defined by the following form.

$$w(x,t) = Y(x)T(t)$$
(10)

Consider the free vibration problem for a uniform beam, *i.e. EI* is constant. The governing equation for this specific case previously was given in Eq.(8). The free vibration solution will be obtained by inserting Eq.(10) into Eq.(8) and rearranging it as

$$\frac{c^2}{Y(x)}\frac{\partial^4 Y(x)}{\partial x^4} = -\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \omega^2$$
(11)

where c is defined in Eq.(9) and ω^2 is defined as constant. Eq.(11) can be rearranged as two ordinary differential equations as

$$\frac{d^4 Y(x)}{dx^4} - \lambda^4 Y(x) = 0$$
(12)

$$\frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0$$
(13)

where

$$\lambda^4 = \frac{\omega^2}{c^2}$$
(14)

General solution of Eq.(12) is a mode shape and given by

$$Y(x) = C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_4 \sin \lambda x$$
(15)

The constants C_1, C_2, C_3 and C_4 can be found from the end conditions of the beam. Then, the natural frequencies of the beam are obtained from Eq.(14) as

$$\omega = \lambda^2 c$$
(16)

Inserting Eq.(9) into Eq.(16) with rearranging leads to

$$\omega = \left(\lambda L\right)^2 \sqrt{\frac{EI}{\rho A L^4}}$$
(17)

2.3 Boundary conditions
The common boundary conditions related to beam's ends are as follows:

2.3.1 Simply supported (pinned) end

Deflection = 0 $Y = 0$

Bending Moment = 0

$$EI\frac{\partial^2 Y}{\partial x^2} = 0$$

2.3.2 Fixed (clamped) end

Deflection = 0

$$Y = 0$$

Slope = 0

$$\frac{\partial Y}{\partial x} = 0$$

2.3.3 Free end

Bending Moment = 0

$$EI\frac{\partial^2 Y}{\partial x^2} = 0$$

Shear Force = 0

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 Y}{\partial x^2}\right) = 0$$

2.3.4 Sliding end

Slope = 0

$$\frac{\partial Y}{\partial x} = 0$$

Shear Force = 0

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 Y}{\partial x^2}\right) = 0$$

The exact frequencies for lateral vibration of the beams with different end conditions will not be computed due to the procedure explained here. Since, the motivation of this chapter is the demonstration of the use of analytical approximate techniques in the analysis of bending vibration of beams, available exact results related to the selected case studies will be directly taken from [5,18]. The reader can refer to these references for further details in analytical derivations of the exact results.

2.4 The methods used in the analysis of transverse vibration of beams

Analytical approximate solution techniques are used widely to solve nonlinear ordinary or partial differential equations, integro-differential equations, delay equations, etc. Main advantage of employing such techniques is that the problems are considered in a more realistic manner and the solution obtained is a continuous function which is not the case for the solutions obtained by discretized solution techniques. Hence these methods are computationally much more efficient in the solution of those equations.

The methods that will be used throughout the study are, Adomian Decomposition Method (ADM), Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM). Below, each technique will be explained and then all will be applied to several problems related to the topic of the article.

2.4.1 Adomian Decomposition Method (ADM)

In the ADM a differential equation of the following form is considered

$$Lu + Ru + Nu = g(x) \tag{18}$$

where L is the linear operator which is highest order derivative, R is the remainder of linear operator including derivatives of less order than L, Nu represents the nonlinear terms and g is the source term. Eq.(18) can be rearranged as

$$Lu = g(x) - Ru - Nu \tag{19}$$

Applying the inverse operator L^{-1} to both sides of Eq.(19) employing given conditions we obtain

$$u = L^{-1}\{g(x)\} - L^{-1}(Ru) - L^{-1}(Nu) \tag{20}$$

After integrating source term and combining it with the terms arising from given conditions of the problem, a function $f(x)$ is defined in the equation as

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \tag{21}$$

The nonlinear operator $Nu = F(u)$ is represented by an infinite series of specially generated (Adomian) polynomials for the specific nonlinearity. Assuming Nu is analytic we write

$$F(u) = \sum_{k=0}^{\infty} A_k \tag{22}$$

The polynomials A_k's are generated for all kinds of nonlinearity so that they depend only on u_o to u_k components and can be produced by the following algorithm.

$$A_0 = F(u_0) \tag{23}$$

$$A_1 = u_1 F'(u_0) \tag{24}$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) \tag{25}$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) \tag{26}$$

$$\vdots$$

The reader can refer to [19,20] for the algorithms used in formulating Adomian polynomials. The solution $u(x)$ is defined by the following series

$$u = \sum_{k=0}^{\infty} u_k \tag{27}$$

where the components of the series are determined recursively as follows:

$$u_0 = f(x) \tag{28}$$

$$u_{k+1} = -L^{-1}\left(Ru_k\right) - L^{-1}\left(A_k\right), \quad k \geq 0 \tag{29}$$

2.4.2 Variational Iteration Method (VIM)

According to VIM, the following differential equation may be considered:

$$Lu + Nu = g(x) \tag{30}$$

where L is a linear operator, and N is a nonlinear operator, and $g(x)$ is an inhomogeneous source term. Based on VIM, a correct functional can be constructed as follows:

$$u_{n+1} = u_n + \int_0^x \lambda(\xi)\left\{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right\} d\xi \tag{31}$$

where λ is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{u}$ is considered as a restricted variation *i.e.* $\delta\tilde{u} = 0$. By solving the differential equation for λ obtained from Eq.(31) in view of $\delta\tilde{u} = 0$ with respect to its boundary conditions, Lagrangian multiplier $\lambda(\xi)$ can be obtained. For further details of the method the reader can refer to [21].

2.4.3 Homotopy Perturbation Method (HPM)

HPM provides an analytical approximate solution for problems at hand as other explained techniques. Brief theoretical steps for the equation of following type can be given as

$$L(u) + N(u) = f(r) \ , \ r \in \Omega \tag{32}$$

with boundary conditions $B(u, \partial u/\partial n) = 0$. In Eq.(8) L is a linear operator, N is nonlinear operator, B is a boundary operator, and $f(r)$ is a known analytic function. HPM defines homotopy as

$$v(r,p) = \Omega \times [0,1] \rightarrow R \tag{33}$$

which satisfies following inequalities:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{34}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{35}$$

where $r \in \Omega$ and $p \in [0,1]$ is an imbedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from Eq.(34) and Eq.(35), we have :

$$H(v,0) = L(v) - L(u_0) = 0 \tag{36}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0 \tag{37}$$

As p changing from zero to unity is that of $v(r,p)$ from u_0 to $u(r)$. In topology, this deformation $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are called homotopic. The basic

assumption is that the solutions of Eq.(34) and Eq.(35) can be expressed as a power series in p such that:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots \tag{38}$$

The approximate solution of $L(u) + N(u) = f(r)$, $r \in \Omega$ can be obtained as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \dots \tag{39}$$

The convergence of the series in Eq.(39) has been proved in [22]. The method is described in detail in references [22-25].

2.5 Case studies
2.5.1 Free vibration of a uniform beam
The governing equation for this case was previously given in Eq.(12). ADM, VIM and HPM will be applied to this equation in order to compute the natural frequencies for the free vibration of a beam with constant flexural stiffness, *i.e.* constant *EI*, and its corresponding mode shapes. To this aim, five different beam configurations are defined with its end conditions. These are *PP*, the beam with both ends pinned, *CC*, the beam with both ends clamped, *CP*, the beam with one end clamped and one end pinned, *CF*, the beam with one end clamped and one end free, *CS*, the beam with one end clamped and one and sliding. The boundary conditions associated with these configurations was given previously in text. Below, the formulations by using ADM, VIM and HPM are given and then applied to the governing equation of the problem.

2.5.1.1 Formulation of the algorithms
2.5.1.1.1 ADM
The linear operator and its inverse operator for Eq.(12) is

$$L(\cdot) = \frac{d^4}{dx^4}(\cdot) \tag{40}$$

$$L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x \int_0^x (\cdot) \, dx \, dx \, dx \, dx \tag{41}$$

To keep the formulation a general one for all configurations to be considered, the boundary conditions are chosen as $Y(0) = A$, $Y'(0) = B$, $Y''(0) = C$ and $Y'''(0) = D$. Suitable values should be replaced in the formulation with these constants. For example, $A = 0$ and $C = 0$ should be inserted for the PP beam. Hence, the equation to be solved and the recursive algorithm can be given as

$$LY = \lambda^4 Y \tag{42}$$

$$Y = A + Bx + C\frac{x^2}{2!} + D\frac{x^3}{3!} + L^{-1}(\lambda^4 Y) \tag{43}$$

$$Y_{n+1} = L^{-1}(\lambda^4 Y_n), \quad n \geq 0 \tag{44}$$

Finally, the solution is defined by

$$Y = Y_0 + Y_1 + Y_2 + Y_3 + ... \tag{45}$$

2.5.1.1.2 VIM

Based on the formulation given previously, Lagrange multiplier λ would be obtained for the governing equation, *i.e.* Eq.(12), as

$$\lambda(\xi) = \frac{(\xi - x)^3}{3!} \tag{46}$$

An iterative algorithm can be constructed inserting Lagrange multiplier and governing equation into the formulation given in Eq.(31) as

$$Y_{n+1} = Y_n + \int_0^x \lambda(\xi) \left\{ Y_n^{iv}(\xi) - \lambda^4 \tilde{Y}_n(\xi) \right\} d\xi \tag{47}$$

Initial approximation for the algorithm is chosen as the solution of $LY = 0$ which is a cubic polynomial with four unknowns which will be determined by the end conditions of the beam.

2.5.1.1.3 HPM

Based on the formulation, Eq.(12) can be divided into two parts as

$$LY = Y^{iv} \tag{48}$$

$$NY = -\lambda^4 Y \tag{49}$$

The solution can be expressed as a power series in p such that

$$Y = Y_0 + pY_1 + p^2 Y_2 + p^3 Y_3 + ... \tag{50}$$

Inserting Eq.(50) into Eq.(35) provides a solution algorithm as

$$Y_0^{iv} - y_0^{iv} = 0 \tag{51}$$

$$Y_1^{iv} + y_0^{iv} - \lambda^4 Y_0 = 0 \tag{52}$$

$$Y_n - \lambda^4 Y_{n-1} = 0, \quad n \geq 2 \tag{53}$$

Hence, an approximate solution would be obtained as

$$Y = Y_0 + Y_1 + Y_2 + Y_3 + ... \tag{54}$$

Initial guess is very important for the convergence of solution in HPM. A cubic polynomial with four unknown coefficients can be chosen as an initial guess which was shown previously to be an effective one in problems related to Euler beams and columns [26-31].

2.5.1.2 Computation of natural frequencies

By the use of described algorithms, an iterative procedure is conducted and a polynomial including the unknown coefficients coming from the initial guess is produced as a solution to the governing equation. Besides four unknowns from initial guess, an additional unknown λ also exists in the solution. Applying each boundary condition to the solution produces a linear algebraic system of equations which can be defined in matrix form as

$$[M(\lambda)]\{\alpha\} = \{0\} \tag{55}$$

where $\{\alpha\} = \langle A, B, C, D \rangle^T$. For a nontrivial solution, determinant of coefficient matrix must be zero. Determinant of matrix $[M(\lambda)]$ yields a characteristic equation in terms of λ. Positive real roots of this equation are the natural free vibration frequencies for the beam with specified end conditions.

2.5.1.3 Determination of vibration mode shapes

Vibration mode shapes for the beams can also be obtained from the polynomial approximations by the methods considered in this study. Introducing, the natural frequencies into the solution, normalized polynomial eigenfunctions for the mode shapes are obtained from

$$\bar{Y}_j = \frac{Y_N(x, \lambda_j)}{\left[\int_0^1 \left| Y_N(x, \lambda_j) \right|^2 dx \right]^{1/2}} \quad , \ j = 1, 2, 3, \ldots \tag{56}$$

The same approach can be employed to predict mode shapes for the cases including variable flexural stiffness.

2.5.1.4 Orthogonality of mode shapes

Normalized mode shapes obtained from Eq.(56) should be orthogonal. These modes can be shown to satisfy the following condition.

$$\int Y_i Y_j \ dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{57}$$

2.5.1.5 Results of the analysis

After applying the procedures explained in the text, the following results are obtained for the natural frequencies and mode shapes. Comparison with the exact solutions is also provided that one can observe an excellent agreement between the exact results and computed results.

Ten iterations are conducted for each method and computed λL values are compared with the corresponding exact values for the first three modes of vibration in the following table.

From the table it can be seen that computed values are highly accurate which show that the techniques used in the analysis are very effective. Natural frequencies can be easily obtained by inserting the values in Table 1 into Eq.(17).

The free vibration mode shapes of uniform beam for the first three mode are also depicted in the following figures. Since the obtained mode shapes coincide with the exact ones, to prevent a possible confusion to the reader, the exact mode shapes and the computed ones are not shown separately in these figures. The mode shapes for the free vibration of a uniform beam for five different configurations are given between Figs.2-6.

Beam	Mode	Exact	ADM	VIM	HPM
P-P	1	3.14159265 (π)	3.14159265	3.14159265	3.14159265
	2	6.28318531 (2π)	6.28318531	6.28318531	6.28318531
	3	9.42477796 (3π)	9.42477796	9.4247796	9.4247796
C-C	1	4.730041	4.73004074	4.73004074	4.73004074
	2	7.853205	7.85320462	7.85320462	7.85320462
	3	10.995608	10.99560784	10.99560784	10.99560784
C-P	1	3.926602	3.92660231	3.92660231	3.92660231
	2	7.068583	7.06858275	7.06858275	7.06858275
	3	10.210176	10.21017612	10.21017612	10.21017612
C-F	1	1.875104	1.87510407	1.87510407	1.87510407
	2	4.694091	4.69409113	4.69409113	4.69409113
	3	7.854757	7.85475744	7.85475744	7.85475744
C-S	1	2.365020	2.36502037	2.36502037	2.36502037
	2	5.497806	5.49780392	5.49780392	5.49780392
	3	8.639380	8.63937983	8.63937983	8.63937983

Table 1. Comparison of λL values for the uniform beam
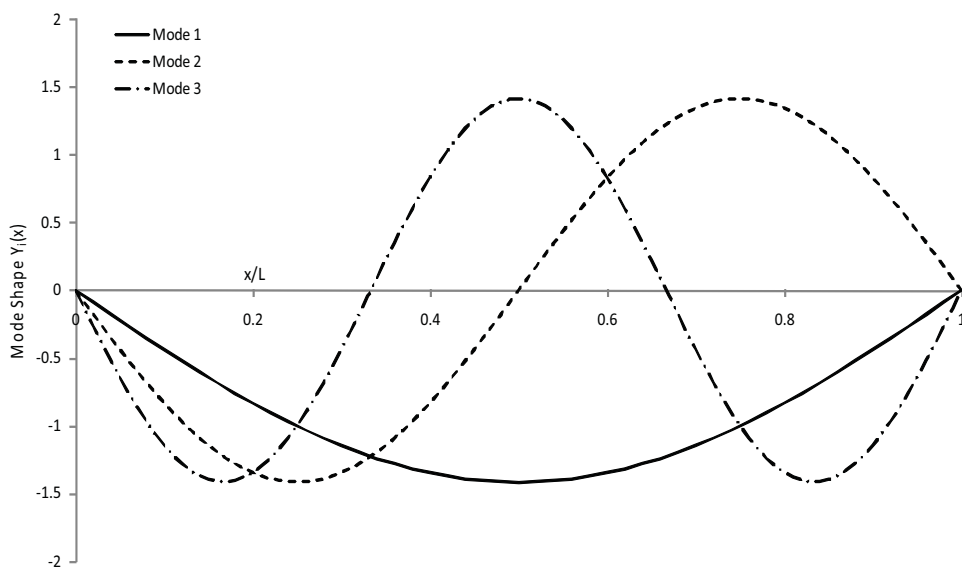
Fig. 2. Free vibration modes of PP beam.
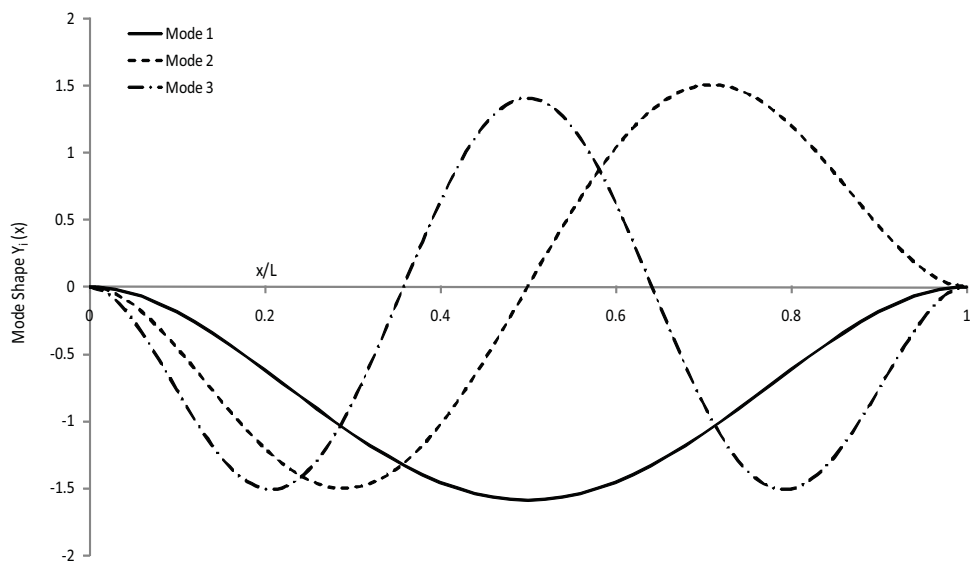
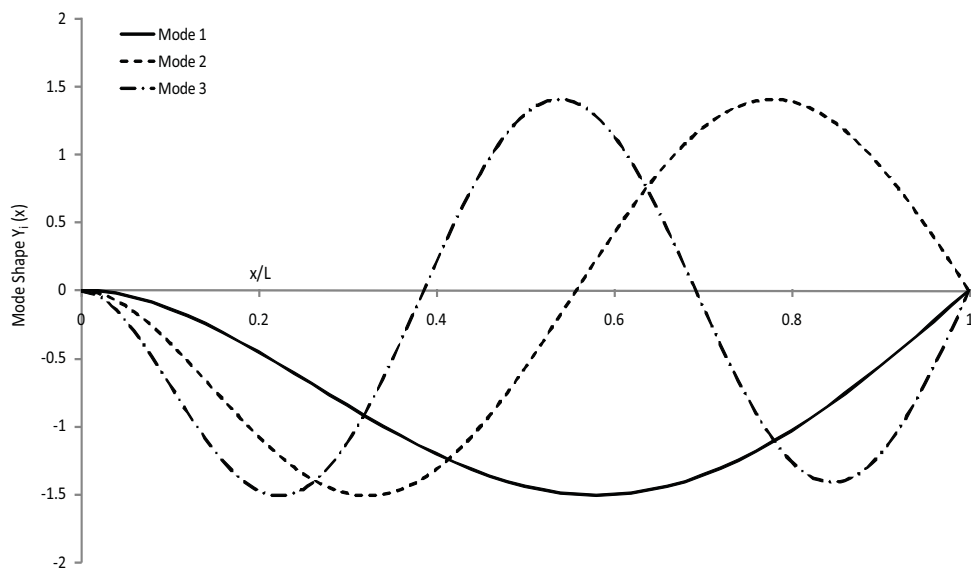Fig. 3. Free vibration modes of CC beam.

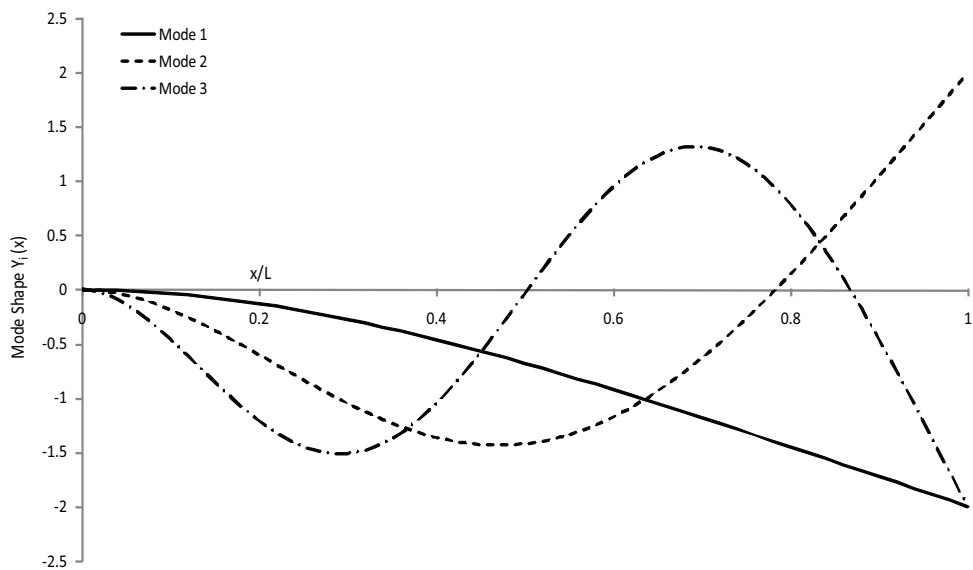Fig. 4. Free vibration modes of CP beam.
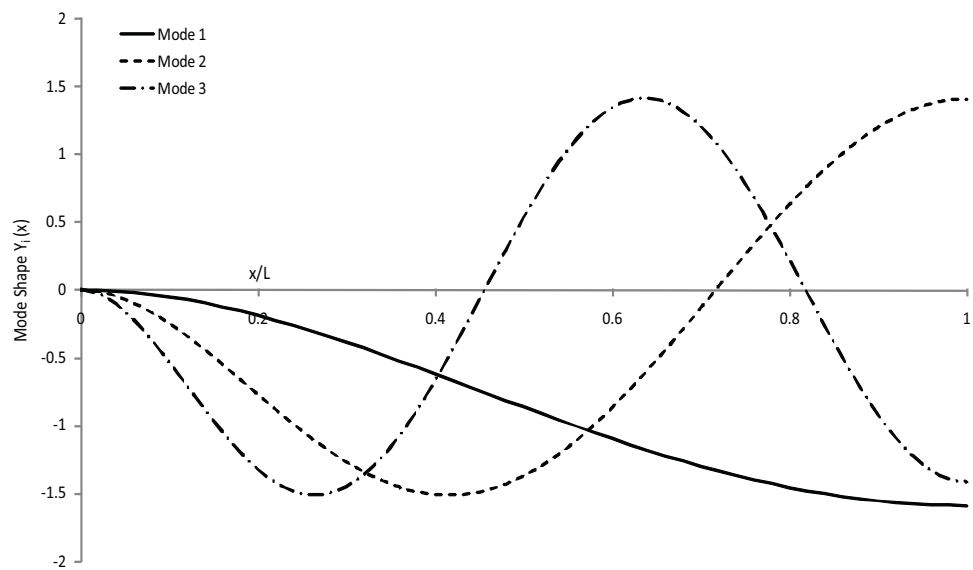
Fig. 5. Free vibration modes of CF beam.

Fig. 6. Free vibration modes of CF beam.

Orthogonality condition given in Eq.(57) for each mode will also be shown to be satisfied. To this aim, the resulting polynomials representing normalized eigenfunctions are integrated according to the orthogonality condition and following results are obtained.
The PP Beam:

$$\int Y_i Y_j \ dx = \begin{bmatrix} 1.0000000000000018 & 3.133937506642793*10^{-14} & 1.1716394903869283*10^{-12} \\ & 1.0000000000011495 & -1.2402960384615706*10^{-11} \\ & & 1.0000000002542724 \end{bmatrix}$$

The CC Beam:

$$\int Y_i Y_j \ dx = \begin{bmatrix} 1.0000000000000218 & -3.2594265231428034*10^{-13} & 3.0586251883350275*10^{-11} \\ & 0.9999999999825311 & -4.152039340197406*10^{-10} \\ & & 0.9999999986384138 \end{bmatrix}$$

The CP Beam:

$$\int Y_i Y_j \ dx = \begin{bmatrix} 1.0000000000000027 & -1.1266760906960104*10^{-13} & 3.757083743946838*10^{-12} \\ & 0.9999999999991402 & -5.469593759847241*10^{-11} \\ & & 1.000000001594055 \end{bmatrix}$$

The CF Beam:

$$\int Y_i Y_j \ dx = \begin{bmatrix} 1.0000000000000000 & 1.134001985461197*10^{-15} & 5.844267022420876*10^{-14} \\ & 1.0000000000000178 & 4.1094000558822104*10^{-13} \\ & & 0.9999999999969831 \end{bmatrix}$$

The CS Beam:

$$\int Y_i Y_j \ dx = \begin{bmatrix} 1.0000000000000009 & -1.067231239470151*10^{-15} & -2.57978811982526*10^{-13} \\ & 1.0000000000002232 & -2.422143056441983*10^{-13} \\ & & 1.0000000000643874 \end{bmatrix}$$

From these results it can be clearly observed that the orthogonality condition is perfectly satisfied for each configuration of the beam.
The analysis for the lateral free vibration of the uniform beam is completed. Now, these techniques will be applied to a circular rod having variable cross-section along its length.

2.5.2 Free vibration of a rod with variable cross-section

A circular rod having a radius changing linearly is considered in this case. Such a rod is shown below in Fig.7. The function representing the radius would be as

$$R(x) = R_0(1 - bx) \tag{58}$$

where R_o is the radius at the left end, L is the length of the rod and $bL \leq 1$.
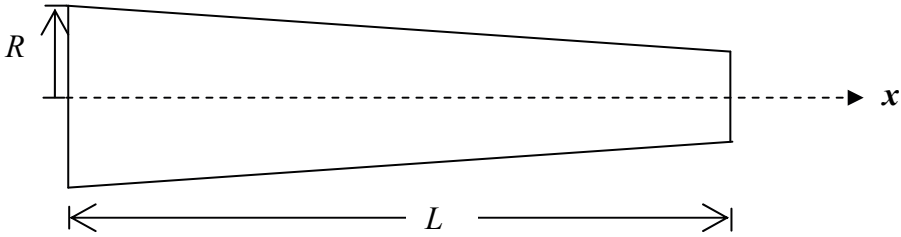
Fig. 7. Circular rod with variable cross-section

Employing Eq.(58), cross-sectional area and moment of inertia for a section at an arbitrary point x becomes:

$$A(x) = A_0(1 - bx)^2 \tag{59}$$

$$I(x) = I_0(1 - bx)^4 \tag{60}$$

where

$$A_0 = \pi R_0^2 \tag{61}$$

$$I_0 = \frac{\pi R_0^4}{4} \tag{62}$$

Free vibration equation of the rod was previously given in Eq.(7) as

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2} \right) + \rho A \frac{\partial^2 w}{\partial t^2} = 0$$

After the application of separation of variables technique by defining the displacement function as $w(x,t) = Y(x)T(t)$, the equation to obtain natural frequencies and mode shapes becomes

$$\frac{d^2}{dx^2}\left(EI(x) \frac{d^2 Y}{dx^2} \right) - \omega^2 \rho A(x) Y = 0 \tag{63}$$

2.5.2.1 Formulation of the algorithms
2.5.2.1.1 ADM

Application of ADM to Eq.(63) leads to the following

$$Y^{iv} - 8b\psi_1(x)Y''' + 12b^2\psi_2(x)Y'' - \lambda_0^4 \psi_2(x)Y = 0 \tag{64}$$

where

$$\psi_1(x) = \frac{1}{1 - bx} \tag{65}$$

$$\psi_2(x) = \frac{1}{(1-bx)^2} \tag{66}$$

$$\lambda_0^4 = \frac{\omega^2}{c_0^2} \tag{67}$$

$$c_0 = \sqrt{\frac{EI_0}{\rho A_0}} \tag{68}$$

Once λ_o is provided by ADM, natural vibration frequencies for the rod can be easily found from the equation below.

$$\omega = (\lambda_0 L)^2 \sqrt{\frac{EI_0}{\rho A_0 L^4}} \tag{69}$$

ADM gives the following formulation with the previously defined fourth order linear operator.

$$Y = A + Bx + C\frac{x^2}{2!} + D\frac{x^3}{3!} + L^{-1}\left(8b\psi_1(x)Y''' - 12b^2\psi_2(x)Y'' + \lambda_0^4\psi_2(x)Y\right) \tag{70}$$

2.5.2.1.2 VIM

Lagrange multiplier is the same as used in the uniform beam case due to the fourth order derivative in Eq.(64). Hence an algorithm by using VIM can be constructed as

$$Y_{n+1} = Y_n + \int_0^x \lambda(\xi)\left\{Y_n^{iv} - 8b\psi_1(x)\tilde{Y}_n''' + 12b^2\psi_2(x)\tilde{Y}_n'' - \lambda_0^4\psi_2(x)\tilde{Y}_n\right\}d\xi \tag{71}$$

2.5.2.1.3 HPM

Application of HPM to Eq.(64) produce following set of recursive equations as the solution algorithm.

$$Y_0^{iv} - y_0^{iv} = 0 \tag{72}$$

$$Y_1^{iv} + y_0^{iv} - 8b\psi_1(x)Y_0''' + 12b^2\psi_2(x)Y_0'' - \lambda_0^4\psi_2(x)Y_0 = 0 \tag{73}$$

$$Y_n - 8b\psi_1(x)Y_{n-1}''' + 12b^2\psi_2(x)Y_{n-1}'' - \lambda_0^4\psi_2(x)Y_{n-1} = 0, \quad n \geq 2 \tag{74}$$

2.5.2.2 Results of the analysis

After applying the proposed formulations, the following results are obtained for the natural frequencies and mode shapes. Ten iterations are conducted for each method and computed $\lambda_o L$ values are given for the first three modes of vibration in the following table.

The free vibration mode shapes of the rod for the first three modes are also depicted in the following figures. The mode shapes for predefined five different configurations are given

between Figs. 8-12. To demonstrate the effect of variable cross-section in the results, a comparison is made with normalized mode shapes for a uniform rod which are given between Figs.2-6.

Beam	Mode	ADM	VIM	HPM
P-P	1	2.97061902	2.97061902	2.97061902
	2	5.95530352	5.95530352	5.95530352
	3	8.93099026	8.93099026	8.93099026
C-C	1	4.48292606	4.48292606	4.48292606
	2	7.44076320	7.44076320	7.44076320
	3	10.41682600	10.41682600	10.41682600
C-P	1	3.80402043	3.80402043	3.80402043
	2	6.74289447	6.74289447	6.74289447
	3	9.70480586	9.70480586	9.70480586
C-F	1	1.96344512	1.96344512	1.96344512
	2	4.58876313	4.58876313	4.58876313
	3	7.52531208	7.52531208	7.52531208
C-S	1	2.35500726	2.35500726	2.35500726
	2	5.26125511	5.26125511	5.26125511
	3	8.21783948	8.21783948	8.21783948

Table 2. Comparison of $\lambda_o L$ values for the variable cross-section rod
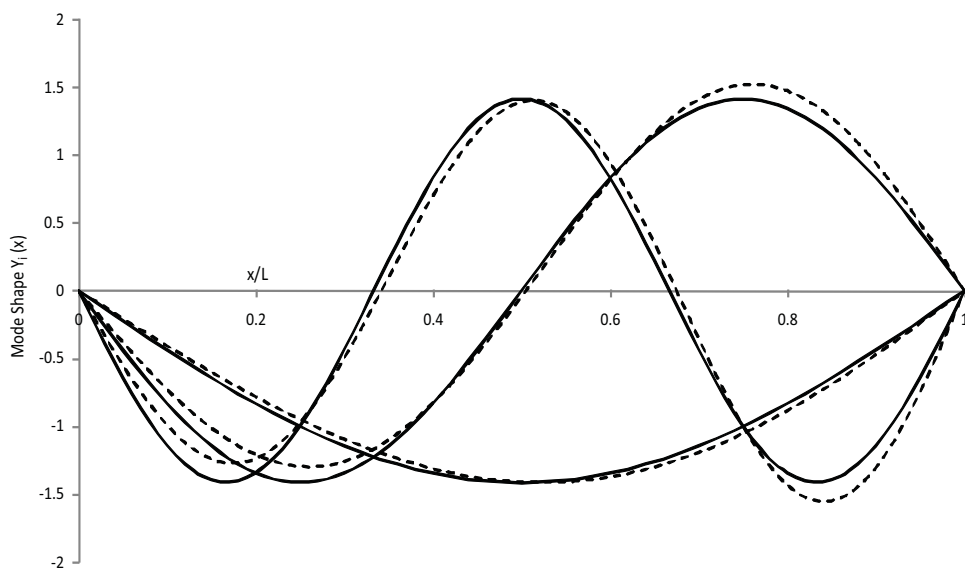
Fig. 8. Free vibration modes of PP rod (- - - -variable cross section ⎯⎯ uniform rod).
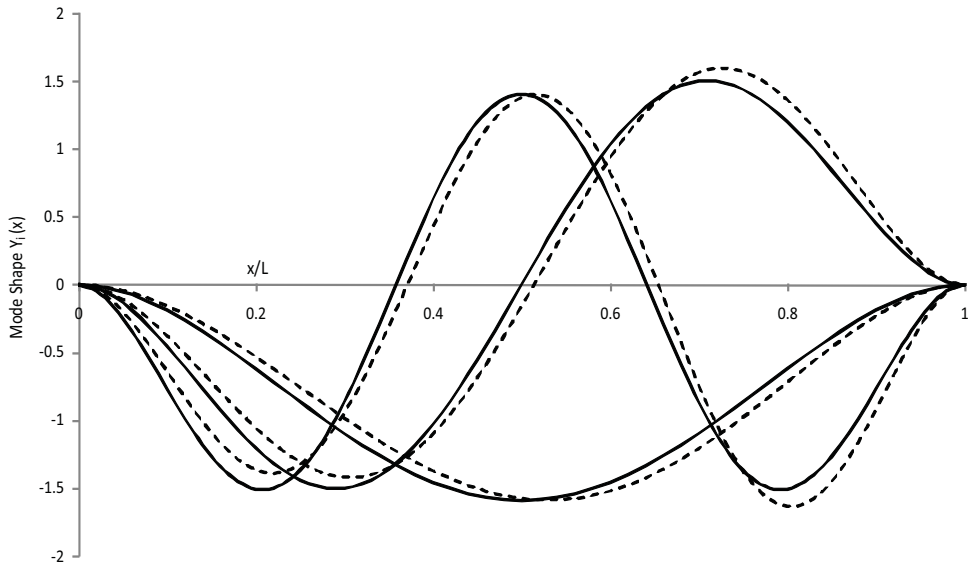
Fig. 9. Free vibration modes of CC rod (_ _ _ _ variable cross section ____ uniform rod).
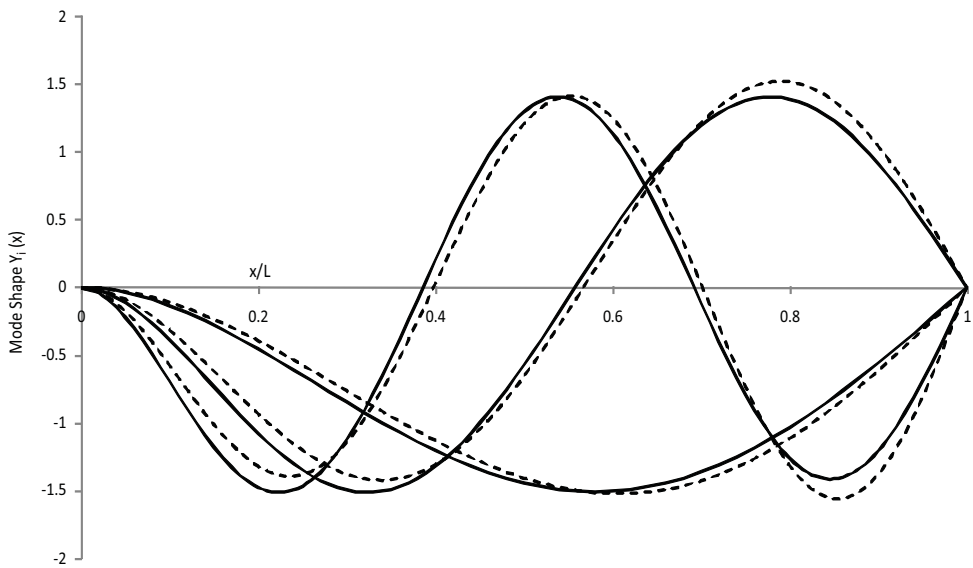
Fig. 10. Free vibration modes of CP rod (_ _ _ _ variable cross section ____ uniform rod).
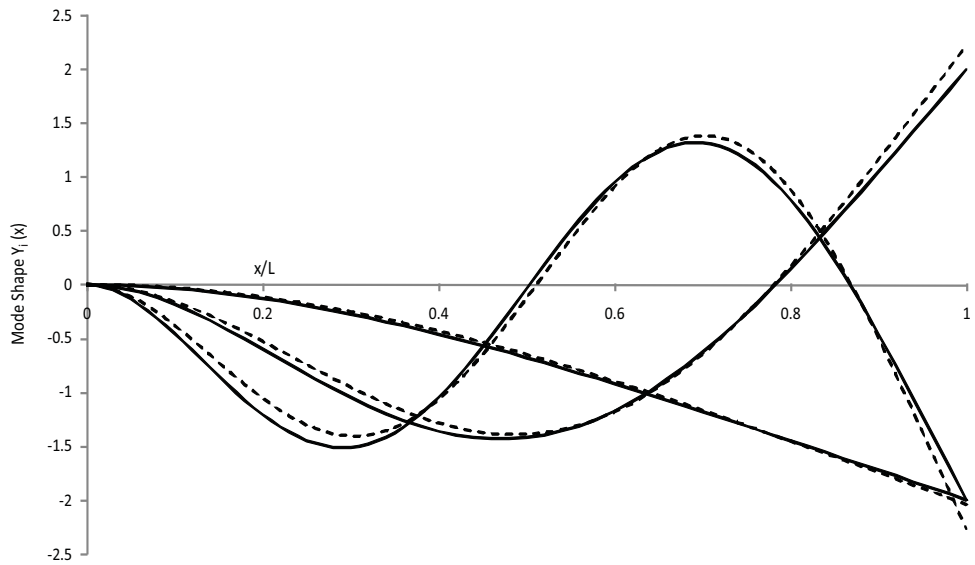
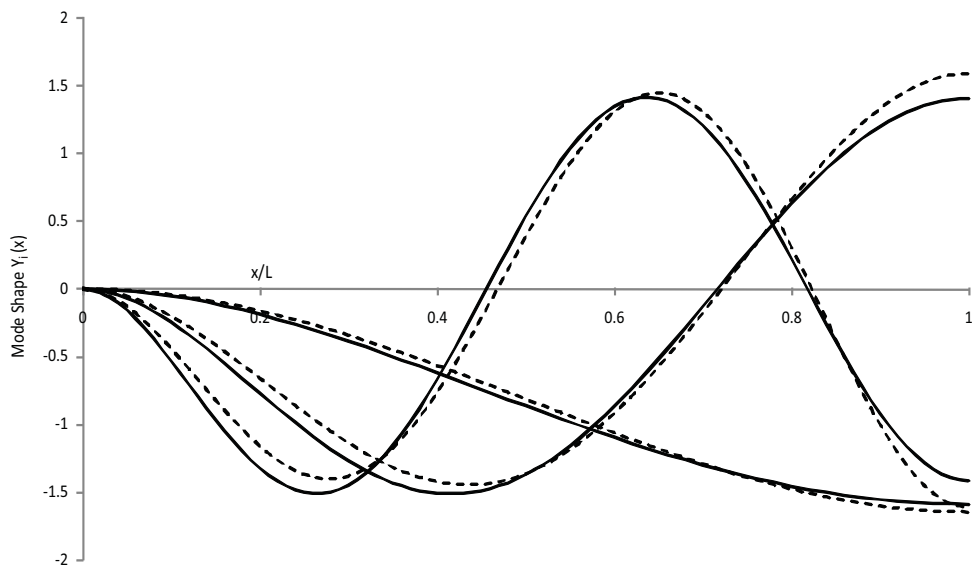Fig. 11. Free vibration modes of CF rod (---- variable cross section ____ uniform rod).

Fig. 12. Free vibration modes of CS rod (----variable cross section —— uniform rod).

3. Conclusion

In this article, some analytical approximation techniques were employed in the transverse vibration analysis of beams. In a variety of such techniques, the most used ones, namely ADM, VIM and HPM were chosen for use in the computations. First, a brief theoretical knowledge was given in the text and then all of the methods were applied to selected cases. Since the exact values for the free vibration of a uniform beam was available, the analyses were started for that case. Results showed an excellent agreement with the exact ones that all three methods were highly effective in the computation of natural frequencies and vibration mode shapes. Orthogonality of the mode shapes was also proven. Finally, ADM, VIM and HPM were applied to the free vibration analysis of a rod having variable cross section. To this aim, a rod with linearly changing radius was chosen and natural frequencies with their corresponding mode shapes were obtained easily.

The study has shown that ADM, VIM and HPM can be used effectively in the analysis of vibration problems. It is possible to construct easy-to-use algorithms which are highly accurate and computationally efficient.

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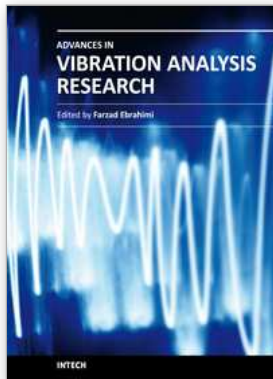
Advances in Vibration Analysis Research

Edited by Dr. Farzad Ebrahimi

Vibrations are extremely important in all areas of human activities, for all sciences, technologies and industrial applications. Sometimes these Vibrations are useful but other times they are undesirable. In any case, understanding and analysis of vibrations are crucial. This book reports on the state of the art research and development findings on this very broad matter through 22 original and innovative research studies exhibiting various investigation directions. The present book is a result of contributions of experts from international scientific community working in different aspects of vibration analysis. The text is addressed not only to researchers, but also to professional engineers, students and other experts in a variety of disciplines, both academic and industrial seeking to gain a better understanding of what has been done in the field recently, and what kind of open problems are in this area.

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