Chapter

Variable, Fractional-Order PID Controller Synthesis Novelty Method

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Abstract

The novelty method of the discrete variable, fractional order PID controller is proposed. The PID controllers are known for years. Many tuning continuous time PID controller methods are invented. Due to different performance criteria there are optimized three parameters: proportional, integral and differentiation gains. In the fractional order PID controllers there are two additional parameters: fractional order integration and differentiation. In the variable, fractional order PID controller fractional orders are generalized to functions. Nowadays all PID controllers are realized by microcontrollers in a discrete time version. Hence, the order functions are discrete variable bounded ones. Such controllers offer better transient characteristics of the closed loop systems. The choice of the order functions is still the open problem. In this Section a novelty intuitive idea is proposed. As the order functions one applies two spline functions with bounded functions defined for every time subinterval. The main idea is that in the final time interval the variable, fractional order PID controller transforms itself to the classical one preserving the stability conditions and zero steady-state error signal. This means that in the last time interval the discrete integration order is -1 and differentiation is 1.

Keywords: fractional-Calculus, PID controller, discrete system

1. Introduction

A continuous-time proportional–integral–derivative controller (PID controller) [1] invented almost 100 years ago is one of the most widely applied controllers in the closed-loop systems [2] with many industrial applications [3–5]. Currently the continuous-time control is successively replaced by discrete-time one in which the integration is replaced by a summation and differentiation by a difference evaluation. So, in the discrete PID controller the classical integral is replaced by a sum and the derivative by a backward difference, [6]. The discrete controller's PID algorithm is mainly realized by micro-controllers [7].

At 70s of the 20-th Century the Fractional Calculus [8] with a great success started a considerable attention in mathematics and engineering [9–12]. Now, the fractional-order backward-difference (FOBD) and the fractional-order backward sum (FOBS) [6, 13] are applied in the dynamical system modeling [14] and discrete control algorithms. The continuous-time FOPID controllers are more difficult in a practical realization [15–18].

There are numerous continuous and discrete-time PID and FOPID controller synthesis methods [16, 19–31]. One should mention that the optimisation of the closed-loop system in this case is more complicated because of the controller optimization. Apart from the three classical gains there are two additional parameters, namely, a fractional order of differentiation and summation [32]. The FOPID control characterizes by slow achieving the steady state and growing calculation “tail” [12].

In the paper a novelty variable, the fractional-order PID (VFOPID) [6, 28, 33–41] controller synthesis is proposed. It consists of dividing the closed-loop system discrete-transient time division into the finite time intervals over which are defined fractional orders summation and differentiation functions. The main idea is that for the final infinite interval $[k_L, +\infty)$ the difference order equals 0 and the summation is $-!$ preserving quick reaching the zero steady state value. Thus, in the VFOPID control the disadvantages of FOPID are extracted. One should admit that in the FOPID or VFOPID control the microcontrollers are numerically loaded.

Fractional-orders systems are characterized by the so called system “memory”. This, in practice, means that in every step the FOPID controller computes its output signals taking into account step-by-step linearly computed number of samples. This causes in practice the micro-controllers realization problems. It is known as “Finite memory principle” [12].

The paper is organized as follows. In Section 2 the basic information related to the fractional calculus and variable, fractional order Grünwald-Letnikov backward difference is given. The main result of the paper includes Section 3. It contains the proposed VFOPID controller synthesis method with the proposal of the order functions form. The brief description of the controller parameters evaluation algorithm is given. The investigations are supported by a numerical example presented in Chapter 4.

2. Mathematical preliminaries

In the paper the following notation will be used. $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}_l = \{l, l+1, l+2, \dots\}$ $\mathbb{R}_+ = [0, +\infty)$. 0_k will denote the zero column vector of dimensions $(k+1) \times 1$ whereas $\mathbf{0}_{k,k}$ is $(k+1) \times (k+1)$ zero matrix. Similarly will be denoted a $(k+1) \times (k+1)$ unit matrix $\mathbf{1}_k$.

In general, a fractional-order functions will be denoted by Greek letters $\nu(\cdot) : \mathbb{N}_0 \rightarrow \mathbb{R}\ \nu(\cdot)$ whereas the integer orders will be denoted by Latin ones $n \in \mathbb{R}_+$. In practice, for $l \in \mathbb{N}_0$: $0 < \nu(l) \leq 1$. For $k, l \in \mathbb{N}_0$ and a given order function $\nu(l)$ the function of two discrete variables $k, l \in \mathbb{N}_0$ is defined by the following formula: $a^{[\nu(l)]}(k)$ as follows:

Definition 2.1. For $k, l \in \mathbb{N}_0$ and a given order function $\nu(\cdot)$ one defines the coefficients function of two 13 discrete variables as

$$a^{[\nu(l)]}(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ (-1)^k \dfrac{\nu(l)(\nu(l)-1)\cdots(\nu(l)-k+1)}{k!} & \text{for} \quad k \in \mathbb{N}_1 \end{cases} \tag{1}$$

One should mention that function (1) for $\nu(l) = n(l) = \text{const} \in \mathbb{N}_0$

$$a^{[n]}(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ \dfrac{n(n-1)\cdots(n-k+1)}{(-1)^k!} & \text{for} \quad k \in [1, n] \\ 0 & \text{for} \quad k \in \mathbb{N}_{n+1} \end{cases} \tag{2}$$

The above function will be named as: the “oblivion function” or “decay function”.

2.1 Variable, fractional-order backward difference

Next one defines the Grünwald–Letnikov variable, fractional-order backward difference (VFOBD). For a discrete-variable bounded real-valued function $f(\cdot)$ defined over a discrete interval $[0, k]$ the VFOBD is defined as a sum (see for instance [6, 9]).

Definition 2.2. The VFOBD with an order function ν, with values $\nu(k) \in [0, 1]$, is defined as a finite sum, provided that the series is convergent

$$
\begin{aligned}
{}_{k_0}\Delta_k^{\nu(k)} f(k) &= \sum_{i=0}^{k-k_0} a^{\nu(k)}(i) f(k-i) \\
&= \begin{bmatrix} 1 & a^{\nu(k)}(1) & a^{\nu(k)}(2) & \cdots & a^{\nu(k)}(k_0) \end{bmatrix} \begin{bmatrix} f(k) \\ f(k-1) \\ \vdots \\ f(k-k_0) \end{bmatrix}
\end{aligned} \tag{3}
$$

Relating to (2) as the first special case of the defined above VFOBD and a constant order function $\nu(k) = \nu = \text{const}$ from (2.1) one gets the fractional-order backward difference (FOBD). The second special case is for a constant integer order function $\nu(k) = \nu = n = \text{const}$ where the integer-order backward difference (IOBD) is a classical one.

Equality (3) is valid for $k, k-1, k-2, \dots, k_0 + 1, k_0$. Hence, one gets a finite set of equations. Collecting them in a vector matrix form one gets

$$
{}_{k_0}^{GL}\Delta_k^{[\nu(k)]} \mathbf{f}(k) = {}_{k_0}\mathbf{A}_k^{[\nu(k)]} \mathbf{f}(k), \tag{4}
$$

where

$$
{}_{k_0}\mathbf{A}_k^{[\nu(k)]} = \begin{bmatrix} 1 & a^{[\nu(k)]}(1) & \cdots & a^{[\nu(k)]}(k-k_0) \\ 0 & 1 & \cdots & a^{[\nu(k-1)]}(k-k_0-1) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a^{[\nu(k_0+1)]}(1) \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{5}
$$

$$
\mathbf{f}(k) = \begin{bmatrix} f(k) \\ f(k-1) \\ \vdots \\ f(k_0) \end{bmatrix}, \tag{6}
$$

$$
{}_{k_0}^{GL}\Delta_k^{[\boldsymbol{n}(k)]} \mathbf{f}(k) = \begin{bmatrix} {}_{k_0}^{GL}\Delta_k^{[\nu(k)]} f(k) \\ \vdots \\ {}_{k_0}^{GL}\Delta_{k_0}^{[\nu(k_0)]} f(k_0) \end{bmatrix}.
$$

2.2 Variable, fractional-order linear time-invariant difference equations

On the base of the Grünwald-Letnikov variable, fractional-order linear time-invariant backward-difference the difference Eqs. (GL-VFOBE) for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$ representing discrete models of real dynamical systems or discrete control strategies are defined by the variable, fractional-order linear time-invariant difference equation (VFODE). $h > 0$ denotes the sampling time.

$$\sum_{l=0}^{n_i} a_{i,l} {}_{k_0}^{GL}\Delta_k^{[\nu_{i,l}(k)]} y(kh) = \sum_{l=0}^{m_i} b_{i,l} {}_{k_0}^{GL}\Delta_k^{[\mu_{i,l}(k)]} u(kh) \tag{7}$$

where $m_i \le n_i$, $\nu_{n_{i,l}}(k) \ge \nu_{n_{i,l-1}}(k) \ge \cdots \nu_{i,1}(k) \ge \nu_{i,0}(k) = 0$, $\mu_{m_{i,l}}(k) \ge \mu_{m_{i,l-1}-1}(k) \ge \cdots$ $\mu_{i,1}(k) \ge \mu_{i,0}(k) \ge 0$, $a_{i,l}$ and $b_{i,l}$ are constant coefficients for $l = 0, 1, \cdots, n_i$ and $l = 0, 1, \cdots, m_i$, respectively. It is assumed that $a_{0,n_0} = 1$.

According to the notation (5) Eq. (7) takes the form

$$\sum_{l=0}^{n_i} a_{i,lk_0} \mathbf{A}_k^{[\nu_{i,l}(k)]} \mathbf{y}(k) = \sum_{l=0}^{m_i} a_{i,lk_0} \mathbf{A}_k^{[\mu_{i,l}(k)]} \mathbf{u}(k) \tag{8}$$

The vector $\mathbf{u}_j(k)$ satisfies the condition $\mathbf{u}_j(k) = \mathbf{0}_k$ for $k < k_0$. In the general solution of (8) to the assumed $\mathbf{u}_j(k)$ and initial conditions vector $\mathbf{y}_{i,k_0-1} = \begin{bmatrix} y_{i,k_0-1} & y_{i,k_0-2} & \cdots \end{bmatrix}^{\mathrm{T}}$ (T denotes the transposition) must be taken into account with $-\infty = k_0' < 0 \le k_0 \le k$. Then, the infinite number of initial conditions (8) are formed in the following vector

$$\mathbf{y}_{i,k_0-1} = \begin{bmatrix} y_{i,k_0-1} \\ y_{i,k_0-2} \\ \vdots \end{bmatrix} \tag{9}$$

and the combined Eq. (8) is of the form

$$\begin{aligned} &\begin{bmatrix} \sum_{l=0}^{n_i} a_{ij,lk_0} \mathbf{A}_k^{[\nu_{i,l}(k)]} & \sum_{l=0}^{n_i} a_{i,l-\infty} \mathbf{A}_{k_0-1}^{[\nu_{i,l}(k)]} \end{bmatrix} \\ &\times \begin{bmatrix} \mathbf{y}_i(kh) \\ \mathbf{y}_{i,k_0-1} \end{bmatrix} = \sum_{l=0}^{m_i} b_{ij,lk_0} \mathbf{A}_k^{[\mu_{i,l}(k)]} \mathbf{u}(kh) \end{aligned} \tag{10}$$

or after simple transformation

$$\begin{aligned} \sum_{l=0}^{n_i} a_{i,lk_0} \mathbf{A}_k^{[\nu_{i,l}(k)]} \mathbf{y}(kh) = &\sum_{l=0}^{m_i} b_{i,lk_0} \mathbf{A}_k^{[\mu_{i,l}(k)]} \mathbf{u}(kh) \\ &- \sum_{l=0}^{n_i} a_{i,l-\infty} \mathbf{A}_{k_0-1}^{[\nu_{i,l}(k)]} \mathbf{y}_{i,k_0-1} \end{aligned} \tag{11}$$

2.3 Main assumptions

To preserve the VFOBDE order one assumes that

$$1 + \sum_{l=0}^{n_i-1} a_{i,l} \ne 0 \quad \text{for} \quad i = 1, 2, \cdots, p \tag{12}$$

In the transfer functions defined by the one-sided **Z** transform one assumes zero initial conditions. Following this assumption equality (11) simplifies to

$$\sum_{l=0}^{n_i} a_{ik_0} \mathbf{A}_k^{[\nu_i(k)]} \mathbf{y}(k) = \sum_{l=0}^{m_i} b_{ik_0} \mathbf{A}_k^{[\mu_i(k)]} \mathbf{u}(k) \tag{13}$$

Defining matrices

$${}_{\mathbf{k_0}}\mathbf{D}_{\mathbf{k}}^{[\nu_\mathbf{P}(\mathbf{k})]} = \sum_{l=0}^{n_i} a_{ik_0} \mathbf{A}_k^{[\nu_i(k)]} \tag{14}$$

$${}_{\mathbf{k_0}}\mathbf{N}_{\mathbf{k}}^{[\mu_\mathbf{P}(\mathbf{k})]} = \sum_{l=0}^{m_i} b_{ik_0} \mathbf{A}_k^{[\mu_i(k)]} \tag{15}$$

one gets

$${}_{k_0}D_k^{[\nu_P(k)]} \mathbf{y}(kh) = {}_{k_0}N_k^{[\mu_P(k)]} \mathbf{u}(kh) \tag{16}$$

Under assumption (12) ${}_{k_0}D_k^{[\nu(k)]}$ is invertible, so for $k_0 = 0$ one can write

$$\mathbf{y}(kh) = \left[{}_0D^{[\nu_P(k)]_k}\right]^{-1} {}_0N^{[\mu_P(k)]_k} \mathbf{u}(kh) \tag{17}$$

Denoting

$${}_0G_k^{[\nu_P(k)]} = \left[{}_0D_k^{[\nu_P(k)]}\right] {}_0N_k^{[\mu(k)]} \tag{18}$$

one gets similar to the transfer function description

$$\mathbf{y}(kh) = {}_0G_k^{[\nu_P(k),\mu_P(k)]} \mathbf{u}(kh) \tag{19}$$

or for simplicity

$$\mathbf{G}_o(kh) = {}_0G_k^{[\nu_P(k),\mu_P(k)]} \tag{20}$$

Remark 2.1. Though the relation (19) looks similar to the classical discrete transfer function it is different by the real discrete variables. It relates discrete SISO systems by vectors and matrices related to its dimensions $k + 1 \in \mathbb{N}_0$.

2.4 VFO linear system description

One considers a closed-loop system illustrated in **Figure 1**. Where a plant is described by (19) where $e(kh)$ and $u(kh)$.

2.4.1 VFO_PID

The classical PID controller output is desribed by three terms

$$u(kh) = K_P e(kh) + K_{I0} \Delta_k^{-\mu(k)} e(kh) + K_{D0} \Delta_k^{\nu(k)} e(kh) \tag{21}$$

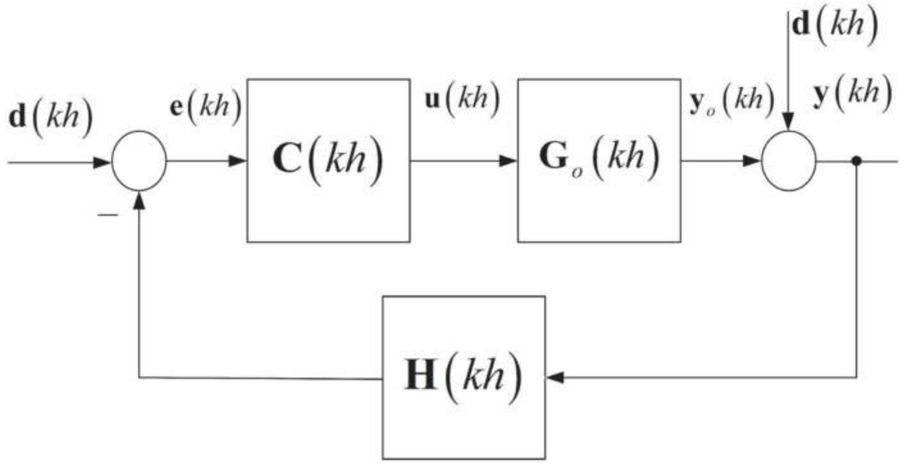

Figure 1.
Closed-loop system.

and in the convention proposed above as

$$\mathbf{u}(kh) = K_P \mathbf{1}_k \mathbf{e}(kh) + K_{D0} \mathbf{G}_k^{[\nu_C(k)]} \mathbf{e}(kh) + K_{I0} \mathbf{G}_k^{[-\mu_C(k)]} \mathbf{e}(kh) \tag{22}$$

which may be expressed as

$$\mathbf{u}(kh) = \left[K_P \mathbf{1}_k + K_{D0} \mathbf{G}_k^{[\nu_C(k)]} + K_{I0} \mathbf{G}_k^{[-\mu_C(k)]} \right] \mathbf{e}(kh) \tag{23}$$

where $\nu_C(k), \mu_C(k) \geq 0$ and controlling and error signals are denoted as $\mathbf{u}(k)$ and $\mathbf{e}(k)$, respectively. Then, denoting.

Remark 2.2. The plant may be described by classical integer order, fractional or even variable, fractional - order difference equations. The matrix - vector description used makes it possible.

$$\mathbf{C}(kh) = K_P \mathbf{1}_k + K_{D0} \mathbf{G}_k^{[\nu_C(k)]} + K_{I0} \mathbf{G}_k^{[-\mu_C(k)]} \tag{24}$$

one gets a VFOPID controller transfer function-like description

$$\mathbf{u}(kh) = \mathbf{C}(kh) \mathbf{e}(kh) \tag{25}$$

To simplify the description one assumes a sensor matrix as

$$\mathbf{H}(kh) = \mathbf{1}_k \tag{26}$$

The closed-loop system is presented in **Figure 1** from which one gets the following relations

$$\begin{aligned} \mathbf{y}(kh) = {} & [\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)\mathbf{H}(kh)]^{-1} \mathbf{G}_o(kh)\mathbf{C}(kh)\mathbf{r}(kh) \\ & + [\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)\mathbf{H}(kh)]^{-1} \mathbf{d}(kh) \end{aligned} \tag{27}$$

where

- $\mathbf{r}(kh)$ - a reference signal vector,
- $\mathbf{d}(kh)$ - an external disturbance signal vector,

- $\mathbf{y}_o(kh)$ - a plant output signal vector,
- $\mathbf{y}(kh)$ - a closed-loop system output signal vector,
- $\mathbf{e}(kh)$ - a closed-loop system error signal,

A system error is evaluated by the formula

$$\begin{aligned}\mathbf{e}(kh) = {} & [\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)\mathbf{H}(kh)]^{-1}\mathbf{r}(kh) \\ & -[\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)\mathbf{H}(kh)]^{-1}\mathbf{H}(kh)\mathbf{d}(kh)\end{aligned} \tag{28}$$

3. Variable, fractional-order PID controller synthesis

In the synthesis of the classical PID controller there are three parameters to evaluate. Namely, $K_,K_I, K_D$ known as the proportional, integral and differential gains. In the fractional-order PID controllers there are two additional parameters: the differentiation order $\nu(kh) \in \mathbb{R}_+$ and the integration one $-\mu(kh) \in \mathbb{R}_+$. In the variable, fractional-order PID controller the mentioned orders are generalized to functions. This means that there are three constant coefficients and two discrete variable functions to find

$$K_P, K_I, K_D, \nu(kh), \mu(kh) \tag{29}$$

In the rejection of the external disturbation one can assume that $\mathbf{r}(kh) = \mathbf{0}$ so Eq. (29) simplifies to

$$\mathbf{e}(kh) = -[\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)\mathbf{H}(kh)]^{-1}\mathbf{H}(kh)\mathbf{d}(kh) \tag{30}$$

Usually the sensor matrix $\mathbf{H}(kh)$ is treated as constant, by assumption that sensors do not introduce its own dynamics to the system. Hence, $\mathbf{H}(kh) = \mathbf{H} = \text{const}$. It may be assumed that $\mathbf{H} = h_0\mathbf{1}_k$ or further, for $h_0 = 1$, formula (30) takes a form

$$\mathbf{e}(kh) = -[\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)]^{-1}\mathbf{d}(kh) \tag{31}$$

The optimal parameters (29) are evaluated due to the assumed optymality criterion. The most popular is so called ISE one (Integral of the Squared Error) or in the discrete-system case: Sum of the Squared Error (SSE).

$$SSE[K_P, K_I, K_D, \nu(kh), \mu(kh)] = \sum_{i=0}^{k_max} e(ih)^2 h = \mathbf{e}(kh)^{\mathrm{T}}\mathbf{e}(kh)h \tag{32}$$

Substitution of (31) into (32) gives

$$\begin{aligned}& SSE[K_P, K_I, K_D, \nu(kh), \mu(kh)] \\ & = \mathbf{d}(kh)^T[\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)]^{-T}[\mathbf{1}_k + \mathbf{G}_o(kh)\mathbf{C}(kh)]^{-1}\mathbf{d}(kh)\end{aligned} \tag{33}$$

In the proposed VFOPID controller synthesis method with partially intuitive and supported by closed-loop systems synthesis experience the classical optimisation due to the performance criterion (32) is performed. The pre-defined differentiation and integration order functions orders are as follows

$$\nu(kh) \geq 0 \tag{34}$$

$$\nu(kh) = \begin{cases} \nu_1(kh) & \text{for} \quad k \in [0, k_{N1}) \\ \nu_2(kh) & \text{for} \quad k \in [k_{N1}, k_{N2}) \\ & \vdots \\ \nu_N(kh) & \text{for} \quad k \in [k_{NN-1}, k_N N) \\ 0 & \text{for} \quad k \in [k_{NN}, +\infty) \end{cases} \tag{35}$$

and

$$\mu(kh) \le 0 \tag{36}$$

$$\mu(kh) = \begin{cases} \mu_1(kh) & \text{for} \quad k \in [0, k_{M1}) \\ \mu_2(kh) & \text{for} \quad k \in [k_{M1}, k_{M2}) \\ & \vdots \\ \mu_N(kh) & \text{for} \quad k \in [k_{MM-1}, k_M M) \\ -1 & \text{for} \quad k \in [k_{MM}, +\infty) \end{cases} \tag{37}$$

Every function $\nu_i(kh)$ for $i = 1, 2, \cdots, N$ and $\mu_i(kh)$ for $i = 1, 2, \cdots, M$ is characterized by a sets of parameters c_{ij} and d_{ij}, respectively.

In the classical closed-loop system with PID controller there is introduced the integration part preserving the steady - state error signal tending to zero. So, in (38) there is a constant order -1 for $k \ge [K_{MM}, +\infty)$.

Now, for initially assumed order functions one applies the following algorithm based on well known Gauss method.

1. Chose a starting set of coefficients $K_P, K - I, K_D, c_{11}, \cdots$ and $d_{11}, \cdots$,

2. Applying the classical Gauss algorithm find a minimal SSE performance index value alongside the first variable (eg. K_P),

3. Repeat step 2 for the next parameter,

4. If the SSE value is satisfactory stop else return to step 2.

Remark 3.1. Algorithm described above can be applied also to the classical discrete PID controller with three parameters.

4. Numerical example

One considers a closed-loop system depicted in **Figure 1**.

A plant is described by a transfer function

$$G_o(s) = \frac{b_0}{s^2 + a_1 s + a_0} \tag{38}$$

where

- $a_1 = 0.5$
- $a_0 = 0.1$
- $b_0 = a_0$

The plant is discretized with the sampling time $h = 0.5$ and a VFOPID controller is applied

$$\nu(kh) = \begin{cases} \nu_1(kh) = 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \in [1, +\infty) \end{cases} \tag{39}$$

$$\mu(kh) = \begin{cases} \mu_1(kh) = -1 + d_1 e^{d_2(kh-1h)} & \text{for} \quad k = [0, 10] \\ -1 & \text{for} \quad k \in (10, +\infty) \end{cases} \tag{40}$$

and controller gains K_P, k_I, K_D and order function parameters d_1, d_2.

Hence, there are 5 parameters to evaluate. Due to the performance index (33) the optimal parameters are as follows

- $K_P = 1.000$
- $K_i = 0.514$
- $K_D = 0.890$

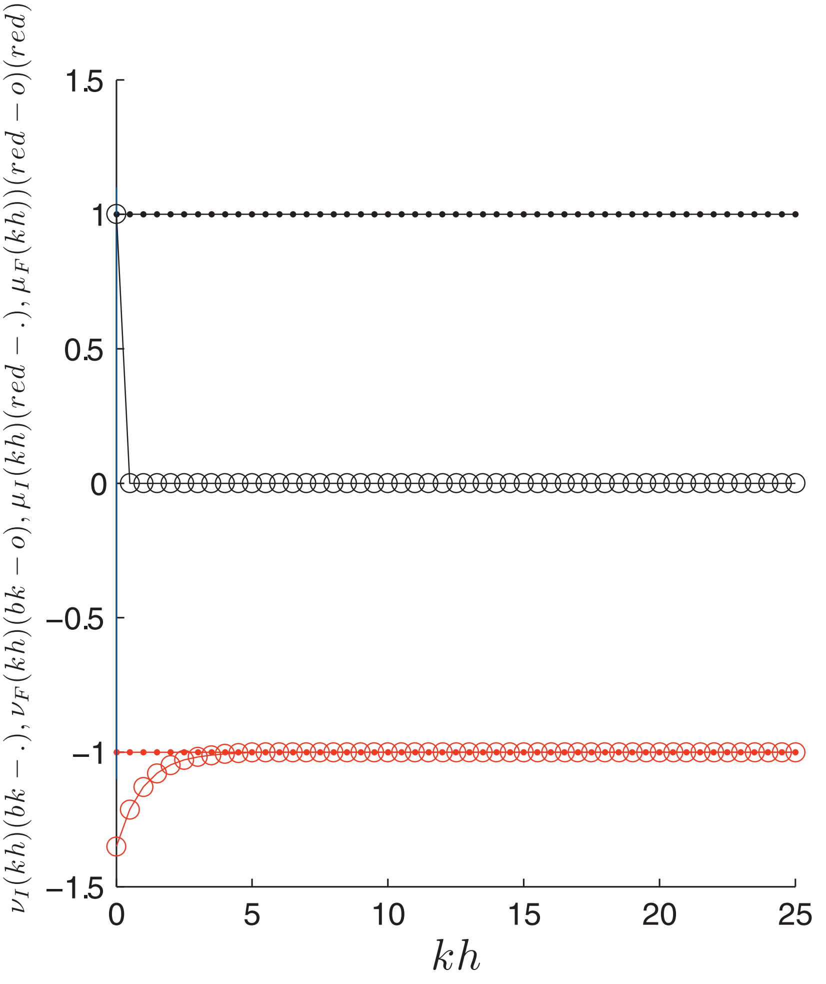

Figure 2.
VFOPID controller order functions: $\nu(kh)$ (in black) and $\mu(kh)$ (in red).

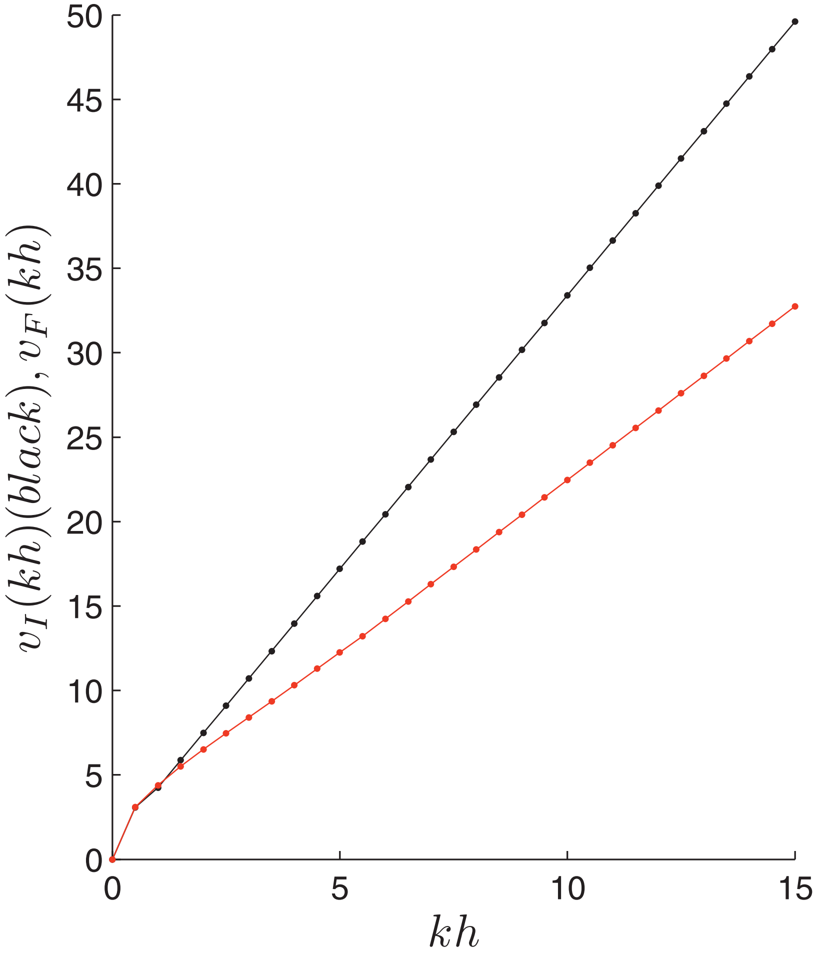

Figure 3.
VFOPID (in red) and IOPID (in black) controller unit step response.

- $d_1 = -0.35$
- $d_2 = -0.5$

The VFOPID controller order functions are plotted in **Figure 2** whereas the PID and VFOPID controllers unite step responses are given in **Figure 3**.

The achieved VFOPID controller synthesis result is compared with the classical discrete-time PID controller optimized due to criterion (30). The optimal parameters are

- $K_P = 1.00$
- $K_i = 0.81$
- $K_D = 0.90$

Figure 4 contains the closed - loop systems with PID (in blue) and VFOPID (in red) controllers unit step responses. There is included a plant unit step response of the plant (in black.)

In **Figure 5** the controlling signals are presented (PID - in black, VFOPID - in red). The controlling signals have typical shapes: first differentiation action and

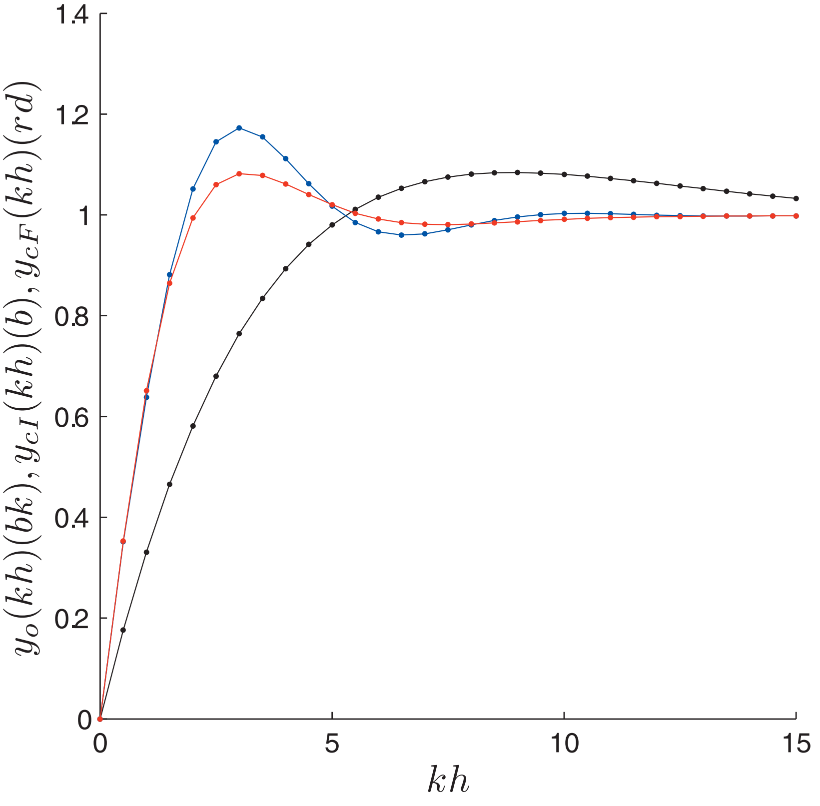

Figure 4.
The closed-loop system response with the VFOPIS (in red) and IOPID controllers (in blue).

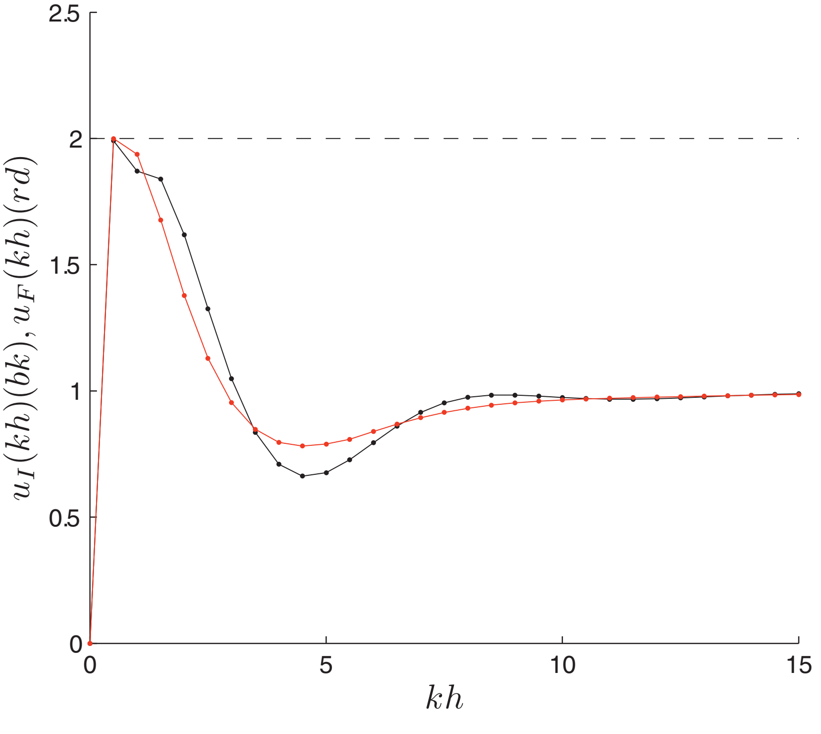

Figure 5.
The closed-loop controlling signals.

finally the classical integration preserving zero steady - state closed - loop system error.

Remark 4.1. In the Numerical example proposed here the VFOPID and the classical PID controllers maximal control signal values are the same reaching assumed bounding value $\max[u_I(kl)],\ \max[u_F(kh)] = 2$.

Remark 4.2. In the Numerical example

$$\begin{aligned} SSE[K_P, K_I, K_D, 1, -1)] &= 1.3312 \\ SSE[K_P, K_I, K_D, \nu(kh), \mu(kh)] &= 1.2899 \end{aligned} \tag{41}$$

5. Conclusions

One should emphasize that the proposed solution of the VFOPID controller do not guarantee the absolute optimum of the closed-loop control system synthesis. It proves that the proposal of a physically realizable VFOPID controller by micro-controller (with finite memory) leads to better results due to the assumed performance criterion.

The main idea of the proposed method is to assume *a priori* the order functions with unknown parameters. In the VFOPID controller synthesis essential is an assumption that the summation order equals 1 One can express the action as the assumption of skeleton order functions with unknown parameters evaluated further in an SSE optimization algorithm.

Here, it is worth mentioning that there are still open problems of the VFOPID controllers tuning.

- One should define a program evaluating the order functions.
- For evaluation of the VFOPID controller parameters one can apply another optimization methods. It seems that optimization methods based on the artificial intelligence will be very effective.
- Another performance index may be applied. Some penalty functions may be introduced to SSE as well a term taking into account the minimal value of the error signal.

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Abbreviations

PID	proportional-integral-derivative controller
FOBD	fractional-order backward difference
FOBS	fractional-order backward sum
FOPID	fractional-Order proportional, fractional-order integral and differential controller
VFOPID	variable, fractional-order PID controller
SSE	squared sum of the error

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