Chapter 4

Nullspace of Compound Magic Squares

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we consider special compound $4n \times 4n$ magic squares. We determine a $2n-3$ dimensional subspace of the nullspace of the $4n \times 4n$ squares. All vectors in the subspaces possess the property that the sum of all entries of each vector equals zero.

Keywords: null space, magic squares, mathematical induction

1. Introduction

A semi-magic square is an $n \times n$ matrix such that the sum of the entries in each row and column is the same. The common value is called the magic constant. If, in addition, the sum of all entries in each left-broken diagonal and each right-broken diagonal is the magic constant, then we call the matrix a pandiagonal magic square. Rosser and Walker show that a pandiagonal 4×4 magic square with magic constant $2s$ has in general the following structure.

A	B	C	ω
E	θ	ς	ρ
s − C	$s-\omega$	s − A	s − B
$s-\varsigma$	$s-\rho$	s − E	$s-\theta$

where

$$\omega = 2s - A - B - C;$$

$$\theta = 2s - A - B - E;$$

$$\varsigma = A + E - C;$$

$$\rho = B + C - E.$$

This result was developed by Rosser and Walker. Hendricks proved that the determinant of a pandiagonal magic square is zero. We note that every antipodal pair of elements add up to one-half of the magic constant. Al-Amerie considered in his M.Sc thesis some of the results here. There are three fundamental primitive pandiagonal squares which are 4 × 4. Kraitchik (see [3, 8]) has shown how to derive all pandiagonal squares from three particular ones.

We define a certain class of 6 × 6 magic squares, which has a similar structure to the structure of a pandiagonal 4 × 4 magic square. In this class each antipodal pair will add up to one-third of the magic constant. Precisely, we have:

Definition 1: A 6 × 6 magic square with $3s$ as a magic constant is called panmagic if

$$a_{ij} + a_{kl} = s, \text{ for each } i, j, k, l \text{ such that } i \equiv k \pmod 3 \text{ and } j \equiv l \pmod 3.$$

The following matrix is a possible form for this kind of squares:

M	R	W	T	L	K
Q	J	I	H	G	F
P	E	D	C	B	A
$s-T$	$s-L$	$s-K$	$s-M$	$s-R$	$s-W$
$s-H$	$s-G$	$s-F$	$s-Q$	$s-J$	$s-I$
$s-C$	$s-B$	$s-A$	$s-P$	$s-E$	$s-D$

where

$$M = J + I + H + E + D + C - L - K - \frac{3s}{2},$$

$$W = K - I + F - D + A,$$

$$P = 3s - E - D - C - B - A$$

$$Q = 3s - J - I - H - G - F,$$

$$R = L - J + G - E + B,$$

$$T = \frac{9s}{2} - L - K - H - G - F - C - B - A.$$

Note that we have the following relations:

$$\begin{aligned} M + Q + P &= T + H + C, \\ R + J + E &= L + G + B, \\ W + I + D &= K + F + A. \end{aligned} \tag{1}$$

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Using Maple we can show that the 6×6 panmagic square possesses a nontrivial null space, which can be written in the following form:

$$\left\{z(x_1, x_2, x_3, -x_1, -x_2, -x_3)' : z \in R\right\}$$

where

$$x_1 = (A - D)(G - J)(B - E)(I - F),$$
$$x_2 = (F - I)(B + 2C + E - 3s) + 2FD - 2AI + (D - A)(G + J + 2H - 3s),$$
$$x_3 = (B - E)(F + I + 2H) + (A + D + 2C + 2B + 2E - 3s)(J - G).$$

Note that the sum of all entries of the vectors is zero. For example:

−51	39	26	0	9	13
54	−10	−2	−5	4	−5
−5	1	2	3	17	18
12	3	−1	63	−27	−14
17	8	17	−42	22	14
9	−5	−6	17	11	10

has as nullspace $\left\{z(34, 115, -132, -34, -115, 132)^t : z \in R\right\}$.

Definition 2: A 8×8 square consisting of 4 pandiagonal magic squares $A_{11}, A_{12}, A_{21}, A_{22}$ having the same magic sum in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is called a compound magic square if the following relation holds:

$$A_{22} + A_{11} = A_{12} + A_{21}.$$

It is easy to check if the last relation guarantees that the square is a magic 8×8 square. In the same manner we can combine four panmagic squares in a magic square.

Definition 3: Let $B_{22}, B_{11}, B_{12}, B_{21}$ be panmagic squares having the same magic constant. Assume that $B_{22} + B_{11} = B_{12} + B_{21}$. Then the matrix

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

is called the compound 12×12 magic square.

The condition $B_{22} + B_{11} = B_{12} + B_{21}$ ensures that the compound 12×12 magic square is magic.

2. Main results

We prove first a simple result for a compound square of 4×4 squares. We then generalize this result for an arbitrary number of squares.

Proposition 1: The compound 8×8 magic square processes a three-dimensional subspace of its nullspace.

Proof: First we note that the vector

$$(1, 1, 1, 1, -1, -1, -1, -1)'$$

is a nonzero vector, which belongs to the nullspace of the square, since the squares have the same magic constant.

Now, the square A_{11} (res. A_{12}) has a nonzero vector v_{11} (res. v_{12}), which belongs to the nullspace of the square, since A_{11}(res. A_{12}) is a pandiagonal magic square. We look for four numbers $f_{11}, f_{12}, f_{21}, f_{22}$ such that the vector

$$\begin{pmatrix} f_{11}v_{11} + f_{12}v_{12} \\ f_{21}v_{11} + f_{22}v_{12} \end{pmatrix}$$

belongs to the nullspace of the square. To do this we compute the following matrix multiplication:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} f_{11}v_{11} + f_{12}v_{12} \\ f_{21}v_{11} + f_{22}v_{12} \end{pmatrix} = \begin{pmatrix} A_{11}\left(f_{11}v_{11} + f_{12}v_{12}\right) + A_{12}\left(f_{21}v_{11} + f_{22}v_{12}\right) \\ A_{21}\left(f_{11}v_{11} + f_{12}v_{12}\right) + A_{22}\left(f_{21}v_{11} + f_{22}v_{12}\right) \end{pmatrix}$$

According to the choice of v_{11} and v_{12} we obtain the vector $\left(g_1, g_2\right)'$ as the result of matrix multiplication, where:

$$g_1 = A_{11}f_{12}v_{12} + A_{12}f_{21}v_{11},$$
$$g_2 = A_{21}v_{11}f_{11} + A_{21}v_{12}f_{12} + (A_{12}v_{11} + A_{21}v_{11})f_{21} + (A_{21}v_{12} - A_{11}v_{12})f_{22}.$$

Note that we used the relation $A_{22} = A_{12} + A_{21} - A_{11}$. We can rewrite the vector $\left(g_1, g_2\right)'$ in the form.

$$\begin{bmatrix} 0 & A_{11}v_{12} & A_{12}v_{11} & 0 \\ A_{21}v_{11} & A_{21}v_{12} & (A_{12} + A_{21})v_{11} & (A_{21} - A_{11})v_{12} \end{bmatrix} \begin{pmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \end{pmatrix} \tag{2}$$

According to Al-Ashhab (see [3]) we can assume that the vectors in the nullspace of the pandiagonal magic square are

$$v_{ij} = \left(v_{ij}^{*}, v_{ij}^{**}, -v_{ij}^{*}, -v_{ij}^{**}\right)', \text{for } i = 1, j = 1, 2$$

Further, we can assume that

$$A_{ij} = \begin{bmatrix} a_{ij} & b_{ij} & c_{ij} & d_{ij} \\ e_{ij} & f_{ij} & g_{ij} & h_{ij} \\ s - c_{ij} & s - d_{ij} & s - a_{ij} & s - b_{ij} \\ s - g_{ij} & s - h_{ij} & s - e_{ij} & s - f_{ij} \end{bmatrix}, i, j = 1, 2$$

Hence, we can assume that:

$$A_{ij}v_{ij} = \begin{pmatrix} a_{ij}v_{ij}^{*} + b_{ij}v_{ij}^{**} - c_{ij}v_{ij}^{*} - d_{ij}v_{ij}^{**} \\ e_{ij}v_{ij}^{*} + f_{ij}v_{ij}^{**} - g_{ij}v_{ij}^{*} - h_{ij}v_{ij}^{**} \\ -c_{ij}v_{ij}^{*} - d_{ij}v_{ij}^{**} + a_{ij}v_{ij}^{*} + b_{ij}v_{ij}^{**} \\ -g_{ij}v_{ij}^{*} - h_{ij}v_{ij}^{**} + e_{ij}v_{ij}^{*} + f_{ij}v_{ij}^{**} \end{pmatrix} = \begin{pmatrix} (a_{ij} - c_{ij})v_{ij}^{*} + (b_{ij} - d_{ij})v_{ij}^{**} \\ \left(e_{ij} - g_{ij}\right)v_{ij}^{*} + \left(f_{ij} - h_{ij}\right)v_{ij}^{**} \\ -(a_{ij} - c_{ij})v_{ij}^{*} - (b_{ij} - d_{ij})v_{ij}^{**} \\ -\left(e_{ij} - g_{ij}\right)v_{ij}^{*} - \left(f_{ij} - h_{ij}\right)v_{ij}^{**} \end{pmatrix}$$

Since the sum of two pandiagonal magic squares is pandiagonal magic, we deduce that four rows in the matrix in Eq. (2) are redundant. Since we have the relations

$$\begin{aligned} a_{11} + e_{11} = c_{11} + g_{11} \quad &\Rightarrow \quad a_{11} - c_{11} = -(e_{11} - g_{11}) \\ b_{11} + f_{11} = d_{11} + h_{11} \quad &\Rightarrow \quad b_{11} - d_{11} = -(f_{11} - h_{11}) \end{aligned}$$

the application of elementary row operations on the matrix in Eq. (2) yields to

$$\begin{bmatrix} 0 & r_{12} & r_{21} & 0 \\ q_{11} & q_{12} & r_{21} + q_{11} & q_{12} - r_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} r_{12} &= (a_{11} - c_{11})v_{12}^{*} + (b_{11} - d_{11})v_{12}^{**} \\ r_{21} &= (a_{12} - c_{12})v_{11}^{*} + (b_{12} - d_{12})v_{11}^{**} \\ q_{11} &= (a_{21} - c_{21})v_{11}^{*} + (b_{21} - d_{21})v_{11}^{**} \\ q_{12} &= (a_{21} - c_{21})v_{12}^{*} + (b_{21} - d_{21})v_{12}^{**} \end{aligned}$$

This analysis enables us to conclude the following relations from (2):

$$f_{11} = -\frac{r_{12}r_{21} + q_{11}r_{12} - q_{12}r_{21}}{q_{11}r_{12}} f_{21} + \frac{-q_{12} + r_{12}}{q_{11}} f_{22}, f_{12} = -\frac{r_{21}}{r_{12}} f_{21}.$$

If we set

$$f_{12} = 0,\ f_{21} = 0,\ f_{22} = q_{11},\ f_{11} = r_{12} - q_{12},$$

which is consistent with the previous relations, we conclude that the vector

$$\begin{pmatrix} (r_{12} - q_{12})v_{11} \\ q_{11}v_{12} \end{pmatrix}$$

belongs to the nullspace of the square. We can make another choice as follows.

$$f_{22} = 0, f_{21} = r_{12}q_{11}, f_{12} = -r_{21}q_{11}\ , f_{11} = r_{21}q_{12} - r_{12}(r_{21} + q_{11})$$

and we obtain a vector belonging to the nullspace of the square, which is

$$\begin{pmatrix} (r_{21}q_{12} - r_{12}(r_{21} + q_{11}))v_{11} - r_{21}q_{11}v_{12} \\ -r_{21}q_{11}v_{11} \end{pmatrix}$$

Now, the vectors v_{12}, v_{11} are linearly independent, since they correspond to different magic squares. Hence, the last two vectors are linearly independent. Also the vector

$$(1, 1, 1, 1, -1, -1, -1, -1)'$$

is linearly independent with the last two vectors, since its first two entries are not the opposite of the third and fourth entry. □

For example, the following square is a compound 8 × 8 magic square.

0	14	−19	13	10	5	−22	15
−12	6	7	7	−20	13	12	3
23	−9	4	−10	26	−11	−6	−1
−3	−3	16	−2	−8	1	24	−9
−16	25	−17	16	−6	16	−20	18
1	−2	2	7	−7	5	7	3
21	−12	20	−21	24	−14	10	−12
2	−3	3	6	−3	1	11	−1

For this square we can construct as described the following two vectors in its nullspace

$$\left(-\frac{38}{5},\frac{722}{5},\frac{38}{5},-\frac{722}{5},-170,-544,170,544\right)^t$$

$$\left(-\frac{216006}{5},-\frac{85026}{5},\frac{216006}{5},\frac{85026}{5},3774,-71706,-3774,71706\right)^t$$

In fact, its nullity is 3. Thus, these two vectors together with

$$(1,1,1,1,-1,-1,-1,-1)'$$

form a basis of its nullspace.

We prove now a similar result to the previous proposition, where we replace the 4×4 square with a 6×6 one.

Proposition 2: The compound 12×12 magic square possess a three-dimensional subspace of its nullspace.

Proof: First we note that the vector

$$(1,1,1,1,1,1-1,-1,-1,-1,-1,-1)'$$

is a nonzero vector, which belongs to the nullspace of the square, since the squares have the same magic constant.

We look for scalars $v_1, v_2, v_3, v_4, v_5, v_6$ such that

$$\begin{bmatrix} a_{11} & b_{11} & c_{11} & d_{11} & e_{11} & f_{11} & a_{12} & b_{12} & c_{12} & d_{12} & e_{12} & f_{12} \\ g_{11} & h_{11} & i_{11} & j_{11} & k_{11} & l_{11} & g_{12} & h_{12} & i_{12} & j_{12} & k_{12} & l_{12} \\ m_{11} & n_{11} & o_{11} & p_{11} & q_{11} & r_{11} & m_{12} & n_{12} & o_{12} & p_{12} & q_{12} & r_{12} \\ s-d_{11} & s-e_{11} & s-f_{11} & s-a_{11} & s-b_{11} & s-c_{11} & s-d_{12} & s-e_{12} & s-f_{12} & s-a_{12} & s-b_{12} & s-c_{12} \\ s-j_{11} & s-k_{11} & s-l_{11} & s-g_{11} & s-h_{11} & s-i_{11} & s-j_{12} & s-k_{12} & s-l_{12} & s-g_{12} & s-h_{12} & s-i_{12} \\ s-p_{11} & s-q_{11} & s-r_{11} & s-m_{11} & s-n_{11} & s-o_{11} & s-p_{12} & s-q_{12} & s-r_{12} & s-m_{12} & s-n_{12} & s-o_{12} \\ a_{21} & b_{21} & c_{21} & d_{21} & e_{21} & f_{21} & a_{22} & b_{22} & c_{22} & d_{22} & e_{22} & f_{22} \\ g_{21} & h_{21} & i_{21} & j_{21} & k_{21} & l_{21} & g_{22} & h_{22} & i_{22} & j_{22} & k_{22} & l_{22} \\ m_{21} & n_{21} & o_{21} & p_{21} & q_{21} & r_{21} & m_{22} & n_{22} & o_{22} & p_{22} & q_{22} & r_{22} \\ s-d_{21} & s-e_{21} & s-f_{21} & s-a_{21} & s-b_{11} & s-c_{21} & s-d_{22} & s-e_{22} & s-f_{22} & s-a_{22} & s-b_{22} & s-c_{22} \\ s-j_{21} & s-k_{21} & s-l_{21} & s-g_{21} & s-h_{21} & s-i_{21} & s-j_{22} & s-k_{22} & s-l_{22} & s-g_{22} & s-h_{22} & s-i_{22} \\ s-p_{21} & s-q_{21} & s-r_{21} & s-m_{21} & s-n_{21} & s-o_{21} & s-p_{22} & s-q_{22} & s-r_{22} & s-m_{22} & s-n_{22} & s-o_{22} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ -v_1 \\ -v_2 \\ -v_3 \\ v_4 \\ v_5 \\ v_6 \\ -v_4 \\ -v_5 \\ -v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We transform this equation into a linear system, in which we eliminate the redundant equations. The system becomes

$$\begin{aligned} &(a_{11}-d_{11})v_1+(b_{11}-e_{11})v_2+(c_{11}-f_{11})v_3+(a_{12}-d_{12})v_4+(b_{12}-e_{12})v_5+(c_{12}-f_{12})v_6=0 \\ &(g_{11}-j_{11})v_1+(h_{11}-k_{11})v_2+(i_{11}-l_{11})v_3+(g_{12}-j_{12})v_4+(h_{12}-k_{12})v_5+(i_{12}-l_{12})v_6=0 \\ &(m_{11}-p_{11})v_1+(n_{11}-q_{11})v_2+(o_{11}-r_{11})v_3+(m_{12}-p_{12})v_4+(n_{12}-q_{12})v_5+(o_{12}-r_{12})v_6=0 \\ &(a_{21}-d_{21})v_1+(b_{21}-e_{21})v_2+(c_{21}-f_{21})v_3+(a_{22}-d_{22})v_4+(b_{22}-e_{22})v_5+(c_{22}-f_{22})v_6=0 \\ &(g_{21}-j_{21})v_1+(h_{21}-k_{21})v_2+(i_{21}-l_{21})v_3+(g_{22}-j_{22})v_4+(h_{22}-k_{22})v_5+(i_{22}-l_{22})v_6=0 \\ &(m_{21}-p_{21})v_1+(n_{21}-q_{21})v_2+(o_{21}-r_{21})v_3+(m_{22}-p_{22})v_4+(n_{22}-q_{22})v_5+(o_{22}-r_{22})v_6=0 \end{aligned}$$

From the definition of the panmagic square we know that

$$a_{ij} + g_{ij} + m_{ij} = d_{ij} + j_{ij} + p_{ij} \quad \Rightarrow \quad (a_{ij} - d_{ij}) + \left(g_{ij} - j_{ij}\right) = -\left(m_{ij} - p_{ij}\right) \tag{3}$$

$$b_{ij} + h_{ij} + n_{ij} = e_{ij} + k_{ij} + q_{ij} \quad \Rightarrow (b_{ij} - e_{ij}) + (h_{ij} - e_{ij}) = -\left(n_{ij} - q_{ij}\right) \tag{4}$$

$$c_{ij} + l_{ij} + o_{ij} = f_{ij} + l_{ij} + r_{ij} \quad \Rightarrow \left(c_{ij} - f_{ij}\right) + (i_{ij} - l_{ij}) = -(o_{ij} - r_{ij}) \tag{5}$$

Thus, due to Eqs. (3)–(5), we can reduce the linear system to the following

$$\begin{aligned}
&(a_{11} - d_{11})v_1 + (b_{11} - e_{11})v_2 + \left(c_{11} - f_{11}\right)v_3 + (a_{12} - d_{12})v_4 + (b_{12} - e_{12})v_5 + \left(c_{12} - f_{12}\right)v_6 = 0 \\
&\left(g_{11} - j_{11}\right)v_1 + (h_{11} - k_{11})v_2 + (i_{11} - l_{11})v_3 + \left(g_{12} - j_{12}\right)v_4 + (h_{12} - k_{12})v_5 + (i_{12} - l_{12})v_6 = 0 \\
&(a_{21} - d_{21})v_1 + (b_{21} - e_{21})v_2 + \left(c_{21} - f_{21}\right)v_3 + (a_{22} - d_{22})v_4 + (b_{22} - e_{22})v_5 + \left(c_{22} - f_{22}\right)v_6 = 0 \\
&\left(g_{21} - j_{21}\right)v_1 + (h_{21} - k_{21})v_2 + (i_{21} - l_{21})v_3 + \left(g_{22} - j_{22}\right)v_4 + (h_{22} - k_{22})v_5 + (i_{22} - l_{22})v_6 = 0
\end{aligned}$$

We can verify using the computer that the coefficient matrix of this system has in general the rank four. Hence, we deduce that v_1, v_2, v_3, v_4 depends on v_5 and v_6. By letting v_5 and v_6 take the values 0 and 1 we obtain two linearly independent vectors in the nullspace. These two vectors do not possess the property that the first six elements are the opposite of the last six elements. Hence, they are independent of the vector $(1, 1, 1, 1, 1, 1 - 1, -1, -1, -1, -1, -1)'$.□

Remark: We did not here make use of the relation $B_{22} + B_{11} = B_{12} + B_{21}$. It actually does not affect the proof.

For example, the following square is a compound 12×12 magic square.

−51	39	26	0	9	13	6	17	15	−6	0	4
54	−10	−2	−5	4	−5	20	5	2	0	9	0
−5	1	2	3	17	18	−24	6	7	8	19	20
12	3	−1	63	−27	−14	18	12	8	6	−5	−3
17	8	17	−42	22	14	12	3	12	−8	7	10
9	−5	−6	17	11	10	4	−7	−8	36	6	5
2	53	45	−131	33	34	59	31	34	−137	24	25
−10	0	10	11	12	13	−44	15	14	16	17	18
−89	21	22	23	29	30	−108	26	27	28	31	32
143	−21	−22	10	−41	−33	149	−12	−13	−47	−19	−22
1	0	−1	22	12	2	−4	−5	−6	56	−3	−2
−11	−17	−18	101	−9	−10	−16	−19	−20	120	−14	−15

Using the computer we can verify that its nullity is 3. In other words, the constructed subspace is the nullspace itself.

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We can generalize the previous result for an arbitrary number of squares involved in the compound square.

Theorem 1: Let A_{ij} be the distinct pandiagonal magic square with magic constant $2s$ having the structure:

$$A_{ij} = \begin{bmatrix} a_{ij} & b_{ij} & c_{ij} & d_{ij} \\ e_{ij} & f_{ij} & g_{ij} & h_{ij} \\ s - c_{ij} & s - d_{ij} & s - a_{ij} & s - b_{ij} \\ s - g_{ij} & s - h_{ij} & s - e_{ij} & s - f_{ij} \end{bmatrix}$$

such that $A_{ij} = A_{1j} + A_{i1} - A_{11}$ for $i, j = 1, \ldots, n$. Assume that $(a_{11} + c_{12} - c_{11} - a_{12}) \neq 0$. Then, the following $4n \times 4n$ matrix

$$\begin{bmatrix} A_{11}A_{12} & A_{13}\ldots A_{1n} \\ A_{21}A_{22} & A_{23}\ldots A_{2n} \\ A_{31}A_{32} & A_{33}\ldots A_{3n} \\ \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \\ A_{n1}A_{n2} & A_{n3}\ldots A_{nn} \end{bmatrix}$$

possesses a $2n - 3$ dimensional subspace of its nullspace, which is generated by the vectors

$$\begin{pmatrix} \mathrm{b}_{11}\text{-}\mathrm{d}_{11}\text{-}\mathrm{b}_{12} + \mathrm{d}_{12} \\ -(a_{11} + c_{12} - c_{11} - a_{12}) \\ -(\mathrm{b}_{11}\text{-}\mathrm{d}_{11}\text{-}\mathrm{b}_{12} + \mathrm{d}_{12}) \\ (a_{11} + c_{12} - c_{11} - a_{12}) \\ -(\mathrm{b}_{11}\text{-}\mathrm{d}_{11}\text{-}\mathrm{b}_{12} + \mathrm{d}_{12}) \\ a_{11} + c_{12} - c_{11} - a_{12} \\ (\mathrm{b}_{11}\text{-}\mathrm{d}_{11}\text{-}\mathrm{b}_{12} + \mathrm{d}_{12}) \\ -(a_{11} + c_{12} - c_{11} - a_{12}) \\ [0] \\ \vdots \\ [0] \end{pmatrix}$$

and

$$\begin{pmatrix} a_{12}+c_{13}-c_{12}-a_{13} \\ 0 \\ -(a_{12}+c_{13}-c_{12}-a_{13}) \\ 0 \\ -(a_{11}-c_{11}+c_{13}-a_{13}) \\ 0 \\ a_{11}-c_{11}+c_{13}-a_{13} \\ 0 \\ a_{11}+c_{12}-c_{11}-a_{12} \\ 0 \\ -(a_{11}+c_{12}-c_{11}-a_{12}) \\ 0 \\ [0] \\ \vdots \\ [0] \end{pmatrix}, \begin{pmatrix} b_{11}-d_{11}-b_{13}+d_{13} \\ -(a_{11}+c_{12}-c_{11}-a_{12}) \\ -(b_{11}-d_{11}-b_{13}+d_{13}) \\ a_{11}+c_{12}-c_{11}-a_{12} \\ -(b_{11}-d_{11}-b_{13}+d_{13}) \\ 0 \\ b_{11}-d_{11}-b_{13}+d_{13} \\ 0 \\ 0 \\ a_{11}+c_{12}-c_{11}-a_{12} \\ 0 \\ -(a_{11}+c_{12}-c_{11}-a_{12}) \\ [0] \\ \vdots \\ [0] \end{pmatrix}, \ldots, \begin{pmatrix} a_{12}+c_{1n}-c_{12}-a_{1n} \\ 0 \\ -(a_{12}+c_{1n}-c_{12}-a_{1n}) \\ 0 \\ -(a_{11}-c_{11}+c_{1n}-a_{1n}) \\ 0 \\ a_{11}-c_{11}+c_{1n}-a_{1n} \\ 0 \\ [0] \\ \vdots \\ [0] \\ a_{11}+c_{12}-c_{11}-a_{12} \\ 0 \\ -(a_{11}+c_{12}-c_{11}-a_{12}) \\ 0 \end{pmatrix}, \begin{pmatrix} b_{11}-d_{11}-b_{1n}+d_{1n} \\ -(a_{11}+c_{12}-c_{11}-a_{12}) \\ -(b_{11}-d_{11}-b_{1n}+d_{1n}) \\ a_{11}+c_{12}-c_{11}-a_{12} \\ -(b_{11}-d_{11}-b_{1n}+d_{1n}) \\ 0 \\ b_{11}-d_{11}-b_{1n}+d_{1n} \\ 0 \\ [0] \\ \vdots \\ [0] \\ 0 \\ a_{11}+c_{12}-c_{11}-a_{12} \\ 0 \\ -(a_{11}+c_{12}-c_{11}-a_{12}) \end{pmatrix}$$

Proof: We will check first that these vectors belong to the nullspace of the matrix. When we multiply the first vector with the matrix, we obtain a vector having in the first row

$$(a_{11}-c_{11})(b_{11}-d_{11}-b_{12}+d_{12})+(b_{11}-d_{11})(a_{11}-c_{11}-a_{12}+c_{12})-(a_{12}-c_{12})(b_{11}-d_{11}-b_{12}+d_{12})-$$
$$(b_{12}-d_{12})(a_{11}-c_{11}-a_{12}+c_{12})$$
$$=(b_{11}-d_{11}-b_{12}+d_{12})[(a_{11}-c_{11})-(a_{12}-c_{12})]-\{(a_{11}-c_{11}-a_{12}+c_{12})[(b_{11}-d_{11})-(b_{12}-d_{12})]\}=0$$

Since we know that

$$(a_{11}-c_{11})=-(e_{11}-g_{11}),(b_{11}-d_{11})=-(f_{11}-h_{11}).$$

we obtain zero in the second row of the vector. Since the third and fourth rows of the squares are complementary to the first two rows, we deduce that the third and fourth rows of the vector are also zero. Now, the fifth entry of the vector is

$$(a_{21}-c_{21})(b_{11}-d_{11}-b_{12}+d_{12})+(b_{21}-d_{21})(a_{11}-c_{11}-a_{12}+c_{12})-$$
$$(a_{22}-c_{22})(b_{11}-d_{11}-b_{12}+d_{12})-(b_{22}-d_{22})(a_{11}-c_{11}-a_{12}+c_{12})=$$
$$(b_{11}-d_{11}-b_{12}+d_{12})[(a_{21}-c_{21})-(a_{22}-c_{22})]-\{(a_{11}-c_{11}-a_{12}+c_{12})[(b_{21}-d_{21})-(b_{22}-d_{22})]\}$$

We use the following relations according to our assumption

$$a_{22}=a_{12}+a_{21}-a_{11}, b_{22}=b_{12}+b_{21}-b_{11},$$
$$c_{22}=c_{12}+c_{21}-c_{11}, d_{22}=d_{12}+d_{21}-d_{11}.$$

and obtain

$$(b_{11}-d_{11}-b_{12}+d_{12})[(a_{21}-c_{21})-(a_{12}+a_{21}-a_{11}-c_{12}-c_{21}+c_{11})]$$
$$-\{(a_{11}-c_{11}-a_{12}+c_{12})[(b_{21}-d_{21})-(b_{12}+b_{21}-b_{11}-d_{12}-d_{21}+d_{11})]\}$$
$$=(b_{11}-d_{11}-b_{12}+d_{12})[-(a_{12}-a_{11}-c_{12}+c_{11})]-(a_{11}-c_{11}-a_{12}+c_{12})[-(b_{12}-b_{11}-d_{12}+d_{11})]=0$$

We continue checking all rows until we reach the last entry, which is

$$(a_{n1}-c_{n1})(b_{11}-d_{11}-b_{12}+d_{12})+(b_{n1}-d_{n1})(a_{11}-c_{11}-a_{12}+c_{12})-$$
$$(a_{n2}-c_{n2})(b_{11}-d_{11}-b_{12}+d_{12})-(b_{n2}-d_{n2})(a_{11}-c_{11}-a_{12}+c_{12})=$$
$$(b_{11}-d_{11}-b_{12}+d_{12})[(a_{n1}-c_{n1})-(a_{n2}-c_{n2})]-(a_{11}-c_{11}-a_{12}+c_{12})[(b_{n1}-d_{n1})-(b_{n2}-d_{n2})]$$

We use

$$a_{n2}=a_{12}+a_{n1}-a_{11}, b_{n2}=b_{12}+b_{n1}-b_{11},$$
$$c_{n2}=c_{12}+c_{n1}-c_{11}, d_{n2}=d_{12}+d_{n1}-d_{11}.$$

in order to obtain this value of the entry

$$(b_{11}-d_{11}-b_{12}+d_{12})[(a_{n1}-c_{n1})-(a_{12}+a_{n1}-a_{11}-c_{12}-c_{n1}+c_{11})]$$
$$-\{(a_{11}-c_{11}-a_{12}+c_{12})[(b_{n1}-d_{n1})-(b_{12}+b_{n1}-b_{11}-d_{12}-d_{n1}+d_{11})]\}$$
$$=(b_{11}-d_{11}-b_{12}+d_{12})[-(a_{12}-a_{11}-c_{12}+c_{11})]-(a_{11}-c_{11}-a_{12}+c_{12})[-(b_{12}-b_{11}-d_{12}+d_{11})]=0$$

Hence, we finished checking the first vector.

Now, we turn our attention to the second vector. When we multiply the matrix with it, we obtain in the first entry.

$$(a_{11}-c_{11})(a_{12}-c_{12}-a_{13}+c_{13})-(a_{12}-c_{12})(a_{11}-c_{11}-a_{13}+c_{13})+(a_{13}-c_{13})(a_{11}-c_{11}-a_{12}+c_{12})=$$
$$(a_{11}-c_{11})[(a_{12}-c_{12})-(a_{13}-c_{13})]-(a_{12}-c_{12})[(a_{11}-c_{11})-(a_{13}-c_{13})]+(a_{13}-c_{13})[(a_{11}-c_{11})-(a_{12}-c_{12})]$$
$$=(a_{11}-c_{11})(a_{12}-c_{12})-(a_{11}-c_{11})(a_{13}-c_{13})-(a_{12}-c_{12})(a_{11}-c_{11})+(a_{12}-c_{12})(a_{13}-c_{13})+(a_{13}-c_{13})(a_{11}-c_{11})$$
$$-(a_{13}-c_{13})(a_{12}-c_{12})=0$$

Using the relations

$$(a_{11}-c_{11})=-(e_{11}-g_{11})$$
$$(b_{11}-d_{11})=-(f_{11}-h_{11})$$

we deduce that the second entry is also zero. In a similar manner we can deal with the third and fourth entries. The fifth entry will be

$$(a_{21}-c_{21})(a_{12}-c_{12}-a_{13}+c_{13})-(a_{22}-c_{22})(a_{11}-c_{11}-a_{13}+c_{13})+(a_{23}-c_{23})(a_{11}-c_{11}-a_{12}+c_{12})$$

We use the relations

$$a_{22}=a_{12}+a_{21}-a_{11}, c_{22}=c_{12}+c_{21}-c_{11}$$
$$a_{23}=a_{13}+a_{21}-a_{11}, c_{23}=c_{13}+c_{21}-c_{11}$$

to obtain for the fifth entry.

$$=(a_{21}-c_{21})[(a_{12}-c_{12})-(a_{13}-c_{13})]-(a_{12}+a_{21}-a_{11}-c_{12}-c_{21}+c_{11})[(a_{11}-c_{11})-(a_{13}-c_{13})]$$
$$+(a_{13}+a_{21}-a_{11}-c_{13}-c_{21}+c_{11})[(a_{11}-c_{11})-(a_{12}-c_{12})]$$
$$=(a_{21}-c_{21})(a_{12}-c_{12})-(a_{21}-c_{21})(a_{13}-c_{13})-(a_{21}-c_{21})(a_{11}-c_{11})+(a_{21}-c_{21})(a_{13}-c_{13})$$
$$-(a_{12}-c_{12})(a_{11}-c_{11})+(a_{12}-c_{12})(a_{13}-c_{13})+(a_{11}-c_{11})^2+(a_{11}-c_{11})(a_{13}-c_{13})+(a_{13}-c_{13})(a_{11}-c_{11})$$
$$-(a_{13}-c_{13})(a_{12}-c_{12})+(a_{21}-c_{21})(a_{11}-c_{11})-(a_{21}-c_{21})(a_{12}-c_{12})-(a_{11}-c_{11})^2+(a_{12}-c_{12})(a_{11}-c_{11})=0$$

We continue checking the entries until we reach the last entry, which is

$$(a_{n1}-c_{n1})(a_{12}-c_{12}-a_{13}+c_{13})-(a_{n2}-c_{n2})(a_{11}-c_{11}-a_{13}+c_{13})+(a_{n3}-c_{n3})(a_{11}-c_{11}-a_{12}+c_{12})$$

Using the relations

$$a_{n2}=a_{12}+a_{n1}-a_{11}, c_{n2}=c_{12}+c_{n1}-c_{11}$$
$$a_{n3}=a_{13}+a_{n1}-a_{11}, c_{n3}=c_{13}+c_{n1}-c_{11}$$

we get

$$=(a_{n1}-c_{n1})[(a_{12}-c_{12})-(a_{13}-c_{13})]-(a_{12}+a_{n1}-a_{11}-c_{12}-c_{n1}+c_{11})[(a_{11}-c_{11})-(a_{13}-c_{13})]$$
$$+(a_{13}+a_{n1}-a_{11}-c_{13}-c_{n1}+c_{11})[(a_{11}-c_{11})-(a_{12}-c_{12})]$$
$$=(a_{n1}-c_{n1})(a_{12}-c_{12})-(a_{n1}-c_{n1})(a_{13}-c_{13})-(a_{n1}-c_{n1})(a_{11}-c_{11})+(a_{n1}-c_{n1})(a_{13}-c_{13})$$
$$-(a_{12}-c_{12})(a_{11}-c_{11})+(a_{12}-c_{12})(a_{13}-c_{13})+(a_{11}-c_{11})^2+(a_{11}-c_{11})(a_{13}-c_{13})+$$
$$(a_{13}-c_{13})(a_{11}-c_{11})-(a_{13}-c_{13})(a_{12}-c_{12})+(a_{n1}-c_{n1})(a_{11}-c_{11})-(a_{n1}-c_{n1})(a_{12}-c_{12})-$$
$$(a_{11}-c_{11})^2+(a_{12}-c_{12})(a_{11}-c_{11})=0$$

Hence, the second vector belongs to the nullspace of the $(4n\times 4n)$-matrix.

Similarly, we can check that all the other vectors are included in the nullspace of the $(4n\times 4n)$-matrix. We check the last vector (the $(2n-3)$-th vector) belongs to the nullspace of the $(4n\times 4n)$-matrix. The first entry by matrix multiplication is:

$$(a_{11}-c_{11})(b_{11}-d_{11}-b_{1n}+d_{1n})+(b_{11}-d_{11})(a_{11}-c_{11}-a_{12}+c_{12})-$$
$$(a_{12}-c_{12})(b_{11}-d_{11}-b_{1n}+d_{1n})-(b_{1n}-d_{1n})(a_{11}-c_{11}-a_{12}+c_{12})=$$
$$(b_{11}-d_{11}-b_{1n}+d_{1n})[(a_{11}-c_{11})-(a_{12}-c_{12})]-(a_{11}-c_{11}-a_{12}+c_{12})[(b_{11}-d_{11})-(b_{1n}-d_{1n})]=0$$

As before we deduce also that the second, third, and fourth entries are zero. The fifth entry is

$$(a_{21}-c_{21})(b_{11}-d_{11}-b_{1n}+d_{1n})+(b_{21}-d_{21})(a_{11}-c_{11}-a_{12}+c_{12})-$$
$$(a_{22}-c_{22})(b_{11}-d_{11}-b_{1n}+d_{1n})-(b_{2n}-d_{2n})(a_{11}-c_{11}-a_{12}+c_{12})=$$
$$(b_{11}-d_{11}-b_{1n}+d_{1n})[(a_{21}-c_{21})-(a_{22}-c_{22})]-(a_{11}-c_{11}-a_{12}+c_{12})[(b_{21}-d_{21})-(b_{2n}-d_{2n})]=(b_{11}-d_{11}-b_{1n}+d_{1n})$$
$$[(a_{21}-c_{21})-(a_{12}+a_{21}-a_{11}-c_{12}-c_{21}+c_{11})]$$

We use the relations

$$a_{22}=a_{12}+a_{21}-a_{11}$$
$$b_{2n}=b_{1n}+b_{21}-b_{11}$$
$$c_{22}=c_{12}+c_{21}-c_{11}$$
$$d_{2n}=d_{1n}+d_{21}-d_{11}$$

Therefore, this entry is

$$(b_{11}-d_{11}-b_{1n}+d_{1n})[(a_{21}-c_{21})-(a_{12}+a_{21}-a_{11}-c_{12}-c_{21}+c_{11})]$$
$$-\{(a_{11}-c_{11}-a_{12}+c_{12})[(b_{21}-d_{21})-(b_{1n}+b_{21}-b_{11}-d_{1n}-d_{21}+d_{11})]\}$$
$$=(b_{11}-d_{11}-b_{1n}+d_{1n})[-(a_{12}-a_{11}-c_{12}+c_{11})]-\{(a_{11}-c_{11}-a_{12}+c_{12})[-(b_{1n}-b_{11}-d_{1n}+d_{11})]\}=0$$

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When we reach the $(2n-3)$th entry, we find that it is

$$\begin{aligned}&(a_{n1}-c_{n1})(b_{11}-d_{11}-b_{1n}+d_{1n})+(b_{n1}-d_{n1})(a_{11}-c_{11}-a_{12}+c_{12})-\\&(a_{nn}-c_{nn})(b_{11}-d_{11}-b_{1n}+d_{1n})-(b_{nn}-d_{nn})(a_{11}-c_{11}-a_{12}+c_{12})=\\&(b_{11}-d_{11}-b_{1n}+d_{1n})[(a_{n1}-c_{n1})-(a_{nn}-c_{nn})]-\\&\{(a_{11}-c_{11}-a_{12}+c_{12})[(b_{n1}-d_{n1})-(b_{nn}-d_{nn})]\}\end{aligned}$$

We use the relations

$$\begin{aligned}a_{nn}&=a_{1n}+a_{n1}-a_{11}\\b_{nn}&=b_{1n}+b_{n1}-b_{11}\\c_{nn}&=c_{1n}+c_{n1}-c_{11}\\d_{nn}&=d_{1n}+d_{n1}-d_{11}\end{aligned}$$

to prove that this entry is

$$\begin{aligned}&(b_{11}-d_{11}-b_{1n}+d_{1n})[(a_{n1}-c_{n1})-(a_{12}+a_{n1}-a_{11}-c_{12}-c_{n1}+c_{11})]\\&-\{(a_{11}-c_{11}-a_{12}+c_{12})[(b_{n1}-d_{n1})-(b_{1n}+b_{n1}-b_{11}-d_{1n}-d_{n1}+d_{11})]\}\\&=(b_{11}-d_{11}-b_{1n}+d_{1n})[-(a_{12}-a_{11}-c_{12}+c_{11})]-(a_{11}-c_{11}-a_{12}+c_{12})[-(b_{1n}-b_{11}-d_{1n}+d_{11})]=0\end{aligned}$$

We prove now that the vectors are linearly independent. Let $k_1, k_2, k_3, \dots, k_{2n-4}, k_{2n-3} \in R$ such that

$$k_1\begin{pmatrix}b_{11}-d_{11}-b_{12}+d_{12}\\-(a_{11}+c_{12}-c_{11}-a_{12})\\-(b_{11}-d_{11}-b_{12}+d_{12})\\(a_{11}+c_{12}-c_{11}-a_{12})\\-(b_{11}-d_{11}-b_{12}+d_{12})\\a_{11}+c_{12}-c_{11}-a_{12}\\(b_{11}-d_{11}-b_{12}+d_{12})\\-(a_{11}+c_{12}-c_{11}-a_{12})\\0\\0\\0\\0\\\vdots\\0\\0\\0\\0\end{pmatrix}+k_2\begin{pmatrix}a_{12}+c_{13}-c_{12}-a_{13}\\0\\-(a_{12}+c_{13}-c_{12}-a_{13})\\0\\-(a_{11}-c_{11}+c_{13}-a_{13})\\0\\a_{11}-c_{11}+c_{13}-a_{13}\\0\\a_{11}+c_{12}-c_{11}-a_{12}\\0\\-(a_{11}+c_{12}-c_{11}-a_{12})\\0\\\vdots\\0\\0\\0\\0\end{pmatrix}+\dots+k_{2n-3}\begin{pmatrix}b_{11}-d_{11}-b_{1n}+d_{1n}\\-(a_{11}+c_{12}-c_{11}-a_{12})\\-(b_{11}-d_{11}-b_{1n}+d_{1n})\\a_{11}+c_{12}-c_{11}-a_{12}\\-(b_{11}-d_{11}-b_{1n}+d_{1n})\\0\\b_{11}-d_{11}-b_{1n}+d_{1n}\\0\\\vdots\\0\\0\\0\\0\\0\\a_{11}+c_{12}-c_{11}-a_{12}\\0\\-(a_{11}+c_{12}-c_{11}-a_{12})\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\\vdots\\0\\0\\0\\0\end{pmatrix}$$

This leads us to the following vector which is a zero vector.

$$\begin{pmatrix}
k_1(b_{11}-d_{11}-b_{12}+d_{12})+k_2(a_{12}+c_{13}-c_{12}-a_{13})+k_3(b_{11}-d_{11}-b_{13}+d_{13})+\ldots+ \\
k_{2n-4}(a_{12}+c_{1n}-c_{12}-a_{1n})+k_{2n-3}(b_{11}-d_{11}-b_{1n}+d_{1n}) \\
-k_1(a_{11}+c_{12}-c_{11}-a_{12})-k_3(a_{11}+c_{12}-c_{11}-a_{12})-k_{2n-3}(a_{11}+c_{12}-c_{11}-a_{12}) \\
-k_1(b_{11}-d_{11}-b_{12}+d_{12})-k_2(a_{12}+c_{13}-c_{12}-a_{13})-k_3(b_{11}-d_{11}-b_{13}+d_{13})-\ldots- \\
k_{2n-4}(a_{12}+c_{1n}-c_{12}-a_{1n})-k_{2n-3}(b_{11}-d_{11}-b_{14}+d_{1n}) \\
k_1(a_{11}+c_{12}-c_{11}-a_{12})+k_3(a_{11}+c_{12}-c_{11}-a_{12})+k_{2n-3}(a_{11}+c_{12}-c_{11}-a_{12}) \\
-k_1(b_{11}-d_{11}-b_{12}+d_{12})-k_2(a_{11}-c_{11}+c_{13}-a_{13})-k_3(b_{11}-d_{11}-b_{13}+d_{13})-\ldots- \\
k_{2n-4}(a_{11}-c_{11}+c_{1n}-a_{1n})-k_{2n-3}(b_{11}-d_{11}-b_{1n}+d_{1n}) \\
k_1(a_{11}+c_{12}-c_{11}-a_{12}) \\
k_1(b_{11}-d_{11}-b_{12}+d_{12})+k_2(a_{11}-c_{11}+c_{13}-a_{13})+k_3(b_{11}-d_{11}-b_{13}+d_{13})+\ldots+ \\
k_{2n-4}(a_{11}-c_{11}+c_{1n}-a_{1n})+k_{2n-3}(b_{11}-d_{11}-b_{1n}+d_{1n}) \\
-k_1(a_{11}+c_{12}-c_{11}-a_{12}) \\
k_2(a_{11}+c_{12}-c_{11}-a_{12}) \\
k_3(a_{11}+c_{12}-c_{11}-a_{12}) \\
-k_2(a_{11}+c_{12}-c_{11}-a_{12}) \\
-k_3(a_{11}+c_{12}-c_{11}-a_{12}) \\
\vdots \\
k_{2n-4}(a_{11}+c_{12}-c_{11}-a_{12}) \\
k_{2n-3}(a_{11}+c_{12}-c_{11}-a_{12}) \\
-k_{2n-4}(a_{11}+c_{12}-c_{11}-a_{12}) \\
-k_{2n-3}(a_{11}+c_{12}-c_{11}-a_{12})
\end{pmatrix}$$

From the $(4n-2)$-th row of this vector we obtain the equation

$$k_{2n-3}(a_{11}+c_{12}-c_{11}-a_{12})=0$$

According to our assumptions we must have $k_{2n-3}=0$. Similarly, we obtain $k_{2n-4}=0$ from the $(4n-3)$-th row. We continue checking all the rows up to the tenth row, which looks like this

$$k_3(a_{11}+c_{12}-c_{11}-a_{12})=0$$

Hence, we conclude that $k_3=0$. From the ninth (res. eighth) row we obtain $k_2=0$ (res. $k_1=0$). Since all $k_1, k_2, k_3, \ldots, k_{2n-4}, k_{2n-3}$ are zero, we are done.□

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