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Some Complicating Effects in the Vibration of Composite Beams

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1. Introduction

In the last 50-60 years, use of composite structures in engineering applications has increased. Due to this fact many studies have been conducted related with composite structures (such as: shells, plates and beams).

Bending, buckling and free vibration analysis of composite structures has taken considerable attention. Beams are one of these structures that are used in mechanical, civil and aeronautical engineering applications (such: robot arms, helicopter rotors and mechanisms). Considering these applications free vibration problem of the composite beams are studied in the previous studies. Kapania & Raciti, 1989 investigated the nonlinear vibrations of un-symmetrically laminated composite beams. Chandashekhara et al., 1990 studied the free vibration of symmetric composite beams. Chandrashekhara & Bangera, 1993 investigated the free vibration of angle-ply composite beams by a higher-order shear deformation theory. They used the shear flexible finite element method. Krishnaswamy et al., 1992 solved the generally layered composite beam vibration problems. Chen et al., 2004 used the state-space based differential quadrature method to study the free vibration of generally laminated composite beams. Solution methods for composite beam vibration problems depend on the boundary conditions, some analytical (Chandrashekhara et al., 1990, Abramovich, 1992, Krishnaswamy et al., 1992, Abramovic & Livshits, 1994, Khdeir & Reddy, 1994, Eisenberger et al., 1995, Marur & Kant, 1996, Kant et al., 1998, Shi & Lam, 1999, Yıldırım et al., 1999, Yıldırım, 2000, Matsunaga, 2001, Kameswara et al., 2001, Banerjee, 2001, Chandrashekhara & Bangera, 1992, Ramtekkar et al., 2002, Murthy et al., 2005, Arya, 2003, Karama et al., 1998, Aydogdu, 2005, 2006) solution procedures have been used.

Many factors can affect the vibrations of beams, in particular the attached springs and masses, axial loads and dampers. This type of complicating effects is considered in the vibration problem of isotropic beams. Gürgöze and his collogues studied vibration of isotropic beam with attached mass, spring and dumpers (Gürgöze, 1986, Gürgöze, 1996, Gürgöze & Erol, 2004). Vibration of Euler-Bernoulli beam carrying two particles and several particles investigated by Naguleswaran, 2001, 2002. Nonlinear vibrations of beam-mass system with different boundary conditions are investigated by Ozkaya & Pakdemirli, 1999, Ozkaya et. al., 1997. They used multiscale perturbation technique in their solutions.

It is interesting to note that, although mass or spring attached composite beams are used or can be used in some engineering applications, their vibration problem is not generally considered in the previous studies. Vibration of symmetrically laminated clamped-free beam with a mass at the free end is studied by Chandrashekhara & Bangera, 1993.
The aim of present study is to fill this gap. Therefore in this study vibration of composite beams with attached mass or springs is studied. After driving equations of motion different boundary conditions, lamination angles, attached mass or spring are considered in detail.

2. Equation of motion

In this study, equations of motion of composite beams will be derived from Classical Laminated Plate Theory (CLPT). For CLPT following displacement field is generally assumed:

$$\begin{aligned} U(x,z;t) &= u(x,t) - zw_{,x} \\ V(x,z;t) &= v(x,t) - zw_{,y} \\ W(x,z;t) &= w(x,t) \end{aligned} \tag{1}$$

where U,V and W are displacement components of a point of the plate in the x, y and z directions respectively and u, v and w are the displacement components of a point of the beam in the midplane again in the x, y and z directions respectively. The comma after a letter denotes partial derivative with respect to x and y. The Hooke's law can be written in the following form using CLPT:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{2}$$

where σ_x and σ_y are the in-plane normal stress components in the x and y directions respectively, τ_{xy} is the shear stress in the x-y plane, ε_x, ε_y and γ_{xy} are normal strains and shear strain respectively and Q_{ij} are the reduced transformed rigidities (Jones, 1975). These strains are defined in the following form:

$$\varepsilon_x = \frac{\partial U}{\partial x}, \quad \varepsilon_y = \frac{\partial V}{\partial y} \qquad \gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \tag{3}$$

Applying Hamilton principle leads to the following equations of motion for laminated composite plate.

$$\begin{aligned} N_{x,x} + N_{xy,y} &= \rho u_{,tt} \\ N_{xy,x} + N_{y,y} &= \rho v_{,tt} \\ M_{x,xx} + 2M_{xy,xy} + M_{y,yy} &= \rho w_{,tt} \end{aligned} \tag{4}$$

where the force and moment resultants are defined in the following form.

$$(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) dz \tag{5}$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z dz \tag{6}$$

These force and moment results can also be written in the following form:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{12} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,x} + v_{,y} \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{bmatrix} \tag{7}$$

where extensional, coupling and bending rigidities are defined as follows:

$$\begin{aligned} A_{ij} &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} dz \\ Bij &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} z dz \\ Dij &= \int_{-h/2}^{h/2} Q_{ij}^{(k)} z^2 dz \end{aligned} \tag{8}$$

Now, consider a laminated composite beam with length L, width b and thickness h. Equations of motion of laminated composite beams can be derived from Eq.(4) assuming $N_y=N_{xy}=M_y=M_{xy}=0$.

$$\begin{aligned} N_{x,x} &= \rho u_{,tt} \\ M_{x,xx} &= \rho w_{,tt} \end{aligned} \tag{9}$$

Eq.(7) can be inverted in the following form:

$$\begin{bmatrix} u_{,x} \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} A^*_{11} & A^*_{12} & A^*_{16} & B^*_{11} & B^*_{12} & B^*_{16} \\ A^*_{12} & A^*_{22} & A^*_{26} & B^*_{12} & B^*_{22} & B^*_{26} \\ A^*_{16} & A^*_{26} & A^*_{66} & B^*_{16} & B^*_{26} & B^*_{66} \\ B^*_{11} & B^*_{12} & B^*_{16} & D^*_{11} & D^*_{12} & D^*_{16} \\ B^*_{12} & B^*_{22} & B^*_{26} & D^*_{12} & D^*_{22} & D^*_{26} \\ B^*_{16} & B^*_{26} & B^*_{66} & D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} \begin{bmatrix} N_x \\ 0 \\ 0 \\ M_x \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

where A_{ij}^* ,B_{ij}^* ,D_{ij}^* are the members of inverse of rigidity matrix given in Eq.(7). Eq.(10) can be written in the following form.

$$u_{,x} = A_{11}^{*} N_x + B_{11}^{*} M_x$$
$$w_{,xx} = B_{11}^{*} N_x + D_{11}^{*} M_x \tag{11}$$

Eq.(11) can be solved in term of N_x and M_x.

$$M_x = \frac{B_{11}^{*}}{[(B_{11}^{*})^2 - (A_{11}^{*} D_{11}^{*})]} u_{,x} + \frac{A_{11}^{*}}{[(B_{11}^{*})^2 - (A_{11}^{*} D_{11}^{*})]} w_{,xx} \tag{12}$$

$$N_x = \frac{B_{11}^{*} u_{,x} - B_{11}^{*\,2}\left[\frac{B_{11}^{*}}{[(B_{11}^{*})^2 - (A_{11}^{*} D_{11}^{*})]} u_{,x} + \frac{A_{11}^{*}}{[(B_{11}^{*})^2 - (A_{11}^{*} D_{11}^{*})]} w_{,xx}\right]}{A_{11}^{*} B_{11}^{*}} \tag{13}$$

Inserting equations (12)-(13) in equation (9) yields to:

$$\bar{A} \frac{\partial^2 u}{\partial x^2} + \bar{B} \frac{\partial^3 w}{\partial x^3} = \rho \frac{\partial^2 u}{\partial t^2}$$
$$\bar{D} \frac{\partial^4 w}{\partial x^4} + \bar{B} \frac{\partial^3 u}{\partial x^3} = -\rho \frac{\partial^2 w}{\partial t^2} \tag{14}$$

Where $\bar{A}$, $\bar{B}$ and $\bar{D}$ are defined in the following form.

$$\bar{A} = \frac{1}{A_{11}^{*}} - \frac{(B_{11}^{*})^2}{[(B_{11}^{*})^2 - (A_{11}^{*} D_{11}^{*})]} \tag{15}$$

$$\bar{B} = -\frac{B_{11}^{*}}{[(B_{11}^{*})^2 - (A_{11}^{*} D_{11}^{*})]} \tag{16}$$

$$\bar{D} = \frac{A_{11}^{*}}{[(B_{11}^{*})^2 - (A_{11}^{*} D_{11}^{*})]} \tag{17}$$

Eqs. (14) are the equations of motion of generally laminated composite beam for the assumptions $N_y=N_{xy}=M_y=M_{xy}=0$. Boundary conditions of the generally laminated composite beams can be written in the following form:

$$\mathrm{S}: \mathrm{w} = \mathrm{M_x} = \mathrm{N_x} = 0$$
$$\mathrm{C}: \mathrm{w} = \mathrm{w}_{,x} = \mathrm{u} = 0 \tag{18}$$
$$\mathrm{F}: \mathrm{M_x} = \mathrm{Q_x} = \mathrm{N_x} = 0$$

2.1 Symmetrically laminated composite beams

For symmetrically laminated composite beams coupling terms B_{ij}'s are zero. Then Eq. (14) takes the following form.

$$\bar{D}\frac{\partial^4 w}{\partial x^4} = -\rho\frac{\partial^2 w}{\partial t^2} \tag{19}$$

General solution of Eq.(19) can be written in the following form:

$$w(x) = A\sin(\Omega x) + B\cos(\Omega x) + C\sinh(\Omega x) + D\cosh(\Omega x) \tag{20}$$

Where A,B,C and D are undetermined coefficients, $\Omega^4 = \rho\omega^2 L^4 / E_2 h^3$ is non-dimensional frequency parameter. Using boundary conditions given in Eq.(18) following Eigenvalue determinants are obtained for different boundary conditions:

H-H boundary condition:

Following condition exists between undetermined coefficients given in Eq.(20): B=D=0

$$\begin{vmatrix} \sin(\Omega) & \sinh(\Omega) \\ -\Omega^2\sin(\Omega) & \Omega^2\sinh(\Omega) \end{vmatrix} = 0 \tag{21}$$

C-H boundary condition:

Following condition exists between undetermined coefficients given in Eq.(20): D=-B, C=-A:

$$\begin{vmatrix} \sin(\Omega)-\sinh(\Omega) & \cos(\Omega)-\cosh(\Omega) \\ -\Omega^2\sin(\Omega)-\Omega^2\sinh(\Omega) & -\Omega^2\cos(\Omega)-\Omega^2\cosh(\Omega) \end{vmatrix} = 0 \tag{22}$$

C-C boundary condition:

Following condition exists between undetermined coefficients given in Eq.(20): D=-B, C=-A:

$$\begin{vmatrix} \sin(\Omega)-\sinh(\Omega) & \cos(\Omega)-\cosh(\Omega) \\ \Omega\cos(\Omega)-\Omega\cosh(\Omega) & -\Omega\sin(\Omega)-\Omega\sinh(\Omega) \end{vmatrix} = 0 \tag{23}$$

C-F boundary condition:

Following condition exists between undetermined coefficients given in Eq.(20): D=-B, C=-A

$$\begin{vmatrix} -\Omega^2\sin(\Omega)-\Omega^2\sinh(\Omega) & -\Omega^2\cos(\Omega)-\Omega^2\cosh(\Omega) \\ -\Omega^3\cos(\Omega)-\Omega^3\cosh(\Omega) & -\Omega^3\sin(\Omega)-\Omega^3\sinh(\Omega) \end{vmatrix} = 0 \tag{24}$$

F-F boundary condition:

Following condition exists between undetermined coefficients given in Eq.(20): D=B, C=A:

$$\begin{vmatrix} -\Omega^2\sin(\Omega) & \Omega^2\sinh(\Omega) \\ -\Omega^2\cos(\Omega) & \Omega^2\cosh(\Omega) \end{vmatrix} = 0 \tag{25}$$

H-F boundary condition:

Following condition exists between undetermined coefficients given in Eq.(20): B=D=0:

$$\begin{vmatrix} -\Omega^2\sin(\Omega)+\Omega^2\sinh(\Omega & -\Omega^2\cos(\Omega)+\Omega^2\cosh(\Omega) \\ -\Omega^3\cos(\Omega)+\Omega^3\cosh(\Omega) & \Omega^3\sin(\Omega)+\Omega^3\sinh(\Omega) \end{vmatrix} = 0 \tag{26}$$

Solution of each determinant equation given in Eq.(21)-Eq.(26) gives frequency parameter of symmetrically laminated composite beams.

2.2 Symmetrically laminated beams with attached mass or spring

Now consider a symmetrically laminated composite beam with attached mass or spring (figure 1). Where η is length of first part of the beam. In order to investigate vibration of two portion composite beam Eq.(20) is written for each portion in the following form:

$$\begin{aligned} w_1(x) &= A_1\sin(\Omega x) + B_1\cos(\Omega x) + C_1\sinh(\Omega x) + D_1\cosh(\Omega x) \\ w_2(x) &= A_2\sin(\Omega x) + B_2\cos(\Omega x) + C_2\sinh(\Omega x) + D_2\cosh(\Omega x) \end{aligned} \tag{27}$$

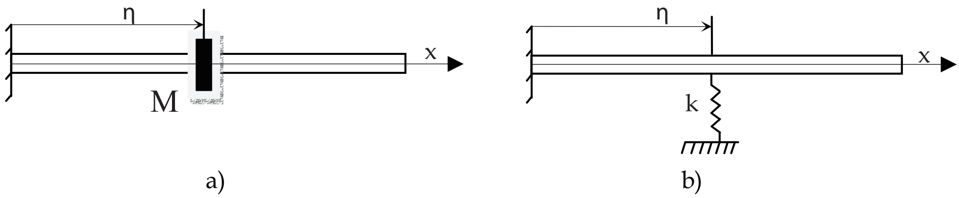

Fig. 1. Composite beam with attached mass (a) and spring (b).

Continuity conditions of the beam at x=η can be written in the following form:

$$\begin{aligned} & w_1(\eta,t) = w_2(\eta,t), \\ & w_1^{'}(\eta,t) = w_2^{'}(\eta,t) \\ & w_1^{''}(\eta,t) = w_2^{''}(\eta,t), \\ & w_1^{'''}(\eta,t) - w_2^{'''}(\eta,t) + \alpha_m\Omega^4 w_1(\eta,t) = 0 \qquad \text{(mass)} \\ & \Omega^3 w_1^{'''}(\eta,t) - \Omega^3 w_2^{'''}(\eta,t) - \alpha_s w_1(\eta,t) = 0 \qquad \text{(spring)} \end{aligned} \tag{28}$$

Where dimensionless mass and spring parameter are defined in the following form:

$$\alpha_m = \frac{M}{\rho_0 L}, \quad \alpha_s = \frac{kL}{AE}$$

Using boundary conditions Eq.(18) and continuity conditions Eq.(28) following equations are obtained for different boundary conditions and composite beams with attached mass and spring at different position.

H-H boundary condition:

Following condition exists between undetermined coefficients given in Eq.(27): B_1=D_1=0:

$$\begin{vmatrix} 0 & 0 & S(\Omega) & C(\Omega) & Sh(\Omega) & Ch(\Omega) \\ 0 & 0 & -S(\Omega) & -C(\Omega) & Sh(\Omega) & Ch(\Omega) \\ S(\eta\Omega) & Sh(\eta\Omega) & -S(\eta\Omega) & -C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ C(\eta\Omega) & Ch(\eta\Omega) & -C(\eta\Omega) & -S(\eta\Omega) & -Ch(\eta\Omega) & -S(\eta\Omega) \\ -S(\eta\Omega) & Sh(\eta\Omega) & S(\eta\Omega) & C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ A61 & A62 & A63 & A64 & A65 & A66 \end{vmatrix} = 0 \tag{29}$$

With spring

$$A61 = -\Omega^3 C(\eta\Omega) - \alpha_S S(\eta\Omega)$$
$$A62 = \Omega^3 Ch(\eta\Omega) - \alpha_S Sh(\eta\Omega)$$
$$A63 = \Omega^3 C(\eta\Omega)$$
$$A64 = -\Omega^3 S(\eta\Omega)$$
$$A65 = -\Omega^3 Ch(\eta\Omega)$$
$$A66 = -\Omega^3 Sh(\eta\Omega)$$

With mass

$$A61 = -C(\eta\Omega) + \alpha_m \Omega S(\eta\Omega)$$
$$A62 = Ch(\eta\Omega) + \alpha_m \Omega Sh(\eta\Omega)$$
$$A63 = Ch(\eta\Omega)$$
$$A64 = -S(\eta\Omega)$$
$$A65 = -Ch(\eta\Omega)$$
$$A66 = -Sh(\eta\Omega)$$

H-C boundary condition:
Following condition exists between undetermined coefficients given in Eq.(27): $B_1=D_1=0$:

$$\begin{vmatrix} 0 & 0 & S(\Omega) & C(\Omega) & Sh(\Omega) & Ch(\Omega) \\ 0 & 0 & C(\Omega) & -S(\Omega) & Ch(\Omega) & Sh(\Omega) \\ S(\eta\Omega) & Sh(\eta\Omega) & -S(\eta\Omega) & -C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ C(\eta\Omega) & Ch(\eta\Omega) & -C(\eta\Omega) & -S(\eta\Omega) & -Ch(\eta\Omega) & -S(\eta\Omega) \\ -S(\eta\Omega) & Sh(\eta\Omega) & S(\eta\Omega) & C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ A61 & A62 & A63 & A64 & A65 & A66 \end{vmatrix} = 0 \quad (30)$$

With mass

$$A61 = -C(\eta\Omega) + \alpha_m \Omega S(\eta\Omega)$$
$$A62 = Ch(\eta\Omega) + \alpha_m \Omega Sh(\eta\Omega)$$
$$A63 = C(\eta\Omega) \qquad A64 = -S(\eta\Omega)$$
$$A65 = -Ch(\eta\Omega) \qquad A66 = -Sh(\eta\Omega)$$

With spring

$$A61 = -\Omega^3 C(\eta\Omega) - \alpha_S S(\eta\Omega)$$
$$A62 = \Omega^3 Ch(\eta\Omega) - \alpha_S Sh(\eta\Omega)$$
$$A63 = \Omega^3 C(\eta\Omega) \qquad A64 = -\Omega^3 S(\eta\Omega)$$
$$A65 = -\Omega^3 Ch(\eta\Omega) \qquad A66 = -\Omega^3 Sh(\eta\Omega)$$

C-C boundary condition:
Following condition exists between undetermined coefficients given in Eq.(27): $D_1=-B_1$, $C_1=-A_1$:

$$\begin{vmatrix} 0 & 0 & S(\Omega) & C(\Omega) & Sh(\Omega) & Ch(\Omega) \\ 0 & 0 & C(\Omega) & -S(\Omega) & Ch(\Omega) & Sh(\Omega) \\ S(\eta\Omega)-Sh(\eta\Omega) & C(\eta\Omega)-Ch(\eta\Omega) & -S(\eta\Omega) & -C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ C(\eta\Omega)-Ch(\eta\Omega) & -S(\eta\Omega)-Sh(\eta\Omega) & -C(\eta\Omega) & -S(\eta\Omega) & -Ch(\eta\Omega) & -S(\eta\Omega) \\ -S(\eta\Omega)-Sh(\eta\Omega) & -C(\eta\Omega)-Ch(\eta\Omega) & S(\eta\Omega) & C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ A61 & A62 & A63 & A64 & A65 & A66 \end{vmatrix} = 0 \quad (31)$$

With mass

$$A61 = -C(\eta\Omega) - Ch(\eta\Omega) + \alpha_m \Omega S(\eta\Omega) - \alpha_m \Omega Sh(\eta\Omega)$$
$$A62 = S(\eta\Omega) - Sh(\eta\Omega) + \alpha_m \Omega C(\eta\Omega) - \alpha_m \Omega Ch(\eta\Omega)$$
$$A63 = C(\eta\Omega) \qquad A64 = -S(\eta\Omega)$$
$$A65 = -Ch(\eta\Omega) \qquad A66 = -Sh(\eta\Omega)$$

With spring

$$A61 = -\Omega^3 C(\eta\Omega) - \Omega^3 Ch(\eta\Omega) - \alpha_s S(\eta\Omega) + \alpha_s Sh(\eta\Omega)$$
$$A62 = \Omega^3 S(\eta\Omega) - \Omega^3 Sh(\eta\Omega) - \alpha_s C(\eta\Omega) + \alpha_s Ch(\eta\Omega)$$
$$A63 = \Omega^3 C(\eta\Omega) \qquad A64 = -\Omega^3 S(\eta\Omega)$$
$$A65 = -\Omega^3 Ch(\eta\Omega) \qquad A66 = -\Omega^3 Sh(\eta\Omega)$$

C-F boundary condition:
Following condition exists between undetermined coefficients given in Eq.(27): $D_1=-B_1$, $C_1=-A_1$:

$$\begin{vmatrix} 0 & 0 & -S(\Omega) & -C(\Omega) & Sh(\Omega)) & Ch(\Omega) \\ 0 & 0 & -C(\Omega) & S(\Omega) & Ch(\Omega) & Sh(\Omega) \\ S(\eta\Omega) - Sh(\eta\Omega) & C(\eta\Omega) - Ch(\eta\Omega) & -S(\eta\Omega) & -C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ C(\eta\Omega) - Ch(\eta\Omega) & -S(\eta\Omega) - Sh(\eta\Omega) & -C(\eta\Omega) & -S(\eta\Omega) & -Ch(\eta\Omega) & -S(\eta\Omega) \\ -S(\eta\Omega) - Sh(\eta\Omega) & -C(\eta\Omega) - Ch(\eta\Omega) & S(\eta\Omega) & C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ A61 & A62 & A63 & A64 & A65 & A66 \end{vmatrix} = 0 \qquad (32)$$

With mass

$$A61 = -C(\eta\Omega) - Ch(\eta\Omega) + \alpha_m \Omega S(\eta\Omega) - \alpha_m \Omega Sh(\eta\Omega)$$
$$A62 = S(\eta\Omega) - Sh(\eta\Omega) + \alpha_m \Omega C(\eta\Omega) - \alpha_m \Omega Ch(\eta\Omega)$$
$$A63 = C(\eta\Omega)$$
$$A64 = -S(\eta\Omega)$$
$$A65 = -Ch(\eta\Omega)$$
$$A66 = -Sh(\eta\Omega)$$

With spring

$$A61 = -\Omega^3 C(\eta\Omega) - \Omega^3 Ch(\eta\Omega) - \alpha_s S(\eta\Omega) + \alpha_s Sh(\eta\Omega)$$
$$A62 = \Omega^3 S(\eta\Omega) - \Omega^3 Sh(\eta\Omega) - \alpha_s C(\eta\Omega) + \alpha_s Ch(\eta\Omega)$$
$$A63 = \Omega^3 C(\eta\Omega)$$
$$A64 = -\Omega^3 S(\eta\Omega)$$
$$A65 = -\Omega^3 Ch(\eta\Omega)$$
$$A66 = -\Omega^3 Sh(\eta\Omega)$$

F-F boundary condition
Following condition exists between undetermined coefficients given in Eq.(27): $D_1=B_1$, $C_1=A_1$:

$$\begin{vmatrix} 0 & 0 & -S(\Omega) & -C(\Omega) & Sh(\Omega)) & Ch(\Omega) \\ 0 & 0 & -C(\Omega) & S(\Omega) & Ch(\Omega) & Sh(\Omega) \\ S(\eta\Omega) + Sh(\eta\Omega) & C(\eta\Omega) + Ch(\eta\Omega) & -S(\eta\Omega) & -C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ C(\eta\Omega) + Ch(\eta\Omega) & -S(\eta\Omega) + Sh(\eta\Omega) & -C(\eta\Omega) & S(\eta\Omega) & -Ch(\eta\Omega) & -Sh(\eta\Omega) \\ -S(\eta\Omega) + Sh(\eta\Omega) & -C(\eta\Omega) + Ch(\eta\Omega) & S(\eta\Omega) & C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ A61 & A62 & A63 & A64 & A65 & A66 \end{vmatrix} = 0 \qquad (33)$$

With mass

$$A61 = -C(\eta\Omega) - Ch(\eta\Omega) + \alpha_m \Omega S(\eta\Omega) + \alpha_m \Omega Sh(\eta\Omega)$$

$$A62 = S(\eta\Omega) - Sh(\eta\Omega) + \alpha_m \Omega C(\eta\Omega) + \alpha_m \Omega Ch(\eta\Omega)$$

$$A63 = C(\eta\Omega) \qquad A64 = -S(\eta\Omega)$$

$$A65 = -Ch(\eta\Omega) \qquad A66 = -Sh(\eta\Omega)$$

With spring

$$A61 = -\Omega^3 C(\eta\Omega) + \Omega^3 Ch(\eta\Omega) - \alpha_s S(\eta\Omega) - \alpha_s Sh(\eta\Omega)$$

$$A62 = \Omega^3 S(\eta\Omega) + \Omega^3 Sh(\eta\Omega) - \alpha_s C(\eta\Omega) + \alpha_s Ch(\eta\Omega)$$

$$A63 = \Omega^3 C(\eta\Omega) \qquad A64 = -\Omega^3 S(\eta\Omega)$$

$$A65 = -\Omega^3 Ch(\eta\Omega) \qquad A66 = -\Omega^3 Sh(\eta\Omega)$$

H-F boundary condition:
Following condition exists between undetermined coefficients given in Eq.(27): $B_1=D_1=0$:

$$\begin{vmatrix} 0 & 0 & -S(\Omega) & -C(\Omega) & Sh(\Omega)) & Ch(\Omega) \\ 0 & 0 & -C(\Omega) & S(\Omega) & Ch(\Omega) & Sh(\Omega) \\ S(\eta\Omega) & Sh(\eta\Omega) & -S(\eta\Omega) & -C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ C(\eta\Omega) & Ch(\eta\Omega) & -C(\eta\Omega) & S(\eta\Omega) & -Ch(\eta\Omega) & -Sh(\eta\Omega) \\ -S(\eta\Omega) & Sh(\eta\Omega) & S(\eta\Omega) & C(\eta\Omega) & -Sh(\eta\Omega) & -Ch(\eta\Omega) \\ A61 & A62 & A63 & A64 & A65 & A66 \end{vmatrix} = 0 \tag{34}$$

With mass

$$A61 = -C(\eta\Omega) + \alpha_m \Omega S(\eta\Omega)$$

$$A62 = Ch(\eta\Omega) + \alpha_m \Omega Sh(\eta\Omega)$$

$$A63 = C(\eta\Omega)$$

$$A64 = -S(\eta\Omega)$$

$$A65 = -Ch(\eta\Omega)$$

$$A66 = -Sh(\eta\Omega)$$

With spring

$$A61 = -\Omega^3 C(\eta\Omega) - \alpha_s S(\eta\Omega)$$

$$A62 = \Omega^3 Ch(\eta\Omega) - \alpha_s Sh(\eta\Omega)$$

$$A63 = \Omega^3 C(\eta\Omega)$$

$$A64 = -\Omega^3 S(\eta\Omega)$$

$$A65 = -\Omega^3 Ch(\eta\Omega)$$

$$A66 = -\Omega^3 Sh(\eta\Omega)$$

Solution of each determinant equation given in Eq.(29)-Eq.(34) gives frequency parameter of symmetrically laminated beams with attached point mass or spring at the different location of the beam.

3. Numerical results

In this section, firstly, numerical results are given for vibration of composite beams with or without attached mass or springs. In order to check validity of present results first five flexural vibration frequencies of laminated composite beams are compared with previous results (Reddy, 1997) and good agreement is observed between two results. After checking validity of present formulation, vibration of composite beams with attached mass or spring is investigated for different boundary conditions. Material properties are chosen as: $E_1=25E_2$, $G_{12}=0.5E_2$ and $\nu_{12}=0.3$. Obtained parametrical results are given in figures. In order to completeness of present study, first five frequency of symmetric three layer (θ/-θ/θ) composite beams are given in Fig.2. According to Fig. 2, dimensionless frequency parameters decrease with increasing lamination angle θ. This is due to decrease in rigidities D_{ij} with increasing θ. The frequency gap is narrowing for higher θ, so this type of beams should be carefully designed. Highest frequencies are obtained for C-C and F-F boundary conditions where as lowest one is obtained for C-F boundary condition.

Variation of frequency ratio of composite beams with attached mass to composite beam without mass (Ω_m/Ω_0) is depicted in Fig.3 for different boundary conditions. According to this figure, ratio of frequencies is insensitive to lamination angle θ. The lowest frequencies generally are most affected by attached mass. Influence of attached mass is decreasing with increasing mode number. This fact can be explained by considering mode shapes of vibrating composite beams. For H-H, C-C, H-C and F-F beams η=0.25 is a nodal point for fourth frequency, therefore this frequency is not affected by attached mass as expected. Highest %40 and lowest %20 changes are observed for frequencies for different boundary conditions.

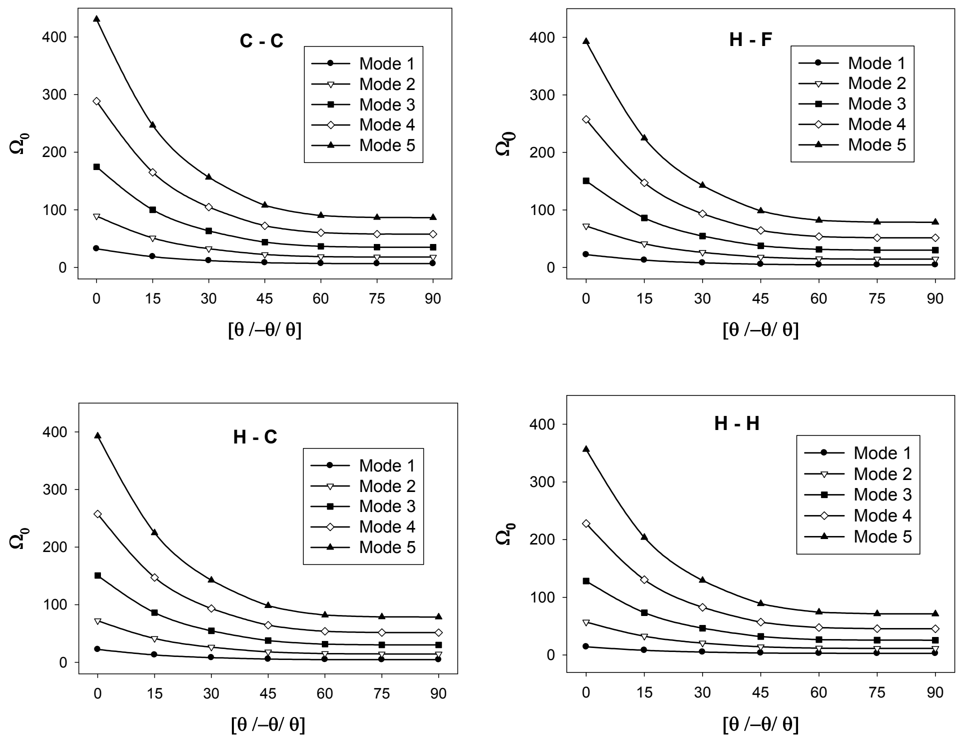

Fig. 2. Variation of frequency parameter of composite beam with lamination angle θ.

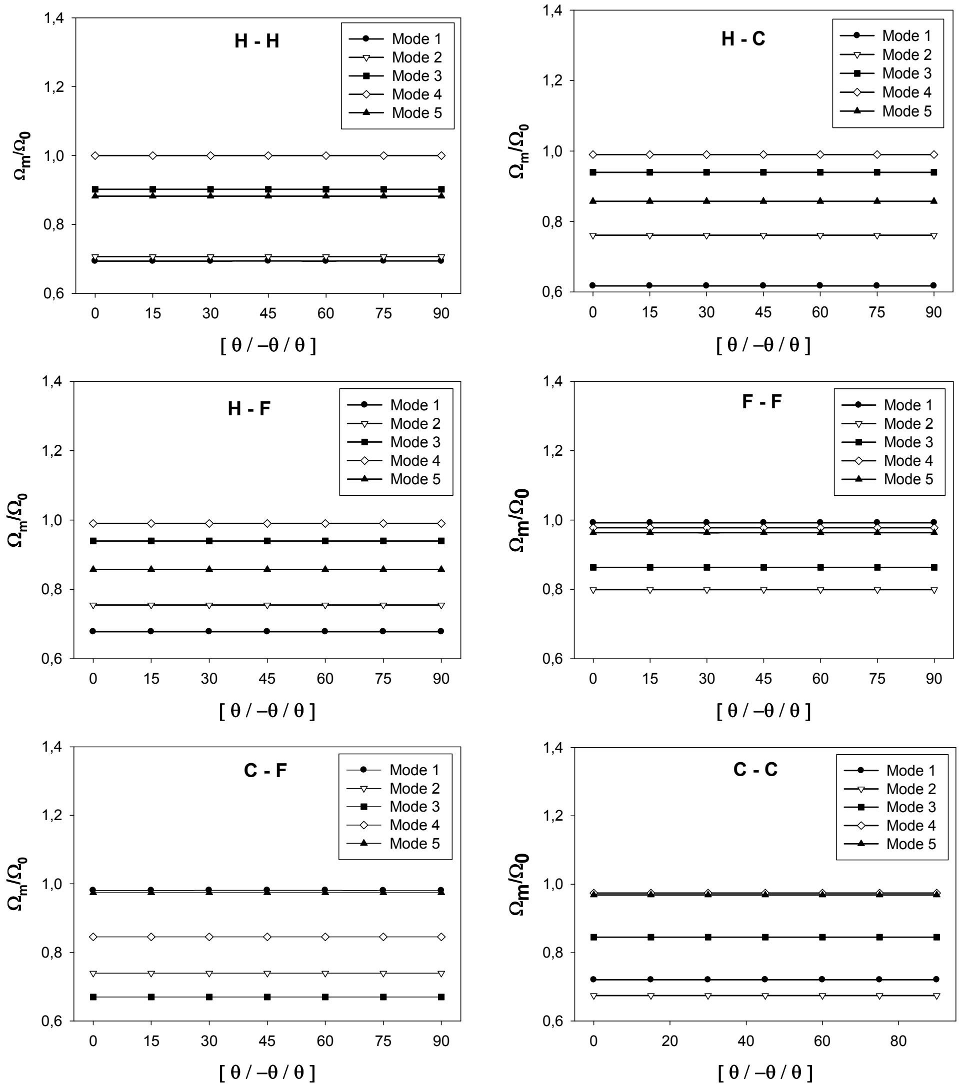

Fig. 3. Variation of frequency ratio of composite beam with lamination angle for α_m=1 and η=0,25 .

Variation of frequency ratio of composite beams with attached spring to composite beam without spring (Ω_s/Ω_0) is given in Fig.4 for different boundary conditions. According to this figure, ratio of frequencies is insensitive to lamination angle θ. Effect of attached spring on the frequency ratio is negligible for composite beams with at least one clamped edge. The beams with F-F and H-F boundary conditions are most affected by attached mass. For these boundary conditions spring behaves like a hinged boundary condition and decreases frequency of composite beam.

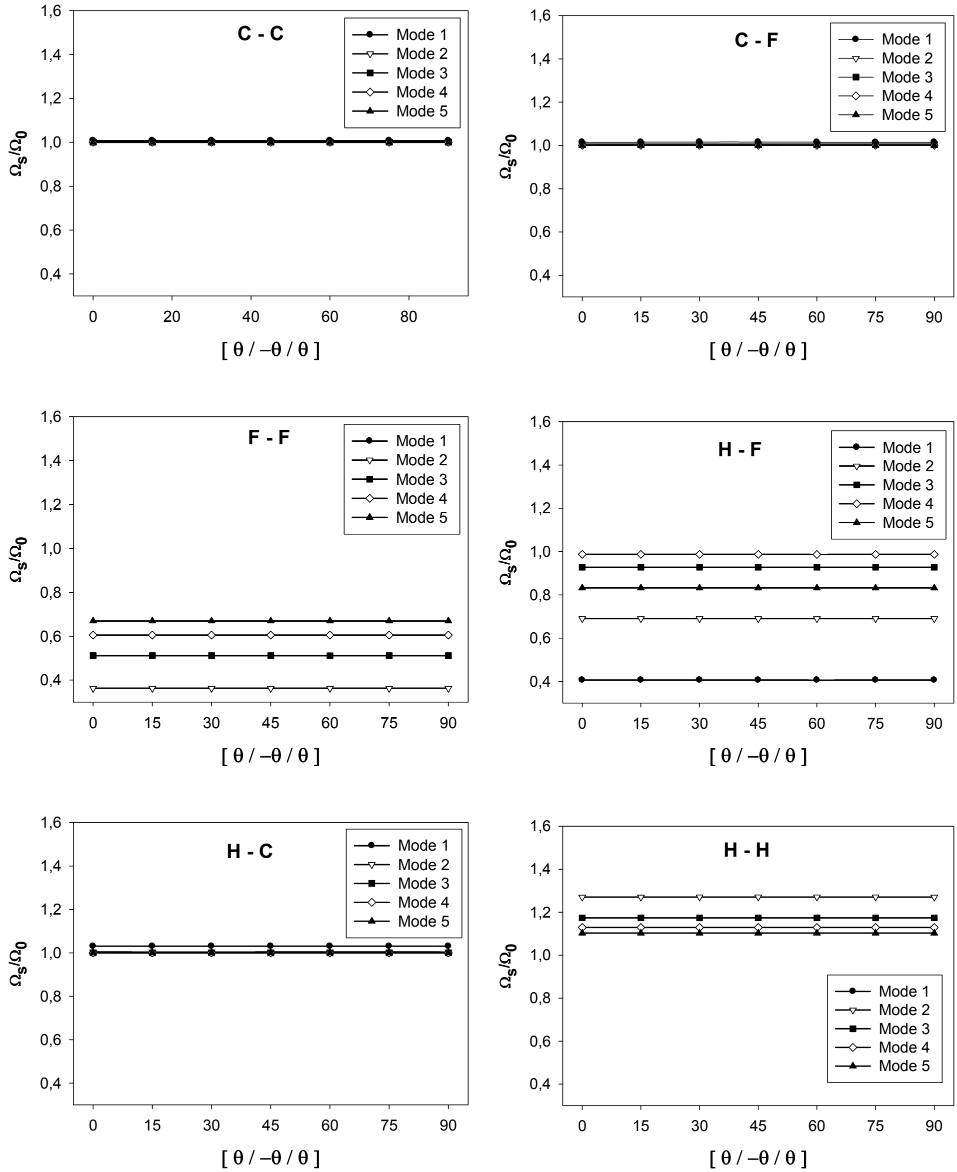

Fig. 4. Variation of frequency ratio of symmetric angle-ply composite beam with lamination angle for α_s=10 and η=0,25.

In Fig. 5, variation of frequency ratio with α_m is given for three layer symmetric angle-ply (30^0/-30^0/30^0) composite beams. Increasing α_m decreases frequency of the composite beam. Different decreases are observed for different boundary conditions.

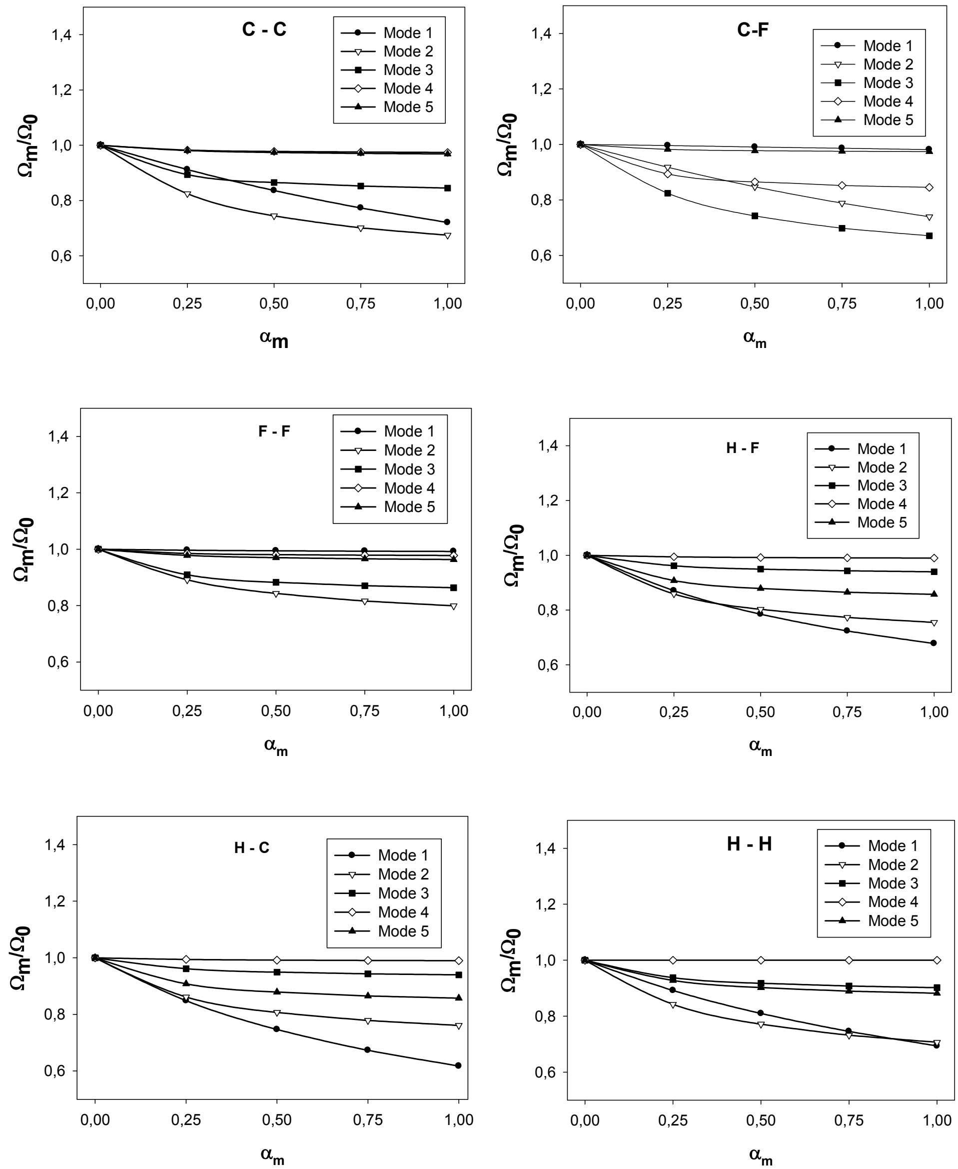

Fig. 5. Variation of frequency ratio of symmetric angle-ply composite beam ($30^0/-30^0/30^0$) with α_m for η=0,25.

Variation of frequency ratio with α_s is given in Fig. 6 for three layer symmetric angle-ply ($30^0/-30^0/30^0$) composite beams. Increasing α_s decreases frequency of the composite beam for F-F and H-F boundary conditions. For these two boundary conditions zero frequencies exist for rigid body motions. Attaching a spring prevents from rigid body motion and these zero frequencies turn two none zero frequencies. Other boundary conditions are insensitive to increase of α_s for given range.

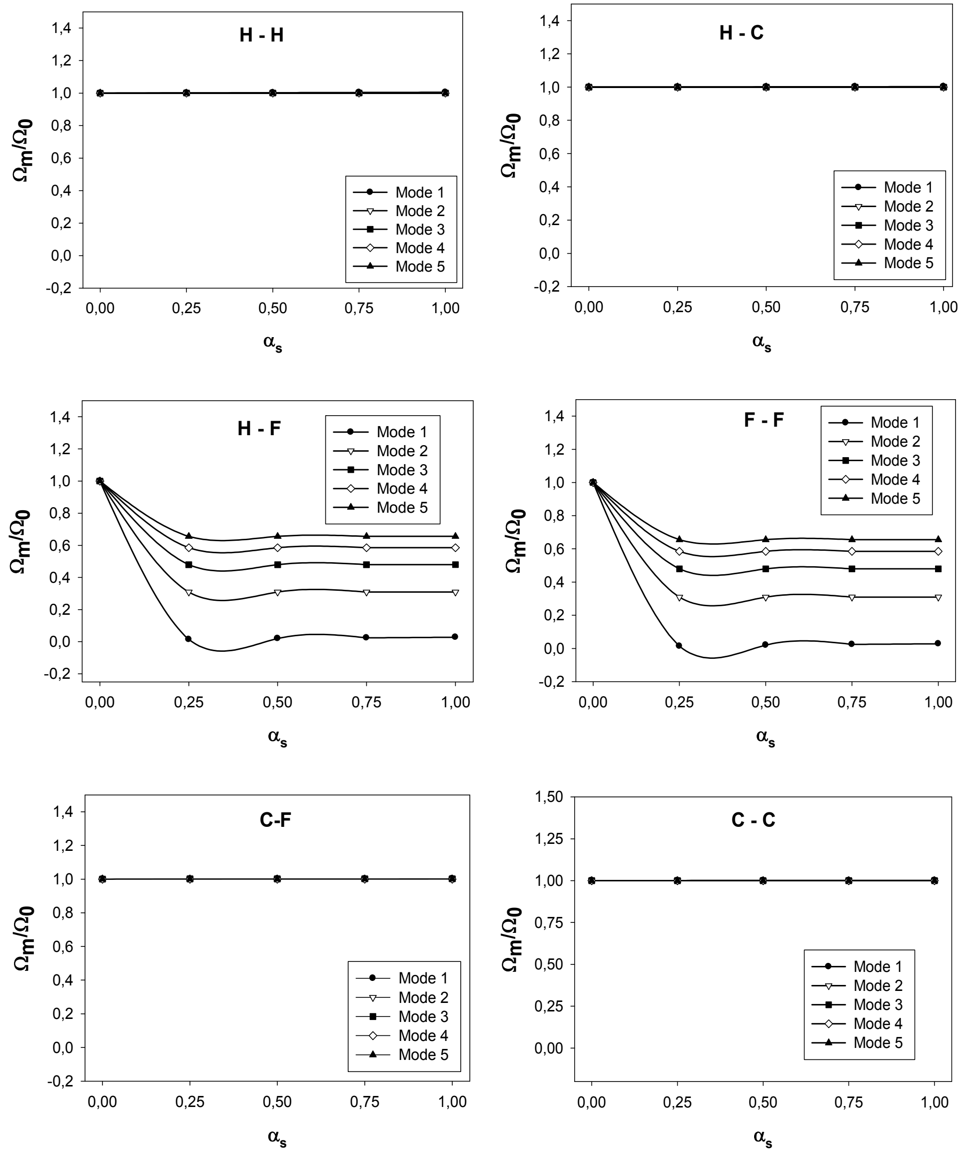

Fig. 6. Variation of frequency ratio of symmetric angle-ply composite beam ($30^0/-30^0/30^0$) with α_s for η=0,25.

In Fig. 7, variation of frequency ratio of composite beam with η for α_m=1 and η=0.25 are given for three layer symmetric angle-ply ($30^0/-30^0/30^0$) composite beams. Generally, lower frequencies are most affected by position of attached mass. Forth frequency is not affected by position of attached mass for boundary conditions other than F-F and F-H. This is due to nodal points coincides with position of attached masses.

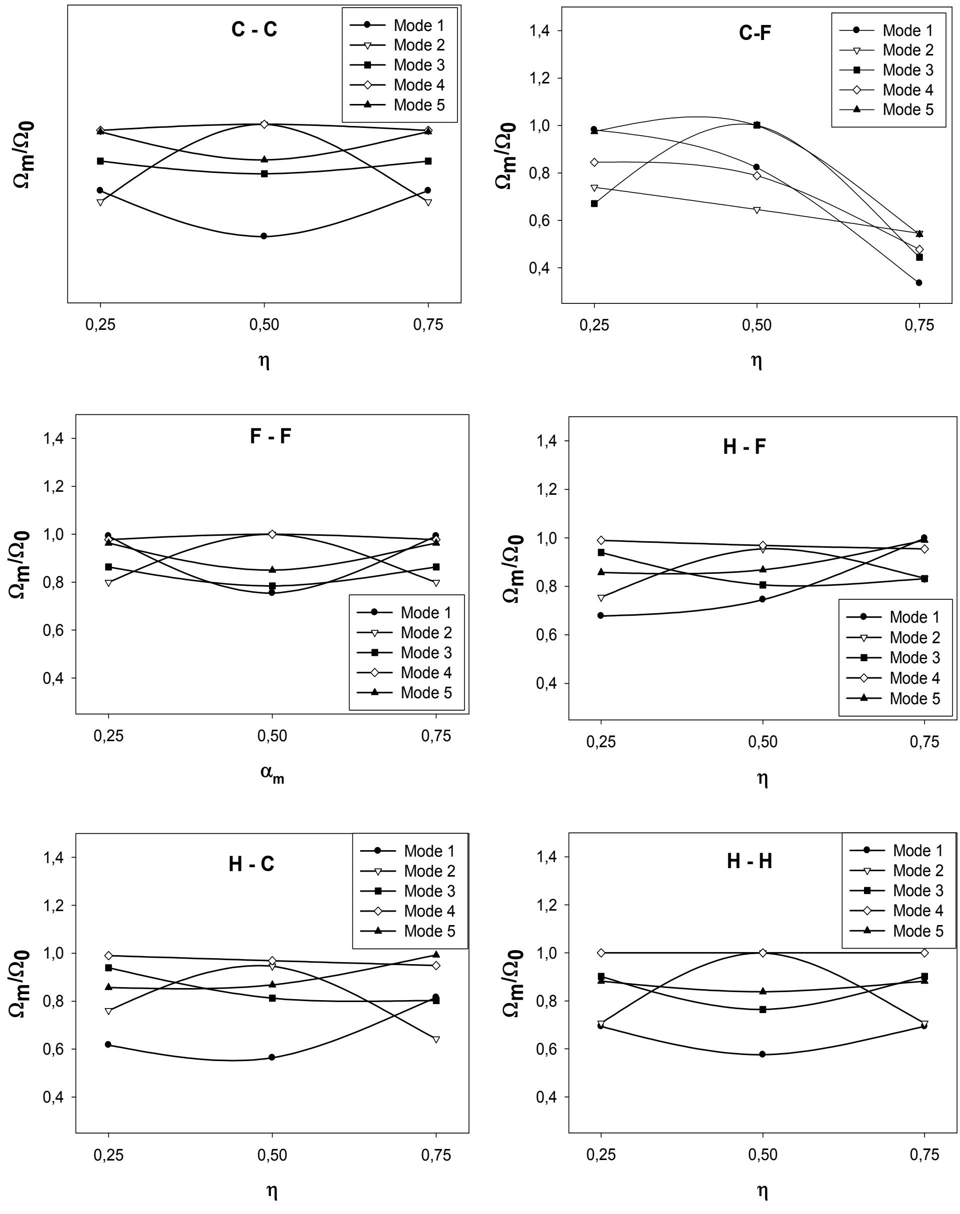

Fig. 7. Variation of frequency ratio of symmetric angle-ply composite beam ($30^0/-30^0/30^0$) with η for α_m=1and η=0,25.

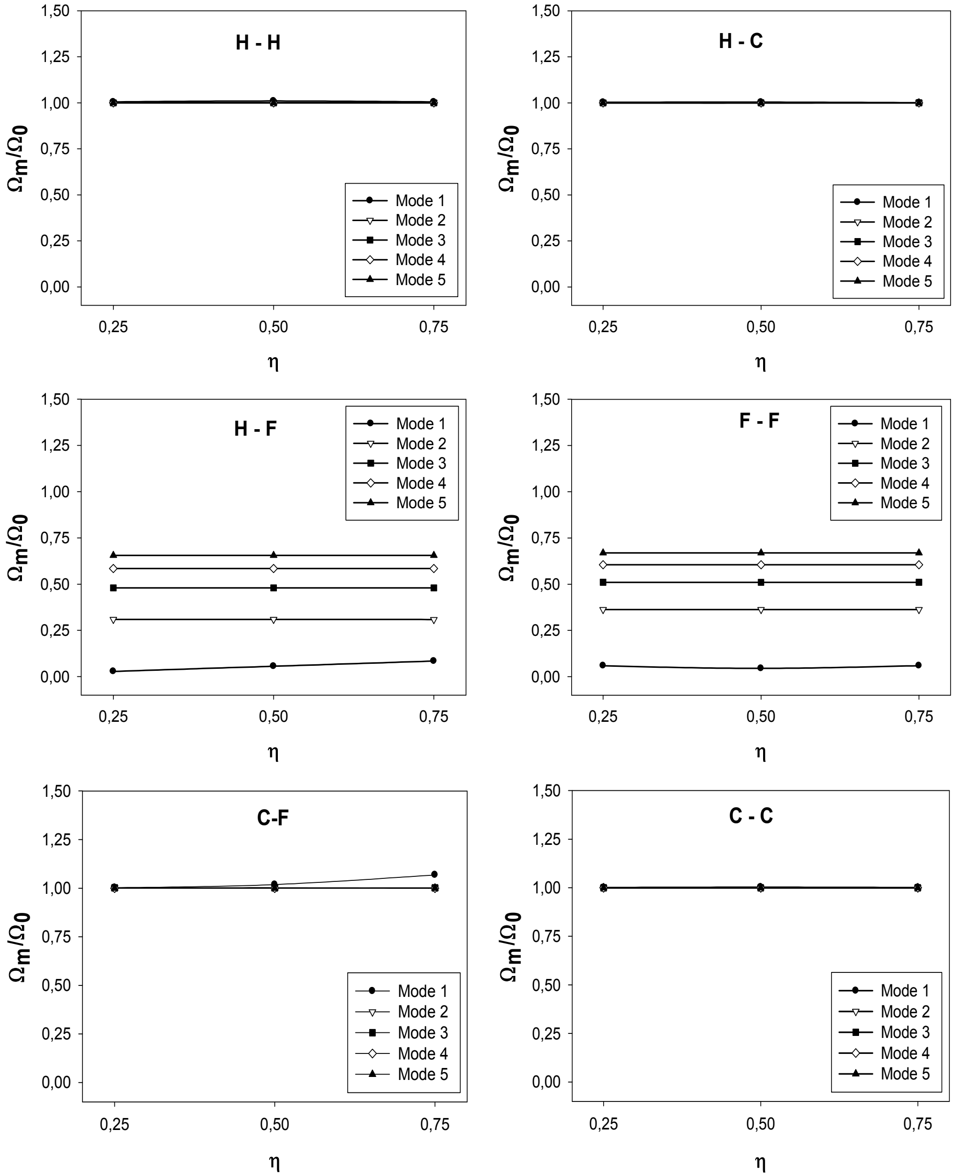

Fig. 8. Variation of frequency ratio of symmetric angle-ply composite beam ($30^0/-30^0/30^0$) with η for α_s=1 and η=0,25.

Variation of frequency ratio of composite beam with η for α_s=1 and η=0.25 are given in Fig. 8 for three layer symmetric angle-ply ($30^0/-30^0/30^0$) composite beams. Similar to Fig. 7 generally lower frequencies are most affected by position of attached spring. Forth frequency is not affected by position of attached spring for boundary conditions other than F-F and F-H. This is due to nodal points coincides with position of attached spring.

In Fig. 9-10, variation of frequency parameter of composite beam with lamination angle for α_s=1, α_m=1 and η=0.25 for different number of layers (single, three and four layer) are given respectively. First frequencies are insensitive to number of layers but for the fourth frequencies higher frequencies are obtained with increasing number of layers.

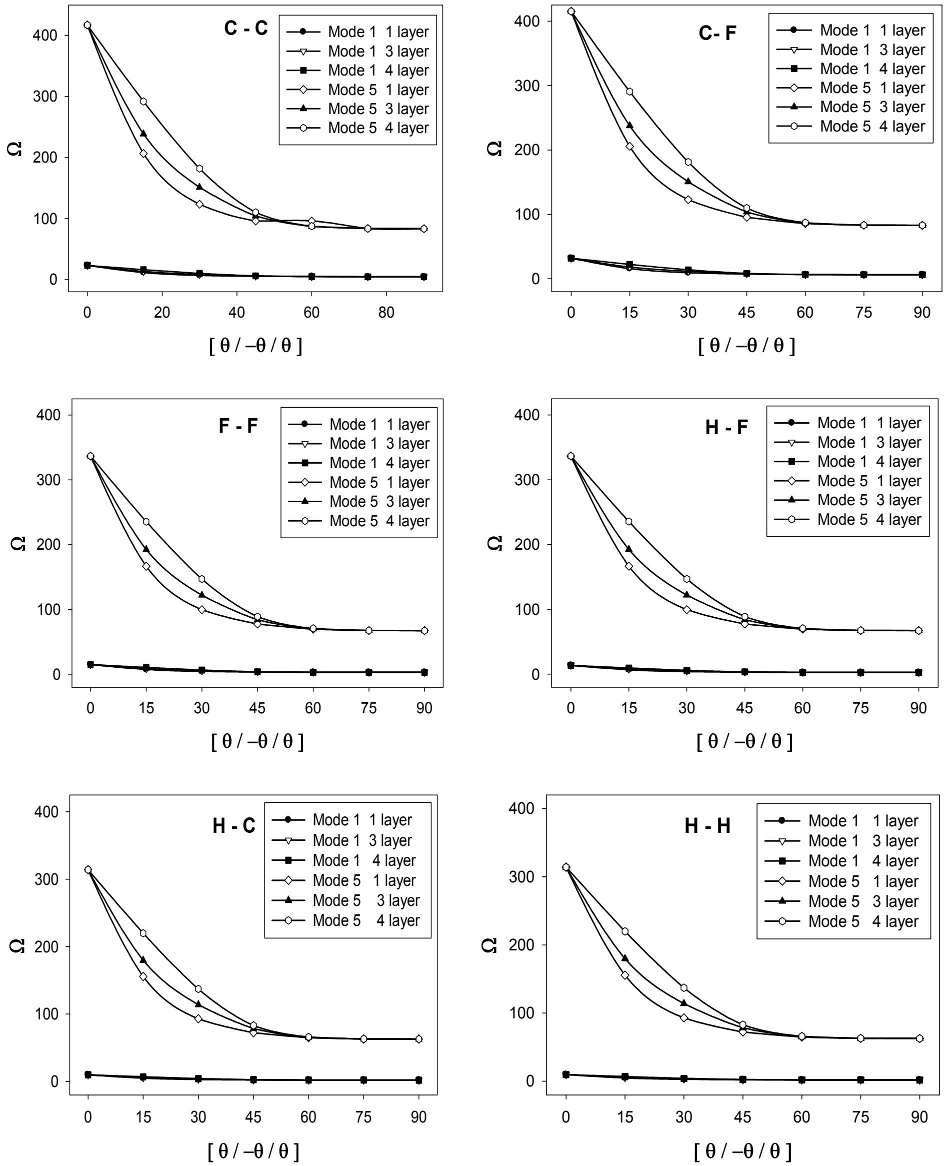

Fig. 9. Variation of frequency parameter of symmetric angle-ply composite beam with lamination angle for α_s=1 and η=0,25 for different number of layers.

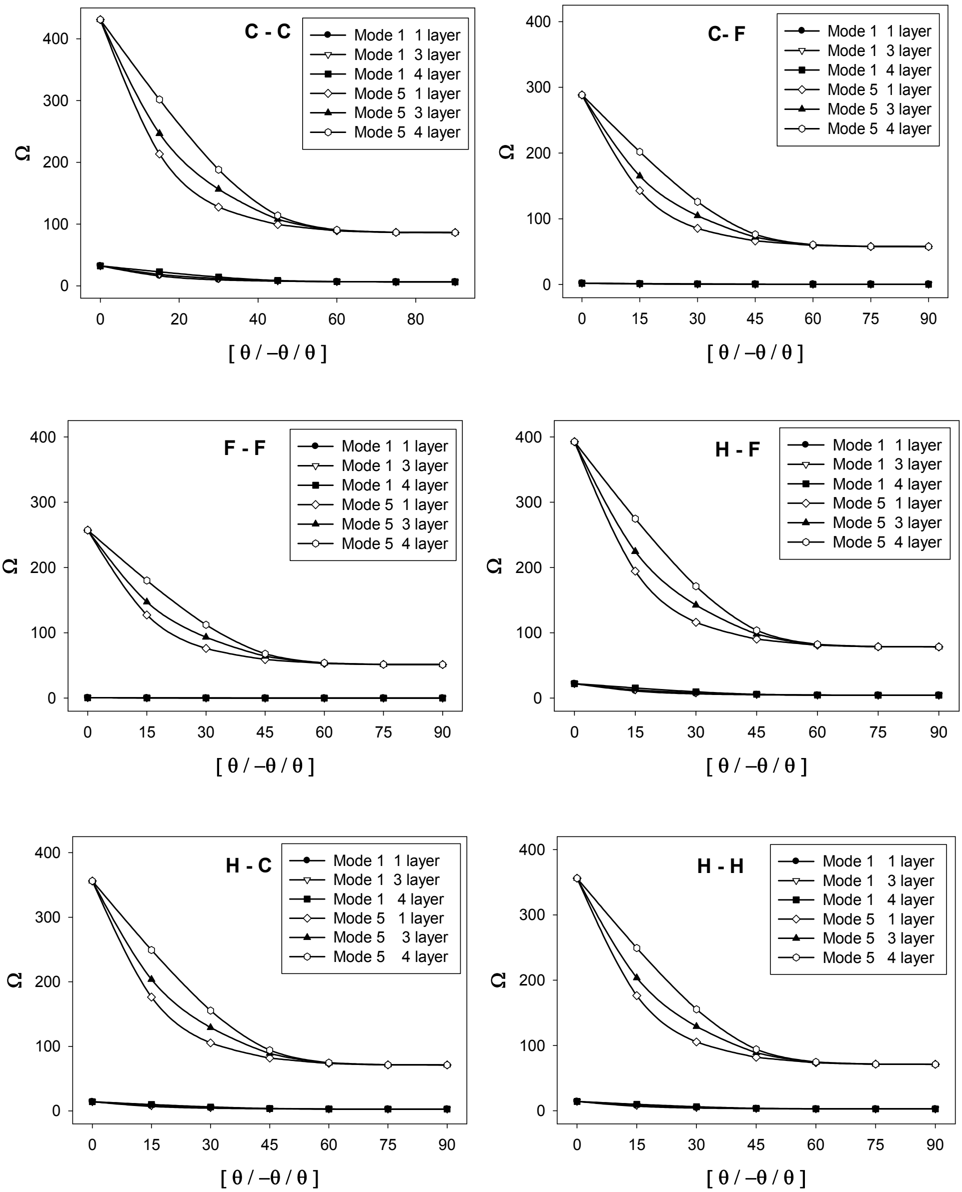

Fig. 10. Variation of frequency parameter of symmetric angle-ply composite beam with lamination angle for α_m=1 and η=0,25 for different number of layers.

In Fig 11-12 variation of frequency ratio of three layer cross-ply composite beams with η is given for α_m=1 and α_s=1 respectively. Generally similar behavior is observed with symmetric angle-ply and cross-ply composite beams.

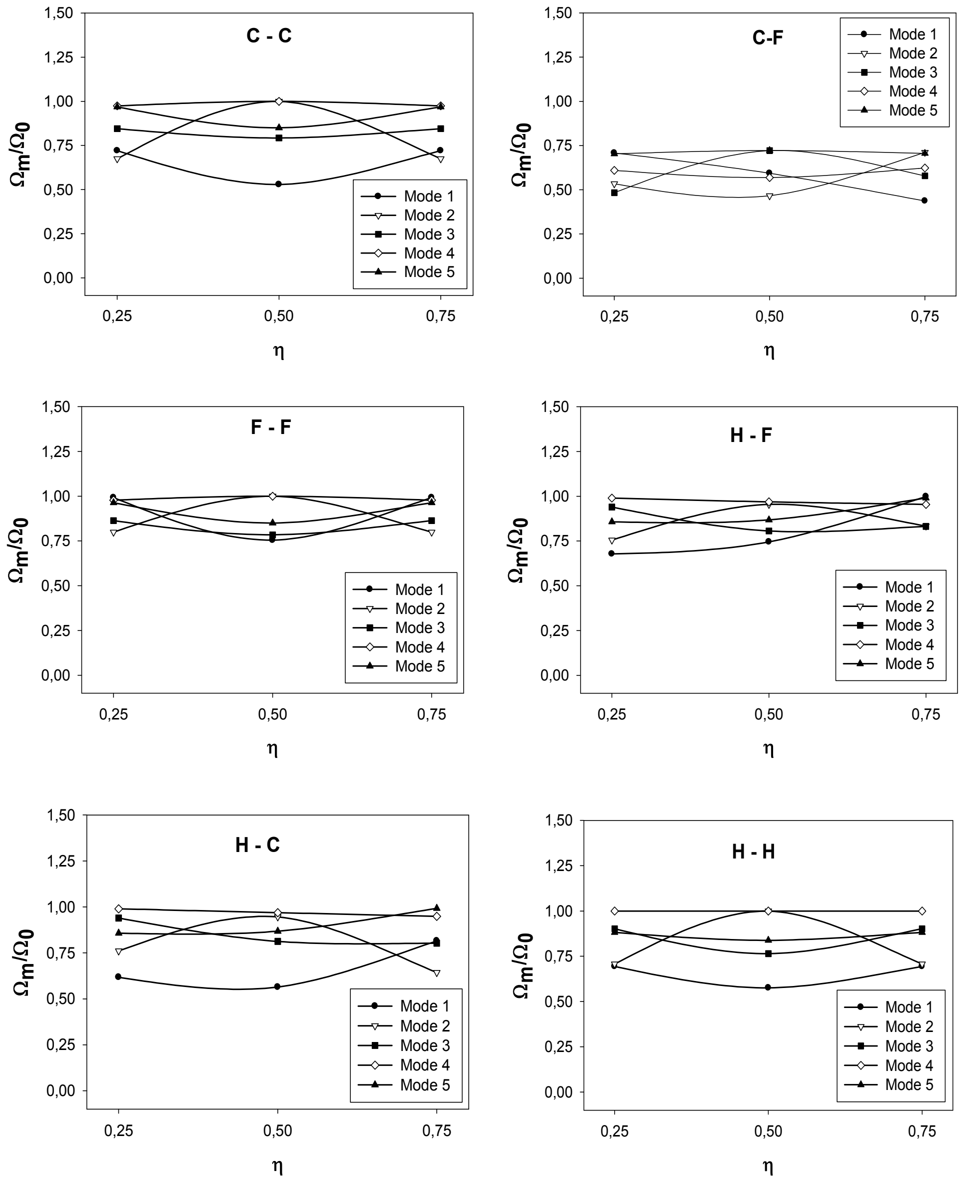

Fig. 11. Variation of frequency ratio of symmetric cross-ply composite beam ($0^0/90^0/0^0$) with η for α_m=1and η=0,25 .

4. Conclusion

In this study, vibration of laminated composite beams with attached mass or spring is studied using classical lamination theory. First five flexural frequencies of composite beams

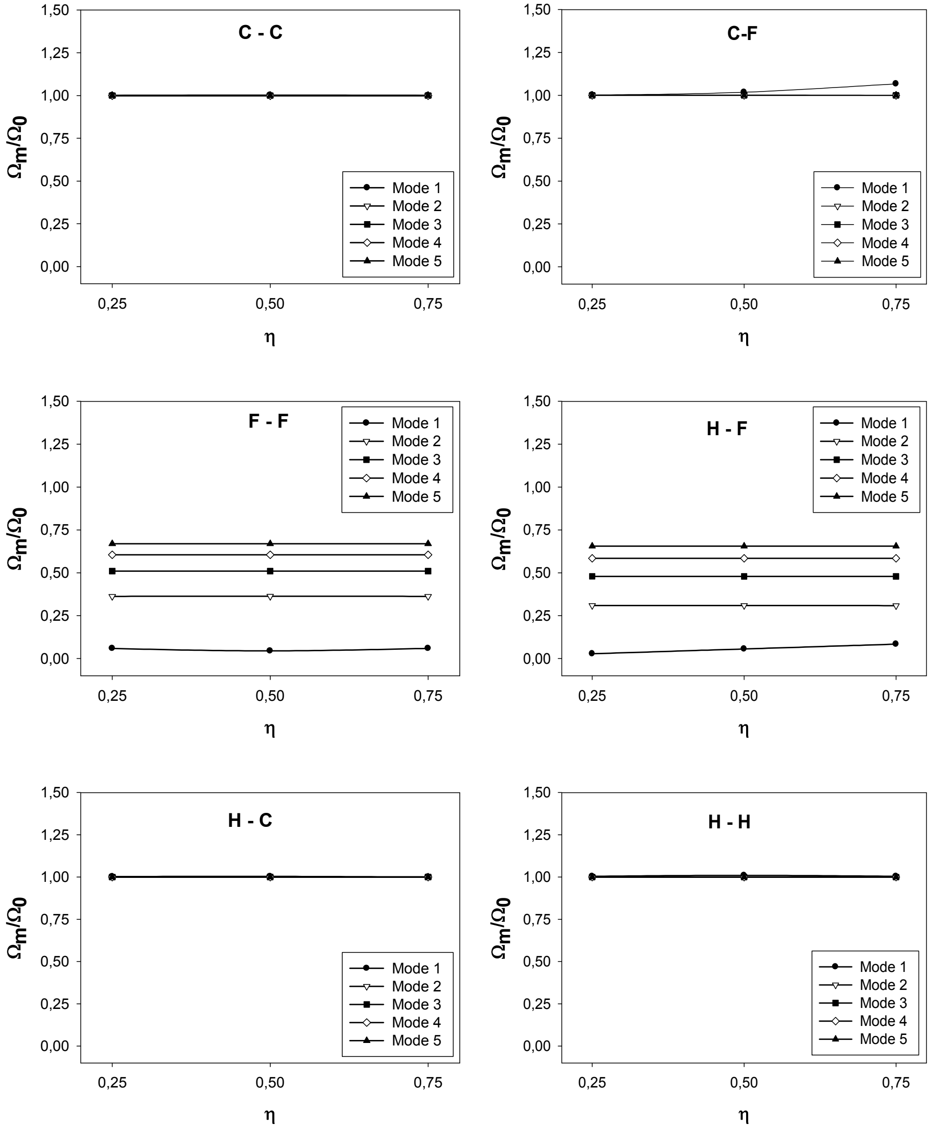

Fig. 12. Variation of frequency ratio of symmetric cross-ply composite beam ($0^0/90^0/0^0$) with η for α_s=1and η=0,25.

are obtained for different boundary conditions, attached mass, spring and their different positions. It is obtained that attaching mass reduces frequency of composite beams whereas attaching spring increases frequency of composite beams. Some modes do not change depending on position of attached spring or mass. This study can be extended to anti-symmetric composite beams and shear deformation effects can be added in the future studies.

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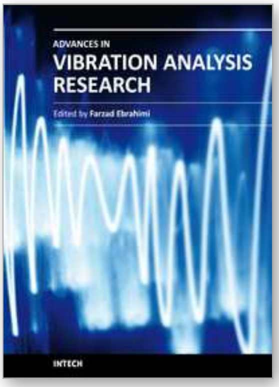

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Vibrations are extremely important in all areas of human activities, for all sciences, technologies and industrial applications. Sometimes these Vibrations are useful but other times they are undesirable. In any case, understanding and analysis of vibrations are crucial. This book reports on the state of the art research and development findings on this very broad matter through 22 original and innovative research studies exhibiting various investigation directions. The present book is a result of contributions of experts from international scientific community working in different aspects of vibration analysis. The text is addressed not only to researchers, but also to professional engineers, students and other experts in a variety of disciplines, both academic and industrial seeking to gain a better understanding of what has been done in the field recently, and what kind of open problems are in this area.