Chapter 4

# Phase Portraits of Cubic Dynamic Systems in a Poincare Circle

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Additional information is available at the end of the chapter

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**Abstract**

In the proposed chapter, we are going to outline the results of a study on an arithmetical plane of a broad family of dynamic systems having polynomial right parts. Let these polynomials be of cubic and square reciprocal forms. The task of our investigation is to find out all the different (in the topological sense) phase portraits in a Poincare circle and indicate the coefficient criteria of their appearance. To achieve this goal, we use the Poincare method of central and orthogonal consecutive displays (or mappings). As a result of this thorough investigation, we have constructed more than 250 topologically different phase portraits in total. Every portrait we present using a special table called a descriptive phase portrait. Each line of such a special table corresponds to one invariant cell of the phase portrait and describes its boundaries, as well as a source of its phase flow and a sink of it.

**Keywords:** dynamic systems, phase portraits, phase flows, Poincare sphere, Poincare circle, singular points, separatrices, trajectories

## 1. Introduction

A dynamic system appears to be a mathematical model of some process or phenomenon, in which fluctuations and other so-called statistical events are not taken into consideration. It can be characterized by its initial state and a law according to which the system goes into a different state. A *phase space* of a dynamic system is the totality of all admissible states of this system.

It is necessary to distinguish dynamic systems with the discrete time and with the continuous time. For dynamic systems with the discrete time (they are called *cascades*), a system's behavior is described with a sequence of its states. For dynamic systems with continuous time (which are called *flows*), a state of the system is defined for each moment of time on a real or an imaginary axis. Cascades and flows are the main subject of study in symbolic and topological dynamics.

Dynamic systems, both with discrete and continuous time, can be usually described by an autonomous system of differential equations, defined in a certain domain and satisfying in it the conditions of the Cauchy theorem of existence and uniqueness of solutions of the differential equations.

Singular points of differential equations correspond to equilibrium positions of dynamic systems, and periodical solutions of differential equations correspond to closed phase curves of dynamic systems.

The main task of the theory of dynamic systems is a study of curves, defined by differential equations. This process includes splitting of a phase space into trajectories and studying their limit behavior—finding and classifying the equilibrium positions, and revealing the attracting and repulsive manifolds (i.e., attractors and repellers; sinks and sources). The most important notions of the theory of dynamic systems are the notion of stability of equilibrium states, which means the ability of a system under considerably small changes of initial data to remain near an equilibrium state (or on a given manifold) for an arbitrary long period of time, as well as the notion of roughness of a system (i.e., the saving of a system's properties under small changes of a model itself). A rough dynamic system is a system that preserves its qualitative character of motion under small changes of parameters.

The research methods proposed in this chapter are new and effective; they can also be used for the study of applied dynamic systems of the second order with polynomial right parts.

According to Jules H. Poincare, a normal autonomous second-order differential system with polynomial right parts, in principle, allows its full qualitative investigation on an extended arithmetical plane $\overline{R}^2_{x,y}$ [1]. Inspired by the great Poincare's works, mathematicians of the next generations, including contemporary researchers, have studied some of such systems, for example, quadratic dynamic systems [2], ones containing nonzero linear terms, homogeneous cubic systems, and dynamic systems with nonlinear homogeneous terms of the odd degrees (3, 5, 7) [3], which have a center or a focus in a singular point O (0, 0) [4], as well as other particular kinds of systems.

We consider in the present chapter a family of dynamic systems on a real plane *x, y*.

$$\frac{dx}{dt} = X(x,y), \frac{dy}{dt} = Y\,(x,y) \tag{1}$$

such that *X (x, y), Y (x, y)* are reciprocal forms of *x* and *y, X* is a cubic, *Y* a square form, and *X (0,1) > 0, Y (0, 1) > 0*. Our objective is to depict in a Poincare circle all kinds (different in the topological sense) of possible for systems phase portraits for Eq. (1), and also to indicate the criteria of every portrait realization close to coefficient ones. With this aim, we apply Poincare's method of consecutive mappings: first, the central mapping of a plane *x, y* (from a center (0, 0, 1) of a sphere $\sum$), augmented with a line at infinity (i.e., $\overline{R}^2_{x,y}$ plane) on a sphere $\sum$: $X^2 + Y^2 + Z^2 = 1$ with identified diametrically opposite points, and second, the orthogonal mapping of a lower enclosed semi-sphere of a sphere $\sum$ to a circle $\overline{\Omega}$: $x^2 + y^2 \leq 1$ with identified diametrically opposite points of its boundary $\Gamma$. We will now describe this process in more detail.

The circle $\overline{\Omega}$ and the sphere $\sum$ in this process are called the Poincare circle and the Poincare sphere, respectively [1].

## 2. Basic definitions and notation

$\varphi(t, p)$, $p = (x, y)$—a fixed point: = a solution (a motion) of Eq. (1) – system with initial data$(0, p)$.

$L_p$:$\varphi = \varphi(t, p), t \in I_{\max}, -$ a trajectory of motion $\varphi(t, p)$.

$L_p^{+(-)}$:= + (−) – a semi-trajectory of a trajectory $L_p$.

*O*-curve of a system := the system's semi-trajectory $L^s{}_p(p \neq O, s \in \{+, -\})$ adjoining to a point *O* under a condition such that $st \to +\infty$.

$O^{+(-)}$- curve of a system: = the system's *O*-curve $L_p^{+(-)}$.

$O_{+(-)}$-curve of a system: = the system's *O*-curve adjoining to a point *O* from a domain $x > 0$ ($x < 0$).

*TO*-curve of a system: = the system's *O*-curve, which, being supplemented by a point *O*, touches some ray in it.

A nodal bundle of *NO*-curves of a system := an open continuous family of the system's *TO*-curves $L^s{}_p$, where $s \in \{+, -\}$ is a fixed index, $p \in \wedge$, $\wedge$ a simple open arc, $L^s{}_p \cap \wedge = \{p\}$.

A saddle bundle of *SO*-curves of a system, a separatrix of the point *O*:= a fixed *TO*-curve, which is not included in some bundle of *NO*-curves of a system.

*E*, *H*, *P*-*O*-sectors of a system: an elliptical, a hyperbolic, a parabolic sector.

A topological type (T-type) of a singular point *O* of a system:= a word $A_O$ consisting of letters *N*, *S* (a word $B_O$ consisting of letters *E*, *H*, *P*), which describes a circular order of bundles *N*, *S* of its *O*-curves (of its *O*-sectors *E*, *H*, *P*) when traversing the point *O* in the " + "-direction, i. e., counterclockwise, starting with some of them.

$$P(u) \coloneqq X(1, u) \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3,$$

$$Q(u) \coloneqq Y(1, u) \equiv a + bu + cu^2.$$

**Note 1.** For every Eq. (1) system:

1) T-type of a singular point $O$ in its form $B_O$ is easy to construct using its T-type in the form $A_O$, and going backward (we need to determine both forms, see Corollary 1);

2) Real roots of a polynomial $P(u)$ (polynomial $Q(u)$) are in fact angular coefficients of isoclines of infinity (isoclines of a zero));

3) When we write out the real roots of the system's polynomials $P(u)$, $Q(u)$, separately or all together, we always number the roots of each one of them in an ascending order.

## 3. Topological type (T-type) of a singular point $O(0, 0)$

In order to find all $O$-curves and to split their totality into the bundles $N$, $S$, let us use the method of exceptional directions of a system in the point $O$ [1]. According to this method, the equation of exceptional directions for the point $O$ of the Eq. (1) system has the form.

$$xY(x,y) \equiv x\left(ax^2 + bxy + cy^2\right) = 0.$$

For this, the following cases are possible:

**1.** When $d \equiv b^2 - 4ac > 0,$ this equation defines simple straight lines $x = 0$ and.

$$y = q_i x, \quad i = 1, 2, \quad q_1 < q_2$$

**2.** When $d = 0$, this equation defines the straight line $x = 0$ and the double straight line.

$$y = qx, q = -\frac{b}{2c}$$

**3.** When $d < 0,$ the equation defines only the straight line $x = 0$.

Theorem 1 is true for the aforementioned cases [5].

**Theorem 1.** Words $A_O$ and $B_O$, which define a topological type (T-type) of a singular point $O$ (0, 0) of the Eq. (1) system*:*

1) in the case of $d > 0$, depending on signs of values $P(q_i) = 1, 2,$ have forms, indicated in a **Table 1**;

| $r$ | $P(q_1)$ | $P(q_2)$ | $A_O$ | $B_O$ |
|---|---|---|---|---|
| 1, 4 | + | + | $S_0 S_+^1 N_+^{\mathbf{2}} S^0 N_-^1 S_-^2 = S_0 S_+^1 N S_-^2$ | $PH^2$ |
| 2 | – | – | $S_0 N_+^1 S_+^{\mathbf{2}} S^0 S_-^1 N_-^2 = N S_+^2 S^0 S_+^1$ | $PH^2$ |
| 3, 6 | – | + | $S_0 N_+^1 N_+^{\mathbf{2}} S^0 S_-^1 S_-^2$ | $PEPH^3$ |
| 5 | + | – | $S_0 S_+^1 S_+^2 S^0 N_-^1 N_-^2$ | $H^3PEP$ |

**Table 1.** T-type of a singular point when $d > 0$ $(r = \overline{1,6})$.

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| $q$ | $P(q)$ | $A_O$ | $B_O$ |
|---|---|---|---|
| + | + | $S_0S_+N_+S^0$ | $H^2P$ |
| – | – | $S_0N_+S_+S^0$ | $PH^2$ |
| + | – | $S_0S^0S_-N_-$ | $H^2P$ |
| – | + | $S_0S^0N_-S_-$ | $PH^2$ |
| 0 | + | $S_0S_+NS_-$ | $H^2P$ |
| 0 | – | $NS_+S^0S_-$ | $PH^2$ |

**Table 2.** T-type of the singular point $O(0, 0)$ when $d = 0$.

2) in the case of $d = 0$ depending on signs of values $q$ and $P(q)$, they have forms, indicated in a **Table 2**,

3) in the case of $d < 0$ they have forms: $A_O = S_0S^0$, $B_O = HH$ (**Table 1**).

**Note 2**. Let us clarify the meaning of the new symbols introduced in Theorem 1.

$S_0$ ($S^0$) means a bundle $S$, adjoining to point $O(0,0)$ from the domain $x > 0$ along a semi-axis $x = 0, y < 0$, when $t \to +\infty$ (along a semi-axis $x = 0$, $y > 0$, when $t \to -\infty$).

The lower sign index " + " or "–" on every bundle $N$ or $S$, different from $S_0$ and $S^0$, indicates whether the bundle consists of $O_+$-curves or of $O_-$-curves. Upper index 1 or 2 on every such a bundle indicates whether its $O$-curves are adjoining to point $O$ along a straight line $y = q_1x$ or along a straight line $y = q_2x$.

In **Table 2**, row 5, 6, a bundle $N$ does not have a lower sign index because it contains both $O_+$ -curves and $O_-$ -curves simultaneously.

**Corollary 1.** From Theorem 1, it follows, that Eq. (1) systems do not have limit cycles on the $R^2_{x;y}$ plane.

Indeed, such a cycle could surround a singular point $O$ (0,0) of an Eq. (1) system, and then the Poincare index of this singular point must be equal to 1 [1]. However, Bendixon's formula for the index of an isolated singular point of a smooth dynamic system is as follows:

$$I(O) = 1 + \frac{e - h}{2}$$

where $e(h)$ is the number of elliptical (hyperbolic) $O$-sectors of the system. This formula and our Theorem 1 give: for the singular point $O$ (0, 0) of every Eq. (1) system, Poincare index $I(O) = 0$.

**Corollary 2.** For the singular point $O$ (0, 0) of an Eq. (1) system, 11 different topological types (T-types) are possible, and from the analysis of these 11 T-types we can conclude:

for every Eq. (1) system, the singular point $O(0, 0)$ has not more than four separatrices (actually 2, 3, or 4 ones).

## 4. Infinitely remote singular points (IR points)

Now it is time to discuss the behavior of trajectories of the Eq. (1) systems in a neighborhood of infinity. For the investigation of this question we use the method of Poincare consecutive transformations, or mappings [1].

The first Poincare transformation

$$x = \frac{1}{z}, \quad y = \frac{u}{z} \quad \left(u = \frac{y}{x}, \quad z = \frac{1}{x}\right).$$

unambiguously maps a phase plane $R^2_{x,y}$ of the Eq. (1) system onto a Poincare sphere $\sum$: $x^2 + y^2 + z^2 = 1$ (where $z = -Z$ [1]) with the diametrically opposite points identified, which is considered without its equator $E$, and an infinitely remote straight line of a plane $\overline{R^2_{x,y}}$. The first Poincare transformation maps onto the equator $E$ of the sphere $\sum$; the diametrically opposite points are also considered to be identified.

The Eq. (1) system in this mapping transforms into a system, which in the Poincare coordinates $u, z$ after a time change $dt = -z^2 d\tau$ looks like the following:

$$\frac{du}{d\tau} = P(u)u - Q(u)z, \qquad \frac{dz}{d\tau} = P(u)z,$$

where $P(u) :\equiv X(1, u)$ and $Q(u) :\equiv Y(1, u)$ are reciprocal polynomials.

This new system is determined on the whole sphere $\sum$, including its equator, and on the whole $(u, z) -$ plane $\alpha^*$, which is tangent to a sphere $\sum$ at point C = (1, 0, 0). We shall study this system, namely on a plane $\overline{R^2_{u,z}}$, and project the received results onto a closed circle $\overline{\Omega}$, sequentially mapping, first, a plane $R^2_{u,z}$ onto the sphere $\sum$ from its center, and second, its lower semi-sphere $\overline{H}$ onto the Poincare circle $\overline{\Omega}$, i. e., onto a closed unit circle of a plane $R^2_{x,y}$ through the orthogonal mapping.

For our new system, the axis $z = 0$ is invariant (consists of this system's trajectories). On this axis, lie its singular points $O_i(u_i, 0)$, $i = \overline{0, m}$, where $u_i$, $i = \overline{1, m}$ are all real roots of the polynomial $P(u)$, and $u_0 = 0$; at the same time, there may exist $i_0 \in \{1, \ldots, m\}$: $u_{i_0} = 0$. Let us call such points IR points of the first kind for the Eq. (1) system.

The second Poincare transformation

$$x = \frac{v}{z}, \quad y = \frac{1}{z} \left(v = \frac{x}{y}, \quad z = \frac{1}{y}\right)$$

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also unambiguously maps a phase plane $R^2_{x,y}$ onto a Poincare sphere $\sum$ with the diametrically opposite points identified, considered without its equator. Every Eq. (1) system transforms into a system, which in the coordinates $\tau, v, z$ looks like the following:

$$\frac{dv}{d\tau} = -X(v,1) + Y(v,1)vz, \qquad \frac{dz}{d\tau} = Y(v,1)z^2.$$

This last system is determined on the whole sphere $\sum$, and on the whole $(v,z)$ – plane $\widehat{\alpha}$, which is tangent to a sphere $\sum$ at point $D = (0,1,0)$ [1]. A set $z = 0$ is invariant for this last system. On this set, lie its singular points$(v_0, 0)$, where $v_0$ is any real root of the polynomial $X(v,1) \equiv p_3 + p_2 v + p_1 v^2 + p_0 v^3$. It would be natural to call such points IR points of the second kind for Eq. (1) systems, but each of these points, for which $v_0 \neq 0$, obviously coincides with one of the IR-points of the first kind, namely with the point $\left(\frac{1}{v_0}, 0\right)$,

while $v_0 = 0$ is not a root of the polynomial $X(x, 1)$, because $X(0, 1) = p_3 \neq 0$ for the Eq. (1) system. Consequently, the following corollary is correct.

**Corollary 3.** The infinitely remote singular points of any Eq. (1) system are only IR-points of the first kind.

With the orthogonal projection of a closed lower semi-sphere $\overline{H}$ of a Poincare sphere $\sum$ onto a plane $x$, $y$, its open part $H$ one-to-one maps onto an open Poincare circle $\Omega$, while its boundary $E$ (an equator of the Poincare sphere $\sum$) maps onto the boundary of the Poincare circle $\Gamma = \partial\Omega$, which implies the following. 1) Trajectories of any Eq.(– (including its singular point $O$ (0, 0)) are displayed in a circle $\Omega$, filling it.

2) Such a system's infinitely remote trajectories (including IR points) are displayed on the boundary $\Gamma$ of a circle $\Omega$, filling it.

Following Poincare, we call the first trajectories of the Eq. (1) system in $\Omega$, and the second, we call trajectories of the Eq. (1) system on $\Gamma$.

As it follows from the aforementioned conclusions, to each IR point $O_i(u_i, 0)$, of the Eq. (1) system, $i \in \{1, \ldots, m\}$, correspond two diametrically opposite points situated on the $\Gamma$ circle.

$$O_i^{\pm}(u_i, 0) : O_i^{+}\left(O_i^{-}\right) \in \Gamma^{+(-)} := \Gamma\big|_{x>0\ (x<0)}.$$

$\forall i \in \{1, \ldots, m\}$ for the point $O_i^{+}\left(O_i^{-}\right)$, we shall introduce the following notation.

1. Let a $O_i^{+}\left(O_i^{-}\right)$—curve be a semi-trajectory of the Eq. (1) system in $\Omega$, starting in an ordinary point $p \in \Omega$ and adjacent to a point $O_i^{+(-)}$.
2. A notation for bundles $N$, $S$, adjacent to a point $O_i^{+}\left(O_i^{-}\right)$ from the circle $\Omega$, similar to the notation introduced for the point $O$ (0, 0).

**3.** A notation of a word $A_i^+(A_i^-)$ consisting of letters $N$,$S$, which fixes an order of bundles of $O_i^+(O_i^-)$-curves at a semi-circumvention of the point $O_i^+(O_i^-)$ in the circle $\Omega$ in the direction of increasing $u$.

We shall describe a T-type of a point $O_i^+(O_i^-)$with a word $A_i^+(A_i^-)$, and a T-type of a point $O_i$ with words $A_i^{\pm}$.

T-types of IR points $O_0^{\pm}(0,0)$ of Eq. (1) systems are described in the following theorem.

**Theorem 2.** Let a number $u=0$ be the multiplicity $k\in\{0,\ldots,3\}$ of the root of a polynomial $P(u)$ of the Eq. (1) system. Then, words $A_0^{\pm}$, which determine the topological types (T-types) of IR points $O_0^{\pm}(0,0)$ of this system, depending on the value of $k$ and a sign of a number $ap_k$ (where $a$ and $p_k$ are coefficients of the system), have the forms as shown in **Table 3** [5].

**Corollary 4**. IR points $O_0^{\pm}$of any Eq.(1)—system do not have separatrices.

T-types of IR points $O_i(u_i,0)\neq O_o(0,0)$, $i=\overline{1,m}$, of Eq. (1) systems are described in the following theorem.

**Theorem 3**. Let a real number $u_i(\neq 0)$ be a multiplicity $k_i\in\{1,2,3\}$ of the root of a polynomial $P(u)$ of an Eq. (1) system. Then for this system, a value $g_i = P^{(ki)}(u_i)Q(u_i) \neq 0$ and words $A_i^{\pm}$, which determine topological types (T-types) of IR points $O_i^{\pm}(u_i,0)$ of this system, depending on the value of $k_i$ and signs of numbers $u_i$ and $g_i$, have forms as shown in **Table 4** [5].

**Corollary 5**. As can be seen from Theorems 2 and 3, for the IR points of Eq. (1) systems, only a finite number (13) of different T-types are possible. The investigation of these T-types shows that IR-points of each Eq. (1) system have only $m$ separatrices: one separatrice for every singular point $O_i(u_i,0)$, $i=\overline{1,m}$.

**Note 3**. In **Tables 3** and **4**, the lower sign index " + " or "–" on every bundle $N$ or $S$, indicates whether the bundle adjusts to the point $O_i^+$ (or to the point $O_i^-$) from the side $u>u_i$ or from the side $u<u_i$ of the isocline $u=u_i$.

In **Table 3**, row 1, a bundle $N$ does not have a lower sign index because as the detailed study of this case shows, it contains $O_i^+$-curves ($O_i^-$-curves) in every domain $|u|>0$ [5].

| $k$ | $ap_k$ | $A_0^+$ | $A_0^-$ |
|---|---|---|---|
| 0 | 0 | $N$ | $N$ |
| 0, 2 | + (−) | $N_+(N_-)$ | $N_-(N_+)$ |
| 1, 3 | + (−) | $N_-N_+(\varnothing)$ | $\varnothing(N_-N_+)$ |

**Table 3.** T-types of IR points $O_0^{\pm}(0,0)$.

| $u_i$ | $k_i$ | $g_i$ | $A_i^+$ | $A_i^-$ |
|---|---|---|---|---|
| +(−) | 1, 3 | + | $N_+(N_-)$ | $S_-(S_+)$ |
| +(−) | 1, 3 | − | $S_-(S_+)$ | $N_+(N_-)$ |
| +(−) | 2 | + | $S_-N_+(\varnothing)$ | $\varnothing(N_-S_+)$ |
| +(−) | 2 | − | $\varnothing(N_-S_+)$ | $S_-N_+(\varnothing)$ |

**Table 4.** T-types of IR points $O_i^{\pm}(u_i, 0),\ \ i \in \{1, \dots, m\}$.

## 5. Systems containing 3 and 2 multipliers in their right parts

In this section, we present a solution to the main assigned problem for those Eq. (1) systems whose decompositions of forms *X (x, y), Y (x, y)* into real forms of lower degrees contain 3 and 2 multipliers, respectively:

$$X(x,y) = p_3(y - u_1x)(y - u_2x)(y - u_3x), Y(x,y) = c(y - q_1x)(y - q_2x) \tag{2}$$

where $p_3 \ \ > 0,\ \ c > 0,\ \ u_1 < u_2 < u_3, q_1 < q_2,\ \ u_i \neq q_j$ for each *i* and *j*.

The solution process contains the follows steps.

### 5.1. Basic concepts and notation

The following notations are introduced for the arbitrary system under consideration in the Section 5.

*P(u), Q(u)* – the system's polynomials *P, Q:*

$$P(u) := X(1,u) \equiv p_3(u - u_1)(u - u_2)(u - u_3),\ \ Q(u) := Y(1,u) \equiv c(u - q_1)(u - q_2)$$

*RSP* (*RSQ*) – an ascending sequence of all real roots of then system's polynomial *P(u) (Q(u)),*
*RSPQ* – an ascending sequence of all real roots of both the system's polynomials *P(u), Q(u).*

### 5.2. The double change (DC) transformation

Let us call a double change of variables in this dynamic system: *(t, y) → (−t, −y).* The double change transformation transforms the system under consideration into another such system, for which numberings and signs of roots of polynomials *P(u), Q(u),* as well as the direction of motion upon trajectories with the increasing of *t* are reversed. Let us agree to call a pair of different Eq. (2) systems mutually inversed in relation to the DC transformation, if this transformation appears to convert one into another, and call them independent of a DC transformation in the opposite case.

Clearly, 10 different types of *RSPQ* are possible for an arbitrary Eq. (2) system, as $C_5^2 = \frac{5!}{3!2!} = 10$.

As we can conclude using the DC transformation of Eq. (2) systems, six of the *RSPQ*s appear to be independent in pairs. Similarly, each of the remaining four systems has the mutually inversed one among the first six Eq. (2)-systems.

Let us assign a specific number $r \in \{1, \ldots, 10\}$ to each one of the different *RSPQ*s of the Eq. (2) system in such a manner that $RSPQr = \overline{1, 6}$ are independent in pairs, while *RSPQ* sequences with numbers $r = \overline{7, 10}$ are mutually inversed to *RSPQ*`s which have numbers $r = \overline{1, 4}$.

It is time to introduce the important notion of a family number *r* of Eq. (2) systems.

An *r family* of Eq. (2) systems $:=$ the totality of systems (belonging to Eq. (2) family) having the *RSPQ* number *r*.

Now following a single plan, we consistently investigate the families of Eq. (2)systems that have numbers $r = \overline{1, 6}$. For families having numbers $r = \overline{7, 10,}$ we obtain data through the DC-transformation of families, $r = \overline{1, 4}$.

A plan of the investigation of each selected Eq. (2) family contains the follows items.

1. We determine a list of singular points of systems of the fixed family in a Poincare circle $\overline{\Omega}$. They appear to be a point $O\ (0, 0) \in \Omega$ and points $O_i^{\pm}\ (u_i, 0) \in \Gamma$, $i = \overline{0, 3}$, $u_0 = 0$. For every point in the list, we use the notions of a saddle (S) and node (N) bundles adjacent to this point's semi-trajectories, of a separatrix of the singular point, and of a topodynamical type of the singular point (TD type).
2. Further, we split the family under consideration to subfamilies with numbers $s = \overline{1, 7}$. For every subfamily, we reveal topodynamical types of singular points and separatrices of them.
3. We investigate the separatrices' behavior for all singular points of systems belonging to the chosen subfamily $\forall\, s \in \{1, \ldots, 7\}$. Very important are the following questions: a question of a uniqueness of a continuation of every given separatrix from a small neighborhood of a singular point to all the lengths of this separatrix, as well as a question about a mutual arrangement of all separatrices in a Poincare circle Ω. We answer these questions for all families of systems under consideration.
4. As a result of all previous studies, we depict phase portraits of dynamic systems of a given family and outline the criteria of every portrait appearance [5, 6].

From this section, we can conclude the following:

Systems of the family number $r = 1$ have 25 different types of phase portraits.

Systems of families number 2 and 3: there are 9 types of phase portraits per family.

Systems of families 4 and 5: there exist 7 types of phase portraits per family.

Systems belonging to the family number $r = 6$ show 36 different types of phase portraits.

Hence, we have obtained 93 different types in total for the systems described in this section—a lot of possible types at first glance. However, it is important to keep this in mind: every given family includes an uncountable number of differential systems.

## 6. Two classes of systems containing various combinations of two different multipliers in both right parts: an A-class

In Sections 6 and 7, the problem has been solved for an Eq. (3) family. The Eq. (3) family of Eq. (1) systems is as follows—the family consists of a totality of all Eq. (1) systems; for each of them, decompositions of forms *X (x, y), Y (x, y)* into real multipliers of the lowest degrees contain two multipliers each:

$$\mathrm{X}(x,y) = p(y-u_1x)^{k_1}(y-u_2x)^{k_2}, \mathrm{Y}(x,y) = \mathrm{q}(y-q_1x)(y-q_2x) \quad (3)$$

where $p, \ q, \ u_1, u_2, q_{1,} q_2 \in R, p > 0, \ q > 0, \ u_1 < u_2, \ q_1 < q_2, \ u_i \neq q_j$ for each $i,j \ \in \{1,2\}$, $k_1, k_2 \in N$, $k_1 + k_2 = 3$.

It is natural to distinguish two classes of Eq. (3) systems. The A class contains systems with $k_1 = 1, \ k_2 = 2;$ and the B class contains systems with $k_1 = 2, k_2 = 1$.

In this section, we give a full solution of the assigned task for systems belonging to the A class of the Eq. (3) family, i.e.,

$$\frac{dx}{dt} = p(y-u_1x)(y-u_2x)^2, \frac{dy}{dt} = \mathrm{q}(y-q_1x)(y-q_2x) \quad (4)$$

The process of forming the solution contains steps similar to the ones described in Section 4 of this chapter.

For an arbitrary Eq.(4)– system, we introduce the following concepts.

Let *P(u), Q(u)* be the system's polynomials *P, Q:*

$$\mathrm{P(u)} := X(1,u) \equiv p(u-u_1)(u-u_2)^2, \quad Q(u) := Y(1,u) \equiv q(u-q_1)(u-q_2),$$

and *RSP* (*RSQ*) be an ascending sequence of all the real roots of the system's polynomial, while *P(u) (Q(u)),RSPQ* is an ascending sequence of all the real roots of both system's polynomials *P(u)* and *Q(u).* There exist 6 different possible variants of *RSPQ* as $C_4^2 = \frac{4!}{2!2!} = 6$. Let us number them from 1 to 6 in some order.

Now let us put into use an important notion:

An *r-family* of Eq.(4) – systems is the totality of Eq. (4) dynamic systems with the *RSPQ* number *r* from the list of six allowable variants.

**A consistent research of families of Eq. (4) dynamic systems**.

The steps of research of every fixed family belonging to Eq. (4) dynamic systems are as follows.

1. For all singular points of a given dynamic system that belongs to the family under consideration, let us introduce notions of S (saddle) and N (node) bundles of semi-trajectories, which are adjacent to a chosen singular point; also let us introduce a notion for its separatrix and a notion for its topodynamical type (TD-type).

2. Now the considered family must be divided into subfamilies numbered $s \in \{1, \ldots, 5\}$.Then it is necessary to determine the TD-types of singular points of systems belonging to the obtained subfamilies, and separatrices of singular points $\forall s = \overline{1, 5}$.
3. For all five subfamilies, we investigate the separatrices` of singular points behavior and find an answer to a question concerning a uniqueness of a global continuation of every chosen separatrix from a tiny neighborhood of a singular point to all the lengths of this separatrix in the Poincare circle Ω, as well as an answer to a question of all separatrices` mutual arrangement in Ω.

The mutual arrangement of all separatrices in the Poincare circle is invariable when, for a given *s*, a global continuation of every separatrix of each singular point of the subfamily number *s* is unique. Consequently, all systems of a chosen subfamily number *s* have, in a Poincare circle, one common type of phase portrait.

But in a different situation, when, for a fixed number *s*, systems of such subfamily have, for example, *m* separatrices with global continuations that are not unique, this subfamily is divided into *m* additional subfamilies (so as to say *subsubfamilies*) of the next order.

As we could understand conducting their further study, for each of *subsubfamilies,* the global continuation of every separatrix is unique, and the mutual arrangement of separatrices in the Poincare circle Ω is invariable.

As a result, the topological type of phase portrait of all systems belonging to this *subsubfamily* in the $\overline{\Omega}$ circle is common for the chosen *subsubfamily.*

**4**. We depict phase portraits in $\overline{\Omega}$ for the systems of Eq.(4) families, $r = \overline{1, 6}$, in the two possible forms (the table and the graphic ones), and indicate for each portrait close to coefficient criteria of its realization.

A conclusion for the Section 6 of our chapter is:

1. Eq. (4)–systems belonging to the number 1 family have in the Poincare circle $\overline{\Omega}$, 13 different topological types of phase portraits.
2. Eq.(4)– systems of the family number 2 have 7 types.
3. Family number 3 have 10 types.
4. Family numbers 4, 5, and 6 have 5 different types of phase portraits per number.

This means that in total, all large families of Eq.(4) dynamic systems of the A class may have 45 different topological types of phase portraits in a Poincare circle.

## 7. Systems with 2 different multipliers in both right parts, belonging to a B class

In this section, the full solution of our task for Eq. (3) systems of the B class is given:

$$\frac{dx}{dt} = p(y - u_1x)^2(y - u_2x), \ \frac{dy}{dt} = \mathrm{q}(y - q_1x)(y - q_2x). \tag{5}$$

For an arbitrary Eq.(5)– system, *P(u), Q(u)* are the system's polynomials *P, Q.*

$$\mathrm{P(u)}: = X(1, u) \equiv p(u - u_1)^2(u - u_2), \quad Q(u): = Y(1, u) \equiv q(u - q_1)(u - q_2),$$

*RSPQ* shows 6 different variants, because $C_4^2 = 6$.

We can thus conclude that all Eq. (5) family of systems is split into 52 different subfamilies, and all systems of each chosen subfamily show in a circle $\overline{\Omega}$, one common type of a phase portrait belonging to this particular subfamily. We have constructed all 52 topologically different phase portraits.

## 8. Systems containing 3 and 1 different multipliers in right parts

In this section, we solve the problem for an Eq. (6) family, i.e., for a family of Eq. (1) systems

$$\frac{dx}{dt} = p_3(y - u_1x)(y - u_2x)(y - u_3x), \ \frac{dy}{dt} = c(y - q_1x)^2 \tag{6}$$

$$p_3 \ > 0, \ \ c > 0, \ \ u_1 < u_2 < u_3, q(\in R) \neq u_{i,} i = \overline{1, 3}.$$

The solution process includes the follows steps. Let us break the Eq. (6) family into subfamilies numbered $r = \overline{1, 4}$.

Each of these is a totality of systems with an *RSPQ* number *r,* where *r* is the system's number in the list of possible *RSPQ*s.

1. $u_1, u_2, u_3, q,$
2. $u_1, u_2, q, u_3,$
3. $u_1, q, u_2, u_3,$
4. $q, u_1, u_2, u_3.$

Applying to the Eq. (6) system, a double change of variables (DC): $(t, y) \to (-t, -y)$, we reveal that it transforms families of these systems having the numbers *r* = 1, 2, 3, 4, into their families with numbers *r* = 4, 3, 2, 1 respectively, and backward. We emphasize: this fact means that families of Eq. (6) systems having numbers 1 and 2 are not connected with the DC transformation, and that families having numbers 3 and 4 are not related to each other; at the same time, family number 3 is mutually inversed by the DC transformation to the family number 2, and family number 4 is mutually inversed to the family number 1 correspondingly. This conclusion follows from the consideration of their *RSPQ* sequences [5, 6].

1. We study alternately the families of systems, *r* = 1,2, following the common program of Eq. (1) systems study [5], i.e.:

1. We fix $r \in \{1,2\}$, then we break the chosen family into subfamilies numbered $s$ [5, 6], $s = \overline{1,9,}$ and find the topodynamical types (TD-types) of singular points of these systems.

2. We construct for the systems of a fixed subfamily $\forall s = \overline{1,9,}$ the so-called off-road map (ORM) [5–7]. The ORM helps us to find an $\alpha(\omega)-$ limit set of every $\alpha(\omega)-$ separatrix. It also lets us describe the mutual arrangement of all separatrices in the Poincare circle Ω.

3. We depict all possible topologically different phase portraits for Eq. (6) systems.

**2.** We investigate consistently families of Eq. (6) systems, $r = 3, 4$, using the DC transformation of the results obtained for families, $r = 2, 1$. Then, we depict all types of existing phase portraits for the families 3 and 4.

Then, we conclude the following.

For families of Eq. (6) systems with numbers 1, 2, 3, and 4, there exist

$$15 + 11 + 11 + 15 = 52$$

different topological types of phase portraits in a Poincare circle $\overline{\Omega}$.

## 9. Systems containing 2 and 1 different multipliers in right parts

In this section, we give the full solution of the problem for Eq. (7) systems, i.e., for the Eq. (1) systems of the kind

$$\dot{x} = p_0x^3 + p_1x^2y + p_2xy^2 + p_3y^3 \equiv p_3(y - u_1x)^2(y - u_2x) \tag{7}$$

$$\dot{y} = x^2 + bxy + cy^2 \equiv c(y - qx)^2,$$

where $p_3 > 0, \;\; c > 0, \;\; u_1 < u_2, q\ (\in R) \neq u_{1,2}$.

The process of study of these systems is quite similar to that previously described for other families of Eq. (1) systems. For an arbitrary Eq. (7) system, *P(u), Q(u)* are the system's polynomials *P, Q:*

$$P(u) := X(1,u) \equiv p_3(u - u_1)^2(u - u_2), \qquad Q(u) := Y(1,u) \equiv c(u - q)^2,$$

and there exists 3 different variants for their *RSPQ*s.

A conclusion from our research for this particular type of systems is the following.

We`ve revealed, that for every possible family of Eq. (7) systems, 7 different topological types of their phase portraits are being implemented. This means that for all three existing families of such systems, $r = \overline{1,3,}$ the number of different phase portraits is 21 [8, 9].

## 10. Conclusions

The presented work is devoted to the original study.

The main task of the work was to depict and describe all the different, in the topological meaning, phase portraits in a Poincare circle, possible for the dynamical differential systems belonging to a broad family of Eq. (1) systems, and to its numerical subfamilies. The authors have constructed all such phase portraits in two ways—in a descriptive (table) and in a graphic form. Each table contains 5–6 rows. Every row describes one invariant cell of the phase portrait in detail—it describes its boundary, source, and sink of its phase flow. The table was the descriptive phase portrait.

The second objective of this work was to develop, outline, and successfully apply some new effective methods of investigation [8–10].

This was a theoretical work, but due to aforementioned new methods, the chapter may be useful for applied studies of dynamic systems of the second order with polynomial right parts. The authors hope that this work may be interesting and useful for researchers and for both students and postgraduates.

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