Chapter

Some Results on the Non-Homogeneous Hofmann Process

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Abstract

The classical counting processes (Poisson and negative binomial) are the most traditional discrete counting processes (*DCPs*); however, these are based on a set of rigid assumptions. We consider a non-homogeneous counting process (which we name non-homogeneous Hofmann process – *NHP*) that can generate the classical counting processes (*CCPs*) as special cases, and also allows modeling counting processes for event history data, which usually exhibit under- or over-dispersion. We present some results of this process that will allow us to use it in other areas and establish both the probability mass function (*pmf*) and the cumulative distribution function (*cdf*) using transition intensities. This counting process (*CP*) will allow other researchers to work on modelling the *CP*, where data dispersion exists in an efficient and more flexible way.

Keywords: mixed Poisson Process, Hofmann process, variance-to-mean ratio, transition intensity

1. Introduction

In ref. [1], Hofmann introduced a new class of infinitely divisible mixed Poisson process (*MPP*), this broader class of *CP* allows obtaining other *CCP* by simply modifying or choosing its parameters, as well as Poisson, negative binomial, Poisson-Pascal among other distributions (see [2]). The family of distributions defined by Hofmann has been used in many types of applications of modelling and simulation studies that include topics such as accident models [3].

In this chapter, we analysed the event of number process $\{N(t), t \geq 0\}$ and used a broader *CP,* which is based on the Hofmann process. The appeal of this *CP* is that, analogous to the family of frequency distributions, it allows to generate several known *CP*. Through an *NHP,* we can generate the following as special cases: the Poisson counting process (*PCP*), the negative binomial counting process (*NBCP*) and the Poisson-Pascal process among other *CCPs*, and this allows us to obtain models for *CP* with under- or over-dispersion. The *NHP* was introduced by Hofmann [1] and has been used by other researchers [3–5]. Some properties of the *NHP* found by Walhin

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[2] are presented in this chapter, and we used the transition intensities to describe additional properties of the *NHP*.

The objective of this chapter is to present a unified view of related results on the *NHP*. The chapter is organised as follows: in Section 2, we present the *NHP*; in Section 3, we present some statistical properties, such as *pmf* and probability generating function (pgf), and formulas for the mean and variance are derived; in Section 4, we present various approaches for the *NHP* using *CCP*; in Section 5, we present other properties for *NHP*; finally, conclusions are presented.

2. Basic concepts of the *NHP*

Let us take $N(t)$ as the number of events that occurs in the time interval $(0, t]$ with $t > 0$ and $N(0) = 0$. The probability of n events occurring in this time interval is denoted by

$$P_n(t) = P[N(t) = n], \qquad n = 0, 1, 2, \ldots \tag{1}$$

According to Dubourdieu [6], an *MPP* $\{N(t) : t \geq 0\}$ is a *PCP* with rate Λ, where the non-negative random variable Λ is called a structure variable. The *MPP* has been studied by several authors [7–9].

When Λ is a continuous random variable with probability density function (*pdf*), $f(\lambda)$, we can find probability by

$$\begin{aligned} \underbrace{\mathbb{E}[P[N(t) = n|\Lambda]]} &= \int_0^\infty P[N(t) = n|\Lambda = \lambda] f(\lambda) d\lambda \\ P[N(t) = n] &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} f(\lambda) d\lambda. \end{aligned} \tag{2}$$

For $n = 0$ and $t > 0$ we have

$$P_0(t) = \int_0^\infty e^{-\lambda t} f(\lambda) d\lambda, \tag{3}$$

The higher order derivatives of the last expression with respect to t are

$$P_0^{(n)}(t) = \frac{d^n}{dt^n} P_0(t) = (-1)^n \int_0^\infty \lambda^n e^{-\lambda t} f(\lambda) d\lambda. \tag{4}$$

By substituting (4) into (2) we get

$$P_n(t) = \frac{t^n}{n!} \left[(-1)^n P_0^{(n)}(t) \right], \qquad n \geq 1 \tag{5}$$

The expressions (3) and (5) characterize an *MPP* with a continuous structure variable Λ. According to Hofmann [1], for the construction of examples, a special structure function is generally assumed, and from this the *pmf* is calculated by (3), (5). In most cases, this leads to formally complicated expressions. In ref. [1], Hofmann presents a *CP* called Hofmann process as an option to model the event number process given by (2) and whose general expression for (3) is as follows:

$$P_0(t) = \exp\{-\theta(t)\} \qquad \theta(t) = \int_0^t \lambda(\tau; a) d\tau \tag{6}$$

where $P_0(t)$ is a completely monotonic function[1]. And $\lambda(\tau; a)$ is a function of three parameters: $a \geq 0$, $q > 0$ and $\kappa \geq 0$, which is a function infinitely divisible and given by

$$\lambda(\tau; a) = \frac{q}{(1 + \kappa\tau)^a} \qquad \forall \tau > 0. \tag{7}$$

Although $\lambda(\tau; a)$ depends on three parameters, we use this notation given that the parameter a provides various *CCPs*. We denote the *NHP* by $\mathcal{H}(a, q, \kappa)$, if the *pmf* of $N(t)$ satisfies the expressions (5) and (6).

Using the expression (7), we get by integrating that

$$\theta(t) = \begin{cases} \ln\left[(1 + \kappa t)^{q/\kappa}\right] & \text{if} \quad a = 1 \\ \dfrac{q}{\kappa(1 - a)}\left[(1 + \kappa t)^{1-a} - 1\right] & \text{if} \quad a \neq 1 \end{cases} \tag{8}$$

By substituting (8) into (6)

$$P_0(t) = \begin{cases} (1 + \kappa t)^{-\frac{q}{\kappa}} & \text{if} \quad a = 1 \\ \exp\left\{-\dfrac{q}{\kappa \cdot (1 - a)}\left[(1 + \kappa t)^{1-a} - 1\right]\right\} & \text{if} \quad a \neq 1 \end{cases} \tag{9}$$

Remark 1.1: If in the expression (9) for $a = 1$ we take the limit as $\kappa \to 0$, we have:

$$\lim_{\kappa \to 0} (1 + \kappa t)^{-\frac{q}{\kappa}} = e^{-qt}, \tag{10}$$

and the last expression agrees with the adequate $P_0(t)$ of a *PCP* with rate qt.

3. Basic properties of the *NHP*

Theorem 1.2: Let $N(t)$ be an NHP then

i. The pgf of the process is given by

$$G_N(z; t) = \begin{cases} (1 + \kappa(1 - z)t)^{-q/\kappa} & \text{if} \quad a = 1 \\ \exp\left\{-\dfrac{q}{\kappa(1 - a)}\left[(1 + \kappa(1 - z)t)^{1-a} - 1\right]\right\} & \text{if} \quad a \neq 1 \end{cases} \tag{11}$$

[1] We say that a function $g(t)$ with $t \in \mathbb{R}^+$ is completely monotonic if it has derivatives $g^{(n)}(t)$ for all $n \in \mathbb{N}$ and its derivatives have alternating signs, i.e., if $(-1)^n g^{(n)}(t) \geq 0$, $t > 0$.

Note that $G_N(z;t) = P_0((1-z)t)$ with $0 \le z < 1$.

ii. The *pmf* of $N(t)$, for t fixed, satisfies the following recursive formula:

$$P_{n+1}(t) = \frac{t\lambda(t;a)}{n+1} \sum_{i=0}^{n} \binom{a+i-1}{i} \left(\frac{\kappa t}{1+\kappa t}\right)^{i} P_{n-i}(t) \tag{12}$$

where $P_0(t) = G_N(0;t)$ is given by (9) and

$$P_0^{(n+1)}(t) = \lambda(t;a) \sum_{j=0}^{n} \binom{n}{j} (-1)^{j+1} \frac{\Gamma(a+j)}{\Gamma(a)} \left(\frac{\kappa}{1+\kappa t}\right)^{j} P_0^{(n-j)}(t)$$

iii. If $a = 1$ the $P_n(t)$ satisfies the recurrence relation

$$\frac{P_{n+1}(t)}{P_n(t)} = \frac{-t}{n+1} \frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)} = \frac{q+\kappa n}{1+\kappa t} \frac{t}{n+1}. \tag{13}$$

iv. The process $N(t)$ has a mean and variance given by

$$\mathbb{E}[N(t)] = qt \qquad \text{and} \qquad Var[N(t)] = (1+a\kappa t)\mathbb{E}[N(t)] \tag{14}$$

Proof:
See details in [2] or [10].
Note that from (14) we have that if $q \neq 0$ then:

$$\lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} = q. \tag{15}$$

It is possible from (14) to calculate the measure based on the variance-to-mean ratio (VMR) introduced by [11]:

$$ID(t) = \frac{Var[N(t)]}{\mathbb{E}[N(t)]} = 1 + a\kappa t. \tag{16}$$

As $ID(t) > 1$, then using the criterion of the VMR, we have that the *NHP* is an over-dispersed *CP* and hence is an option for modelling over-dispersion in count data.

Using the expression (11), in **Table 1**, we present the functions for qt and κt that allow to obtain some *CP*. We consider the *CCPs* studied in [10], which are special cases of *NHP* when $a = 1$ since this reduces to the Panjer counting process (see [12]). In addition, we consider other processes, such as the Neyman Type A process introduced by [13], the Poisson Pascal process introduced by [14] and the Pólya-Aeppli process introduced by [15].

3.1 *NHP* is infinitely divisible

The following relationships are identical to those of [16] which characterize infinitely divisible *pmf*:

	Counting process	$P_0[(1-z)t]$	Functions	
			qt	κt
Classical (a = 1)	Poisson	$\exp\{-(1-z)\gamma t\}, \quad \kappa \to 0$	γt	0
	Negative binomial (or Pólya)	$\left[\frac{\delta}{\delta+(1-z)t}\right]^{\gamma}, \quad \delta > 0$	$\frac{\gamma}{\delta}t$	$\frac{t}{\delta}$
	Geometric	$\frac{\delta}{\delta+(1-z)t}$	$\frac{t}{\delta}$	$\frac{t}{\delta}$
Other (a > 1)	Neyman Type A	$\exp\{\gamma[\exp\{(z-1)\delta t\}-1]\}, \quad a \to \infty$	$\gamma\delta t$	$\frac{\delta t}{a-1}$
	Poisson-Pascal	$\exp\left\{\gamma\left[(1+(1-z)\delta t)^{-(a-1)}-1\right]\right\}$	$(a-1)\gamma\delta t$	δt
	Pólya-Aeppli	$\exp\left\{\frac{-(1-z)\gamma t}{1-\left[1-(1+\delta t)^{-1}\right]z}\right\}, \quad a=2$	$(1+\delta t)\gamma t$	δt
Source: own elaboration				

Table 1.
Functions qt and κt for some CCPs.

Theorem 1.3: The *pmf* $\{P_n(t)\}$ with $P_0(t) > 0$ is infinitely divisible if and only if satisfies that

$$(n+1)P_{n+1}(t) = \sum_{i=0}^{n} r_i(t)P_{n-i}(t) \qquad \text{for } t \text{ fixed.}$$

where the quantities $r_n(t)$ with $n \in \mathbb{Z}^+$ are nonnegative.

Proof: See details in [16].

Corollary 1.3.1: The *pmf* $\{P_n(t)\}$ of the *NHP* is infinitely divisible.

Proof:

By multiplying (12) by $(n+1)$ we get

$$(n+1)P_{n+1}(t) = \sum_{i=0}^{n} t\lambda(t;a)\binom{a+i-1}{i}\left(\frac{\kappa t}{1+\kappa t}\right)^{i} P_{n-i}(t).$$

We denote

$$r_i(t;a) = qt\binom{a+i-1}{i}\frac{(\kappa t)^i}{(1+\kappa t)^{a+i}} \qquad i = 0, 1, \ldots, n. \tag{17}$$

Note that $r_i(t;a) \geq 0$, which allows to conclude that $P_n(t)$ is infinitely divisible.

The following relationship is given by [17]: all log-convex distributions are infinitely divisible but not all log-concave distributions are infinitely divisible.

Theorem 1.4: Let $N(t)$ be an infinitely divisible $\mathbb{Z}^+$-valued random variable with pmf $P_n(t)$. Then

$$\mathbb{E}[N(t)] = \sum_{i=0}^{\infty} r_i(t;a) \tag{18}$$

Proof:
We know that the expectation of $N(t)$ it is given by

$$\begin{aligned}\mathbb{E}[N(t)] &= \sum_{n=1}^{\infty} nP_n(t) = \sum_{m=0}^{\infty}(m+1)P_{m+1}(t)\\ &= \sum_{m=0}^{\infty}\sum_{i=0}^{m} r_i(t;a)P_{m-i}(t)\end{aligned}$$

Now, by interchanging the order of summation, we get

$$\begin{aligned}\mathbb{E}[N(t)] &= \sum_{i=0}^{\infty}\sum_{m=i}^{\infty} r_i(t;a)P_{m-i}(t) = \sum_{i=0}^{\infty} r_i(t;a)\sum_{m=i}^{\infty} P_{m-i}(t)\\ &\underset{j=m-i}{=} \sum_{i=0}^{\infty} r_i(t;a)\sum_{j=0}^{\infty} P_j(t) = \sum_{i=0}^{\infty} r_i(t;a).\end{aligned}$$

which completes the proof.

4. *NHP* in terms of *CCP*s

In this section, we present various approaches for the *NHP* using *CCP*.

4.1 *NHP* as a non-homogeneous pure birth process

We use logarithmic differentiation to find the derivative of (5) and we get

$$\frac{P_{n'}(t)}{P_n(t)} = \frac{n}{t} + \frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)}$$

Then

$$P_{n'}(t) = \frac{n}{t}P_n(t) + \frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)}P_n(t) \tag{19}$$

From (5), we obtain

$$\begin{aligned}\frac{n}{t}P_n(t) &= -\frac{(-1)^{n-1}}{(n-1)!}t^{n-1}P_0^{(n)}(t) = \frac{(-1)^{n-1}}{(n-1)!}t^{n-1}P_0^{(n)}(t)\left(-\frac{P_0^{(n-1)}(t)}{P_0^{(n-1)}(t)}\right)\\ &= -\frac{P_0^{(n)}(t)}{P_0^{(n-1)}(t)}P_{n-1}(t)\end{aligned}$$

By substituting in (19), we have

$$P_{n'}(t) = \left(-\frac{P_0^{(n)}(t)}{P_0^{(n-1)}(t)}\right)P_{n-1}(t) - \left(-\frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)}\right)P_n(t). \tag{20}$$

We denote

$$\lambda_n(t;a) = -\frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)} = -\frac{d}{dt}\ln\left[(-1)^n P_0^{(n)}(t)\right]. \tag{21}$$

In ref. [18], Lundberg shows that this corresponds to the transition intensities. Then from (20) and (21), we can derive the following system of Kolmogorov differential equations that must be satisfied by the *NHP*:

$$\begin{aligned} P_0'(t) &= -\lambda_0(t;a)P_0(t) \\ P_n'(t) &= \lambda_{n-1}(t;a)P_{n-1}(t) - \lambda_n(t;a)P_n(t) \qquad \text{for} \qquad n \geq 1. \end{aligned} \tag{22}$$

By notation, we denote $\lambda_0(t;a) = \theta'(t) = \frac{q}{(1+\kappa t)^a}$. With initial conditions

$$P_0(0) = 1 \quad \text{and} \quad P_n(0) = 0 \quad \forall n \geq 1 \tag{23}$$

Using the method given in ref. [18], we find that the solution of (22) is given by

$$P_n(t) = \int_0^t \lambda_{n-1}(\tau;a)P_{n-1}(\tau)\exp\left\{-\int_\tau^t \lambda_{n-1}(\nu;a)d\nu\right\}d\tau \qquad \text{for} \qquad n \geq 1.$$

From the system of equations given in (22), we have that the *NHP* is a non-homogeneous pure birth process (*NHPBP*), which agrees with the definition given by Seal in ref. [19]. So, if $N(t)$ satisfies (6), then $N(t)$ is an *NHPBP* with transition intensities given by (21).

4.2 *NHP as MPP*

The list of equivalences provided by Lundberg in ref. [18] is satisfied by the *NHP* defined in (6), which is presented in the following theorem:

Theorem 1.5: Let $N(t)$ be an *NHP* with marginal *pmf*, given by (5) and transition intensities, given by (21). Then:

i. $\lambda_n(t;a)$ satisfy $\lambda_{n+1}(t;a) = \lambda_n(t;a) - \frac{\lambda_n'(t;a)}{\lambda_n(t;a)}$ for $n = 0, 1, \ldots$

ii. $P_n(t)$ and $\lambda_n(t;a)$ satisfy the relation

$$\frac{P_n(t)}{P_{n-1}(t)} = \frac{t}{n}\lambda_{n-1}(t;a) \quad \text{for} \quad n = 1, 2, \ldots \tag{24}$$

Proof:

i. By finding the derivative of function (21) with respect to t, we obtain

$$\begin{aligned} \lambda_n'(t;a) &= -\left[\frac{P_0^{(n+2)}(t)P_0^{(n)}(t) - P_0^{(n+1)}(t)P_0^{(n+1)}(t)}{\left(P_0^{(n)}(t)\right)^2}\right] \\ &= -\frac{P_0^{(n+2)}(t)}{P_0^{(n+1)}(t)}\frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)} + \left(-\frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)}\right)^2 \\ &= -\lambda_{n+1}(t;a)\lambda_n(t;a) + [\lambda_n(t;a)]^2 \end{aligned}$$

By dividing by $\lambda_n(t;a)$, we have

$$\frac{\lambda_n'(t;a)}{\lambda_n(t;a)} = \lambda_n(t;a) - \lambda_{n+1}(t;a) \tag{25}$$

ii. By substituting (21) into (13), we get:

$$\frac{P_n(t)}{P_{n-1}(t)} = \frac{\frac{(-1)^n}{n!} t^n P_0^{(n)}(t)}{\frac{(-1)^{n-1}}{(n-1)!} t^{n-1} P_0^{(n-1)}(t)} = -\frac{t}{n} \frac{P_0^{(n)}(t)}{P_0^{(n-1)}(t)} = \frac{t}{n} \lambda_{n-1}(t;a),$$

which completes the proof. □

In ref. [7], it is proved that the above three statements are equivalent.

Corollary 1.5.1: Let $N(t)$ be an *NHP* with transition intensities given by (21), then

$$\frac{P_n(t)}{P_0(t)} = \prod_{j=1}^{n} \frac{t \lambda_{j-1}(t;a)}{j} \tag{26}$$

Proof:

Note that

$$\frac{P_n(t)}{P_0(t)} = \prod_{j=1}^{n} \frac{P_j(t)}{P_{j-1}(t)}.$$

Substituting (24) in the above expression completes the proof.

Corollary 1.5.2: Let $N(t)$ be an *NHP* with transition intensities given by (21), then

$$\prod_{j=0}^{n-1} \lambda_j(t;a) = (-1)^n \frac{P_0^{(n)}(t)}{P_0(t)} \qquad n \geq 1. \tag{27}$$

Proof:

From (21), we get

$$\prod_{j=0}^{n-1} \lambda_j(t;a) = \prod_{j=0}^{n-1} \left(-\frac{P_0^{(j+1)}(t)}{P_0^{(j)}(t)} \right) = (-1)^n \frac{P_0^{(n)}(t)}{P_0(t)}.$$

This finishes the proof of Corollary.

The following additional properties set in ref. [9] are also satisfied by *NHP*:

Proposition 1.6: Let $\{N(t); t \geq 0\}$ be an *NHP* and Λ the continuous structure variable of the MPP. Then:

1. The transition intensities are such that

$$\mathbb{E}[\Lambda | N(t) = n] = \lambda_n(t;a). \tag{28}$$

and

$$Var[\Lambda | N(t) = n] = -\lambda_n'(t;a). \tag{29}$$

2. The mean of $N(t)$ is given by

$$\mathbb{E}[N(t)] = t\mathbb{E}[\Lambda]. \tag{30}$$

3. The mean of Λ is given by

$$\mathbb{E}[\Lambda] = -P_0'(0). \tag{31}$$

Proof:
1. From (2), taking the expected value of Λ, conditioning on $N(t)$, we get

$$\mathbb{E}[\Lambda|N(t) = n] = \int_0^\infty \frac{\lambda e^{-\lambda t}(\lambda t)^n f(\lambda)}{n! P[N(t) = n]} d\lambda = \frac{n+1}{t} \frac{P_{n+1}(t)}{P_n(t)}. \tag{32}$$

By substituting (24) into (32), we have

$$\mathbb{E}[\Lambda|N(t) = n] = \lambda_n(t; a).$$

Analogously, we can show that

$$\mathbb{E}\left[\Lambda^2|N(t) = n\right] = \int_0^\infty \frac{\lambda^2 e^{-\lambda t}(\lambda t)^n f(\lambda)}{n! P[N(t) = n]} d\lambda = \frac{(n+2)(n+1)}{t^2} \frac{P_{n+2}(t)}{P_n(t)}. \tag{33}$$

By substituting (24) into (33), we have

$$\mathbb{E}\left[\Lambda^2|N(t) = n\right] = \lambda_{n+1}(t; a)\lambda_n(t; a).$$

Then the conditional variance of Λ, given that $N(t) = n$, is

$$Var[\Lambda|N(t) = n] = \lambda_{n+1}(t; a)\lambda_n(t; a) - \lambda_n^2(t; a),$$

and substituting Eq. (25) into the above yields the result.

2. We use the law of total expectation to find the expected value

$$\mathbb{E}[\Lambda] = \mathbb{E}[|\mathbb{E}(\Lambda|N(t) = n)] = \sum_{n=0}^\infty \mathbb{E}(|\Lambda|N(t) = n) P[N(t) = n]$$
$$= \sum_{n=0}^\infty \lambda_n(t; a) P_n(t)$$

By substituting (24) into the above expression, we get

$$\mathbb{E}[\Lambda] = \sum_{n=0}^\infty \frac{n+1}{t} P_{n+1}(t) = \sum_{j=0}^\infty \frac{r_j(t; a)}{t} = \frac{1}{t}\mathbb{E}[N(t)].$$

And the proof is completed.

3. The pgf of $N(t)$ is defined as

$$\begin{aligned} \underbrace{G_N(z; t)}_{} &= \sum_{n=0}^\infty z^n P_n(t) = \sum_{n=0}^\infty z^n \int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} f(\lambda) d\lambda \\ P_0[(1-z)t] &= \int_0^\infty \left[\sum_{n=0}^\infty \frac{(z\lambda t)^n}{n!}\right] e^{-\lambda t} f(\lambda) d\lambda = \int_0^\infty e^{\lambda(z-1)t} f(\lambda) d\lambda \\ &= M_\Lambda[(z-1)t]. \end{aligned} \tag{34}$$

We make $z = 0$ in the above expression and we have

$$P_0(t) = M_\Lambda(-t)$$

Now, if we differentiate both sides with respect to t, we obtain

$$P_0'(t) = -M_\Lambda'(-t)$$

We complete the proof by substituting $t = 0$ in the above expression. □

According to Walhin and Paris in ref. [20], the intensity of the stochastic process $N(t)$ in the period $[t, t+1]$ is

$$\mathbb{E}[N(t+1) - N(t)|N(t) = n] = \mathbb{E}[\Lambda|N(t) = n].$$

The moment generating function of the process will uniquely determine the distribution of the process, on comparing expression (34) with $P_0[(1-z)t]$ given for $a = 1$ and as shown in **Table 1**, we find the particular cases: the *PCP* if $\Lambda \sim \delta_\gamma(\lambda)$ (i.e. has a degenerate *cdf* at $\lambda = \gamma$), the *NBCP* if $\Lambda \sim \Gamma(\gamma, \delta)$ and the Geometric Counting Process if $\Lambda \sim \exp(\delta)$.

5. Additional properties

In this Section, we will introduce several other properties of the *NHP*.

5.1 Other expressions for $\mathbf{P_n(t)}$ in terms of $\boldsymbol{\lambda_n(t; a)}$

Theorem 1.7: Let $N(t)$ be an *NHP* with transition intensities given by (21), then

$$P_n(t) = Q_n(t) - Q_{n+1}(t) \quad \text{for} \quad n \geq 1,$$

where $Q_0(t)$ is Heaviside's step function and

$$Q_{n+1}(t) = \int_0^t \lambda_n(v; a) P_n(v) dv. \tag{35}$$

Proof:

We write the expression (22) as

$$\frac{d[P_n(\tau)]}{d\tau} = \lambda_{n-1}(\tau; a) P_{n-1}(\tau) - \lambda_n(\tau; a) P_n(\tau) \qquad \text{for} \qquad n \geq 1.$$

By integration of the above expression with respect to τ between 0 and t, we get

$$\begin{aligned} \int_0^t d[P_n(\tau)] &= \int_0^t \lambda_{n-1}(\tau; a) P_{n-1}(\tau) d\tau - \int_0^t \lambda_n(\tau; a) P_n(\tau) d\tau \\ P_n(\tau)\big|_0^t &= Q_n(t) - Q_{n+1}(t) \qquad \text{for} \qquad n \geq 1. \end{aligned} \tag{36}$$

Since $P_n(0) = 0, \forall n \geq 1$, so the proof is completed.

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Corollary 1.7.1: Let $N(t)$ be an *NHP* with transition intensities given by (21), then

$$P[N(t) > n] = Q_{n+1}(t) \qquad \text{for} \qquad n \geq 0 \tag{37}$$

Proof: The proof consists of a direct calculation

$$\begin{aligned} P[N(t) > n] &= 1 - P[N(t) \leq n] \\ &= 1 - \sum_{j=0}^{n} P_j(t) = 1 - P_0(t) - \sum_{j=1}^{n} P_j(t) \end{aligned}$$

Using the previous result:

$$\begin{aligned} P[N(t) > n] &= 1 - P_0(t) - \sum_{j=1}^{n} \left[Q_j(t) - Q_{j+1}(t)\right] \\ &= 1 - P_0(t) - \left[Q_1(t) - Q_{n+1}(t)\right] \end{aligned} \tag{38}$$

Note that

$$Q_1(t) = \int_0^t \lambda_0(v; a) P_0(v) dv = -\int_0^t P_0'(v) dv = -P_0(v)|_0^t = 1 - P_0(t)$$

Replacing $Q_1(t)$ in (38) the proof is completed.
The expression (37) allows to calculate the *cdf* of an *NHP*.
Corollary 1.7.2: The function $Q_{n+1}(t)$ satisfies the following condition:

$$\lim_{t \to \infty} Q_{n+1}(t) = 1 \qquad \text{for} \qquad n \geq 0. \tag{39}$$

Proof:
From (37), we get

$$\lim_{t \to \infty} Q_{n+1}(t) = \lim_{t \to \infty} \left[1 - \sum_{j=0}^{n} P_j(t)\right].$$

As we have for $n \geq 1 : P_n(\infty) = 0$, and using the above relationship

$$\lim_{t \to \infty} Q_{n+1}(t) = 1 - \lim_{t \to \infty} P_0(t).$$

For example, from expression (9) when $a = 1$, we have:

$$P_0(t) = (1 + \kappa t)^{-\frac{q}{\kappa}} \qquad \text{for} \qquad \frac{q}{\kappa} > 0 \tag{40}$$

and we take the limit as $t \to \infty$, we get:

$$\lim_{t \to \infty} Q_{n+1}(t) = 1 - \lim_{t \to \infty} (1 + \kappa t)^{-\frac{q}{\kappa}} = 1. \square$$

Proposition 1.8: Let $N(t)$ be an *NHP* with transition intensities given by (21), then

$$\exp\left\{-\int_{t}^{t+h}\lambda_n(v;a)dv\right\} = \frac{P_0^{(n)}(t+h)}{P_0^{(n)}(t)} \qquad \text{for} \quad h \geq 0. \tag{41}$$

Proof:
By substituting (28) into (40), we have

$$\exp\left\{-\int_{t}^{t+h}\lambda_n(v;a)dv\right\} = \exp\left\{\int_{t}^{t+h}\frac{P_0^{(n+1)}(v)}{P_0^{(n)}(v)}dv\right\}$$

$$= \exp\left\{\int_{t}^{t+h} d\left[\ln\left(P_0^{(n)}(v)\right)\right]\right\}$$

$$= \exp\left\{.\ln\left[P_0^{(n)}(v)\right]\Big|_{t}^{t+h}\right\} = \frac{P_0^{(n)}(t+h)}{P_0^{(n)}(t)}.$$

Corollary 1.8.1: Let $N(t)$ be an NHP. If the probability that no event occurs in a small interval of length h is denoted by $P_0(t, t+h)$, that is $P_0(t, t+h) = P(N(t+h) - N(t) = 0)$, then

$$P_0(t+h) = P_0(t) \cdot P_0(t, t+h) \qquad \text{for} \quad t, h \geq 0. \tag{42}$$

Proof:
According to Lundberg in [18]:

$$P(N(t+h) = 0|N(t) = 0) = \exp\left\{-\int_{t}^{t+h}\lambda_0(u)du\right\} \tag{43}$$

where $\lambda_0(t)$ denotes the intensity function associated with the time-dependent (or nonstationary) *PCP*. If we make $n = 0$ in (40), then we obtain

$$P_0(t, t+h) = \exp\left\{-\int_{t}^{t+h}\lambda_0(v;a)dv\right\} = \frac{P_0(t+h)}{P_0(t)} \tag{44}$$

Thus,

$$P_0(t+h) = P_0(t) \cdot P_0(t, t+h) \qquad \text{for} \quad t, h \geq 0.$$

The expression obtained in (41) may be interpreted as if no event occurred, then the *NHP* has independent increments.

Lemma 1.9: Let $N(t)$ be an *NHP* with transition intensities given by (21). Then this CP satisfies

$$\sum_{j=0}^{m}\frac{\lambda_j'(t;a)}{\lambda_j(t;a)} = \lambda_0(t;a) - \lambda_{m+1}(t;a) \qquad \text{for all} \quad m \geq 0. \tag{45}$$

Proof:

From (25), we have

$$\frac{\lambda_j'(t;a)}{\lambda_j(t;a)} = \lambda_j(t;a) - \lambda_{j+1}(t;a) \qquad \text{for all} \quad j \geq 0. \tag{46}$$

Thus, (44) turns out the mth partial sum of a telescoping series and from here

$$\sum_{j=0}^{m} \frac{\lambda_j'(t;a)}{\lambda_j(t;a)} = \lambda_0(t;a) - \lambda_{m+1}(t;a) \qquad \text{for all} \quad m \geq 0.$$

Now, using the above lemma, we will prove the following proposition:

Proposition 1.10: Let $N(t)$ be an NHP with marginal pmf given by (5), then $P_n(t)$ satisfies that

i. Process with time-dependent increments

$$\lim_{h\to 0} \frac{P_{n,n+1}(t, t+h)}{h} = \lambda_n(t;a)$$

ii. The probability that no event occurs in $(t, t+h]$ is

$$P_0(t, t+h) = 1 - h\lambda_0(t;a) + o(h) \tag{47}$$

iii. The probability that one event occurs in $(t, t+h]$ is

$$P_1(t, t+h) = h\lambda_0(t;a) - o(h) \tag{48}$$

iv. *Faddy's conjecture*[2]: If the transition intensities be an increasing sequence with n, i.e,

$$\lambda_0(t;a) < \lambda_1(t;a) < \ldots < \lambda_n(t;a), \qquad \text{for any fixed } t \tag{49}$$

then $Var[N(t)] > \mathbb{E}[N(t)]$, this last inequality is reversed for a decreasing sequence.

Proof:

i. As the *NHP* is an *MPP* then, according to Lundberg in [18], for $0 \leq u < v, \quad i \leq j$, $N(t)$ satisfies:

$$\underbrace{P(N(v) = j \mid N(u) = i)}_{P_{i,j}(u,v)} = \binom{j}{i} \left(\frac{u}{v}\right)^i \left(1 - \frac{u}{v}\right)^{j-i} \frac{P_j(v)}{P_i(u)} \tag{50}$$

Replacing the expression $P_n(t)$ given in (12), when $\kappa \neq 0$, we obtain in (49) that the transition probabilities for the *NHP* are:

[2] See [21].

$$
\begin{aligned}
P_{i,j}(u,v) &= \binom{j}{i}\left(\frac{u}{v}\right)^{i}\left(1-\frac{u}{v}\right)^{j-i}\frac{P_j(v)}{P_i(u)} \\
&= \binom{j}{i}\left(\frac{u}{v}\right)^{i}\left(\frac{v-u}{v}\right)^{j-i}\left[\frac{\frac{(-1)^j v^j P_0^{(j)}(v)}{j!}}{\frac{(-1)^i u^i P_0^{(i)}(u)}{i!}}\right] \\
&= \frac{(u-v)^{j-i}}{(j-i)!}\frac{P_0^{(j)}(v)}{P_0^{(i)}(u)} \\
&= \prod_{m=1}^{j-i}\left[\frac{v-u}{m}\lambda_{m+i-1}(u;a)\right]\exp\left\{-\int_u^v \lambda_j(w;a)dw\right\}.
\end{aligned}
\tag{51}
$$

We complete the proof of the theorem by the following steps: Rewrite the product in (50) by replacing all instances of $i=n, j=n+1, u=t$ and $v=t+h$, and we make the limit as h approaches zero. Then the transition intensities given by (21) represent the instantaneous transitions probabilities of the *NHP*.

ii. Certainly, the function given by (9) is continuous for $t \geq 0$ and also analytic, due to $P_0^{(n)}(t)$, exists for all $n \geq 1$. Then it is possible to express $P_0(t+h)$ through a Taylor series as follows:

$$
P_0(t+h) = \sum_{m=0}^{\infty}\frac{h^m}{m!}P_0^{(m)}(t). \tag{52}
$$

By substituting the expression for the mth derivative of $P_0(t)$ obtained given by (27) in (51), we have:

$$
P_0(t+h) = P_0(t) + \sum_{m=1}^{\infty}\frac{h^m}{m!}\left[(-1)^m\left(\prod_{j=0}^{m-1}\lambda_j(t;a)\right)P_0(t)\right]. \tag{53}
$$

Notice that $P_0(t+h)$ satisfies (41), then (52) is similar to:4

$$
P_0(t)\cdot P_0(t,t+h) = P_0(t)\left[1+\sum_{m=1}^{\infty}(-1)^m\frac{h^m}{m!}\left(\prod_{j=0}^{m-1}\lambda_j(t;a)\right)\right] \tag{54}
$$

Let $n = m-1$ then:

$$
\begin{aligned}
P_0(t,t+h) &= 1+\sum_{n=0}^{\infty}(-1)^{n+1}\frac{h^{n+1}}{(n+1)!}\left(\prod_{j=0}^{n}\lambda_j(t;a)\right) \\
&= 1-h\sum_{n=0}^{\infty}\frac{(-h)^n}{(n+1)!}\left(\prod_{j=0}^{n}\lambda_j(t;a)\right)
\end{aligned}
\tag{55}
$$

From the expansion of the first terms of (54), we get:

$$
P_0(t,t+h) = 1-h\lambda_0(t;a)+o(h) \tag{56}
$$

where

$$o(h) = \sum_{n=1}^{\infty} \frac{(-h)^{n+1}}{(n+1)!} \prod_{j=0}^{n} \lambda_j(t;a).$$

The last function satisfies that $\lim_{h\to 0} o(h)/h = 0$ ([21, 22]).

iii. From (55) and the fact $P_0(t, t+h) = P(N(t+h) - N(t) = 0)$, we obtain

$$P(N(t+h) - N(t) > 0) = 1 - P_0(t, t+h). \tag{57}$$

Given that the *NHP* $N(t)$ is an *NHPBP* and assuming that we have in a small time interval, then there will be only two cases: there is a birth or not in that period. Thus,

$$P(N(t+h) - N(t) > 0) = P(N(t+h) - N(t) = 1) = P_1(t, t+h).$$

Then, from (56), we obtain:

$$P_1(t, t+h) = h\lambda_0(t;a) - o(h), \tag{58}$$

provided that h is infinitesimal.

iv. According to Steutel et al. in ref. [16], a non-degenerate distribution $\{P_n(t)\}$ is log-convex if and only if $P_n(t) > 0$ for all $n \geq 0$ and $\left\{\frac{P_{n+1}(t)}{P_n(t)}\right\}$ is a nondecreasing sequence. By assumption

$$\frac{P_n(t)}{P_{n-1}(t)} < \frac{P_{n+1}(t)}{P_n(t)} \qquad \text{for some} \quad n \geq 1 \tag{59}$$

By substituting (5) into (58)

$$\begin{aligned} \frac{\frac{t^n}{n!}\left[(-1)^n P_0^{(n)}(t)\right]}{\frac{t^{n-1}}{(n-1)!}\left[(-1)^{n-1} P_0^{(n-1)}(t)\right]} &< \frac{\frac{t^{n+1}}{(n+1)!}\left[(-1)^{n+1} P_0^{(n+1)}(t)\right]}{\frac{t^n}{n!}\left[(-1)^n P_0^{(n)}(t)\right]} \\ \frac{1}{n}\left(-\frac{P_0^{(n)}(t)}{P_0^{(n-1)}(t)}\right) &< \frac{1}{n+1}\left(-\frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)}\right) \\ \frac{1}{n}\lambda_{n-1}(t;a) &< \frac{1}{n+1}\lambda_n(t;a) \end{aligned},$$

we know $1 < \frac{n+1}{n}$ for all n. Hence, we have the following:

$$\lambda_{n-1}(t;a) < \frac{n+1}{n}\lambda_{n-1}(t;a) < \lambda_n(t;a). \tag{60}$$

Thus, we obtain that (48) is satisfied and, therefore, the conjecture holds.

The expression (48) allows to identify under- or over-dispersion of a *CP*, then we can classify the process according to the fixed criteria given in (16).

Corollary 1.10.1: If $a \neq 0$ and $N(t)$ is an *NHP*, then it does not have independent increments.

Proof:

From theorem 1.5, we know that an *NHP* is an *MPP*. According to McFadden in ref. [9], if $\{N(t), t \geq 0\}$ is a *CP* with independent increments, then its transition intensities satisfy that $\lambda_0(t; a) = \lambda_1(t; a)$, but by expression (48), we get

$$\lambda_0(t; a) = \frac{q}{(1 + \kappa t)^a} \neq \frac{a\kappa}{1 + \kappa t} + \frac{q}{(1 + \kappa t)^a} = \lambda_1(t; a) \quad \text{if} \quad a \neq 0 \tag{61}$$

And therefore, $N(t)$ is a *CP* that does not have independent increments.

This was to be expected since that *MPP* has stationary increments but does not meet the condition of independent increments (see [23]).

6. Conclusions

In this chapter, we studied the *NHP* presenting some of its properties indicating that it is a good option for modelling *CP* regardless of the fact that it presents under- or over-dispersion.

Using transition intensities, we found some properties of the *NHP* and provided explicit analytic expressions for its *pmf* and *cdf*.

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