

# Preface

Intermediate Algebra is the second part of a two-part course in Algebra. Written in a clear and concise manner, it carefully builds on the basics learned in Elementary Algebra and introduces the more advanced topics required for further study of applications found in most disciplines. Used as a standalone textbook, it offers plenty of review as well as something new to engage the student in each chapter. Written as a blend of the traditional and graphical approaches to the subject, this textbook introduces functions early and stresses the geometry behind the algebra. While CAS independent, a standard scientific calculator will be required and further research using technology is encouraged.

Intermediate Algebra clearly lays out the steps required to build the skills needed to solve a variety of equations and interpret the results. With robust and diverse exercise sets, students have the opportunity to solve plenty of practice problems. In addition to embedded video examples and other online learning resources, the importance of practice with pencil and paper is stressed. This text respects the traditional approaches to algebra pedagogy while enhancing it with the technology available today. In addition, Intermediate Algebra was written from the ground up in an open and modular format, allowing the instructor to modify it and leverage their individual expertise as a means to maximize the student experience and success.

The importance of Algebra cannot be overstated; it is the basis for all mathematical modeling used in all disciplines. After completing a course sequence based on Elementary and Intermediate Algebra, students will be on firm footing for success in higher-level studies at the college level.

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# Table of Contents

[About the Author](#)

[Acknowledgments](#)

[Preface](#)

## Chapter 1: [Algebra Fundamentals](#)

---

- 1.1: [Review of Real Numbers and Absolute Value](#)
- 1.2: [Operations with Real Numbers](#)
- 1.3: [Square and Cube Roots of Real Numbers](#)
- 1.4: [Algebraic Expressions and Formulas](#)
- 1.5: [Rules of Exponents and Scientific Notation](#)
- 1.6: [Polynomials and Their Operations](#)
- 1.7: [Solving Linear Equations](#)
- 1.8: [Solving Linear Inequalities with One Variable](#)
- 1.9: [Review Exercises and Sample Exam](#)

## Chapter 2: [Graphing Functions and Inequalities](#)

---

- 2.1: [Relations, Graphs, and Functions](#)
- 2.2: [Linear Functions and Their Graphs](#)
- 2.3: [Modeling Linear Functions](#)
- 2.4: [Graphing the Basic Functions](#)
- 2.5: [Using Transformations to Graph Functions](#)
- 2.6: [Solving Absolute Value Equations and Inequalities](#)
- 2.7: [Solving Inequalities with Two Variables](#)
- 2.8: [Review Exercises and Sample Exam](#)

## Chapter 3: [Solving Linear Systems](#)

---

- 3.1: [Linear Systems with Two Variables and Their Solutions](#)
- 3.2: [Solving Linear Systems with Two Variables](#)
- 3.3: [Applications of Linear Systems with Two Variables](#)
- 3.4: [Solving Linear Systems with Three Variables](#)

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- 3.5: [Matrices and Gaussian Elimination](#)
- 3.6: [Determinants and Cramer's Rule](#)
- 3.7: [Solving Systems of Inequalities with Two Variables](#)
- 3.8: [Review Exercises and Sample Exam](#)

#### **Chapter 4: [Polynomial and Rational Functions](#)**

---

- 4.1: [Algebra of Functions](#)
- 4.2: [Factoring Polynomials](#)
- 4.3: [Factoring Trinomials](#)
- 4.4: [Solve Polynomial Equations by Factoring](#)
- 4.5: [Rational Functions: Multiplication and Division](#)
- 4.6: [Rational Functions: Addition and Subtraction](#)
- 4.7: [Solving Rational Equations](#)
- 4.8: [Applications and Variation](#)
- 4.9: [Review Exercises and Sample Exam](#)

#### **Chapter 5: [Radical Functions and Equations](#)**

---

- 5.1: [Roots and Radicals](#)
- 5.2: [Simplifying Radical Expressions](#)
- 5.3: [Adding and Subtracting Radical Expressions](#)
- 5.4: [Multiplying and Dividing Radical Expressions](#)
- 5.5: [Rational Exponents](#)
- 5.6: [Solving Radical Equations](#)
- 5.7: [Complex Numbers and Their Operations](#)
- 5.8: [Review Exercises and Sample Exam](#)

#### **Chapter 6: [Solving Equations and Inequalities](#)**

---

- 6.1: [Extracting Square Roots and Completing the Square](#)
- 6.2: [Quadratic Formula](#)
- 6.3: [Solving Equations Quadratic in Form](#)
- 6.4: [Quadratic Functions and Their Graphs](#)
- 6.5: [Solving Quadratic Inequalities](#)
- 6.6: [Solving Polynomial and Rational Inequalities](#)

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6.7: [Review Exercises and Sample Exam](#)

## **Chapter 7: [Exponential and Logarithmic Functions](#)**

---

- 7.1: [Composition and Inverse Functions](#)
- 7.2: [Exponential Functions and Their Graphs](#)
- 7.3: [Logarithmic Functions and Their Graphs](#)
- 7.4: [Properties of the Logarithm](#)
- 7.5: [Solving Exponential and Logarithmic Equations](#)
- 7.6: [Applications](#)
- 7.7: [Review Exercises and Sample Exam](#)

## **Chapter 8: [Conic Sections](#)**

---

- 8.1: [Distance, Midpoint, and the Parabola](#)
- 8.2: [Circles](#)
- 8.3: [Ellipses](#)
- 8.4: [Hyperbolas](#)
- 8.5: [Solving Nonlinear Systems](#)
- 8.6: [Review Exercises and Sample Exam](#)

## **Chapter 9: [Sequences, Series, and the Binomial Theorem](#)**

---

- 9.1: [Introduction to Sequences and Series](#)
- 9.2: [Arithmetic Sequences and Series](#)
- 9.3: [Geometric Sequences and Series](#)
- 9.4: [Binomial Theorem](#)
- 9.5: [Review Exercises and Sample Exam](#)

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## About the Author John Redden



John Redden earned his degrees at California State University–Northridge and Glendale Community College. He is now a professor of mathematics at the College of the Sequoias, located in Visalia, California. With over a decade of experience working with students to develop their algebra skills, he knows just

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where they struggle and how to present complex techniques in more understandable ways. His student-friendly and commonsense approach carries over to his writing of *Intermediate Algebra* and various other open-source learning resources.

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# Chapter 1

## Algebra Fundamentals

### 1.1 Review of Real Numbers and Absolute Value

#### LEARNING OBJECTIVES

1. Review the set of real numbers.
2. Review the real number line and notation.
3. Define the geometric and algebraic definition of absolute value.

#### Real Numbers

Algebra is often described as the generalization of arithmetic. The systematic use of **variables**, letters used to represent numbers, allows us to communicate and solve a wide variety of real-world problems. For this reason, we begin by reviewing real numbers and their operations.

A **set** is a collection of objects, typically grouped within braces { }, where each object is called an **element**. When studying mathematics, we focus on special sets of numbers.

The three periods (...) are called an ellipsis and indicate that the numbers continue without bound. A **subset**, denoted  $\subset$ , is a set consisting of elements that belong to a given set. Notice that the sets of **natural** and **whole numbers** are both subsets of the set of integers and we can write:

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A set with no elements is called the **empty set** and has its own special notation:

**Rational numbers**, denoted  $\mathbb{Q}$ , are defined as any number of the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b$  is nonzero. We can describe this set using **set notation**:

The vertical line  $|$  inside the braces reads, “*such that*” and the symbol  $\in$  indicates set membership and reads, “*is an element of.*” The notation above in its entirety reads, “*the set of all numbers such that  $a$  and  $b$  are elements of the set of integers and  $b$  is not equal to zero.*”

Decimals that terminate or repeat are rational. For example,

the set of integers is a subset of the set of rational numbers,  $\mathbb{Z} \subseteq \mathbb{Q}$ , because every integer can be expressed as a ratio of the integer and 1. In other words, any integer can be written over 1 and can be considered a rational number. For example,

**Irrational numbers** are defined as any numbers that cannot be written as a ratio of two integers.

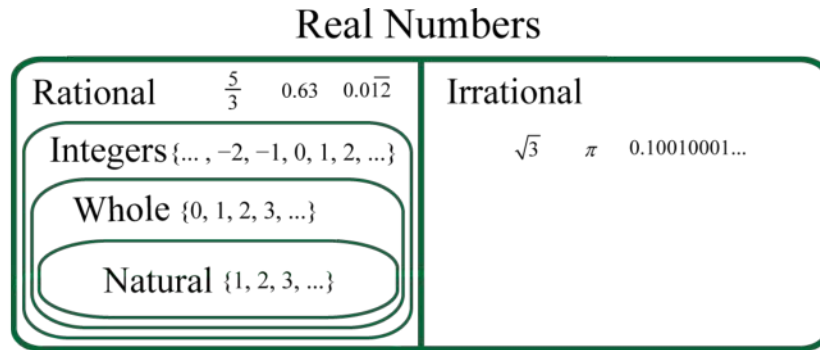
Non terminating decimals that do not repeat are irrational. For example,

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Finally, the set of **real numbers**, denoted  $\mathbb{R}$ , is defined as the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers. In summary,



The set of **even integers** is the set of all integers that are evenly divisible by 2. We can obtain the set of even integers by multiplying each integer by 2.

The set of **odd integers** is the set of all nonzero integers that are not evenly divisible by 2.

A **prime number** is an integer greater than 1 that is divisible only by 1 and itself. The smallest prime number is 2 and the rest are necessarily odd.

Any integer greater than 1 that is not prime is called a composite number and can be uniquely written as a product of primes. When a composite number, such as 42, is written as a product,  $42=2 \cdot 21$ , we say that  $2 \cdot 21$  is a factorization of 42 and that 2 and 21 are factors. Note that

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factors divide the number evenly. We can continue to write composite factors as products until only a product of primes remains.

$$\begin{array}{c} 42 \\ \swarrow \searrow \\ = 2 \cdot 21 \\ \quad \swarrow \searrow \\ = 2 \cdot 3 \cdot 7 \end{array}$$

Therefore, the **prime factorization** of 42 is

### Example 1

Determine the prime factorization of 210.

Solution:

Begin by writing 210 as a product with 10 as a factor. Then continue factoring until only a product of primes remains.

Since the prime factorization is unique, it does not matter how we choose to initially factor the number; the end result will be the same.

Answer:

A **fraction** is a rational number written as a quotient, or ratio, of two integers  $a$  and  $b$  where  $b \neq 0$ .

$$\begin{array}{l} \text{Numerator} \rightarrow \frac{a}{b} \\ \text{Denominator} \rightarrow \end{array}$$

The integer above the fraction bar is called the **numerator** and the integer below is called the **denominator**. Two equal ratios expressed using different numerators and denominators are called **equivalent fractions**. For example,

Consider the following factorizations of 50 and 100:

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The numbers 50 and 100 share the factor 25. A shared factor is called a common factor. Making use of the fact that , we have

Dividing and replacing this factor with a 1 is called **cancelling**. Together, these basic steps for finding equivalent fractions define the process of **reducing**. Since factors divide their product evenly, we achieve the same result by dividing both the numerator and denominator by 25 as follows:

Finding equivalent fractions where the numerator and denominator are **relatively prime**, or have no common factor other than 1, is called **reducing to lowest terms**. This can be done by dividing the numerator and denominator by the **greatest common factor (GCF)**. The GCF is the largest number that divides a set of numbers evenly. One way to find the GCF of 50 and 100 is to list all the factors of each and identify the largest number that appears in both lists. Remember, each number is also a factor of itself.

Common factors are listed in bold, and we see that the greatest common factor is 50. We use the following notation to indicate the GCF of two numbers:  $GCF(50, 100) = 50$ . After determining the GCF, reduce by dividing both the numerator and the denominator as follows:

## Example 2

Reduce to lowest terms:  $\frac{108}{72}$ .

Solution:

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A quick way to find the GCF of the numerator and denominator requires us to first write each as a product of primes. The GCF will be the product of all the common prime factors.

In this case, the product of the common prime factors is 36.

We can convert the improper fraction to a mixed number ; however, it is important to note that converting to a mixed number is not part of the reducing process. We consider improper fractions, such as  $\frac{12}{5}$ , to be reduced to lowest terms. In algebra it is often preferable to work with improper fractions, although in some applications, mixed numbers are more appropriate.

Answer:

Recall the relationship between multiplication and division:

In this case, the **dividend** 12 is evenly divided by the **divisor** 6 to obtain the **quotient** 2. It is true in general that if we multiply the divisor by the quotient we obtain the dividend. Now consider the case where the dividend is zero and the divisor is nonzero:

This demonstrates that zero divided by any nonzero real number must be zero. Now consider a nonzero number divided by zero:

Zero times anything is zero and we conclude that there is no real number such that  $0 = \text{number} \times 0$ . Thus, the quotient  $12 \div 0$  is **undefined**. Try it on a calculator, what does it say? For our purposes, we will simply write “undefined.” To summarize, given any real number  $a$ , then

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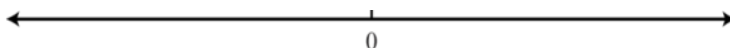
We are left to consider the case where the dividend and divisor are both zero.

Here, any real number seems to work. For example,  $0 \cdot 5 = 0$  and also,  $0 \cdot 3 = 0$ . Therefore, the quotient is uncertain or **indeterminate**.

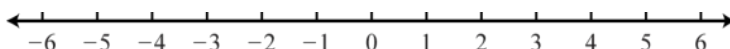
In this course, we state that  $0 \div 0$  is undefined.

## The Number Line and Notation

A **real number line**, or simply **number line**, allows us to visually display real numbers by associating them with unique points on a line. The real number associated with a point is called a **coordinate**. A point on the real number line that is associated with a coordinate is called its **graph**. To construct a number line, draw a horizontal line with arrows on both ends to indicate that it continues without bound. Next, choose any point to represent the number zero; this point is called the **origin**.



Positive real numbers lie to the right of the origin and negative real numbers lie to the left. The number zero (0) is neither positive nor negative. Typically, each tick represents one unit.

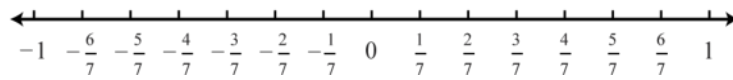
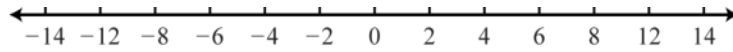


As illustrated below, the scale need not always be one unit. In the first number line, each tick mark represents two units. In the second, each tick mark represents :

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The graph of each real number is shown as a dot at the appropriate point on the number line. A partial graph of the set of integers, follows:



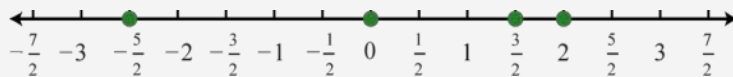
### Example 3

Graph the following set of real numbers:

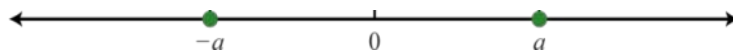
Solution:

Graph the numbers on a number line with a scale where each tick mark represents  $\frac{1}{2}$  unit.

Answer:



The **opposite** of any real number  $a$  is  $-a$ . Opposite real numbers are the same distance from the origin on a number line, but their graphs lie on opposite sides of the origin and the numbers have opposite signs.

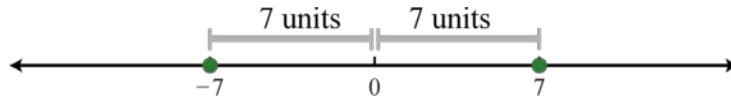


Given the integer  $-7$ , the integer the same distance from the origin and with the opposite sign is  $+7$ , or just 7.

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Therefore, we say that the opposite of  $-7$  is  $-(-7) = 7$ . This idea leads to what is often referred to as the **double-negative property**. For any real number  $a$ ,

### Example 4

Calculate:

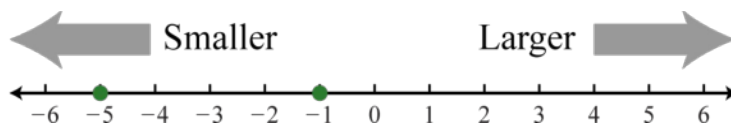
Solution:

Here we apply the double-negative within the innermost parentheses first.

Answer:

In general, an odd number of sequential negative signs results in a negative value and an even number of sequential negative signs results in a positive value.

When comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 15 is greater than 5, but it may not be so clear to see that  $-1$  is greater than  $-5$  until we graph each number on a number line.



We use symbols to help us efficiently communicate relationships between numbers on the number line.

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The relationship between the **integers** in the previous illustration can be expressed two ways as follows:

The symbols  $<$  and  $>$  are used to denote **strict inequalities**, and the symbols  $\leq$  and  $\geq$  are used to denote **inclusive inequalities**. In some situations, more than one symbol can be correctly applied. For example, the following two statements are both true:

In addition, the “*or equal to*” component of an inclusive inequality allows us to correctly write the following:

The logical use of the word “*or*” requires that only one of the conditions need be true: the “*less than*” or the “*equal to*.”

### Example 5

Fill in the blank with  $<$ ,  $=$ , or  $>$ :

Solution:

Use  $>$  because the graph of  $-2$  is to the right of the graph of  $-12$  on a number line. Therefore,  $-2 > -12$ , which reads, “*negative two is greater than negative twelve.*”



Answer:  $-2 > -12$

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An **algebraic inequality**, such as  $x \geq 2$ , is read, “ $x$  is greater than or equal to 2.” Here the letter  $x$  is a variable, which can represent any real number. However, the statement  $x \geq 2$  imposes a condition on the variable. Solutions are the values for  $x$  that satisfy the condition. This inequality has infinitely many solutions for  $x$ , some of which are 2, 3, 4.1, 5, 20, and 20.001. Since it is impossible to list all of the solutions, a system is needed that allows a clear communication of this infinite set. Common ways of expressing solutions to an inequality are by graphing them on a number line, using interval notation, or using set notation.

To express the solution graphically, draw a number line and shade in all the values that are solutions to the inequality. This is called the **graph of the solution set**. Interval and set notation follow:



In this example, there is an inclusive inequality, which means that the lower-bound 2 is included in the solution set. Denote this with a closed dot on the number line and a square bracket in interval notation. The symbol  $\infty$  is read as “**infinity**” and indicates that the set is unbounded to the right on a number line. If using a standard keyboard, use (inf) as a shortened form to denote infinity. Now compare the notation in the previous example to that of the strict, or noninclusive, inequality that follows:

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Strict inequalities imply that solutions may get very close to the boundary point, in this case 3, but not actually include it. Denote this idea with an open dot on the number line and a round parenthesis in interval notation. The symbol  $-\infty$  is read as “**negative infinity**” and indicates that the set is unbounded to the left on a number line. Infinity is a bound to the real numbers, but is not itself a real number: it cannot be included in the solution set and thus is always enclosed with a parenthesis.

Interval notation is textual and is determined after graphing the solution set on a number line. The numbers in interval notation should be written in the same order as they appear on the number line, with smaller numbers in the set appearing first. Set notation, sometimes called set-builder notation, allows us to describe the set using familiar mathematical notation. For example,

Here, describes the type of number. This implies that the variable  $x$  represents a real number. The statement  $x \geq 2$  is the condition that describes the set using mathematical notation. At this point in our study of algebra, it is assumed that all variables represent real numbers. For this reason, you can omit the “”, and write

### Example 6

Graph the solution set and give the interval and set notation equivalents:

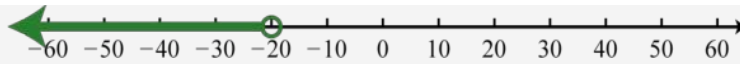
Solution:

Use an open dot at  $-20$ , because of the strict inequality  $<$ , and shade all real numbers to the left.

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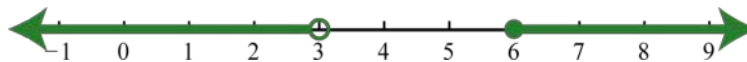
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Answer: Interval notation:  $(-\infty, -20)$ ; set notation:  $\{x|x < -20\}$

A **compound inequality** is actually two or more inequalities in one statement joined by the word “and” or by the word “or”. Compound inequalities with the logical “or” require that either condition must be satisfied. Therefore, the solution set of this type of compound inequality consists of all the elements of the solution sets of each inequality. When we join these individual solution sets it is called the **union**, denoted  $\cup$ . For example,



An inequality such as,

reads, “*negative one is less than or equal to  $x$  and  $x$  is less than three.*” This is actually a compound inequality because it can be decomposed as follows:

The logical “and” requires that both conditions must be true. Both inequalities will be satisfied by all the elements in the intersection, denoted  $\cap$ , of the solution sets of each.

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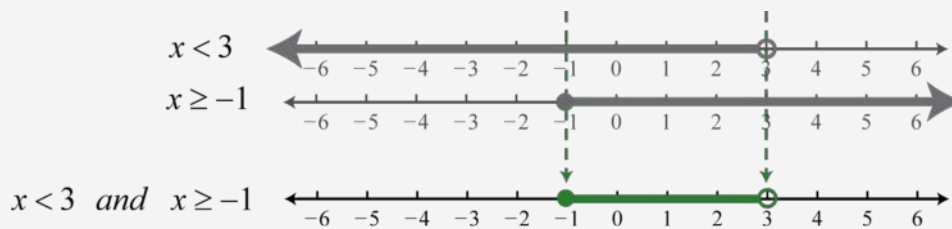


## Example 7

Graph and give the interval notation equivalent:  $-1 \leq x < 3$ .

Solution:

Determine the intersection, or overlap, of the two solution sets to  $x < 3$  and  $x \geq -1$ . The solutions to each inequality are sketched above the number line as a means to determine the intersection, which is graphed on the number line below.



Here, 3 is not a solution because it solves only one of the inequalities. Alternatively, we may interpret as all possible values for  $x$  between, or bounded by,  $-1$  and  $3$  where  $-1$  is included in the solution set.

Answer: Interval notation: ; set notation:

In this text, we will often point out the equivalent notation used to express mathematical quantities electronically using the standard symbols available on a keyboard.

Many calculators, computer algebra systems, and programming languages use the notation presented above, in quotes.

## Absolute Value

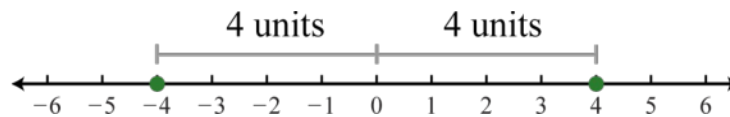
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The **absolute value** of a real number  $a$ , denoted  $|a|$ , is defined as the distance between zero (the origin) and the graph of that real number on the number line. Since it is a distance, it is always positive. For example,

Both 4 and  $-4$  are four units from the origin, as illustrated below:



Also, it is worth noting that,

The algebraic definition of the absolute value of a real number  $a$  follows:

This is called a **piecewise definition**. The result depends on the quantity  $a$ . If  $a$  is nonnegative, as indicated by the inequality  $a \geq 0$ , then the absolute value will be that number  $a$ . If  $a$  is negative, as indicated by the inequality  $a < 0$ , then the absolute value will be the opposite of that number,  $-a$ . The results will be the same as the geometric definition. For example, to determine  $|-4|$  we make note that the value is negative and use the second part of the definition. The absolute value will be the opposite of  $-4$ .

At this point, we can determine what real numbers have certain absolute values.

## Example 8

Determine the values represented by  $x$ :

Solution:

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Think of a real number whose distance to the origin is 6 units. There are two solutions: the distance to the right of the origin and the distance to the left of the origin, namely  $\{\pm 6\}$ . The symbol  $\pm$  is read “*plus or minus*” and indicates that there are two answers, one positive and one negative.

Answer:  $x = \pm 6$

### Example 9

Determine the values represented by  $x$ :  $|x| = -6$ .

Solution:

Here we wish to find a value where the distance to the origin is negative. Since negative distance is not defined, this equation has no solution. Use the empty set  $\emptyset$  to denote this.

Answer:  $\emptyset$

The absolute value can be expressed textually using the notation  $\text{abs}(a)$ . We often encounter negative absolute values, such as  $-|3|$  or  $-\text{abs}(3)$ . Notice that the negative sign is in front of the absolute value symbol. In this case, work the absolute value first and then find the opposite of the result.

Try not to confuse this with the double negative property, which states that  $-(-3) = +3$ .

### Example 10

Simplify:

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Solution:

First, find the absolute value of  $-50$  and then apply the double-negative property.

Answer: 50

## KEY TAKEAWAYS

Algebra is often described as the generalization of arithmetic. The systematic use of variables, used to represent real numbers, allows us to communicate and solve a wide variety of real-world problems. Therefore, it is important to review the subsets of real numbers and their properties.

The number line allows us to visually display real numbers by associating them with unique points on a line.

Special notation is used to communicate equality and order relationships between numbers on a number line.

The absolute value of a real number is defined geometrically as the distance between zero and the graph of that number on a number line. Alternatively, the absolute value of a real number is defined algebraically in a piecewise manner. If a real number  $a$  is nonnegative, then the absolute value will be that number  $a$ . If  $a$  is negative, then the absolute value will be the opposite of that number,  $-a$ .

## TOPIC EXERCISES

### PART A: REAL NUMBERS

**Use set notation to list the described elements.**

1. Every other positive odd number up to 21.
2. Every other positive even number up to 22.
3. The even prime numbers.
4. Rational numbers that are also irrational.
5. The set of negative integers.

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6. The set of negative even integers.
7. Three consecutive odd integers starting with 13.
8. Three consecutive even integers starting with 22.

**Determine the prime factorization of the given composite number.**

1. 195
2. 78
3. 330
4. 273
5. 180
6. 350

**Reduce to lowest terms.**

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.

## **PART B: NUMBER LINE AND NOTATION**

**Graph the following sets of numbers.**

25.  $\{-5, 5, 10, 15\}$
26.  $\{-4, -2, 0, 2, 4\}$

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27.  $\{-32, -12, 0, 1, 2\}$   
28.  $\{-34, -14, 0, 12, 34\}$   
29.  $\{-5, -4, -3, -1, 1\}$   
30.  $\{-40, -30, -20, 10, 30\}$

**Simplify.**

25.  $-(-10)$   
26.  $-(-35)$   
27.  $-(-(-12))$   
28.  $-(-(-53))$   
29.  $-(-(-(-12)))$   
30.  $-(-(-(-(-34))))$

**Fill in the blank with  $<$ ,  $=$ , or  $>$ .**

25.  $-10$  \_\_\_\_\_  $-15$   
26.  $-101$  \_\_\_\_\_  $-100$   
27.  $-33$  \_\_\_\_\_  $0$   
28.  $0$  \_\_\_\_\_  $-50$   
29.  $-(-(-2))$  \_\_\_\_\_  $-(-3)$   
30.  $-(-(-12))$  \_\_\_\_\_  $-14$   
31.  $-(-(-23))$  \_\_\_\_\_  $-(-12)$   
32.  $-(-23)$  \_\_\_\_\_  $-(-(-(-23)))$

**True or False.**

25.  $0=0$



26.  $5 \leq 5$
27.  $-$  is irrational.
28. 0 is a nonnegative number.
29. Any integer is a rational number.
30. The constant  $\pi$  is rational.

**Graph the solution set and give the interval notation equivalent.**

25.  $x < -1$
26.  $x > -3$
27.  $x \geq -8$
28.  $x \leq 6$
29.  $-10 \leq x < 4$
30.  $3 < x \leq 7$
31.  $-40 < x < 0$
32.  $-12 \leq x \leq -4$
33.  $x < 5$  and  $x \geq 0$
34.  $x \leq -10$  and  $x \geq -40$
35.  $x \leq 7$  and  $x < 10$
36.  $x < 1$  and  $x > 3$
37.  $x < -2$  or  $x \geq 5$
38.  $x \leq 0$  or  $x \geq 4$
39.  $x < 6$  or  $x > 2$
40.  $x < 0$  or  $x \leq 5$

**Write an equivalent inequality.**

25. All real numbers less than  $-15$ .

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26. All real numbers greater than or equal to  $-7$ .
27. All real numbers less than  $6$  and greater than zero.
28. All real numbers less than zero and greater than  $-5$ .
29. All real numbers less than or equal to  $5$  or greater than  $10$ .
30. All real numbers between  $-2$  and  $2$ .

**Determine the inequality given the answers expressed in interval notation.**

25.  $(-\infty, 12)$
26.  $[-8, \infty)$
27.  $(-\infty, 0]$
28.  $(0, \infty)$
29.  $(-6, 14)$
30.  $(0, 12]$
31.  $[5, 25)$
32.  $[-30, -10]$
33.  $(-\infty, 2) \cup [3, \infty)$
34.  $(-\infty, -19] \cup [-12, \infty)$
35.  $(-\infty, -2) \cup (0, \infty)$
36.  $(-\infty, -15] \cup (-5, \infty)$

## PART C: ABSOLUTE VALUE

**Simplify.**

85.  $|-9|$
86.  $|14|$
87.  $-|-4|$

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88.  $-|8|$   
89.  
90.  
91.  $-|(-7)|$   
92.  $-|(-10)|$   
93.  $-(-|-2|)$   
94.  $-(-|-10|)$   
95.  $-(-|(-5)|)$   
96.  $-(-(-|-20|))$

**Determine the values represented by  $a$ .**

85.  $|a|=10$   
86.  $|a|=7$   
87.  
88.  
89.  $|a|=0$   
90.  $|a|=-1$

## **PART D: DISCUSSION BOARD**

103. Research and discuss the origins and evolution of algebra.
104. Research and discuss reasons why algebra is a required subject today.
105. Solution sets to inequalities can be expressed using a graph, interval notation, or set notation. Discuss the merits and drawbacks of each method. Which do you prefer?
106. Research and discuss the Fundamental Theorem of Algebra. Illustrate its idea with an example and share your results.

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## ANSWERS

1.  $\{1, 5, 9, 13, 17, 21\}$

2.

3.  $\{2\}$

4.

5.  $\{\dots, -3, -2, -1\}$

6.

7.  $\{13, 15, 17\}$

8.

9.  $3 \cdot 5 \cdot 13$

10.

11.  $2 \cdot 3 \cdot 5 \cdot 11$

12.

13.  $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

14.

15.

16.

17.

18.

19.

20.

21.

22.

23. 0

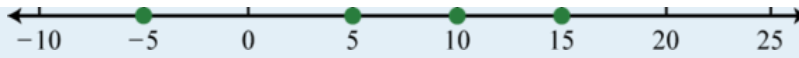
24.

25.

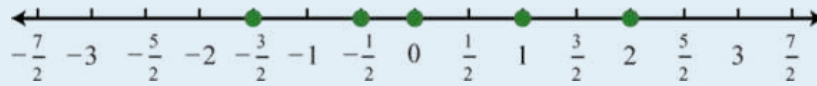
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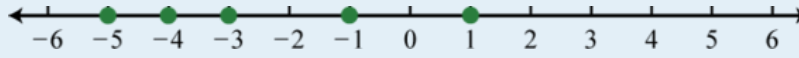


26.



27.

28.



29. 10

30.

31.

32. -12

33.

34.

35.

36. >

37.

38. <

39.

40. <

41.

42. <

43.

44. True

45.

46. False

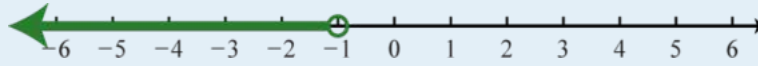
47.



48. True

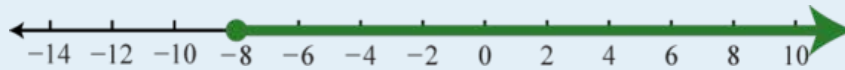
49.

50.  $(-\infty, -1)$ ;



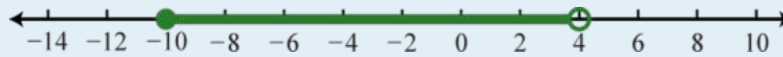
51.

52.  $[8, \infty)$ ;



53.

54.  $[-10, 4)$ ;



55.

56.  $(-40, 0)$ ;

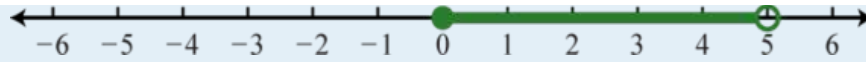


57.

58.  $[0, 5)$ ;







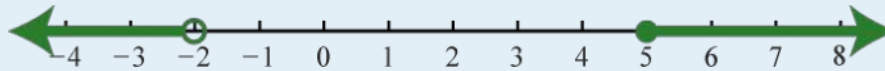
59.

60.  $(-\infty, 7)$ ;



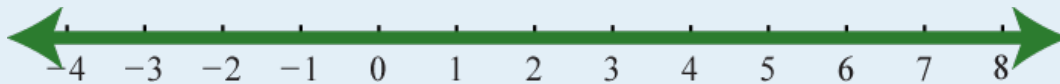
61.

62.  $(-\infty, -2) \cup [5, \infty)$ ;



63.

64.  $(-\infty, \infty) = \mathbb{R}$ ;



65.

66.  $x < -15$

67.

68.  $0 < x < 6$

69.

70.  $x \leq 5$  or  $x > 10$

71.



72.  $x < 12$
- 73.
74.  $x \leq 0$
- 75.
76.  $-6 < x < 14$
- 77.
78.  $5 \leq x < 25$
- 79.
80.  $x < 2$  or  $x \geq 3$
- 81.
82.  $x < -2$  or  $x > 0$
- 83.
85. 9
- 86.
87. -4
- 88.
89. -58
- 90.
91. -7
- 92.
93. 2
- 94.
95. 5
- 96.
97.  $a = \pm 10$
- 98.
99.  $a = \pm 12$
- 100.

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- 101.  $a=0$
- 102.
- 103. Answer may vary
- 104.
- 105. Answer may vary

## 1.2 Operations with Real Numbers

### LEARNING OBJECTIVES

1. Review the properties of real numbers.
2. Simplify expressions involving grouping symbols and exponents.
3. Simplify using the correct order of operations.

### Working with Real Numbers

In this section, we continue to review the properties of real numbers and their operations. The result of adding real numbers is called the **sum** and the result of subtracting is called the **difference**. Given any real numbers  $a$ ,  $b$ , and  $c$ , we have the following properties of addition:

<b>Additive Identity Property:</b>	$a+0=0+a=a$
<b>Additive Inverse Property:</b>	$a+(-a)=(-a)+a=0$
<b>Associative Property:</b>	$(a+b)+c=a+(b+c)$
<b>Commutative Property:</b>	$a+b=b+a$

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It is important to note that addition is commutative and subtraction is not. In other words, the order in which we add does not matter and will yield the same result. However, this is not true of subtraction.

We use these properties, along with the double-negative property for real numbers, to perform more involved sequential operations. To simplify things, make it a general rule to first replace all sequential operations with either addition or subtraction and then perform each operation in order from left to right.

### Example 1

Simplify:  $-10 - (-10) + (-5)$ .

Solution:

Replace the sequential operations and then perform them from left to right.

Answer:  $-5$

Adding or subtracting fractions requires a **common denominator**. Assume the common denominator  $c$  is a nonzero integer and we have



## Example 2

Simplify:

Solution:

First determine the least common multiple (LCM) of 9, 15, and 45. The least common multiple of all the denominators is called the **least common denominator** (LCD). We begin by listing the multiples of each given denominator:

Here we see that the  $\text{LCM}(9, 15, 45) = 45$ . Multiply the numerator and the denominator of each fraction by values that result in equivalent fractions with the determined common denominator.

Once we have equivalent fractions, with a common denominator, we can perform the operations on the numerators and write the result over the common denominator.

And then reduce if necessary,

Answer:

Finding the LCM using lists of multiples, as described in the previous example, is often very cumbersome. For example, try making a list of multiples for 12 and 81. We can streamline the process of finding the LCM by using prime factors.

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The least common multiple is the product of each prime factor raised to the highest power. In this case,

Often we will find the need to translate English sentences involving addition and subtraction to mathematical statements. Below are some common translations.

### Example 3

What is 8 subtracted from the sum of 3 and  $\frac{1}{2}$  ?

Solution:

We know that subtraction is not commutative; therefore, we must take care to subtract in the correct order. First, add 3 and  $\frac{1}{2}$  and then subtract 8 as follows:

$$\overbrace{\left( 3 + \frac{1}{2} \right)}^{\text{"the sum of 3 and } \frac{1}{2} \text{"}} \quad \overbrace{- 8}^{\text{"subtract 8 from the sum"}}$$

Perform the indicated operations.

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Answer:

The result of multiplying real numbers is called the **product** and the result of dividing is called the **quotient**. Given any real numbers  $a$ ,  $b$ , and  $c$ , we have the following properties of multiplication:

<b>Zero Factor Property:</b>	$a \cdot 0 = 0 \cdot a = 0$
<b>Multiplicative Identity Property:</b>	$a \cdot 1 = 1 \cdot a = a$
<b>Associative Property:</b>	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
<b>Commutative Property:</b>	$a \cdot b = b \cdot a$

It is important to note that multiplication is commutative and division is not. In other words, the order in which we multiply does not matter and will yield the same result. However, this is not true of division.

We will use these properties to perform sequential operations involving multiplication and division. Recall that the product of a positive number and a negative number is negative. Also, the product of two negative numbers is positive.

## Example 4

Multiply:  $5(-3)(-2)(-4)$ .

Solution:

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Multiply two numbers at a time as follows:

$$\begin{aligned}5(-3)(-2)(-4) &= \underbrace{5(-3)}(-2)(-4) \\ &= \underbrace{-15(-2)}(-4) \\ &= \underbrace{30(-4)} \\ &= -120\end{aligned}$$

Answer:  $-120$

Because multiplication is commutative, the order in which we multiply does not affect the final answer. However, when sequential operations involve multiplication and division, order does matter; hence we must work the operations from *left to right* to obtain a correct result.

## Example 5

Simplify:  $10 \div (-2)(-5)$ .

Solution:

Perform the division first; otherwise the result will be incorrect.

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<p style="text-align: center; color: green; font-weight: bold;">Correct!</p> $10 \div (-2)(-5) = \underbrace{10 \div (-2)}_{\text{division first}} (-5)$ $= -5(-5)$ $= 25 \quad \checkmark$	<p style="text-align: center; color: red; font-weight: bold;">Incorrect!</p> $10 \div (-2)(-5) = 10 \div \underbrace{(-2)(-5)}_{\text{multiplication first}}$ $= 10 \div 10$ $= 1 \quad \times$
---	---

Notice that the order in which we multiply and divide does affect the result. Therefore, it is important to perform the operations of multiplication and division as they appear from left to right.

Answer: 25

The product of two fractions is the fraction formed by the product of the numerators and the product of the denominators. In other words, to multiply fractions, multiply the numerators and multiply the denominators:

## Example 6

Multiply:

Solution:

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Multiply the numerators and multiply the denominators. Reduce by dividing out any common factors.

Answer:

Two real numbers whose product is 1 are called **reciprocals**. Therefore, and are reciprocals because For example,

Because their product is 1, and are reciprocals. Some other reciprocals are listed below:

This definition is important because dividing fractions requires that you multiply the dividend by the reciprocal of the divisor.

In general,

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## Example 7

Simplify:

Solution:

Perform the multiplication and division from left to right.

In algebra, it is often preferable to work with improper fractions. In this case, we leave the answer expressed as an improper fraction.

Answer:

**Try this!** Simplify:

Answer: 3

## Grouping Symbols and Exponents

In a computation where more than one operation is involved, grouping symbols help tell us which operations to perform first. The **grouping symbols** commonly used in algebra are:

All of the above grouping symbols, as well as absolute value, have the same order of precedence. Perform operations inside the innermost grouping symbol or absolute value first.

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## Example 8

Simplify:

Solution:

Perform the operations within the parentheses first.

Answer: 43

## Example 9

Simplify: Solution:

The fraction bar groups the numerator and denominator. Hence, they should be simplified separately.

Answer: -25

If a number is repeated as a factor numerous times, then we can write the product in a more compact form using **exponential notation**. For example,

The **base** is the factor and the positive integer **exponent** indicates the number of times the base is repeated as a factor. In the above example, the base is 5 and the exponent is 4. Exponents are sometimes indicated with the caret (^) symbol found on the keyboard,  $5^4 = 5 \cdot 5 \cdot 5 \cdot 5$ . In general, if  $a$  is the base that is repeated as a factor  $n$  times, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{n \text{ factors of } a}$$

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When the exponent is 2 we call the result a **square**, and when the exponent is 3 we call the result a **cube**. For example,

If the exponent is greater than 3, then the notation  $a^n$  is read, “ $a$  raised to the  $n$ th power.” The base can be any real number,

Notice that the result of a negative base with an even exponent is positive. The result of a negative base with an odd exponent is negative. These facts are often confused when negative numbers are involved. Study the following four examples carefully:

<b><i>The base is (-3).</i></b>	<b><i>The base is 3.</i></b>
$(-3)^4(-3)^3 = (-3)(-3)(-3)(-3) = +81(-3)(-3)$	$-3^4 - 3^3 = -1 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = -81 - 1 \cdot 3 \cdot 3 \cdot 3 = -$
$(-3) = -27$	$27$

The parentheses indicate that the negative number is to be used as the base.

## Example 10

Calculate:

- 1.
- 2.

Solution:

Here is the base for both problems.

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1. Use the base as a factor three times.
2. Use the base as a factor four times.

Answers:

- 1.
- 2.

**Try this!** Simplify:

- 1.
- 2.

Answers:

1. -16
2. 16

## Order of Operations

When several operations are to be applied within a calculation, we must follow a specific order to ensure a single correct result.

1. Perform all calculations within the innermost **parentheses** or grouping symbol first.
2. Evaluate all **exponents**.
3. Apply **multiplication and division** from left to right.

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4. Perform all remaining **addition and subtraction** operations last from left to right.

Note that multiplication and division *should* be worked from *left to right*. Because of this, it is often reasonable to perform division before multiplication.

## Example 11

Simplify:

Solution:

First, evaluate and then perform multiplication and division as they appear from left to right.

Multiplying first would have led to an incorrect result.

$$\begin{aligned}5^3 - 24 \div 6 \cdot \frac{1}{2} + 2 &= 125 - 24 \div \underbrace{6 \cdot \frac{1}{2}}_{\text{Incorrect}} + 2 \\ &= 125 - \underbrace{24 \div 3} + 2 \\ &= \underbrace{125 - 8} + 2 \\ &= \underbrace{117} + 2 \\ &= 119 \quad \times\end{aligned}$$

Answer: 125

## Example 12

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Simplify:

Solution:

Take care to correctly identify the base when squaring.

Answer: 46

We are less likely to make a mistake if we work one operation at a time. Some problems may involve an absolute value, in which case we assign it the same order of precedence as parentheses.

### Example 13

Simplify:

Solution:

Begin by performing the operations within the absolute value first.

Subtracting  $7-5$  first will lead to incorrect results.

$$\begin{aligned} 7-5|-2^2+(-3)^2| &= \underbrace{7-5}_{\text{Incorrect}}|-2^2+(-3)^2| \\ &= 2|-4+9| \\ &= 2|5| \\ &= 10 \quad \times \end{aligned}$$

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Answer:  $-18$

**Try this!** Simplify:

Answer:  $-45$

## KEY TAKEAWAYS

Addition is commutative and subtraction is not. Furthermore, multiplication is commutative and division is not.

Adding or subtracting fractions requires a common denominator; multiplying or dividing fractions does not.

Grouping symbols indicate which operations to perform first. We usually group mathematical operations with parentheses, brackets, braces, and the fraction bar. We also group operations within absolute values. All groupings have the same order of precedence: the operations within the innermost grouping are performed first.

When using exponential notation  $an$ , the base  $a$  is used as a factor  $n$  times. Parentheses indicate that a negative number is to be used as the base. For example,  $(-2)^3$  is positive and  $-2^3$  is negative.

To ensure a single correct result when applying operations within a calculation, follow the order of operations. First, perform operations in the innermost parentheses or groupings. Next, simplify all exponents. Perform multiplication and division operations from left to right. Finally, perform addition and subtraction operations from left to right.

## TOPIC EXERCISES

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## PART A: WORKING WITH REAL NUMBERS

**Perform the operations. Reduce all fractions to lowest terms.**

1.  $33 - (-15) + (-8)$
2.  $-10 - 9 + (-6)$
3.  $-23 + (-7) - (-10)$
4.  $-1 - (-1) - 1$
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
11. Subtract 3 from 10.
12. Subtract  $-2$  from 16.
13. Subtract from 4.
14. Subtract from
15. Calculate the sum of  $-10$  and 25.
16. Calculate the sum of  $-30$  and  $-20$ .
17. Find the difference of 10 and 5.
18. Find the difference of  $-17$  and  $-3$ .

**The formula  $d = |b - a|$  gives the distance between any two points on a number line. Determine the distance between the given numbers on a number line.**

1. 10 and 15
2. 6 and 22

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3. 0 and 12
4. -8 and 0
5. -5 and -25
6. -12 and -3

**Determine the reciprocal of the following.**

- 1.
- 2.
- 3.
4. -12
5.  $a$  where
- 6.
7.  $ab$  where
- 8.

**Perform the operations.**

1.  $-4(-5) \div 2$
2.  $(-15)(-3) \div (-9)$
3.  $-22 \div (-11)(-2)$
4.  $50 \div (-25)(-4)$
- 5.
- 6.
- 7.
- 8.
- 9.
10. .

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- 11.
- 12.
13. Find the product of 12 and 7.
14. Find the product of and 12.
15. Find the quotient of  $-36$  and 12.
16. Find the quotient of and 9.
17. Subtract 10 from the sum of 8 and  $-5$ .
18. Subtract  $-2$  from the sum of  $-5$  and  $-3$ .
19. Joe earns \$18.00 per hour and “time and a half” for every hour he works over 40 hours. What is his pay for 45 hours of work this week?
20. Billy purchased 12 bottles of water at \$0.75 per bottle, 5 pounds of assorted candy at \$4.50 per pound, and 15 packages of microwave popcorn costing \$0.50 each for his party. What was his total bill?
21. James and Mary carpoled home from college for the Thanksgiving holiday. They shared the driving, but Mary drove twice as far as James. If Mary drove for 210 miles, then how many miles was the entire trip?
22. A foot plank is to be cut into 3 pieces of equal length. What will be the length of each piece?
23. A student earned 72, 78, 84, and 90 points on her first four algebra exams. What was her average test score? (Recall that the average is calculated by adding all the values in a set and dividing that result by the number of elements in the set.)
24. The coldest temperature on Earth,  $-129^{\circ}$  F, was recorded in 1983 at Vostok Station, Antarctica. The hottest temperature on Earth,  $136^{\circ}$  F, was recorded in 1922 at Al' Aziziyah, Libya. Calculate the temperature range on Earth.
- 25.

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## PART B: GROUPING SYMBOLS AND EXPONENTS

**Perform the operations.**

57.

58.

59.

60.

61.

62.

63.

64.

65.

66.

**Perform the operations.**

57.

58.

59.

60.

61.

62.

63.

64.

65.

66.

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- 68.
- 69.
- 70.
- 71.
- 72.

## PART C: ORDER OF OPERATIONS

**Simplify.**

- 83.
- 84.
- 85.
- 86.
- 87.
- 88.
- 89.
- 90.
- 91.
- 92.
- 93.
- 94.
- 95.
- 96.
- 97.
- 98.
- 99.
- 100.  $-3(5-2|-6|)$
- 101.
- 102. .
- 103.



104.  
105.  
106.  
107. Calculate the sum of the squares of the first three consecutive positive odd integers.  
108. Calculate the sum of the squares of the first three consecutive positive even integers.  
109. What is 6 subtracted from the sum of the squares of 5 and 8?  
110. What is 5 subtracted from the sum of the cubes of 2 and 3?

## PART D: DISCUSSION BOARD

111. What is PEMDAS and what is it missing?  
112. Does 0 have a reciprocal? Explain.  
113. Explain why we need a common denominator in order to add or subtract fractions.  
114. Explain why  $\pi$  is positive and  $i$  is negative.

## ANSWERS

1. 40  
2.  
3. -20  
4.  
5.  
6.



- 7.
- 8.
- 9.
- 10.
11. 7
- 12.
13. 296
- 14.
15. 15
- 16.
17. 5
- 18.
19. 5 units
- 20.
21. 12 units
- 22.
23. 20 units
- 24.
25. 3
- 26.
- 27.
- 28.
- 29.
- 30.
- 31.
- 32.
33. 10
- 34.

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- 35. -4
- 36.
- 37. -35
- 38.
- 39. -1
- 40.
- 41. -12
- 42.
- 43.
- 44.
- 45. 84
- 46.
- 47. -3
- 48.
- 49. -7
- 50.
- 51. \$855
- 52.
- 53. 315 miles
- 54.
- 55. 81 points
- 56.
- 57. -12
- 58.
- 59. -1
- 60.
- 61. 75
- 62.



- 63. -3
- 64.
- 65. 0
- 66.
- 67. 144
- 68.
- 69. -144
- 70.
- 71. -625
- 72.
- 73.
- 74.
- 75.
- 76.
- 77. 1
- 78.
- 79. -1
- 80.
- 81. -100
- 82.
- 83. 20
- 84.
- 85. -17
- 86.
- 87. 41
- 88.
- 89. 35
- 90.

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- 91.
- 92.
- 93.
- 94.
- 95.
- 96.
97.     -35
- 98.
99.     11
- 100.
101.    -36
- 102.
103.    150
- 104.
- 105.
- 106.
107.    35
- 108.
109.    83
- 110.
111.    Answer may vary
- 112.
113.    Answer may vary



## 1.3 Square and Cube Roots of Real Numbers

### LEARNING OBJECTIVES

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1. Calculate the exact and approximate value of the square root of a real number.
2. Calculate the exact and approximate value of the cube root of a real number.
3. Simplify the square and cube root of a real number.
4. Apply the Pythagorean theorem.

## The Definition of Square and Cube Roots

A **square root** of a number is a number that when multiplied by itself yields the original number. For example, 4 is a square root of 16, because  $4 \times 4 = 16$ . Since  $4^2 = 16$ , we can say that  $-4$  is a square root of 16 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the **radical sign** to denote the **principal (nonnegative) square root** and a negative sign in front of the radical  $-$  to denote the negative square root.

Zero is the only real number with exactly one square root.

If the **radicand**, the number inside the radical sign, is nonzero and can be factored as the square of another nonzero number, then the square root of the number is apparent. In this case, we have the following property:

It is important to point out that  $a$  is required to be nonnegative. Note that because the radical denotes the principal square root. Instead,

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This distinction will be carefully considered later in the course.

### **Example 1**

Find the square root:

- 1.
- 2.
- 3.

Solution:

- 1.
- 2.
- 3.**

### **Example 2**

Find the negative square root:

- 1.
- 2.

Solution:

- 1.
- 2.

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The radicand may not always be a perfect square. If a positive integer is not a perfect square, then its square root will be irrational. Consider  $\sqrt{5}$ , we can obtain an approximation by bounding it using the perfect squares 4 and 9 as follows:

With this we conclude that  $\sqrt{5}$  is somewhere between 2 and 3. This number is better approximated on most calculators using the square root button,  $\sqrt{5} \approx 2.236$ .

Next, consider the square root of a negative number. To determine the square root of  $-9$ , you must find a number that when squared results in  $-9$ ,

However, any real number squared always results in a positive number,

The square root of a negative number is currently left undefined. Try calculating  $\sqrt{-9}$  on your calculator; what does it say? For now, we will state that  $\sqrt{-9}$  is not a real number. The square root of a negative number is defined later in the course.

A **cube root** of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol  $\sqrt[3]{\phantom{x}}$ , where 3 is called the **index**. For example,

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The product of three equal factors will be positive if the factor is positive, and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

In general, given any real number  $a$ , we have the following property:

When simplifying cube roots, look for factors that are perfect cubes.

### Example 3

Find the cube root:

- 1.
- 2.
- 3.

Solution:

- 1.
- 2.

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3.

#### Example 4

Find the cube root:

1.

2.

Solution:

1.

2.

It may be the case that the radicand is not a perfect cube. If this is the case, then its cube root will be irrational. For example,  $\sqrt[3]{2}$  is an irrational number, which can be approximated on most calculators using the root button  $\sqrt[n]{\phantom{x}}$ . Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$\sqrt[3]{2}$$

Therefore, we have

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We will extend these ideas using any integer as an index later in this course. It is important to point out that a square root has index 2; therefore, the following are equivalent:

In other words, if no index is given, it is assumed to be the square root.

## Simplifying Square and Cube Roots

It will not always be the case that the radicand is a perfect square. If not, we use the following two properties to simplify the expression. Given real numbers and where ,

<b>Product Rule for Radicals:</b>	
<b>Quotient Rule for Radicals:</b>	

A **simplified radical** is one where the radicand does not consist of any factors that can be written as perfect powers of the index. Given a square root, the idea is to identify the largest square factor of the radicand and then apply the property shown above. As an example, to simplify , notice that 12 is not a perfect square. However, 12 does have a perfect square factor,  $12=4\cdot 3$ . Apply the property as follows:

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The number  $\sqrt{135}$  is a simplified irrational number. You are often asked to find an approximate answer rounded off to a certain decimal place. In that case, use a calculator to find the decimal approximation using either the original problem or the simplified equivalent.

As a check, calculate  $\sqrt{135}$  and  $\sqrt{3 \cdot 3 \cdot 3 \cdot 5}$  on a calculator and verify that the results are both approximately 3.46.

## Example 5

Simplify:

Solution:

Begin by finding the largest perfect square factor of 135.

Therefore,

Answer:



## Example 6

Simplify:

Solution:

We begin by finding the prime factorizations of both 108 and 169. This will enable us to easily determine the largest perfect square factors.

Therefore,

Answer:

## Example 7

Simplify:

Solution:

Answer:

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**Try this!** Simplify:

Answer:

A cube root is simplified if it does not contain any factors that can be written as perfect cubes. The idea is to identify the largest cube factor of the radicand and then apply the product or quotient rule for radicals. As an example, to simplify  $\sqrt[3]{80}$ , notice that 80 is not a perfect cube. However,  $80=8 \cdot 10$  and we can write,

## Example 8

Simplify:

Solution:

Begin by finding the largest perfect cube factor of 162.

Therefore,

Answer:

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## Example 9

Simplify:

Solution:

Answer:

**Try this!** Simplify:

Answer:

Consider the following two calculations,

Notice that it does not matter if we apply the exponent first or the square root first. This is true for any positive real number. We have the following,

## Example 10

Simplify:

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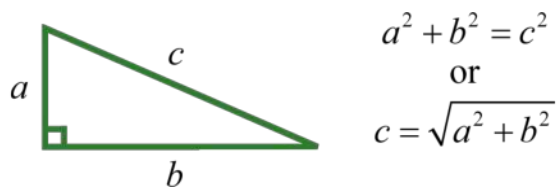
Solution:

Apply the fact that if  $a$  is nonnegative.

## Pythagorean Theorem

A **right triangle** is a triangle where one of the angles measures  $90^\circ$ . The side opposite the right angle is the longest side, called the **hypotenuse**, and the other two sides are called **legs**. Numerous real-world applications involve this geometric figure.

The **Pythagorean theorem** states that given any right triangle with legs measuring  $a$  and  $b$  units, the square of the measure of the hypotenuse  $c$  is equal to the sum of the squares of the measures of the legs. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.



## Example 11

Calculate the diagonal of a square with sides measuring 5 units.

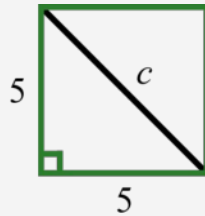
Solution:

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The diagonal of a square will form an isosceles right triangle where the two equal legs measure 5 units each.



We can use the Pythagorean theorem to determine the length of the hypotenuse.

Answer: units

The Pythagorean theorem actually states that having side lengths satisfying the property is a necessary and sufficient condition of right triangles. In other words, if we can show that the sum of the squares of the lengths of the legs of the triangle is equal to the square of the hypotenuse, then it must be a right triangle.

### Example 12

Determine whether or not a triangle with legs  $a=1$  cm and  $b=2$  cm and hypotenuse cm is a right triangle.

Solution:

If the legs satisfy the condition then the Pythagorean theorem guarantees that the triangle is a right triangle.



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Answer: Yes, the described triangle is a right triangle.

## KEY TAKEAWAYS

The square root of a number is a number that when squared results in the original number. The principal square root of a positive real number is the positive square root. The square root of a negative number is currently left undefined.

When simplifying the square root of a number, look for perfect square factors of the radicand. Apply the product or quotient rule for radicals and then simplify.

The cube root of a number is a number that when cubed results in the original number. Every real number has only one real cube root.

When simplifying cube roots, look for perfect cube factors of the radicand. Apply the product or quotient rule for radicals and then simplify.

The Pythagorean theorem gives us a necessary and sufficient condition of right triangles: if and only if  $a$ ,  $b$  and  $c$  represent the lengths of the sides of a right triangle.

## TOPIC EXERCISES

### PART A: SQUARE AND CUBE ROOTS

**Simplify.**

- 1.
- 2.
- 3.
- 4.
- 5.

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- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.
- 15.
- 16.
- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.
- 25.
- 26.
- 27.
- 28.
- 29.
- 30.
- 31.
- 32.
- 33.

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- 34.
- 35.
- 36.
- 37.
- 38.

**Use a calculator to approximate to the nearest hundredth.**

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.

$$3\sqrt{3}$$

- Determine the set consisting of the squares of the first twelve positive integers.
- Determine the set consisting of the cubes of the first twelve positive integers.

## **PART B: SIMPLIFYING SQUARE ROOTS AND CUBE ROOTS**

**Simplify.**

- 53.



- 54.
- 55.
- 56.
- 57.
- 58.
- 59.
- 60.
- 61.
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
- 69.
- 70.
- 71.
- 72.
- 73.
- 74.
- 75.
- 76.
- 77.
- 78.

## PART C: PYTHAGOREAN THEOREM

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79. If the two legs of a right triangle measure 3 units and 4 units, then find the length of the hypotenuse.
80. If the two legs of a right triangle measure 6 units and 8 units, then find the length of the hypotenuse.
81. If the two equal legs of an isosceles right triangle measure 7 units, then find the length of the hypotenuse.
82. If the two equal legs of an isosceles right triangle measure 10 units, then find the length of the hypotenuse.
83. Calculate the diagonal of a square with sides measuring 3 centimeters.
84. Calculate the diagonal of a square with sides measuring 10 centimeters.
85. Calculate the diagonal of a square with sides measuring centimeters.
86. Calculate the diagonal of a square with sides measuring centimeters.
87. Calculate the length of the diagonal of a rectangle with dimensions 4 centimeters by 8 centimeters.
88. Calculate the length of the diagonal of a rectangle with dimensions 8 meters by 10 meters.
89. Calculate the length of the diagonal of a rectangle with dimensions meters by 2 meters.
90. Calculate the length of the diagonal of a rectangle with dimensions meters by meters.
91. To ensure that a newly built gate is square, the measured diagonal must match the distance calculated using the Pythagorean theorem. If the gate measures 4 feet by 4 feet, what must the diagonal measure in inches? (Round off to the nearest tenth of an inch.)
92. If a doorframe measures 3.5 feet by 6.6 feet, what must the diagonal measure to ensure that the frame is a perfect rectangle?

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**Determine whether or not the given triangle with legs  $a$  and  $b$  and hypotenuse  $c$  is a right triangle or not.**

- 79.  $a=3$ ,  $b=7$ , and  $c=10$
- 80.  $a=5$ ,  $b=12$ , and  $c=13$
- 81.  $a=8$ ,  $b=15$ , and  $c=17$
- 82.  $a=7$ ,  $b=24$ , and  $c=30$
- 83.  $a=3$ ,  $b=2$ , and
- 84.  $a=5$ ,  $b=4$ , and
- 85.  $a=4$ ,  $b=3$ , and

100.  $a=5$ ,  $b=12$ , and  $c=13$

### PART D: DISCUSSION BOARD

- 101. What does your calculator say after taking the square root of a negative number? Share your results on the discussion board and explain why it says that.
- 102. Research and discuss the history of the Pythagorean theorem.
- 103. Research and discuss the history of the square root.
- 104. Discuss the importance of the principal square root. Why is it that the same issue does not come up with cube roots? Provide some examples with your explanation.

### ANSWERS

- 1. 9
- 2.
- 3. -4
- 4.
- 5.
- 6.

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- 7.
- 8.
9. Not a real number.
- 10.
11. 0.6
- 12.
13. 5
- 14.
15. 16
- 16.
17. -20
- 18.
19. 4
- 20.
21. -3
- 22.
23. 0
- 24.
25. 0.4
- 26.
27. -10
- 28.
29. -15
- 30.
- 31.
- 32.
- 33.
- 34.

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- 35. 32
- 36.
- 37. 24
- 38.
- 39. 1.73
- 40.
- 41. 4.36
- 42.
- 43. 6.71
- 44.
- 45. 1.44
- 46.
- 47. 3.04
- 48.
- 49. 8.62
- 50.
- 51. {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}
- 52.
- 53.
- 54.
- 55.
- 56.
- 57.
- 58.
- 59.
- 60.
- 61.
- 62.

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63. Not a real number.

64.

65.

66.

67.

68.

69.

70.

71.

72.

73.

74.

75. 64

76.

77. 2

78.

79. 5 units

80.

81. units

82.

83. centimeters

84.

85. centimeters

86.

87. centimeters

88.

89. meters

90.

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91. The diagonal must measure approximately 67.9 inches.
- 92.
93. Not a right triangle.
- 94.
95. Right triangle.
- 96.
97. Right triangle.
- 98.
99. Right triangle.
- 100.
101. Answer may vary
- 102.
103. Answer may vary

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# 1.4 Algebraic Expressions and Formulas

## LEARNING OBJECTIVES

1. Identify the parts of an algebraic expression.
2. Apply the distributive property.
3. Evaluate algebraic expressions.
4. Use formulas that model common applications.

### Algebraic Expressions and the Distributive Property

In algebra, letters called variables are used to represent numbers. Combinations of variables and numbers along with mathematical operations form **algebraic expressions**, or just **expressions**. The following are some examples of expressions with one variable,  $x$ :

--	--	--	--

**Terms** in an algebraic expression are separated by addition operators and **factors** are separated by multiplication operators. The numerical factor of a term is called the **coefficient**. For example, the algebraic expression  $x + 2y + 3$  can be thought of as  $1x + 2y + 3$  and has three terms. The first term,  $1x$ , represents the quantity  $x$  where 1 is the coefficient and  $x$  and  $y$  are the variables. All of the variable factors with their exponents form



the **variable part of a term**. If a term is written without a variable factor, then it is called a **constant term**. Consider the components of ,

<i>Terms</i>	<i>Coefficient</i>	<i>Variable Part</i>
	1	
$6xy$	6	$xy$
$-3$	$-3$	

The third term in this expression,  $-3$ , is called a constant term because it is written without a variable factor. While a variable represents an unknown quantity and may change, the constant term does not change.

## Example 1

List all coefficients and variable parts of each term:

Solution:

We want to think of the third term in this example

<i>Terms</i>	<i>Coefficient</i>	<i>Variable Part</i>
10	10	
$-5ab$	$-5$	$ab$
	$-1$	

Answer: Coefficients:  $\{-5, -1, 10\}$ ; Variable parts:

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In our study of algebra, we will encounter a wide variety of algebraic expressions. Typically, expressions use the two most common variables,  $x$  and  $y$ . However, expressions may use any letter (or symbol) for a variable, even Greek letters, such as alpha ( $\alpha$ ) and beta ( $\beta$ ). Some letters and symbols are reserved for constants, such as  $e$  and  $i$ . Since there is only a limited number of letters, you will also use subscripts,  $x_1, x_2, \dots$ , to indicate different variables.

The properties of real numbers are important in our study of algebra because a variable is simply a letter that represents a real number. In particular, the **distributive property** states that if given any real numbers  $a, b$  and  $c$ , then,

This property is one that we apply often when simplifying algebraic expressions. To demonstrate how it will be used, we simplify  $2(5-3)$  in two ways, and observe the same correct result.

<i>Working parenthesis first.</i>	<i>Using the distributive property.</i>

Certainly, if the contents of the parentheses can be simplified we should do that first. On the other hand, when the contents of parentheses cannot be simplified any further, we




multiply every term within it by the factor outside of it using the distributive property. Applying the distributive property allows us to multiply and remove the parentheses.

## Example 2

Simplify:

Solution:

Multiply only the terms grouped within the parentheses for which we are applying the distributive property.

$$5(-2a + 5b) - 2c$$


Answer:

Recall that multiplication is commutative and therefore we can write the distributive property in the following manner,

## Example 3

Simplify:

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Solution:

Multiply all terms within the parenthesis by 3.

Answer:

Terms whose variable parts have the same variables with the same exponents are called like **terms**, or **similar terms**. Furthermore, constant terms are considered to be like terms. If an algebraic expression contains like terms, apply the distributive property as follows:

In other words, if the variable parts of terms are *exactly the same*, then we can add or subtract the coefficients to obtain the coefficient of a single term with the same variable part. This process is called **combining like terms**. For example,

Notice that the variable factors and their exponents do not change. Combining like terms in this manner, so that the expression contains no other similar terms, is

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called simplifying the expression. Use this idea to simplify algebraic expressions with multiple like terms.

### **Example 4**

Simplify:

Solution:

Identify the like terms and add the corresponding coefficients.

Answer:

### **Example 5**

Simplify:

Solution:

Distribute  $-2$  and then combine like terms.

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Answer:

## Evaluating Algebraic Expressions

An algebraic expression can be thought of as a generalization of particular arithmetic operations. Performing these operations after substituting given values for variables is called **evaluating**. In algebra, a variable represents an unknown value. However, if the problem specifically assigns a value to a variable, then you can replace that letter with the given number and evaluate using the order of operations.

### Example 6

Evaluate:

1.            where
2.            where  $y = -4$

Solution:

To avoid common errors, it is a best practice to first replace all variables with parentheses, and then replace, or substitute, the appropriate given value.

a.

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b.

Answer:

1.

2. 14

Often algebraic expressions will involve more than one variable.

### Example 7

Evaluate where  $a = -1$  and

Solution:

After substituting in the appropriate values, we must take care to simplify using the correct order of operations.

Answer:  $-2$

### Example 8

Evaluate where and  $y = -3$ .

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Solution:

At this point we have a complex fraction. Simplify the numerator and then multiply by the reciprocal of the denominator.

Answer:

The answer to the previous example can be written as a mixed number, Unless the original problem has mixed numbers in it, or it is an answer to a real-world application, solutions will be expressed as reduced improper fractions.

## Example 9

Evaluate where  $a=-1$ ,  $b=-7$ , and

Solution:

Substitute in the appropriate values and then simplify.

Answer:

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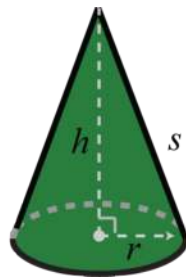


**Try this!** Evaluate where  $V=25\pi$  and  $h=3$ .

Answer: 5

## Using Formulas

The main difference between algebra and arithmetic is the organized use of variables. This idea leads to reusable **formulas**, which are mathematical models using algebraic expressions to describe common applications. For example, the volume of a right circular cone depends on its radius  $r$  and height  $h$  and is modeled by the formula:



In this equation, variables and constants are used to describe the relationship between volume and the length of the base and height. If the radius of the base measures 3 meters and the height measures 5 meters, then the volume can be calculated using the formula as follows:

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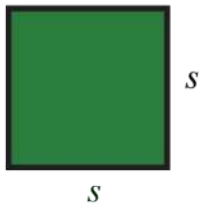
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Using , we can approximate the volume: cubic meters.

A list of formulas that describe the area and perimeter of common plane figures follows. The letter  $P$  represents perimeter and is measured in linear units. The letter  $A$  represents area and is measured in square units.

Square



$$P = 4s$$

$$A = s^2$$

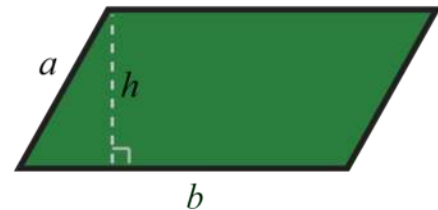
Rectangle



$$P = 2l + 2w$$

$$A = lw$$

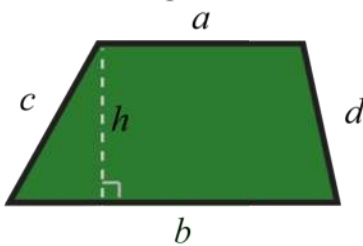
Parallelogram



$$P = 2a + 2b$$

$$A = bh$$

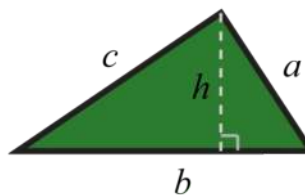
Trapezoid



$$P = a + b + c + d$$

$$A = \frac{1}{2}h(a + b)$$

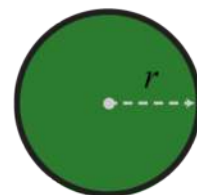
Triangle



$$P = a + b + c$$

$$A = \frac{1}{2}bh$$

Circle



$$C = 2\pi r$$

$$A = \pi r^2$$

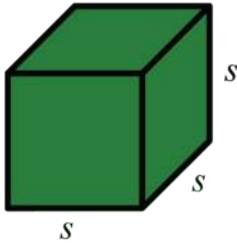
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A list of formulas that describe the surface area and volume of common figures follows. Here  $SA$  represents surface area and is measured in square units. The letter  $V$  represents volume and is measured in cubic units.

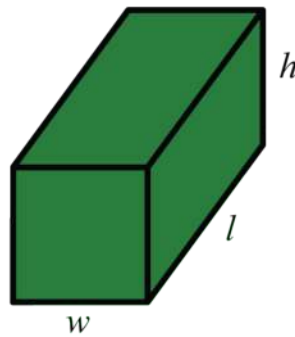
Cube



$$SA = 6s^2$$

$$V = s^3$$

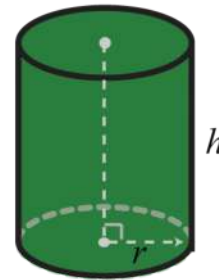
Rectangular Solid



$$SA = 2lw + 2lh + 2wh$$

$$V = lwh$$

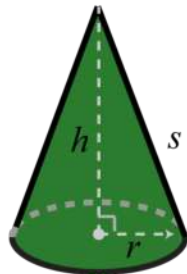
Right Circular Cylinder



$$SA = 2\pi r^2 + 2\pi rh$$

$$V = \pi r^2 h$$

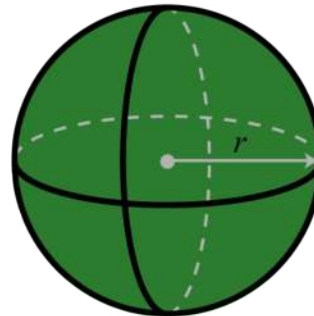
Right Circular Cone



$$SA = \pi r^2 + \pi rs$$

$$V = \frac{1}{3} \pi r^2 h$$

Sphere



$$SA = 4\pi r^2$$

$$V = \frac{4}{3} \pi r^3$$

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## Example 10

The diameter of a spherical balloon is 10 inches. Determine the volume rounded off to the nearest hundredth.

Solution:

The formula for the volume of a sphere is

This formula gives the volume in terms of the radius,  $r$ . Therefore, divide the diameter by 2 and then substitute into the formula. Here, inches and we have

Answer: The volume of the balloon is approximately 523.60 cubic inches.

Formulas can be found in a multitude of subjects. For example, **uniform motion** is modeled by the formula , which expresses distance  $D$ , in terms of the average rate, or speed,  $r$  and the time traveled at that rate,  $t$ . This formula, , is used often and is read, “*distance equals rate times time.*”

## Example 11

Jim’s road trip took hours at an average speed of 66 miles per hour. How far did he travel?

Solution:

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Substitute the appropriate values into the formula and then simplify.

Answer: Jim traveled 165 miles.

**Simple interest**  $I$  is given by the formula , where  $p$  represents the principal amount invested at an annual interest rate  $r$  for  $t$  years.

## Example 12

Calculate the simple interest earned on a 2-year investment of \$1,250 at an annual interest rate of

Solution:

Convert to a decimal number before using it in the formula.

Use this and the fact that  $p = \$1,250$  and  $t = 2$  years to calculate the simple interest.

Answer: The simple interest earned is \$93.75.

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## KEY TAKEAWAYS

Think of algebraic expressions as generalizations of common arithmetic operations that are formed by combining numbers, variables, and mathematical operations.

The distributive property, is used when multiplying grouped algebraic expressions. Applying the distributive property allows us to remove parentheses.

Combine like terms, or terms whose variable parts have the same variables with the same exponents, by adding or subtracting the coefficients to obtain the coefficient of a single term with the same variable part. Remember that the variable factors and their exponents do not change.

To avoid common errors when evaluating, it is a best practice to replace all variables with parentheses and then substitute the appropriate values.

The use of algebraic expressions allows us to create useful and reusable formulas that model common applications.

## TOPIC EXERCISES

### PART A: ALGEBRAIC EXPRESSIONS AND THE DISTRIBUTIVE PROPERTY

**List all of the coefficients and variable parts of each term.**

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.

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## Multiply.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.

## Combine like terms.

1.  $18x - 5x + 3x$
2.  $30x - 50x + 10x$
3.  $3y - 4 + 2y - 12$
4.  $12y + 7 - 15y - 6$
- 5.
- 6.
- 7.
- 8.
- 9.



10.

11.

12.

13.

14.

15.

16.

17.

18.

**Simplify.**

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.  $a^2$

13.

14.

15.

16.

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## PART B: EVALUATING ALGEBRAIC EXPRESSIONS

### Evaluate.

55. where  $x=-2$
56. where  $x=-1$
57. where  $x=-5$
58. where  $x=3$
59. where
60. where
61. where
62. where
63. where  $x=-1$
64. where  $y=-2$
65. where  $y=-1$
66. where  $y=-2$
67. where  $a=-2$  and  $b=-1$
68. where  $a=-3$  and  $b=2$
69. where  $x=2$  and  $y=-5$
70. where  $x=-4$  and  $y=-3$
71. where  $a=-1$  and  $b=-2$
72. where  $x=-3$  and  $y=-2$
73. where  $a=-2$  and  $b=-3$
74. where  $a=2$  and  $b=-1$

### Evaluate given the following values.

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55.  $a=6, b=1$  and  $c=-1$   
56.  $a=15, b=4$  and  $c=-4$   
57.  $a=1, b=-2$  and  $c=-4$   
58.  $a=1, b=-2$  and  $c=-30$   
59.  $a=1, b=2$  and  $c=-1$   
60.  $a=1, b=-4$  and  $c=-50$   
61.  $a=1, b=-1$  and  $c=-1$   
62.  $a=-2, b=-13$  and  $c=1$

## PART C: USING FORMULAS

**Convert the following temperatures to degrees Celsius given  $F$ , where  $F$  represents degrees Fahrenheit.**

83.  $95^{\circ}\text{F}$   
84.  $86^{\circ}\text{F}$   
85.  $32^{\circ}\text{F}$   
86.  $-40^{\circ}\text{F}$
87. Calculate the perimeter and area of a rectangle with dimensions 12 feet by 5 feet.  
88. Calculate the perimeter and area of a rectangle with dimensions 5 meters by 1 meter.  
89. Calculate the surface area and volume of a sphere with radius 6 centimeters.  
90. The radius of the base of a right circular cylinder measures 4 inches and the height measures 10 inches. Calculate the surface area and volume.  
91. Calculate the volume of a sphere with a diameter of 18 centimeters.

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92. The diameter of the base of a right circular cone measures 6 inches. If the height is 112 feet, then calculate its volume.
93. Given that the height of a right circular cylinder is equal to the radius of the base, derive a formula for the surface area in terms of the radius of the base.
94. Given that the area of the base of a right circular cylinder is  $25\pi$  square inches, find the volume if the height is 1 foot.
95. Jose was able to drive from Tucson to Phoenix in 2 hours at an average speed of 58 mph. How far is Phoenix from Tucson?
96. If a bullet train can average 152 mph, then how far can it travel in of an hour?
97. Margaret traveled for hour at an average speed of 68 miles per hour. How far did she travel?
98. The trip from Flagstaff, AZ to the Grand Canyon national park took 112 hours at an average speed of 54 mph. How far is the Grand Canyon national park from Flagstaff?
99. Calculate the simple interest earned on a 3-year investment of \$2,500 at an annual interest rate of
100. Calculate the simple interest earned on a 1-year investment of \$5,750 at an annual interest rate of
101. What is the simple interest earned on a 5-year investment of \$20,000 at an annual interest rate of 6%?
102. What is the simple interest earned on a 1-year investment of \$50,000 at an annual interest rate of 4.5%?
103. The time  $t$  in seconds an object is in free fall is given by the formula , where  $s$  represents the distance in feet the object has fallen. How long does it take an object to fall 32 feet? (Give the exact answer and the approximate answer to the nearest hundredth.)

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104. The current  $I$  measured in amperes, is given by the formula , where  $P$  is the power usage measured in watts, and  $R$  is the resistance measured in ohms. If a light bulb uses 60 watts of power and has 240 ohms of resistance, then how many amperes of current are required?

## PART D: DISCUSSION BOARD

105. Find and post a useful mathematical model. Demonstrate its use with some values.

106. Research and discuss the history of the variable. What can we use if we run out of letters?

107. Find and post a link to a useful resource describing the Greek alphabet.

108. Given the algebraic expression , explain why we do not subtract 5 and 3 first.

109. Do we need a separate distributive property for more than two terms? For example, Explain.

110. How can we check to see if we have simplified an expression correctly?

## ANSWERS

1. Coefficients:  $\{-5,1,-1\}$ ; variable parts:

2.

3. Coefficients:  $\{5,-3,1\}$ ; variable parts:

4.

5. Coefficients:  $\{1,-3,9\}$ ; variable parts:

6.

7.

8.

9.

10.

11. 6

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- 12.
- 13.
- 14.
- 15.
- 16.
- 17.
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- 22.
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- 24.
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- 29.
- 30.
- 31.
- 32.
- 33.
- 34.
- 35.
- 36.
- 37.
- 38.
- 39.

16x

27.

29.

$8a^2b^2 - ab + 10$

$-5x^2y + 17xy - 3xy^2$





- 40.
- 41.
- 42.
- 43.
- 44.
- 45.
- 46.
- 47.
- 48.
- 49.
- 50.
- 51.
- 52.
- 53.
- 54.
- 55. 7
- 56.
- 57. 35
- 58.
- 59.
- 60.
- 61. 0
- 62.
- 63. 36
- 64.
- 65. 4
- 66.
- 67. -1



- 68.
69.  $-96$
- 70.
71.  $3$
- 72.
73.  $-65$
- 74.
75.  $5$
- 76.
77.  $4$
- 78.
- 79.
- 80.
- 81.
- 82.
83.  $35^{\circ}\text{C}$
- 84.
85.  $0^{\circ}\text{C}$
- 86.
87.  $P = 34$  feet;  $A = 60$  square feet
- 88.
89.  $SA = 144\pi$  square centimeters;  $V = 288\pi$  cubic centimeters
- 90.
91.  $972\pi$  cubic centimeters
- 92.
- 93.
- 94.
95.  $116$  miles

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- 96.
- 97. 119 miles
- 98.
- 99. \$393.75
- 100.
- 101. \$6,000
- 102.
- 103. seconds
- 104.
- 105. Answer may vary
- 106.
- 107. Answer may vary
- 108.
- 109. Answer may vary



## 1.5 Rules of Exponents and Scientific Notation

### LEARNING OBJECTIVES

1. Review the rules of exponents.
2. Review the definition of negative exponents and zero as an exponent.
3. Work with numbers using scientific notation.

### Review of the Rules of Exponents

In this section, we review the rules of exponents. Recall that if a factor is repeated multiple times, then the product can be written in exponential form . The positive integer exponent  $n$  indicates the number of times the base  $x$  is repeated as a factor.

$$x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ times}}$$

Consider the product of and ,

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$$x^4 \cdot x^6 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{10 \text{ factors of } x} = x^{10}$$

Expanding the expression using the definition produces multiple factors of the base which is quite cumbersome, particularly when  $n$  is large. For this reason, we have useful rules to help us simplify expressions with exponents. In this example, notice that we could obtain the same result by adding the exponents.

In general, this describes the **product rule for exponents**. In other words, when multiplying two expressions with the same base we add the exponents. Compare this to raising a factor involving an exponent to a power, such as

$$\begin{aligned} (x^6)^4 &= \underbrace{x^6 \cdot x^6 \cdot x^6 \cdot x^6}_{4 \text{ factors of } x^6} \\ &= x^{6+6+6+6} \\ &= x^{24} \end{aligned}$$

Here we have 4 factors of  $x^6$ , which is equivalent to multiplying the exponents.

This describes the **power rule for exponents**. Now we consider raising grouped products to a power. For example,

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After expanding, we are left with four factors of the product . This is equivalent to raising each of the original grouped factors to the fourth power and applying the power rule.

In general, this describes the use of the power rule for a product as well as the power rule for exponents. In summary, the rules of exponents streamline the process of working with algebraic expressions and will be used extensively as we move through our study of algebra. Given any positive integers  $m$  and  $n$  where we have

<b>Product rule for exponents:</b>	
<b>Quotient rule for exponents:</b>	
<b>Power rule for exponents:</b>	
<b>Power rule for a product:</b>	



<b>Power rule for a quotient:</b>	
-----------------------------------	--

These rules allow us to efficiently perform operations with exponents.

### Example 1

Simplify:

Solution:

Answer:

In the previous example, notice that we did not multiply the base 10 times itself. When applying the product rule, add the exponents and leave the base unchanged.

### Example 2

Simplify:

Solution:

Recall that the variable  $x$  is assumed to have an exponent of one,

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Answer:

The base could in fact be any algebraic expression.

### **Example 3**

Simplify:

Solution:

Treat the expression as the base.

Answer:

The commutative property of multiplication allows us to use the product rule for exponents to simplify factors of an algebraic expression.

### **Example 4**

Simplify:

Solution:

Multiply the coefficients and add the exponents of variable factors with the same base.

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Answer:

Division involves the quotient rule for exponents.

### **Example 5**

Simplify:

Solution:

Answer:

The power rule for a quotient allows us to apply that exponent to the numerator and denominator. This rule requires that the denominator is nonzero and so we will make this assumption for the remainder of the section.

### **Example 6**

Simplify:

Solution:

First apply the power rule for a quotient and then the power rule for a product.

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Answer:

Using the quotient rule for exponents, we can define what it means to have zero as an exponent. Consider the following calculation:

Twenty-five divided by twenty-five is clearly equal to one, and when the quotient rule for exponents is applied, we see that a zero exponent results. In general, given any nonzero real number  $x$  and integer  $n$ ,

This leads us to the definition of **zero as an exponent**,

It is important to note that  $0^0$  is indeterminate. If the base is negative, then the result is still positive one. In other words, any nonzero base raised to the zero power is defined to be equal to one. In the following examples assume all variables are nonzero.

## Example 7

Simplify:

- 1.
- 2.

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Solution:

1. Any nonzero quantity raised to the zero power is equal to 1.
2. In the example,  $-2x^0$ , the base is  $x$ , not  $-2x$ .

Noting that we can write,

In general, given any nonzero real number  $x$  and integer  $n$ ,

This leads us to the definition of negative exponents:

An expression is completely simplified if it does not contain any negative exponents.

## Example 8

Simplify:

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Solution:

Rewrite the entire quantity in the denominator with an exponent of 2 and then simplify further.

Answer:

Sometimes negative exponents appear in the denominator.

## Example 9

Simplify:

Solution:

Answer:

The previous example suggests a property of quotients with negative exponents. Given any integers  $m$  and  $n$  where  $n \neq 0$ , then

This leads us to the property

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In other words, negative exponents in the numerator can be written as positive exponents in the denominator and negative exponents in the denominator can be written as positive exponents in the numerator.

## Example 10

Simplify:

Solution:

Take care with the coefficient  $-5$ , recognize that this is the base and that the exponent is actually positive one:  $5^{-1}$ . Hence, the rules of negative exponents do not apply to this coefficient; leave it in the numerator.

Answer:

In summary, given integers  $m$  and  $n$  where we have

<b>Zero exponent:</b>	
<b>Negative exponent:</b>	

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<b>Quotients with negative exponents:</b>	
---	--

Furthermore, all of the rules of exponents defined so far extend to any integer exponents. We will expand the scope of these properties to include any real number exponents later in the course.

**Try this!** Simplify:

Answer:

## Scientific Notation

Real numbers expressed using **scientific notation** have the form,

where  $n$  is an integer and  $1 \leq a < 10$ . This form is particularly useful when the numbers are very large or very small. For example,

It is cumbersome to write all the zeros in both of these cases. Scientific notation is an alternative, compact representation of these numbers. The factor  $10^n$  indicates the power of ten to multiply the coefficient by to convert back to decimal form:

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$$9.46 \times 10^{15} = 9.46 \times \overbrace{1,000,000,000,000,000}^{15 \text{ zeros}} = 9,460,000,000,000,000$$

This is equivalent to moving the decimal in the coefficient fifteen places to the right. A negative exponent indicates that the number is very small:

$$2.5 \times 10^{-11} = 2.5 \times \frac{1}{10^{11}} = \frac{2.5}{\underbrace{100,000,000,000}_{11 \text{ zeros}}} = 0.000000000025$$

This is equivalent to moving the decimal in the coefficient eleven places to the left.

Converting a decimal number to scientific notation involves moving the decimal as well. Consider all of the equivalent forms of 0.00563 with factors of 10 that follow:

While all of these are equal, is the only form expressed in correct scientific notation. This is because the coefficient 5.63 is between 1 and 10 as required by the definition. Notice that we can convert back to decimal form, as a check, by moving the decimal three places to the left.

## Example 11

Write using scientific notation.

Solution:

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Here we count twelve decimal places to the left of the decimal point to obtain the number 1.075.

Answer:

## Example 12

Write 0.000003045 using scientific notation.

Solution:

Here we count six decimal places to the right to obtain 3.045.

Answer

Often we will need to perform operations when using numbers in scientific notation. All the rules of exponents developed so far also apply to numbers in scientific notation.

## Example 13

Multiply:

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Solution:

Use the fact that multiplication is commutative, and apply the product rule for exponents.

Answer:

## Example 14

Divide:

Solution:

Answer:

## Example 15

The speed of light is approximately miles per hour. Express this speed in miles per second.

Solution:

A unit analysis indicates that we must divide the number by 3,600.

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Answer: The speed of light is approximately miles per second.

## Example 16

The Sun moves around the center of the galaxy in a nearly circular orbit. The distance from the center of our galaxy to the Sun is approximately 26,000 light-years. What is the circumference of the orbit of the Sun around the galaxy in meters?

Solution:

One light-year measures meters. Therefore, multiply this by 26,000 or to find the length of 26,000 light years in meters.

The radius  $r$  of this very large circle is approximately  $2.46 \times 10^{20}$  meters. Use the formula to calculate the circumference of the orbit.

Answer: The circumference of the Sun's orbit is approximately meters.

**Try this!** Divide:

Answer:

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## KEY TAKEAWAYS

When multiplying two quantities with the same base, add exponents:

When dividing two quantities with the same base, subtract exponents:

When raising powers to powers, multiply exponents:

When a grouped quantity involving multiplication and division is raised to a power, apply that power to all of the factors in the numerator and the denominator: and

Any nonzero quantity raised to the 0 power is defined to be equal to 1:

Expressions with negative exponents in the numerator can be rewritten as expressions with positive exponents in the denominator:

Expressions with negative exponents in the denominator can be rewritten as expressions with positive exponents in the numerator:

Take care to distinguish negative coefficients from negative exponents.

Scientific notation is particularly useful when working with numbers that are very large or very small.

## TOPIC EXERCISES

### PART A: RULES OF EXPONENTS

**Simplify. (Assume all variables represent nonzero numbers.)**

1.

2.

3.

4.

5.

6.

7.

8.

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	9.
	10.
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34.	
	35.
	36.

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- 37.
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- 60.
- 61.
- 62.
- 63.
- 64.



- 65.
- 66.
- 67.
- 68.
- 69.
- 70.
- 71.
- 72.
- 73.
- 74.

The value in dollars of a new mobile phone can be estimated by using the formula , where  $t$  is the number of years after purchase.

1. How much was the phone worth new?
2. How much will the phone be worth in 1 year?
3. How much will the phone be worth in 3 years?
4. How much will the phone be worth in 10 years?
5. How much will the phone be worth in 100 years?
6. According to the formula, will the phone ever be worthless? Explain.
7. The height of a particular right circular cone is equal to the square of the radius of the base, Find a formula for the volume in terms of  $r$ .
8. A sphere has a radius Find the volume in terms of  $x$ .

## PART B: SCIENTIFIC NOTATION

Convert to a decimal number.

- 83.
- 84.
- 85.

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86.

**Rewrite using scientific notation.**

83. 7,050,000

84. 430,000,000,000

85. 0.00005001

86. 0.000000231

**Perform the operations.**

83.

84.

85.

86.

87.

88.

89.

90.

91.  $987,000,000,000,000 \times 23,000,000$

92.  $0.00000000024 \times 0.00000004$

93.  $0.000000000522 \div 0.0000009$

94.  $81,000,000,000 \div 0.0000648$

95. The population density of Earth refers to the number of people per square mile of land area. If the total land area on Earth is square miles and the population in 2007 was estimated to be people, then calculate the population density of Earth at that time.

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96. In 2008 the population of New York City was estimated to be 8.364 million people. The total land area is 305 square miles. Calculate the population density of New York City.
97. The mass of Earth is kilograms and the mass of the Moon is kilograms. By what factor is the mass of Earth greater than the mass of the Moon?
98. The mass of the Sun is kilograms and the mass of Earth is kilograms. By what factor is the mass of the Sun greater than the mass of Earth? Express your answer in scientific notation.
99. The radius of the Sun is miles and the average distance from Earth to the Moon is miles. By what factor is the radius of the Sun larger than the average distance from Earth to the Moon?
100. One light year, meters, is the distance that light travels in a vacuum in one year. If the distance from our Sun to the nearest star, Proxima Centauri, is estimated to be meters, then calculate the number of years it would take light to travel that distance.
101. It is estimated that there are about 1 million ants per person on the planet. If the world population was estimated to be 6.67 billion people in 2007, then estimate the world ant population at that time.
102. The radius of the earth is meters and the radius of the sun is meters. By what factor is the radius of the Sun larger than the radius of the Earth?
103. A gigabyte is bytes and a megabyte is bytes. If the average song in the MP3 format consumes about 4.5 megabytes of storage, then how many songs will fit on a 4-gigabyte memory card?
104. Water weighs approximately 18 grams per mole. If one mole is about molecules, then approximate the weight of each molecule of water.

## PART C: DISCUSSION BOARD

113. Use numbers to show that

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114. Why is indeterminate?
115. Explain to a beginning algebra student why
116. René Descartes (1637) established the usage of exponential form: , and so on. Before this, how were exponents denoted?

## ANSWERS

- 1.
- 2.
3. 10
- 4.
- 5.
- 6.
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- 9.
- 10.
- 11.
- 12.
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- 41.
- 42.
- 43.
- 44.
- 45.
- 46.
47. 1
- 48.
49. -5
- 50.



- 51.
- 52.
- 53.
- 54.
- 55.
- 56.
- 57.  $a$
- 58.
- 59.
- 60.
- 61.
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
- 69.
- 70.
- 71.
- 72.
- 73.
- 74.
- 75.
- 76. \$210
- 77.
- 78. \$30



- 79.
- 80. \$1.04
- 81.
- 82.
- 83. 520,000,000
- 84.
- 85. 0.00000102
- 86.
- 87.
- 88.
- 89.
- 90.
- 91.
- 92.
- 93.
- 94.
- 95.
- 96.
- 97.
- 98.
- 99.
- 100.
- 101.
- 102.
- 103. About 116 people per square mile
- 104.
- 105. 81.2



- 106.
- 107. 1.807
- 108.
- 109. ants
- 110.
- 111. Approximately 889 songs
- 112.
- 113. Answer may vary
- 114.
- 115. Answer may vary

## 1.6 Polynomials and Their Operations

### LEARNING OBJECTIVES

1. Identify a polynomial and determine its degree.
2. Add and subtract polynomials.
3. Multiply and divide polynomials.

### Definitions

A **polynomial** is a special algebraic expression with terms that consist of real number coefficients and variable factors with whole number exponents. Some examples of polynomials follow:

--	--	--	--

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The **degree of a term** in a polynomial is defined to be the exponent of the variable, or if there is more than one variable in the term, the degree is the sum of their exponents. Recall that any constant term can be written as a product of and itself. Hence the degree of a constant term is 0.

<i>Term</i>	<i>Degree</i>
	2
	$2+1=3$
	$2+3=5$
8	0, since
$2x$	1, since

The **degree of a polynomial** is the largest degree of all of its terms.

<i>Polynomial</i>	<i>Degree</i>
	5

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<i>Polynomial</i>	<i>Degree</i>
	4, because has degree 4.
	1, because

Of particular interest are **polynomials with one variable**, where each term is of the form  $ax^n$ . Here  $a$  is any real number and  $n$  is any whole number. Such polynomials have the standard form:

Typically, we arrange terms of polynomials in descending order based on the degree of each term. The **leading coefficient** is the coefficient of the variable with the highest power, in this case,

### Example 1

Write in standard form:

Solution:

Since terms are defined to be separated by addition, we write the following:

In this form, we can see that the subtraction in the original corresponds to negative coefficients. Because addition is commutative, we can write the terms in descending order based on the degree as follows:

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Answer:

We classify polynomials by the number of terms and the degree:

<i>Expression</i>	<i>Classification</i>	<i>Degree</i>
	<b>Monomial</b> (one term)	1
	<b>Binomial</b> (two terms)	2
	<b>Trinomial</b> (three terms)	3
	<b>Polynomial</b> (many terms)	4

We can further classify polynomials with one variable by their degree:

<i>Polynomial</i>	<i>Name</i>
5	<b>Constant</b> (degree 0)
$2x+1$	<b>Linear</b> (degree 1)
	<b>Quadratic</b> (degree 2)
	<b>Cubic</b> (degree 3)
	Fourth-degree polynomial

In this text, we call any polynomial of degree  $n$  an  $n$ th-degree polynomial. In other words, if the degree is 4, we call the polynomial a fourth-degree polynomial. If the degree is 5, we call it a fifth-degree polynomial, and so on.

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## Example 2

State whether the following polynomial is linear or quadratic and give the leading coefficient:

Solution:

The highest power is 2; therefore, it is a quadratic polynomial. Rewriting in standard form we have

Here and thus the leading coefficient is  $-1$ .

Answer: Quadratic; leading coefficient:  $-1$

## Adding and Subtracting Polynomials

We begin by simplifying algebraic expressions that look like  $+(a+b)$  or  $-(a+b)$ . Here, the coefficients are actually implied to be  $+1$  and  $-1$  respectively and therefore the distributive property applies. Multiply each term within the parentheses by these factors as follows:

Use this idea as a means to eliminate parentheses when adding and subtracting polynomials.

## Example 3

Add:

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Solution:

The property  $+(a+b)=a+b$  allows us to eliminate the parentheses, after which we can then combine like terms.

Answer:

## Example 4

Add:

Solution:

Remember that the variable parts have to be exactly the same before we can add the coefficients.

Answer:

When subtracting polynomials, the parentheses become very important.

## Example 5

Subtract:

Solution:

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The property  $-(a+b)=-a-b$  allows us to remove the parentheses after subtracting each term.

Answer:


Subtracting a quantity is equivalent to multiplying it by  $-1$ .

## Example 6

Subtract:

Solution:

Distribute the  $-1$ , remove the parentheses, and then combine like terms. Multiplying the terms of a polynomial by  $-1$  changes all the signs.

$$(3x^2 - 2xy + y^2) - (2x^2 - xy + 3y^2)$$


Answer:

**Try this!** Subtract:

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Answer:

## Multiplying Polynomials

Use the product rule for exponents, , to multiply a monomial times a polynomial. In other words, when multiplying two expressions with the same base, add the exponents. To find the product of monomials, multiply the coefficients and add the exponents of variable factors with the same base. For example,


To multiply a polynomial by a monomial, apply the distributive property, and then simplify each term.

### Example 7

Multiply:

Solution:

Apply the distributive property and then simplify.

$$5xy^2(2x^2y^2 - xy + 1)$$


Answer:

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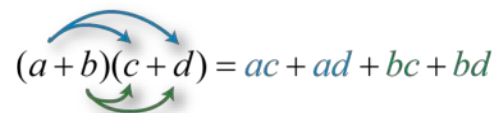
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To summarize, multiplying a polynomial by a monomial involves the distributive property and the product rule for exponents. Multiply all of the terms of the polynomial by the monomial. For each term, multiply the coefficients and add exponents of variables where the bases are the same.

In the same manner that we used the distributive property to distribute a monomial, we use it to distribute a binomial.

Here we apply the distributive property multiple times to produce the final result. This same result is obtained in one step if we apply the distributive property to  $a$  and  $b$  separately as follows:

$$(a + b)(c + d) = ac + ad + bc + bd$$


This is often called the FOIL method. Multiply the first, outer, inner, and then last terms.

## Example 8

Multiply:

Solution:

Distribute  $6x$  and  $-1$  and then combine like terms.

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Answer:

Consider the following two calculations:

--	--

This leads us to two formulas that describe **perfect square trinomials**:

We can use these formulas to quickly square a binomial.

## Example 9

Multiply:

Solution:

Here  $a=3x$  and  $b=5$ . Apply the formula:

$$\begin{aligned}(a+b)^2 &= a^2 + 2a \cdot b + b^2 \\ &\quad \downarrow \quad \quad \downarrow \downarrow \quad \quad \downarrow \\ (3x+5)^2 &= (3x)^2 + 2 \cdot (3x)(5) + (5)^2 \\ &= 9x^2 + 30x + 25\end{aligned}$$

Answer:

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This process should become routine enough to be performed mentally. Our third special product follows:

This product is called **difference of squares**:

The binomials  $(a+b)$  and  $(a-b)$  are called **conjugate binomials**. When multiplying conjugate binomials the middle terms are opposites and their sum is zero; the product is itself a binomial.

### **Example 10**

Multiply:  $(3xy+1)(3xy-1)$ .

Solution:

Answer:

**Try this!** Multiply:

Answer:

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## Example 11

Multiply:

Solution:

Here we perform one product at a time.

$$\begin{aligned}(5x - 2)^3 &= (5x - 2) \underbrace{(5x - 2)(5x - 2)}_{\text{Multiply first.}} \\ &= (5x - 2)(25x^2 - 10x - 10x + 4) \\ &= (5x - 2)(25x^2 - 20x + 4) \\ &= 125x^3 - 100x^2 + 20x - 50x^2 + 40x - 8 \\ &= 125x^3 - 150x^2 + 60x - 8\end{aligned}$$

Answer:

## Dividing Polynomials

Use the quotient rule for exponents, , to divide a polynomial by a monomial. In other words, when dividing two expressions with the same base, subtract the exponents. In this section, we will assume that all variables in the denominator are nonzero.

## Example 12

Divide:

Solution:

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Divide the coefficients and apply the quotient rule by subtracting the exponents of the like bases.

Answer:

When dividing a polynomial by a monomial, we may treat the monomial as a common denominator and break up the fraction using the following property:

Applying this property will result in terms that can be treated as quotients of monomials.

### **Example 13**

Divide:

Solution:

Break up the fraction by dividing each term in the numerator by the monomial in the denominator, and then simplify each term.

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Answer:

We can check our division by multiplying our answer, the quotient, by the monomial in the denominator, the divisor, to see if we obtain the original numerator, the dividend.

or	or
$Dividend = Divisor \cdot Quotient$	✓

The same technique outlined for dividing by a monomial *does not* work for polynomials with two or more terms in the denominator. In this section, we will outline a process called **polynomial long division**, which is based on the division algorithm for real numbers. For the sake of clarity, we will assume that all expressions in the denominator are nonzero.

## Example 14

Divide:

Solution:

Here  $x-2$  is the divisor and is the dividend. To determine the first term of the quotient, divide the leading term of the dividend by the leading term of the divisor.

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$$x - 2 \overline{) x^3 + 3x^2 - 8x - 4}$$

*We begin by dividing the leading terms.*

$$x^3 \div x = x^2$$

Multiply the first term of the quotient by the divisor, remembering to distribute, and line up like terms with the dividend.

$$x - 2 \overline{) x^3 + 3x^2 - 8x - 4}$$

$$\underline{x^3 - 2x^2}$$

*Distribute and line up like terms.*

$$x^2 \cdot (x - 2) = x^3 - 2x^2$$

Subtract the resulting quantity from the dividend. Take care to subtract both terms.

$$x - 2 \overline{) x^3 + 3x^2 - 8x - 4}$$

$$\underline{-(x^3 - 2x^2)}$$

$$5x^2$$

*Subtract.*

$$-(x^3 - 2x^2) = -x^3 + 2x^2$$

Bring down the remaining terms and repeat the process.

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$$\begin{array}{r}
 \phantom{x-2} \overline{)x^3 + 3x^2 - 8x - 4} \quad \begin{array}{l} x^2 \quad ? \end{array} \\
 \underline{-(x^3 - 2x^2)} \\
 5x^2 - 8x - 4
 \end{array}
 \quad \text{Bring down the remaining terms.}$$

Notice that the leading term is eliminated and that the result has a degree that is one less. The complete process is illustrated below:

$$\begin{array}{r}
 \phantom{x-2} \overline{)x^3 + 3x^2 - 8x - 4} \quad \begin{array}{l} x^2 + 5x + 2 \end{array} \\
 \underline{-(x^3 - 2x^2)} \\
 5x^2 - 8x - 4 \\
 \underline{-(5x^2 - 10x)} \\
 2x - 4 \\
 \underline{-(2x - 4)} \\
 0
 \end{array}$$

Polynomial long division ends when the degree of the remainder is less than the degree of the divisor. Here, the remainder is 0. Therefore, the binomial divides the polynomial evenly and the answer is the quotient shown above the division bar.

To check the answer, multiply the divisor by the quotient to see if you obtain the dividend as illustrated below:

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This is left to the reader as an exercise.

Answer:

Next, we demonstrate the case where there is a nonzero remainder.

$$\begin{array}{r} \textit{Quotient} \\ \textit{Divisor} \overline{) \textit{Dividend}} \\ \hline \textit{Remainder} \end{array}$$

Just as with real numbers, the final answer adds to the quotient the fraction where the remainder is the numerator and the divisor is the denominator. In general, when dividing we have:

If we multiply both sides by the divisor we obtain,

## Example 15

Divide:

Solution:

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Since the denominator is a binomial, begin by setting up polynomial long division.

$$2x-1 \overline{) 6x^2 - 5x + 3}$$

To start, determine what monomial times  $2x-1$  results in a leading term. This is the quotient of the given leading terms: Multiply 3 times the divisor  $2x-1$ , and line up the result with like terms of the dividend.

$$2x-1 \overline{) 6x^2 - 5x + 3}$$
$$6x^2 - 3x$$

Subtract the result from the dividend and bring down the constant term +3.

$$2x-1 \overline{) 6x^2 - 5x + 3}$$
$$-(6x^2 - 3x)$$
$$\hline -2x + 3$$

Subtracting eliminates the leading term. Multiply  $2x-1$  by  $-1$  and line up the result.

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$$\begin{array}{r}
 2x-1 \overline{) 6x^2 - 5x + 3} \\
 \underline{-(6x^2 - 3x)} \\
 -2x + 3 \\
 \underline{-(-2x + 1)} \\
 2
 \end{array}$$

Subtract again and notice that we are left with a remainder.

$$\begin{array}{r}
 2x-1 \overline{) 6x^2 - 5x + 3} \\
 \underline{-(6x^2 - 3x)} \\
 -2x + 3 \\
 \underline{-(-2x + 1)} \\
 2
 \end{array}$$

The constant term 2 has degree 0 and thus the division ends. Therefore,

To check that this result is correct, we multiply as follows:



Answer:



Occasionally, some of the powers of the variables appear to be missing within a polynomial. This can lead to errors when lining up like terms. Therefore, when first learning how to divide polynomials using long division, fill in the missing terms with zero coefficients, called **placeholders**.

## Example 16

Divide:

Solution:

Notice that the binomial in the numerator does not have terms with degree 2 or 1. The division is simplified if we rewrite the expression with placeholders:

Set up polynomial long division:

$$3x + 4 \overline{) 27x^3 + 0x^2 + 0x + 64}$$

We begin with and work the rest of the division algorithm.

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$$\begin{array}{r}
 9x^2 - 12x + 16 \\
 3x + 4 \overline{) 27x^3 + 0x^2 + 0x + 64} \\
 \underline{-(27x^3 + 36x^2)} \phantom{+ 0x + 64} \\
 -36x^2 + 0x + 64 \\
 \underline{-(-36x^2 - 48x)} \phantom{+ 64} \\
 48x + 64 \\
 \underline{-(48x + 64)} \\
 0
 \end{array}$$

Answer:

## Example 17

Divide:

Solution:

$$x^2 - 2x + 5 \overline{) 3x^4 - 2x^3 + 6x^2 + 23x - 7}$$

Begin the process by dividing the leading terms to determine the leading term of the quotient. Take care to distribute and line up the like terms. Continue the process until the remainder has a degree less than 2.

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$$\begin{array}{r}
 \phantom{x^2 - 2x + 5} \overline{3x^2 + 4x - 1} \\
 x^2 - 2x + 5 \overline{)3x^4 - 2x^3 + 6x^2 + 23x - 7} \\
 \underline{-(3x^4 - 6x^3 + 15x^2)} \\
 4x^3 - 9x^2 + 23x - 7 \\
 \underline{-(4x^3 - 8x^2 + 20x)} \\
 -x^2 + 3x - 7 \\
 \underline{-(-x^2 + 2x - 5)} \\
 x - 2
 \end{array}$$

The remainder is  $x-2$ . Write the answer with the remainder:

Answer:

Polynomial long division takes time and practice to master. Work lots of problems and remember that you may check your answers by multiplying the quotient by the divisor (and adding the remainder if present) to obtain the dividend.

**Try this!** Divide:

Answer:

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## KEY TAKEAWAYS

Polynomials are special algebraic expressions where the terms are the products of real numbers and variables with whole number exponents.

The degree of a polynomial with one variable is the largest exponent of the variable found in any term. In addition, the terms of a polynomial are typically arranged in descending order based on the degree of each term.

When adding polynomials, remove the associated parentheses and then combine like terms. When subtracting polynomials, distribute the  $-1$ , remove the parentheses, and then combine like terms.

To multiply polynomials apply the distributive property; multiply each term in the first polynomial with each term in the second polynomial. Then combine like terms.

When dividing by a monomial, divide all terms in the numerator by the monomial and then simplify each term.

When dividing a polynomial by another polynomial, apply the division algorithm.

## TOPIC EXERCISES

### PART A: DEFINITIONS

**Write the given polynomials in standard form.**

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.

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**Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.**

- 1.
- 2.
- 3.
- 4.
5.  $x^4 - 1$
6.  $5$

**State whether the polynomial is linear or quadratic and give the leading coefficient.**

- 1.
- 2.
- 3.
- 4.
- 5.
6.  $x - 1$
- 7.
8.  $1 - 5x$

## **PART B: ADDING AND SUBTRACTING POLYNOMIALS**

**Simplify.**

- 21.
- 22.
- 23.
- 24.



- 25.
- 26.
- 27.
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- 29.
- 30.
- 31.
- 32.
- 33.
- 34.
- 35.
- 36.
- 37.
- 38.
- 39.
- 40.
41. Subtract  $4y-3$  from .
42. Subtract from
43. A right circular cylinder has a height that is equal to the radius of the base,  $h=r$ . Find a formula for the surface area in terms of  $h$ .
44. A rectangular solid has a width that is twice the height and a length that is 3 times that of the height. Find a formula for the surface area in terms of the height.

## PART C: MULTIPLYING POLYNOMIALS

### Multiply.

- 45.
- 46.
- 47.



48.

49.

50.

51.

52.

53.

54.

55.

56.

57.  $(x+4)(x-5)$

58.  $(x-7)(x-6)$

59.  $(2x-3)(3x-1)$

60.  $(9x+1)(3x+2)$

61.

62.

63.  $(3x+5)(3x-5)$

64.  $(x+6)(x-6)$

65.

66.  $(ab+7)(ab-7)$

67.

68.  $(xy+5)(x-y)$

69.

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72.

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74.

75.

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94.

95. Find the product of  $3x-2$  and

96. Find the product of and

97. Each side of a square measures units. Determine the area in terms of  $x$ .

98. Each edge of a cube measures units. Determine the volume in terms of  $x$ .

## PART D: DIVIDING POLYNOMIALS

**Divide.**

99.

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100.  
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127.

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128.  
129.  
130.  
131. Find the quotient of and  $2x-1$ .  
132. Find the quotient of and  $3x-2$ .

## ANSWERS

1.  
2.  
3.  
4.  
5.  
6.  
7. Trinomial; degree 2  
8.  
9. Trinomial; degree 4  
10.  
11. Binomial; degree 4  
12.  
13. Quadratic,  $-9$   
14.  
15. Linear, 2  
16.  
17. Quadratic, 5  
18.  
19. Quadratic,  $-2$   
20.  
21.



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- 76.
- 77.

$6x^2 - 11x + 3$



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- 101.
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- 103.
- 104.
- 105.

square units

$5x$



- 106.
- 107.
- 108.
- 109.
- 110.
- 111.
- 112.
- 113.
- 114.
- 115.
- 116.
- 117.
- 118.
- 119.
- 120.
- 121.
- 122.
- 123.  $2x-3$
- 124.
- 125.
- 126.
- 127.
- 128.
- 129.
- 130.
- 131.  $5x-3$

121.



## 1.7 Solving Linear Equations

### LEARNING OBJECTIVES

1. Use the properties of equality to solve basic linear equations.
2. Identify and solve conditional linear equations, identities, and contradictions.
3. Clear fractions from equations.
4. Set up and solve linear applications.

### Solving Basic Linear Equations

An **equation** is a statement indicating that two algebraic expressions are equal. A **linear equation with one variable**,  $x$ , is an equation that can be written in the standard form  $ax+b=0$  where  $a$  and  $b$  are real numbers and  $a \neq 0$ . For example,

$$3x-12=0$$

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A **solution** to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation  $3x-12=0$  is  $x$  and the solution is  $x=4$ . To verify this, substitute the value 4 in for  $x$  and check that you obtain a true statement.



Alternatively, when an equation is equal to a constant, we may verify a solution by substituting the value in for the variable and showing that the result is equal to that constant. In this sense, we say that solutions “satisfy the equation.”

## Example 1

Is a solution to ?

Solution:

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, and then substitute the appropriate values. By making use of parentheses, we avoid some common errors when working the order of operations.



Answer: No, does not satisfy the equation.

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Developing techniques for solving various algebraic equations is one of our main goals in algebra. This section reviews the basic techniques used for solving linear equations with one variable. We begin by defining **equivalent equations** as equations with the same solution set.

Here we can see that the three linear equations are equivalent because they share the same solution set, namely,  $\{7\}$ . To obtain equivalent equations, use the following **properties of equality**. Given algebraic expressions  $A$  and  $B$ , where  $c$  is a nonzero number:

<b>Addition property of equality:</b>	If $A=B$ , then $A + c=B + c$
<b>Subtraction property of equality:</b>	If $A=B$ , then $A - c=B - c$
<b>Multiplication property of equality:</b>	If $A=B$ , then $cA=cB$
<b>Division property of equality:</b>	

**Note:** Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation  $0 = 0$ .

We solve algebraic equations by isolating the variable with a coefficient of 1. If given a linear equation of the form  $ax+b=c$ , then we can solve it in two steps. First, use the

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appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking the solution in the following examples is left to the reader.

## Example 2

Solve:

Solution:

Answer: The solution is 3.

## Example 3

Solve:  $56=8+12y$ .

Solution:

When no sign precedes the term, it is understood to be positive. In other words, think of this as  $56=+8+12y$ . Therefore, we begin by subtracting 8 on both sides of the equal sign.

It does not matter on which side we choose to isolate the variable because the **symmetric property** states that  $4=y$  is equivalent to  $y=4$ .

Answer: The solution is 4.

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## Example 4

Solve:

Solution:

Isolate the variable term using the addition property of equality, and then multiply both sides of the equation by the reciprocal of the coefficient

Answer: The solution is  $-6$ .

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation.

**Try this!** Solve:

Answer:  $x = -2$

## General Guidelines for Solving Linear Equations

Typically linear equations are not given in standard form, and so solving them requires additional steps. When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation. Do this by isolating the variable using the following steps:

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**Step 1:** Simplify both sides of the equation using the order of operations and combine all like terms on the same side of the equal sign.

**Step 2:** Use the appropriate properties of equality to combine like terms on opposite sides of the equal sign. The goal is to obtain the variable term on one side of the equation and the constant term on the other.

**Step 3:** Divide or multiply as needed to isolate the variable.

**Step 4:** Check to see if the answer solves the original equation.

We will often encounter linear equations where the expressions on each side of the equal sign can be simplified. If this is the case, then it is best to simplify each side first before solving. Normally this involves combining same-side like terms.

**Note:** At this point in our study of algebra the use of the properties of equality should seem routine. Therefore, displaying these steps in this text, usually in blue, becomes optional.

## Example 5

Solve:  $-4a+2-a=1$ .

Solution:

First combine the like terms on the left side of the equal sign.

Always use the original equation to check to see if the solution is correct.



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Answer: The solution is

Given a linear equation in the form  $ax+b=cx+d$ , we begin the solving process by combining like terms on opposite sides of the equal sign. To do this, use the addition or subtraction property of equality to place like terms on the same side so that they can be combined. In the examples that remain, the check is left to the reader.

## Example 6

Solve:  $-2y-3=5y+11$ .

Solution:

Subtract  $5y$  on both sides so that we can combine the terms involving  $y$  on the left side.

From here, solve using the techniques developed previously.

Answer: The solution is  $-2$ .

Solving will often require the application of the distributive property.

## Example 7

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Solve:

Solution:

Simplify the linear expressions on either side of the equal sign first.

Answer: The solution is

## Example 8

Solve:

Solution:

Begin by applying the distributive property.

Here we point out that  $-a$  is equivalent to  $-1a$ ; therefore, we choose to divide both sides of the equation by  $-1$ .

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Alternatively, we can multiply both sides of  $-a=-2$  by negative one and achieve the same result.

Answer: The solution is 2.

**Try this!** Solve:

Answer:  $x=1$

There are three different types of equations. Up to this point, we have been solving **conditional equations**. These are equations that are true for particular values. An **identity** is an equation that is true for all possible values of the variable. For example,

$$x = x \quad \textit{Identity}$$

has a solution set consisting of all real numbers, . A **contradiction** is an equation that is never true and thus has no solutions. For example,

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$$x + 1 = x \quad \text{Contradiction}$$

has no solution. We use the empty set,  $\emptyset$ , to indicate that there are no solutions. If the end result of solving an equation is a true statement, like  $0 = 0$ , then the equation is an identity and any real number is a solution. If solving results in a false statement, like  $0 = 1$ , then the equation is a contradiction and there is no solution.

### Example 9

Solve:  $4(x+5)+6=2(2x+3)$ .

Solution:

*x*

Solving leads to a false statement; therefore, the equation is a contradiction and there is no solution.

Answer:  $\emptyset$

### Example 10

Solve:  $3(3y+5)+5=10(y+2)-y$ .

Solution:

*✓*



Solving leads to a true statement; therefore, the equation is an identity and any real number is a solution.

Answer:

The coefficients of linear equations may be any real number, even decimals and fractions. When this is the case it is possible to use the multiplication property of equality to clear the fractional coefficients and obtain integer coefficients in a single step. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD).

## Example 11

Solve:

Solution:

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, it is the  $LCD(3,5)=15$ .

Answer: The solution is  $-9$ .

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It is important to know that this technique only works for equations. *Do not try to clear fractions when simplifying expressions.* As a reminder:

Expression	Equation

We simplify expressions and solve equations. If you multiply an expression by 6, you will change the problem. However, if you multiply both sides of an equation by 6, you obtain an equivalent equation.

<b>Incorrect</b>	<b>Correct</b>
✗	✓

## Applications Involving Linear Equations

Algebra simplifies the process of solving real-world problems. This is done by using letters to represent unknowns, restating problems in the form of equations, and by offering systematic techniques for solving those equations. To solve problems using

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algebra, first translate the wording of the problem into mathematical statements that describe the relationships between the given information and the unknowns. Usually, this translation to mathematical statements is the difficult step in the process. The key to the translation is to carefully read the problem and identify certain key words and phrases.

Key Words	Translation
<b>Sum</b> , increased by, more than, plus, added to, total	+
<b>Difference</b> , decreased by, subtracted from, less, minus	−
<b>Product</b> , multiplied by, of, times, twice	·
<b>Quotient</b> , divided by, ratio, per	÷
<b>Is</b> , total, result	=

When translating sentences into mathematical statements, be sure to read the sentence several times and parse out the key words and phrases. It is important to first identify the variable, “*let x represent...*” and state in words what the unknown quantity is. This step not only makes our work more readable, but also forces us to think about what we are looking for.

## Example 12

When 6 is subtracted from twice the sum of a number and 8 the result is 5. Find the number.

Solution:

Let  $n$  represent the unknown number.

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$$\underbrace{2 \cdot}_{\text{"twice"}} \underbrace{(n+8)}_{\text{"the sum of a number and 8"}} \underbrace{- 6}_{\text{"6 is subtracted from"}} \underbrace{=}_{\text{"result is"}} 5$$

To understand why we included the parentheses in the set up, you must study the structure of the following two sentences and their translations:

<i>"twice the sum of a number and 8"</i>	$2(n+8)$
<i>"the sum of <b>twice a number</b> and 8"</i>	$2n+8$

The key was to focus on the phrase *"twice the sum,"* this prompted us to group the sum within parentheses and then multiply by 2. After translating the sentence into a mathematical statement we then solve.

Check.



Answer: The number is

General guidelines for setting up and solving word problems follow.

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**Step 1:** Read the problem several times, identify the key words and phrases, and organize the given information.

**Step 2:** Identify the variables by assigning a letter or expression to the unknown quantities.

**Step 3:** Translate and set up an algebraic equation that models the problem.

**Step 4:** Solve the resulting algebraic equation.

**Step 5:** Finally, answer the question in sentence form and make sure it makes sense (check it).

For now, set up all of your equations using only one variable. Avoid two variables by looking for a relationship between the unknowns.

### Example 13

A rectangle has a perimeter measuring 92 meters. The length is 2 meters less than 3 times the width. Find the dimensions of the rectangle.

Solution:

The sentence “*The length is 2 meters less than 3 times the width,*” gives us the relationship between the two variables.

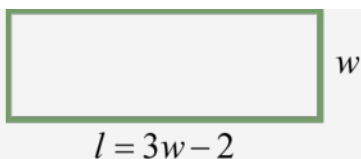
Let  $w$  represent the width of the rectangle.

Let  $3w - 2$  represent the length.

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The sentence “A rectangle has a perimeter measuring 92 meters” suggests an algebraic set up. Substitute 92 for the perimeter and the expression  $3w - 2$  for the length into the appropriate formula as follows:

Once you have set up an algebraic equation with one variable, solve for the width,  $w$ .

Use  $3w - 2$  to find the length.

To check, make sure the perimeter is 92 meters.

Answer: The rectangle measures 12 meters by 34 meters.

## Example 14

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Given a annual interest rate, how long will it take \$2,500 to yield \$437.50 in simple interest?

Solution:

Let  $t$  represent the time needed to earn \$437.50 at . Organize the information needed to use the formula for simple interest,  $I=prt$ .

Given interest for the time period:	$I = \$437.50$
Given principal:	$p = \$2,500$
Given rate:	

Next, substitute all of the known quantities into the formula and then solve for the only unknown,  $t$ .

Answer: It takes 4 years for \$2,500 invested at to earn \$437.50 in simple interest.

## Example 15

Susan invested her total savings of \$12,500 in two accounts earning simple interest. Her mutual fund account earned 7% last year and her CD earned 4.5%. If her total interest for the year was \$670, how much was in each account?

Solution:

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The relationship between the two unknowns is that they total \$12,500. When a total is involved, a common technique used to avoid two variables is to represent the second unknown as the difference of the total and the first unknown.

Let  $x$  represent the amount invested in the mutual fund.

Let  $12,500 - x$  represent the remaining amount invested in the CD.

Organize the data.

Interest earned in the mutual fund:	
Interest earned in the CD:	
Total interest:	

The total interest is the sum of the interest earned from each account.

This equation models the problem with one variable. Solve for  $x$ .

Use to find the amount in the CD.

Answer: Susan invested \$4,300 at 7% in a mutual fund and \$8,200 at 4.5% in a CD.

## KEY TAKEAWAYS

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Solving general linear equations involves isolating the variable, with coefficient 1, on one side of the equal sign. To do this, first use the appropriate equality property of addition or subtraction to isolate the variable term on one side of the equal sign. Next, isolate the variable using the equality property of multiplication or division. Finally, check to verify that your solution solves the original equation.

If solving a linear equation leads to a true statement like  $0 = 0$ , then the equation is an identity and the solution set consists of all real numbers,  $\mathbb{R}$ .

If solving a linear equation leads to a false statement like  $0 = 5$ , then the equation is a contradiction and there is no solution,  $\emptyset$ .

Clear fractions by multiplying both sides of an equation by the least common multiple of all the denominators. Distribute and multiply all terms by the LCD to obtain an equivalent equation with integer coefficients.

Simplify the process of solving real-world problems by creating mathematical models that describe the relationship between unknowns. Use algebra to solve the resulting equations.

## TOPIC EXERCISES

### PART A: SOLVING BASIC LINEAR EQUATIONS

**Determine whether or not the given value is a solution.**

1.  $-5x+4=-1$ ;  $x=-1$
2.  $4x-3=-7$ ;  $x=-1$
3.  $3y-4=5$ ;
4.  $-2y+7=12$
5.  $3a-6=18-a$ ;  $a=-3$
6.  $5(2t-1)=2-t$ ;  $t=2$
7.  $ax-b=0$ ;

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8.  $ax+b=2b$ ;

**Solve.**

1.  $5x-3=27$

2.  $6x-7=47$

3.  $4x+13=35$

4.  $6x-9=18$

5.  $9a+10=10$

6.  $5-3a=5$

7.  $-8t+5=15$

8.  $-9t+12=33$

9.

10.

11.

12.

13.  $7-y=22$

14.  $6-y=12$

15. Solve for  $x$ :  $ax-b=c$

16. Solve for  $x$ :  $ax+b=0$

## PART B: SOLVING LINEAR EQUATIONS

**Solve.**

25.  $6x-5+2x=19$

26.  $7-2x+9=24$

27.  $12x-2-9x=5x+8$

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28.  $16-3x-22=8-4x$
29.  $5y-6-9y=3-2y+8$
30.  $7-9y+12=3y+11-11y$
31.  $3+3a-11=5a-8-2a$
32.  $2-3a=5a+7-8a$
- 33.
- 34.
35.  $1.2x-0.5-2.6x=2-2.4x$
36.  $1.59-3.87x=3.48-4.1x-0.51$
37.  $5-10x=2x+8-12x$
38.  $8x-3-3x=5x-3$
39.  $5(y+2)=3(2y-1)+10$
40.  $7(y-3)=4(2y+1)-21$
41.  $7-5(3t-9)=22$
42.  $10-5(3t+7)=20$
43.  $5-2x=4-2(x-4)$
44.  $2(4x-5)+7x=5(3x-2)$
45.  $4(4a-1)=5(a-3)+2(a-2)$
46.  $6(2b-1)+24b=8(3b-1)$
- 47.
- 48.
49.  $1.2(2x+1)+0.6x=4x$
50.  $6+0.5(7x-5)=2.5x+0.3$
51.  $5(y+3)=15(y+1)-10y$
52.  $3(4-y)-2(y+7)=-5y$
- 53.
- 54.
55.  $6-3(7x+1)=7(4-3x)$

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56.  $6(x-6)-3(2x-9)=-9$
- 57.
- 58.
59.  $-2(3x+1)-(x-3)=-7x+1$
60.  $6(2x+1)-(10x+9)=0$
61. Solve for  $w$ :  $P=2l+2w$
62. Solve for  $a$ :  $P=a+b+c$
63. Solve for  $t$ :  $D=rt$
64. Solve for  $w$ :
65. Solve for  $b$ :
66. Solve for  $a$ :
67. Solve for  $a$ :
68. Solve for  $h$ :
69. Solve for  $F$ :
70. Solve for  $x$ :

## PART C: APPLICATIONS

**Set up an algebraic equation then solve.**

### Number Problems

71. When 3 is subtracted from the sum of a number and 10 the result is 2. Find the number.
72. The sum of 3 times a number and 12 is equal to 3. Find the number.
73. Three times the sum of a number and 6 is equal to 5 times the number. Find the number.
74. Twice the sum of a number and 4 is equal to 3 times the sum of the number and 1. Find the number.
75. A larger integer is 1 more than 3 times another integer. If the sum of the integers is 57, find the integers.

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76. A larger integer is 5 more than twice another integer. If the sum of the integers is 83, find the integers.
77. One integer is 3 less than twice another integer. Find the integers if their sum is 135.
78. One integer is 10 less than 4 times another integer. Find the integers if their sum is 100.
79. The sum of three consecutive integers is 339. Find the integers.
80. The sum of four consecutive integers is 130. Find the integers.
81. The sum of three consecutive even integers is 174. Find the integers.
82. The sum of four consecutive even integers is 116. Find the integers.
83. The sum of three consecutive odd integers is 81. Find the integers.
84. The sum of four consecutive odd integers is 176. Find the integers.

### Geometry Problems

71. The length of a rectangle is 5 centimeters less than twice its width. If the perimeter is 134 centimeters, find the length and width.
72. The length of a rectangle is 4 centimeters more than 3 times its width. If the perimeter is 64 centimeters, find the length and width.
73. The width of a rectangle is one-half that of its length. If the perimeter measures 36 inches, find the dimensions of the rectangle.
74. The width of a rectangle is 4 inches less than its length. If the perimeter measures 72 inches, find the dimensions of the rectangle.
75. The perimeter of a square is 48 inches. Find the length of each side.
76. The perimeter of an equilateral triangle is 96 inches. Find the length of each side.
77. The circumference of a circle measures  $80\pi$  units. Find the radius.

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78. The circumference of a circle measures 25 centimeters. Find the radius rounded off to the nearest hundredth.

### Simple Interest Problems

71. For how many years must \$1,000 be invested at % to earn \$165 in simple interest?

72. For how many years must \$20,000 be invested at % to earn \$3,125 in simple interest?

73. At what annual interest rate must \$6500 be invested for 2 years to yield \$1,040 in simple interest?

74. At what annual interest rate must \$5,750 be invested for 1 year to yield \$333.50 in simple interest?

75. If the simple interest earned for 5 years was \$1,860 and the annual interest rate was 6%, what was the principal?

76. If the simple interest earned for 2 years was \$543.75 and the annual interest rate was %, what was the principal?

77. How many years will it take \$600 to double earning simple interest at a 5% annual rate? (Hint: To double, the investment must earn \$600 in simple interest.)

78. How many years will it take \$10,000 to double earning simple interest at a 5% annual rate? (Hint: To double, the investment must earn \$10,000 in simple interest.)

79. Jim invested \$4,200 in two accounts. One account earns 3% simple interest and the other earns 6%. If the interest after 1 year was \$159, how much did he invest in each account?

80. Jane has her \$6,500 savings invested in two accounts. She has part of it in a CD at 5% annual interest and the rest in a savings account that earns 4% annual

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interest. If the simple interest earned from both accounts is \$303 for the year, then how much does she have in each account?

81. Jose put last year's bonus of \$8,400 into two accounts. He invested part in a CD with 2.5% annual interest and the rest in a money market fund with 1.5% annual interest. His total interest for the year was \$198. How much did he invest in each account?

82. Mary invested her total savings of \$3,300 in two accounts. Her mutual fund account earned 6.2% last year and her CD earned 2.4%. If her total interest for the year was \$124.80, how much was in each account?

83. Alice invests money into two accounts, one with 3% annual interest and another with 5% annual interest. She invests 3 times as much in the higher yielding account as she does in the lower yielding account. If her total interest for the year is \$126, how much did she invest in each account?

84. James invested an inheritance in two separate banks. One bank offered % annual interest rate and the other %. He invested twice as much in the higher yielding bank account than he did in the other. If his total simple interest for 1 year was \$5,760, then what was the amount of his inheritance?

### Uniform Motion Problems

71. If it takes Jim hours to drive the 40 miles to work, then what is Jim's average speed?

72. It took Jill hours to drive the 189 miles home from college. What was her average speed?

73. At what speed should Jim drive if he wishes to travel 176 miles in hours?

74. James and Martin were able to drive the 1,140 miles from Los Angeles to Seattle. If the total trip took 19 hours, then what was their average speed?

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## PART D: DISCUSSION BOARD

111. What is regarded as the main business of algebra? Explain.
112. What is the origin of the word *algebra*?
113. Create an identity or contradiction of your own and share it on the discussion board. Provide a solution and explain how you found it.
114. Post something you found particularly useful or interesting in this section. Explain why.
115. Conduct a web search for “solving linear equations.” Share a link to website or video tutorial that you think is helpful.

## ANSWERS

1. No
- 2.
3. Yes
- 4.
5. No
- 6.
7. Yes
- 8.
9. 6
- 10.
- 11.
- 12.
13. 0
- 14.
- 15.
- 16.

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- 17.
- 18.
19. -3
- 20.
21. -15
- 22.
- 23.
- 24.
25. 3
- 26.
27. -5
- 28.
- 29.
- 30.
31. R
- 32.
- 33.
- 34.
35. 2.5
- 36.
37.  $\emptyset$
- 38.
39. 3
- 40.
41. 2
- 42.
43.  $\emptyset$
- 44.

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- 45.
- 46.
- 47. -81
- 48.
- 49. 1.2
- 50.
- 51. R
- 52.
- 53. 0
- 54.
- 55.  $\emptyset$
- 56.
- 57.
- 58.
- 59. R
- 60.
- 61.
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
- 69.
- 70.
- 71. -5
- 72.



73. 9
- 74.
75. 14, 43
- 76.
77. 46, 89
- 78.
79. 112, 113, 114
- 80.
81. 56, 58, 60
- 82.
83. 25, 27, 29
- 84.
85. Width: 24 centimeters; length: 43 centimeters
- 86.
87. Width: 6 inches; length: 12 inches
- 88.
89. 12 inches
- 90.
91. 40 units
- 92.
93. 3 years
- 94.
95. 8%
- 96.
97. \$6,200
- 98.
99. 20 years
- 100.

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101. He invested \$3,100 at 3% and \$1,100 at 6%.
- 102.
103. Jose invested \$7,200 in the CD and \$1,200 in the money market fund.
- 104.
105. Alice invested \$700 at 3% and \$2,100 at 5%.
- 106.
107. 32 miles per hour
- 108.
109. 64 miles per hour
- 110.
111. Answer may vary
- 112.
113. Answer may vary
- 114.
115. Answer may vary

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## 1.8 Solving Linear Inequalities with One Variable

### LEARNING OBJECTIVES

1. Identify linear inequalities and check solutions.
2. Solve linear inequalities and express the solutions graphically on a number line and in interval notation.
3. Solve compound linear inequalities and express the solutions graphically on a number line and in interval notation.
4. Solve applications involving linear inequalities and interpret the results.

### Linear Inequalities

A **linear inequality** is a mathematical statement that relates a linear expression as either less than or greater than another. The following are some examples of linear inequalities, all of which are solved in this section:

$5x+7<22$	$-2(x+8)+6\geq 20$	$-2(4x-5)<9-2(x-2)$
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A **solution to a linear inequality** is a real number that will produce a true statement when substituted for the variable. Linear inequalities have either infinitely many solutions or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation.

## Example 1

Are  $x=-4$  and  $x=6$  solutions to  $5x+7<22$ ?

Solution:

Substitute the values in for  $x$ , simplify, and check to see if we obtain a true statement.

<i>Check</i> $x=-4$	<i>Check</i> $x=6$
✓	✗

Answer:  $x=-4$  is a solution and  $x=6$  is not.

All but one of the techniques learned for solving linear equations apply to solving linear inequalities. You may add or subtract any real number to both sides of an inequality, and you may multiply or divide both sides by any *positive* real number to create equivalent inequalities. For example:

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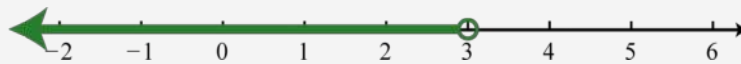


Subtracting 7 from each side and dividing each side by positive 5 results in an inequality that is true.

## Example 2

Solve and graph the solution set:  $5x+7<22$ .

Solution:



It is helpful to take a minute and choose a few values in and out of the solution set, substitute them into the original inequality, and then verify the results. As indicated, you should expect  $x=0$  to solve the original inequality and that  $x=5$  should not.

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<b>Check</b> $x=0$	<b>Check</b> $x=5$
✓	✗

Checking in this manner gives us a good indication that we have solved the inequality correctly.

We can express this solution in two ways: using set notation and interval notation.

{

In this text we will choose to present answers using interval notation.

Answer:  $(-\infty, 3)$

When working with linear inequalities, a different rule applies when multiplying or dividing by a negative number. To illustrate the problem, consider the true statement  $10 > -5$  and divide both sides by  $-5$ .

✗ False

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Dividing by  $-5$  results in a false statement. To retain a true statement, the inequality must be reversed.

✓ True

The same problem occurs when multiplying by a negative number. This leads to the following new rule: *when multiplying or dividing by a negative number, reverse the inequality*. It is easy to forget to do this so take special care to watch for negative coefficients. In general, given algebraic expressions  $A$  and  $B$ , where  $c$  is a positive nonzero real number, we have the following **properties of inequalities**:

<b>Addition property of inequalities:</b>	If $A < B$ then, $A + c < B + c$
<b>Subtraction property of inequalities:</b>	If $A < B$ , then $A - c < B - c$
<b>Multiplication property of inequalities:</b>	If $A < B$ , then $cA < cB$ If $A < B$ , then $-cA > -cB$
<b>Division property of inequalities:</b>	If $A < B$ , then $Ac < Bc$ If $A < B$ , then $A - c > B - c$

We use these properties to obtain an **equivalent inequality**, one with the same solution set, where the variable is isolated. The process is similar to solving linear equations.

### Example 3

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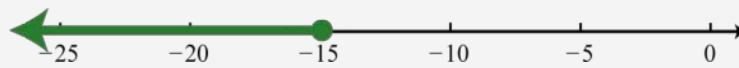
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Solve and graph the solution set:  $-2(x+8)+6 \geq 20$ .

Solution:



Answer: Interval notation  $(-\infty, -15]$

### Example 4

Solve and graph the solution set:  $-2(4x-5) < 9-2(x-2)$ .

Solution:



Answer: Interval notation

### Example 5

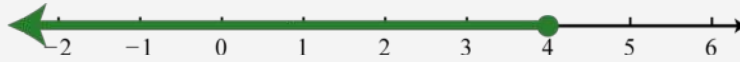
Solve and graph the solution set:

Solution:

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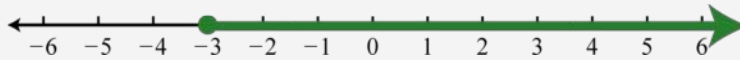




Answer: Interval notation:  $(-\infty, 4]$

**Try this!** Solve and graph the solution set:  $10 - 5(2x + 3) \leq 25$ .

Answer:  $[-3, \infty)$



## Compound Inequalities

Following are some examples of compound linear inequalities:

$-13 < 3x - 7 < 17$	$4x + 5 \leq -15$ or $6x - 11 > 7$
---------------------	------------------------------------

These **compound inequalities** are actually two inequalities in one statement joined by the word *and* or by the word *or*. For example,

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$$-13 < 3x - 7 < 17$$

is a compound inequality because it can be decomposed as follows:

$$-13 < 3x - 7 \text{ and } 3x - 7 < 17$$

We can solve each inequality individually; the intersection of the two solution sets solves the original compound inequality. While this method works, there is another method that usually requires fewer steps. Apply the properties of this section to all three parts of the compound inequality with the goal of *isolating the variable in the middle* of the statement to determine the bounds of the solution set.

## Example 6

Solve and graph the solution set:  $-13 < 3x - 7 < 17$ .

Solution:



Answer: Interval notation:  $(-2, 8)$

## Example 7

Solve and graph the solution set:

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Solution:



Answer: Interval notation  $[-3,4)$

It is important to note that when multiplying or dividing all three parts of a compound inequality by a negative number, you must reverse all of the inequalities in the statement. For example:

The answer above can be written in an equivalent form, where smaller numbers lie to the left and the larger numbers lie to the right, as they appear on a number line.

$$-10 < x < 5$$

Using interval notation, write:  $(-10, 5)$ .

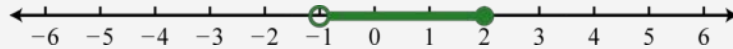
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**Try this!** Solve and graph the solution set:  $-3 \leq -3(2x-3) < 15$ .

Answer:  $(-1, 2]$ ;



For compound inequalities with the word “or” you work both inequalities separately and then consider the union of the solution sets. Values in this union solve either inequality.

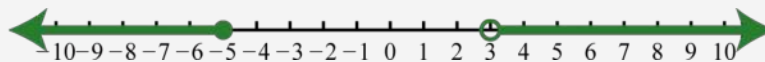
## Example 8

Solve and graph the solution set:  $4x+5 \leq -15$  or  $6x-11 > 7$ .

Solution:

Solve each inequality and form the union by combining the solution sets.

	or	
--	----	--



Answer: Interval notation  $(-\infty, -5] \cup (3, \infty)$

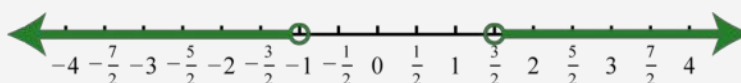
**Try this!** Solve and graph the solution set:

Answer:

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## Applications of Linear Inequalities

Some of the key words and phrases that indicate inequalities are summarized below:

Key Phrases	Translation
A number is <b>at least</b> 5.	
A number is <b>5 or more inclusive</b> .	$x \geq 5$
A number is <b>at most</b> 3.	
A number is <b>3 or less inclusive</b> .	$x \leq 3$
A number is <b>strictly less than</b> 4.	
A number is <b>less than</b> 4, <b>noninclusive</b> .	$x < 4$
A number is <b>greater than</b> 7.	
A number is <b>more than</b> 7, <b>noninclusive</b> .	$x > 7$
A number is <b>in between</b> 2 and 10.	$2 < x < 10$
A number is at least 5 <b>and</b> at most 15.	
A number may <b>range</b> from 5 to 15.	$5 \leq x \leq 15$

As with all applications, carefully read the problem several times and look for key words and phrases. Identify the unknowns and assign variables. Next, translate the wording

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into a mathematical inequality. Finally, use the properties you have learned to solve the inequality and express the solution graphically or in interval notation.

### Example 9

Seven less than 3 times the sum of a number and 5 is at most 11. Find all numbers that satisfy this condition.

Solution:

First, choose a variable for the unknown number and identify the key words and phrases.

Let  $n$  represent the unknown indicated by “a number.”

$$\begin{array}{ccccccc} \textit{three times the sum of} & & \textit{seven less than} & & \textit{is at most} & & \\ \textit{a number and 5} & & & & & & \\ \underbrace{3(n+5)} & & \underbrace{- 7} & & \underbrace{\leq} & & 11 \end{array}$$

Solve for  $n$ .

Answer: Any number less than or equal to 1 will satisfy the statement.

### Example 10

To earn a B in a mathematics course the test average must be at least 80% and less than 90%. If a student earned 92%, 96%, 79%, and 83% on the first four tests, what must she score on the fifth test to earn a B?

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Solution:

Set up a compound inequality where the test average is between 80% and 90%. In this case, include the lower bound, 80.

Let  $x$  represent the score on the fifth test.

Answer: She must earn a score of at least 50% and less than 100%.

In the previous example, the upper bound 100% was not part of the solution set. What would happen if she did earn a 100% on the fifth test?

As we can see, her average would be 90%, which would earn her an A.

## KEY TAKEAWAYS

Inequalities typically have infinitely many solutions. The solutions are presented graphically on a number line or using interval notation or both.

All but one of the rules for solving linear inequalities are the same as solving linear equations. If you divide or multiply an inequality by a negative number, reverse the inequality to obtain an equivalent inequality.

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Compound inequalities involving the word “or” require us to solve each inequality and form the union of each solution set. These are the values that solve at least one of the given inequalities.

Compound inequalities involving the word “and” require the intersection of the solution sets for each inequality. These are the values that solve both or all of the given inequalities.

The general guidelines for solving word problems apply to applications involving inequalities. Be aware of a new list of key words and phrases that indicate a mathematical setup involving inequalities.

## TOPIC EXERCISES

### PART A: LINEAR INEQUALITIES

**Determine whether or not the given value is a solution.**

1.  $5x-1 < -2$ ;  $x = -1$
2.  $-3x+1 > -10$ ;  $x = 1$
3.  $2x-3 < -5$ ;  $x = 1$
4.  $5x-7 < 0$ ;  $x = 2$
5.  $9y-4 \geq 5$ ;  $y = 1$
6.  $-6y+1 \leq 3$ ;  $y = -1$
7.  $12a+3 \leq -2$ ;
8.  $25a-2 \leq -22$ ;
9.  $-10 < 2x-5 < -5$ ;
10.  $3x+8 < -2$  or  $4x-2 > 5$ ;  $x = 2$

**Graph all solutions on a number line and provide the corresponding interval notation.**

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1.  $3x+5>-4$
2.  $2x+1>-1$
3.  $5-6y<-1$
4.  $7-9y>43$
5.  $6-a\leq 6$
6.  $-2a+5>5$
- 7.
- 8.
- 9.
- 10.
11.  $2(3x+14)<-2$
12.  $5(2y+9)>-15$
13.  $5-2(4+3y)\leq 45$
14.  $-12+5(5-2x)<83$
15.  $6(7-2a)+6a\leq 12$
16.  $2a+10(4-a)\geq 8$
17.  $9(2t-3)-3(3t+2)<30$
18.  $-3(t-3)-(4-t)>1$
- 19.
- 20.
21.  $5x-2(x-3)<3(2x-1)$
22.  $3(2x-1)-10>4(3x-2)-5x$
23.  $-3y\geq 3(y+8)+6(y-1)$
24.  $12\leq 4(y-1)+2(2y+1)$
25.  $-2(5t-3)-4>5(-2t+3)$
26.  $-7(3t-4)>2(3-10t)-t$
- 27.
- 28.

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29.  $4(3x+4) \geq 3(6x+5) - 6x$   
 30.  $1 - 4(3x+7) < -3(x+9) - 9x$   
 31.  $6 - 3(2a-1) \leq 4(3-a) + 1$   
 32.  $12 - 5(2a+6) \geq 2(5-4a) - a$

## PART B: COMPOUND INEQUALITIES

**Graph all solutions on a number line and provide the corresponding interval notation.**

43.  $-1 < 2x + 1 < 9$   
 44.  $-4 < 5x + 11 < 16$   
 45.  $-7 \leq 6y - 7 \leq 17$   
 46.  $-7 \leq 3y + 5 \leq 2$   
 47.  
 48.  
 49.  $-4 \leq 11 - 5t < 31$   
 50.  $15 < 12 - t \leq 16$   
 51.  
 52.  
 53.  $5x + 2 < -3$  or  $7x - 6 > 15$   
 54.  $4x + 15 \leq -1$  or  $3x - 8 \geq -11$   
 55.  $8x - 3 \leq 1$  or  $6x - 7 \geq 8$   
 56.  $6x + 1 < -3$  or  $9x - 20 > -5$   
 57.  $8x - 7 < 1$  or  $4x + 11 > 3$   
 58.  $10x - 21 < 9$  or  $7x + 9 \geq 30$   
 59.  $7 + 2y < 5$  or  $20 - 3y > 5$   
 60.  $5 - y < 5$  or  $7 - 8y \leq 23$   
 61.  $15 + 2x < -15$  or  $10 - 3x > 40$

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- 62.
63.  $9-2x \leq 15$  and  $5x-3 \leq 7$
64.  $5-4x > 1$  and  $15+2x \geq 5$
65.  $7y-18 < 17$  and  $2y-15 < 25$
66.  $13y+20 \geq 7$  and  $8+15y > 8$
67.  $5-4x \leq 9$  and  $3x+13 \leq 1$
68.  $17-5x \geq 7$  and  $4x-7 > 1$
69.  $9y+20 \leq 2$  and  $7y+15 \geq 1$
70.  $21-6y \leq 3$  and  $-7+2y \leq -1$
71.  $-21 < 6(x-3) < -9$
72.  $0 \leq 2(2x+5) < 8$
73.  $-15 \leq 5+4(2y-3) < 17$
74.  $5 < 8-3(3-2y) \leq 29$
75.  $5 < 5-3(4+t) < 17$
76.  $-3 \leq 3-2(5+2t) \leq 21$
77.  $-40 < 2(x+5)-(5-x) \leq -10$
78.  $-60 \leq 5(x-4)-2(x+5) \leq 15$
- 79.
- 80.
- 81.
- 82.

## PART C: APPLICATIONS

**Find all numbers that satisfy the given condition.**

83. Three less than twice the sum of a number and 6 is at most 13.
84. Five less than 3 times the sum of a number and 4 is at most 10.
85. Five times the sum of a number and 3 is at least 5.
86. Three times the difference between a number and 2 is at least 12.

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87. The sum of 3 times a number and 8 is between 2 and 20.
88. Eight less than twice a number is between  $-20$  and  $-8$ .
89. Four subtracted from three times some number is between  $-4$  and 14.
90. Nine subtracted from 5 times some number is between 1 and 11.

**Set up an algebraic inequality and then solve.**

83. With a golf club membership, costing \$120 per month, each round of golf costs only \$35.00. How many rounds of golf can a member play if he wishes to keep his costs \$270 per month at most?
84. A rental truck costs \$95 per day plus \$0.65 per mile driven. How many miles can be driven on a one-day rental to keep the cost at most \$120?
85. Mark earned 6, 7, and 10 points out of 10 on the first three quizzes. What must he score on the fourth quiz to average at least 8?
86. Joe earned scores of 78, 82, 88 and 70 on his first four algebra exams. What must he score on the fifth exam to average at least 80?
87. A gymnast scored 13.2, 13.0, 14.3, 13.8, and 14.6 on the first five events. What must he score on the sixth event to average at least 14.0?
88. A dancer scored 7.5 and 8.2 from the first two judges. What must her score from the third judge come in as if she is to average 8.4 or higher?
89. If two times an angle is between 180 degrees and 270 degrees, then what are the bounds of the original angle?
90. The perimeter of a square must be between 120 inches and 460 inches. Find the length of all possible sides that satisfy this condition.
91. A computer is set to shut down if the temperature exceeds  $45^{\circ}\text{C}$ . Give an equivalent statement using degrees Fahrenheit. Hint:
92. A certain antifreeze is effective for a temperature range of  $-35^{\circ}\text{C}$  to  $120^{\circ}\text{C}$ . Find the equivalent range in degrees Fahrenheit.

**PART D: DISCUSSION BOARD**

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101. Often students reverse the inequality when solving  $5x+2<-18$ ? Why do you think this is a common error? Explain to a beginning algebra student why we do not.
102. Conduct a web search for “solving linear inequalities.” Share a link to website or video tutorial that you think is helpful.
103. Write your own 5 key takeaways for this entire chapter. What did you find to be review and what did you find to be new? Share your thoughts on the discussion board.

## ANSWERS

1. Yes

2.

3. No

4.

5. Yes

6.

7. No

8.

9. Yes

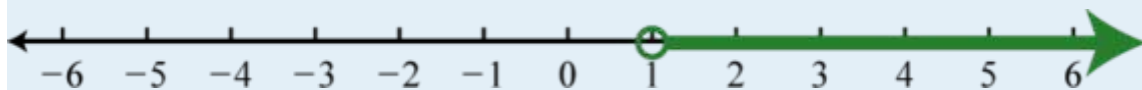
10.

11.  $(-3, \infty)$ ;



12.

13.  $(1, \infty)$ ;



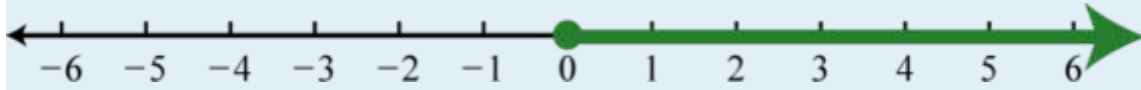
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14.

15.  $[0, \infty);$



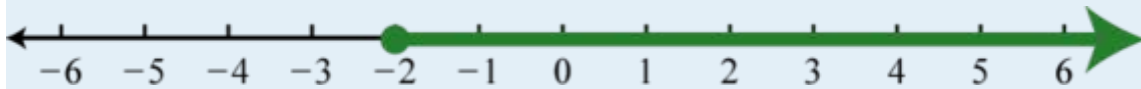
16.

17.  $(-\infty, 3];$



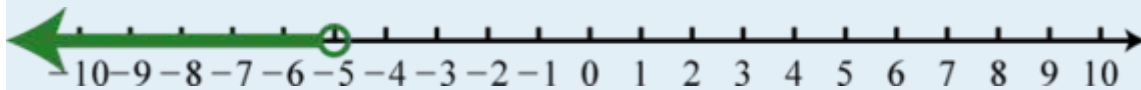
18.

19.  $[-2, \infty);$



20.

21.  $(-\infty, -5);$



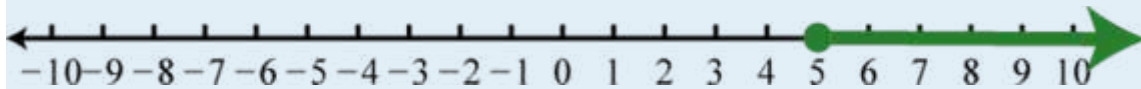
22.

23.  $[-8, \infty);$



24.

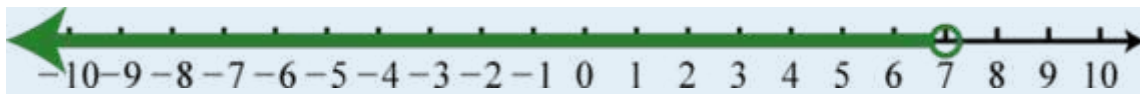
25.  $[5, \infty);$



26.

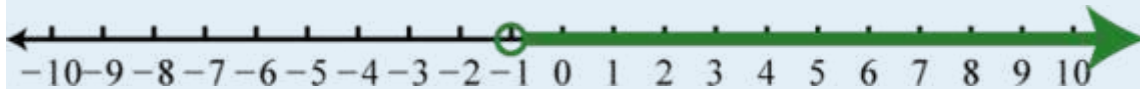
27.  $(-\infty, 7);$





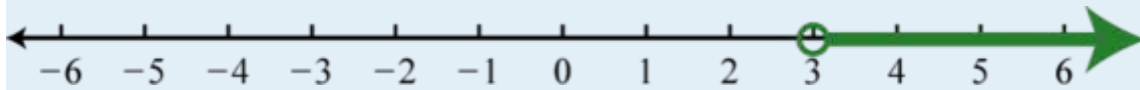
28.

29.  $(-1, \infty)$ ;



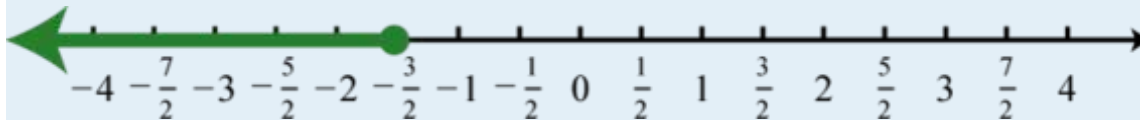
30.

31.  $(3, \infty)$ ;



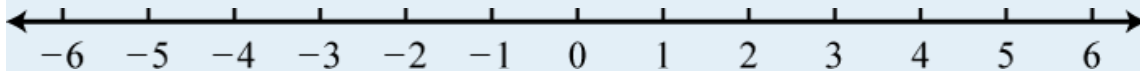
32.

33.  $2]$ ;



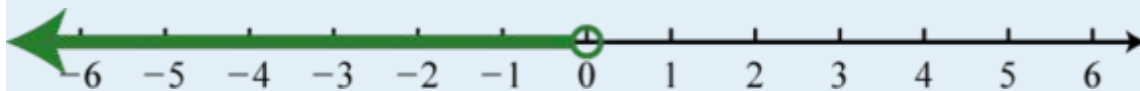
34.

35.  $\emptyset$ ;



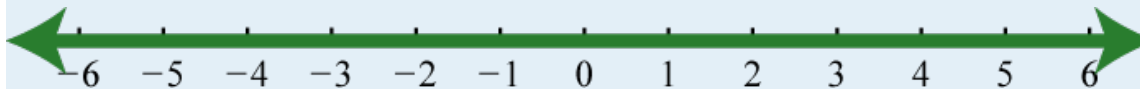
36.

37.  $(-\infty, 0)$ ;



38.

39.  $\mathbb{R}$ ;





40.

41.  $[-2, \infty)$ ;



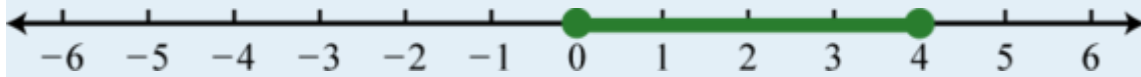
42.

43.  $(-1, 4)$ ;



44.

45.  $[0, 4]$ ;



46.

47.  $(-5, 5]$ ;



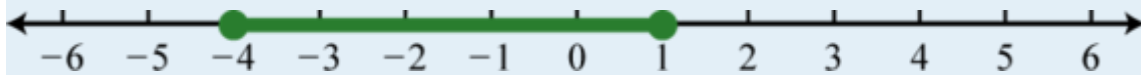
48.

49.  $(-4, 3]$ ;



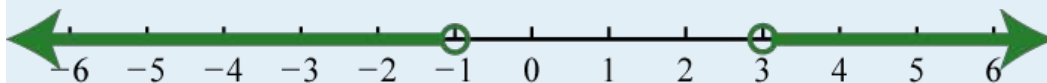
50.

51.  $[-4, 1]$ ;



52.

53.  $(-\infty, -1) \cup (3, \infty)$ ;



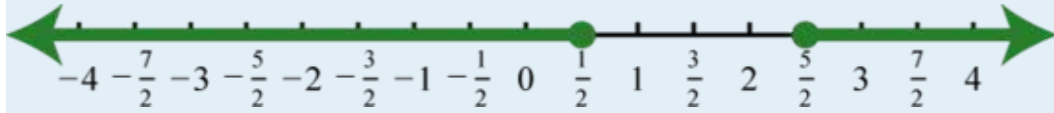
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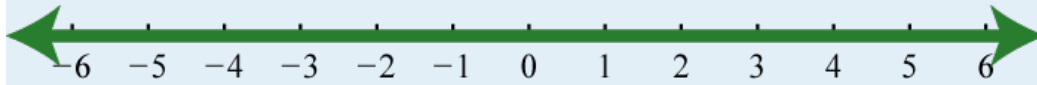
54.

55.



56.

57.  $\mathbb{R}$ ;



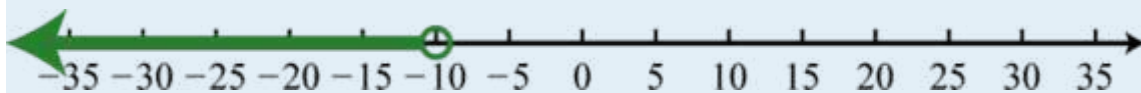
58.

59.  $(-\infty, 5)$ ;



60.

61.  $(-\infty, -10)$ ;



62.

63.  $[-3, 2]$ ;



64.

65.  $(-\infty, 5)$ ;



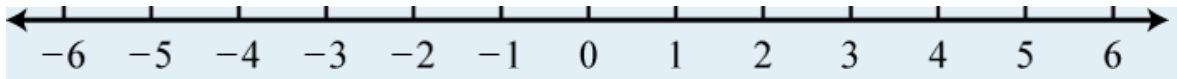
66.

67.  $\emptyset$ ;

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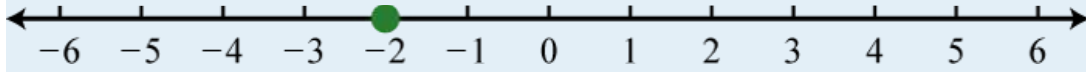
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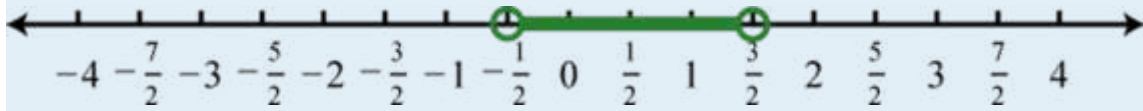
68.

69.  $-2$ ;



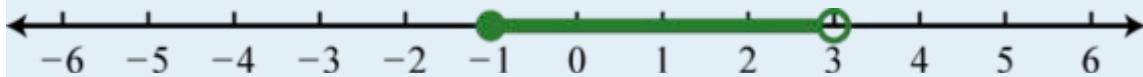
70.

71.



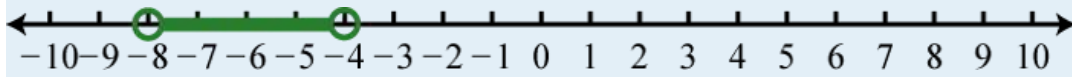
72.

73.  $[-1,3)$ ;



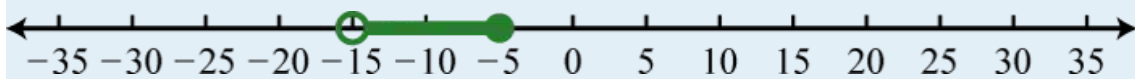
74.

75.  $(-8,-4)$ ;



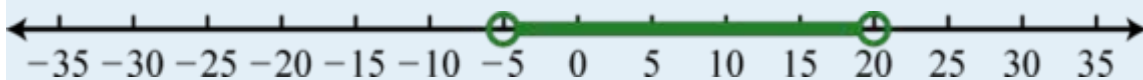
76.

77.  $(-15,-5]$ ;



78.

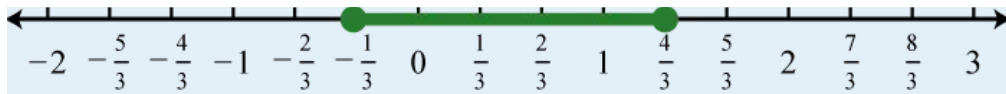
79.  $(-5,20)$ ;



80.

81. ;





82.

83.  $(-\infty, 2]$

84.

85.  $[-2, \infty)$

86.

87.  $(-2, 4)$

88.

89.  $(0, 6)$

90.

91. Members may play 4 rounds or fewer.

92.

93. Mark must earn at least 9 points on the fourth quiz.

94.

95. He must score a 15.1 on the sixth event.

96.

97. The angle is between 90 degrees and 135 degrees.

98.

99. The computer will shut down when the temperature exceeds  $113^\circ\text{F}$ .

100.

101. Answer may vary

102.

103. Answer may vary

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## 1.9 Review Exercises and Sample Exam

### REVIEW EXERCISES

#### REVIEW OF REAL NUMBERS AND ABSOLUTE VALUE

Reduce to lowest terms.

- 1.
- 2.
- 3.
- 4.

Simplify.

- 1.
- 2.
3.  $-(-(-a))$
4.  $-(-(-(-a)))$

Graph the solution set and give the interval notation equivalent.

1.  $x \geq -10$
2.  $x < 0$
3.  $-8 \leq x < 0$

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4.  $-10 < x \leq 4$
5.  $x < 3$  and  $x \geq -1$
6.  $x < 0$  and  $x > 1$
7.  $x < -2$  or  $x > -6$
8.  $x \leq -1$  or  $x > 3$

**Determine the inequality that corresponds to the set expressed using interval notation.**

1.  $[-8, \infty)$
2.  $(-\infty, -7)$
3.  $[12, 32]$
4.  $[-10, 0)$
5.  $(-\infty, 1] \cup (5, \infty)$
6.  $(-\infty, -10) \cup (-5, \infty)$
7.  $(-4, \infty)$
8.  $(-\infty, 0)$

**Simplify.**

- 1.
- 2.
3.  $-(-|-4|)$
4.  $-(-(-|-3|))$

**Determine the values represented by  $a$ .**

1.  $|a| = 6$



2.  $|a|=1$

3.  $|a|=-5$

4.  $|a|=a$

## OPERATIONS WITH REAL NUMBERS

Perform the operations.

33.

34.

35.

36.

37.

38.

39.

40.

41.

42.

43.

44.

45.

46.

47.

48.

49.

50.

## SQUARE AND CUBE ROOTS OF REAL NUMBERS

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**Simplify.**

51.

52.

53.

54.

55.

56.

57.

58.

59.

60.

61.

62.

**Use a calculator to approximate the following to the nearest thousandth.**

51.

52.

53.

54.

55. Find the length of the diagonal of a square with sides measuring 8 centimeters.

56. Find the length of the diagonal of a rectangle with sides measuring 6 centimeters and 12 centimeters.

**ALGEBRAIC EXPRESSIONS AND FORMULAS**

**Multiply.**

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- 69.
- 70.
- 71.
- 72.

**Combine like terms.**

- 69.
- 70.
- 71.
- 72.

**Simplify.**

- 69.
- 70.
- 71.
- 72.

**Evaluate.**

- 69. where
- 70.  $x^2 - x - 1$  where
- 71.  $a^4 - b^4$  where  $a = -3$  and  $b = -1$
- 72.  $a^2 - 3ab + 5b^2$  where  $a = 4$  and  $b = -2$
- 73.  $(2x + 1)(x - 3)$  where  $x = -3$
- 74.  $(3x + 1)(x + 5)$  where  $x = -5$
- 75. where  $a = 2$ ,  $b = -4$ , and  $c = -1$
- 76. where  $a = 3$ ,  $b = -6$ , and  $c = -2$



77. where and  $h=5$
78. where
79. What is the simple interest earned on a 4 year investment of \$4,500 at an annual interest rate of %?
80. James traveled at an average speed of 48 miles per hour for hours. How far did he travel?
81. The period of a pendulum  $T$  in seconds is given by the formula where  $L$  represents its length in feet. Approximate the period of a pendulum with length 2 feet. Round off to the nearest tenth of a foot.
82. The average distance  $d$ , in miles, a person can see an object is given by the formula where  $h$  represents the person's height above the ground, measured in feet. What average distance can a person see an object from a height of 10 feet? Round off to the nearest tenth of a mile.

## RULES OF EXPONENTS AND SCIENTIFIC NOTATION

### Multiply.

- 95.
- 96.
- 97.
- 98.
- 99.
- 100.
- 101.
- 102.
- 103.
- 104.
- 105.
- 106.

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**Perform the operations.**

95.  $(4.3 \times 10^{22})(3.1 \times 10^{-8})$

96.  $(6.8 \times 10^{-33})(1.6 \times 10^7)$

97.

98.

99. The value of a new tablet computer in dollars can be estimated using the formula  $v = 450(t+1)^{-1}$  where  $t$  represents the number of years after it is purchased. Use the formula to estimate the value of the tablet computer  $t$  years after it was purchased.

100. The speed of light is approximately  $6.7 \times 10^8$  miles per hour. Express this speed in miles per minute and determine the distance light travels in 4 minutes.

**POLYNOMIALS AND THEIR OPERATIONS****Simplify.**

113.  $(x^2 + 3x - 5) - (2x^2 + 5x - 7)$

114.  $(6x^2 - 3x + 5) + (9x^2 + 3x - 4)$

115.  $(a^2b^2 - ab + 6) - (ab + 9) + (a^2b^2 - 10)$

116.  $(x^2 - 2y^2) - (x^2 + 3xy - y^2) - (3xy + y^2)$

117.

118.

119.  $(2x + 5)(x - 4)$

120.  $(3x - 2)(x^2 - 5x + 2)$

121.  $(x^2 - 2x + 5)(2x^2 - x + 4)$

122.  $(a^2 + b^2)(a^2 - b^2)$

123.  $(2a + b)(4a^2 - 2ab + b^2)$

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124.  $(2x-3)^2$

125.  $(3x-1)^3$

126.  $(2x+3)^4$

127.  $(x^2-y^2)^2$

128.  $(x^2y^2+1)^2$

129.

130.

131.

132.

133.

134.

135.

136.

## SOLVING LINEAR EQUATIONS

**Solve.**

137.  $6x-8=2$

138.  $12x-5=3$

139.

140.

141.

142.

143.  $3a-5-2a=4a-6$

144.  $8-5y+2=4-7y$

145.  $5x-6-8x=1-3x$

146.  $17-6x-10=5x+7-11x$

147.  $5(3x+3)-(10x-4)=4$

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148.  $6-2(3x-1)=-4(1-3x)$

149.  $9-3(2x+3)+6x=0$

150.  $-5(x+2)-(4-5x)=1$

151.

152.

153. Solve for  $s$ :  $A=\pi r^2+\pi r s$

154. Solve for  $x$ :  $y=mx+b$

155. A larger integer is 3 more than twice another. If their sum divided by 2 is 9, find the integers.

156. The sum of three consecutive odd integers is 171. Find the integers.

157. The length of a rectangle is 3 meters less than twice its width. If the perimeter measures 66 meters, find the length and width.

158. How long will it take \$500 to earn \$124 in simple interest earning 6.2% annual interest?

159. It took Sally hours to drive the 147 miles home from her grandmother's house. What was her average speed?

160. Jeannine invested her bonus of \$8,300 in two accounts. One account earned % simple interest and the other earned % simple interest. If her total interest for one year was \$341.75, how much did she invest in each account?

## SOLVING LINEAR INEQUALITIES WITH ONE VARIABLE

**Solve. Graph all solutions on a number line and provide the corresponding interval notation.**

161.  $5x-7<18$

162.  $2x-1>2$

163.  $9-x\leq 3$

164.  $3-7x\geq 10$

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165.  $61-3(x+3)>13$
166.  $7-3(2x-1)\geq 6$
- 167.
- 168.
- 169.
- 170.
171.  $-4\leq 3x+5<11$
172.  $5<2x+15\leq 13$
173.  $-1<4(x+1)-1<9$
174.  $0\leq 3(2x-3)+1\leq 10$
- 175.
- 176.
177.  $2x+3<13$  and  $4x-1>10$
178.  $3x-1\leq 8$  and  $2x+5\geq 23$
179.  $5x-3<-2$  or  $5x-3>2$
180.  $1-3x\leq -1$  or  $1-3x\geq 1$
181.  $5x+6<6$  or  $9x-2>-11$
182.  $2(3x-1)<-16$  or  $3(1-2x)<-15$
183. Jerry scored 90, 85, 92, and 76 on the first four algebra exams. What must he score on the fifth exam so that his average is at least 80?
184. If 6 degrees less than 3 times an angle is between 90 degrees and 180 degrees, then what are the bounds of the original angle?

## ANSWERS

- 1.
- 2.
- 3.



4.

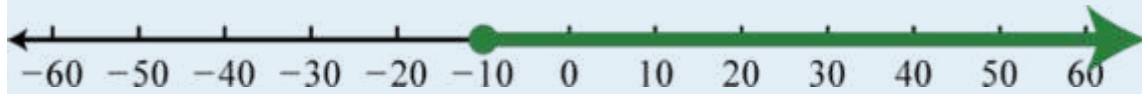
5.

6.

7.  $-a$

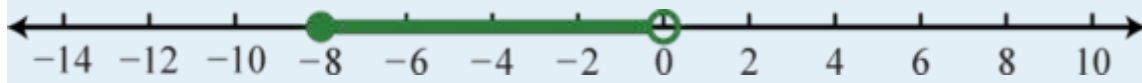
8.

9.  $[-10, \infty)$



10.

11.  $[-8, 0)$



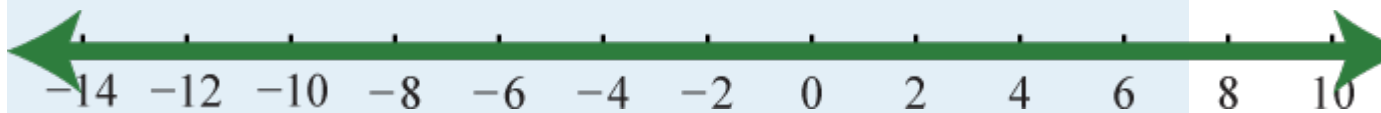
12.

13.  $[-1, 3)$



14.

15.  $\mathbb{R}$



16.

17.  $x \geq -8$

18.

19.  $12 \leq x \leq 32$

20.

21.  $x \leq 1$  or  $x > 5$

22.



23.  $x > -4$

24.

25.

26.

27. 4

28.

29.  $a = \pm 6$

30.

31.  $\emptyset$

32.

33.

34.

35. -4

36.

37.

38.

39. -15

40.

41. -6

42.

43. -24

44.

45.

46.

47. -50

48.

49. 14

50.

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- 51.
- 52.
53. 0
- 54.
- 55.
- 56.
- 57.
- 58.
- 59.
- 60.
- 61.
- 62.
63. 3.464
- 64.
65. 2.621
- 66.
67. centimeters
- 68.
69.  $6x^2+2x-4$
- 70.
71.  $-3a^2+15ab+6b^2$
- 72.
73.  $x^2y-10xy^2$
- 74.
75.  $5ab+1$
- 76.
77.  $-x^2+16x+3$



78.

79.  $-a^2 - ab + 2b^2$

80.

81.

82.

83. 80

84.

85. 30

86.

87.

88.

89.  $60\pi$

90.

91. \$855

92.

93. 1.6 seconds

94.

95.  $x^7$

96.

97.  $-21x^6y^3z^4$

98.

99.

100.

101.

102.

103.

104.

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105.

106.

107.  $1.333 \times 10^{15}$

108.

109.  $7 \times 10^{-23}$

110.

111.  $\$128.57$

112.

113.  $-x^2 - 2x + 2$

114.

115.  $2a^2b^2 - 2ab - 13$

116.

117.  $-12x^2 - 6x + 3$

118.

119.  $2x^2 - 3x - 20$

120.

121.  $2x^4 - 5x^3 + 16x^2 - 13x + 20$

122.

123.  $8a^3 + b^3$

124.

125.  $27x^3 - 27x^2 + 9x - 1$

126.

127.  $x^4 - 2x^2y^2 + y^4$

128.

129.  $9a + 27b - 3$

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- 130.
131.  $x^2 - 3x + 2$
- 132.
- 133.
- 134.
135.  $a^2 - a + 2$
- 136.
- 137.
- 138.
- 139.
- 140.
- 141.
- 142.
- 143.
- 144.
145.  $\emptyset$
- 146.
147.  $-3$
- 148.
149.  $\mathbb{R}$
- 150.
151.  $-$
- 152.
- 153.
- 154.
155.  $5, 13$
- 156.



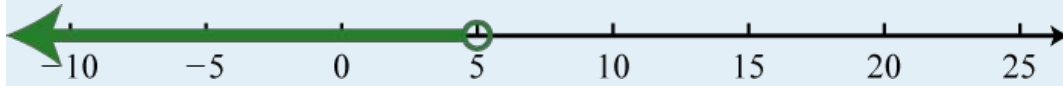
157. Length: 21 meters; Width: 12 meters

158.

159. 42 miles per hour

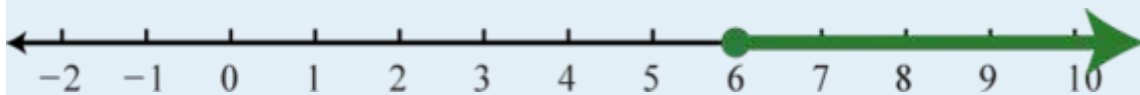
160.

161.  $(-\infty, 5)$ ;



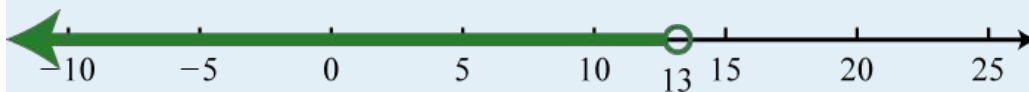
162.

163.  $[6, \infty)$ ;



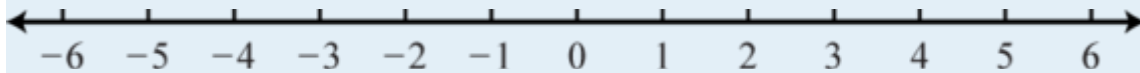
164.

165.  $(-\infty, 13)$ ;



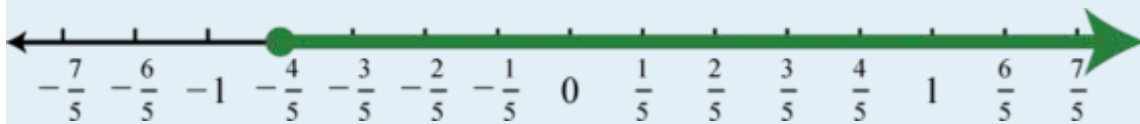
166.

167.  $\emptyset$ ;



168.

169.  $[-, \infty)$ ;



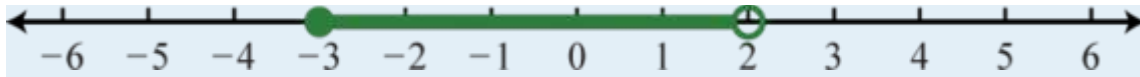
170.

171.  $[-3, 2)$ ;

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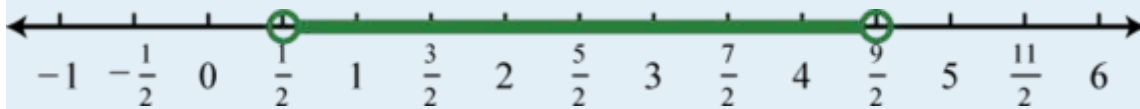
172.

173.  $(-1, )$ ;



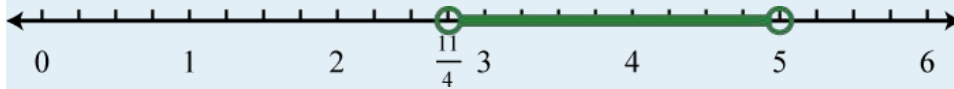
174.

175.  $(, )$ ;



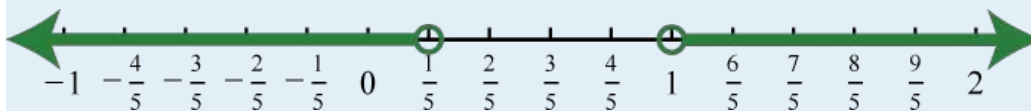
176.

177.  $(, 5)$ ;



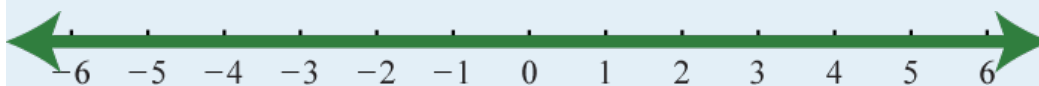
178.

179.  $(-\infty, ) \cup (1, \infty)$ ;



180.

181.  $\mathbb{R}$ ;



182.

183. Jerry must score at least 57 on the fifth exam.

184.



## SAMPLE EXAM

**Simplify.**

- 1.
- 2.
- 3.
- 4.
5. Find the diagonal of a square with sides measuring 6 centimeters.

**Simplify.**

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.

**Solve.**

- 1.
2. .
3.  $12 - 5(3x - 1) = 2(4x + 3)$
- 4.
5. Solve for  $y$ :  $ax + by = c$

**Solve. Graph the solutions on a number line and give the corresponding interval notation.**

1.  $2(3x - 5) - (7x - 3) \geq 0$

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2.  $2(4x-1)-4(5+2x)<-10$
- 3.
4.  $3x-7>14$  or  $3x-7<-14$

**Use algebra to solve the following.**

1. Degrees Fahrenheit  $F$  is given by the formula where  $C$  represents degrees Celsius. What is the Fahrenheit equivalent to  $35^\circ$  Celsius?
2. The length of a rectangle is 5 inches less than its width. If the perimeter is 134 inches, find the length and width of the rectangle.
3. Melanie invested 4,500 in two separate accounts. She invested part in a CD that earned 3.2% simple interest and the rest in a savings account that earned 2.8% simple interest. If the total simple interest for one year was \$138.80, how much did she invest in each account?
4. A rental car costs \$45.00 per day plus \$0.48 per mile driven. If the total cost of a one-day rental is to be at most \$105, how many miles can be driven?

## ANSWERS

1. 38
- 2.
- 3.
- 4.
5. centimeters
- 6.
- 7.
- 8.
- 9.
- 10.

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11.

12.

13.

14.

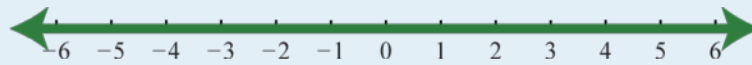
15.

16.

17.

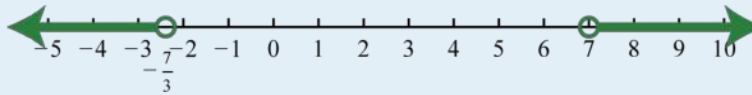
18.

19.  $\mathbb{R}$ ;



20.

21.  $(-\infty, -\frac{7}{3}) \cup (7, \infty)$ ;



22.

23. Length: 31 inches; width: 36 inches

24.

25. The car can be driven at most 125 miles.

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## Chapter 2

# Graphing Functions and Inequalities

This chapter doesn't have an introduction. Please click "Next" to continue to the first section, or select from the Table of Contents to the left.

### 2.1 Relations, Graphs, and Functions

#### LEARNING OBJECTIVES

1. State the domain and range of a relation.
2. Identify a function.
3. Use function notation.

#### Graphs, Relations, Domain, and Range

The **rectangular coordinate system** consists of two real number lines that intersect at a right angle. The horizontal number line is called the **x-axis**, and the vertical number line is called the **y-axis**. These two number lines define a flat surface called a **plane**, and each point on this plane is associated with an **ordered pair** of real numbers  $(x, y)$ . The first number is called the  $x$ -coordinate, and the second number is called the  $y$ -coordinate. The intersection of the two axes is known as the **origin**, which corresponds to the point  $(0, 0)$ .

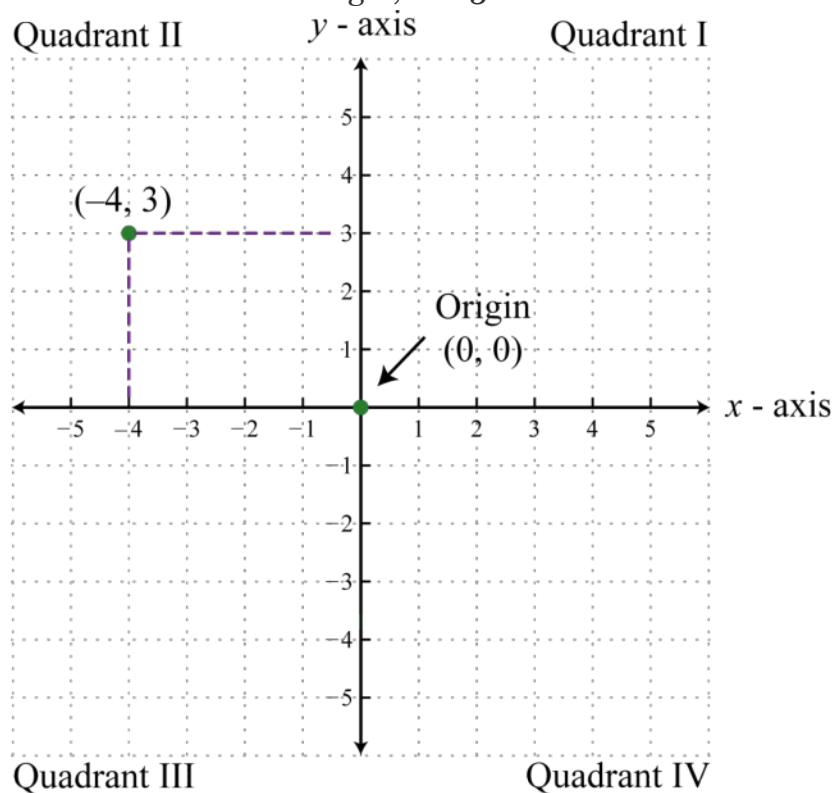
The  $x$ - and  $y$ -axes break the plane into four regions called **quadrants**, named using roman numerals I, II, III, and IV, as pictured. The ordered pair  $(x, y)$  represents the

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position of points relative to the origin. For example, the ordered pair  $(-4, 3)$  represents the position 4 units to the left of the origin, and 3 units above in the second quadrant.



This system is often called the **Cartesian coordinate system**, named after the French mathematician René Descartes (1596–1650).

*Figure 2.1*





*Rene Descartes Wikipedia*

---

Next, we define a **relation** as any set of ordered pairs. In the context of algebra, the relations of interest are sets of ordered pairs  $(x, y)$  in the rectangular coordinate plane. Typically, the coordinates are related by a rule expressed using an algebraic equation. For example, both the algebraic equations  $y = |x| - 2$  and  $x = |y| + 1$  define relationships between  $x$  and  $y$ . Following are some integers that satisfy both equations:

$y =  x  - 2$		$x =  y  + 1$	
$x$	$y$	$x$	$y$
-3	1	4	-3
-2	0	3	-2
-1	-1	2	-1
0	-2	1	0
1	-1	2	1
2	0	3	2
3	1	4	3

Here two relations consisting of seven ordered pair solutions are obtained:

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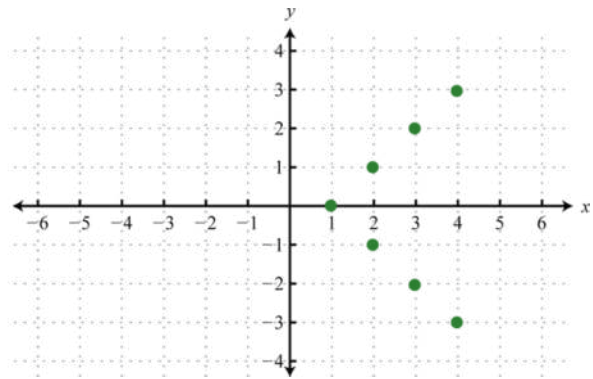
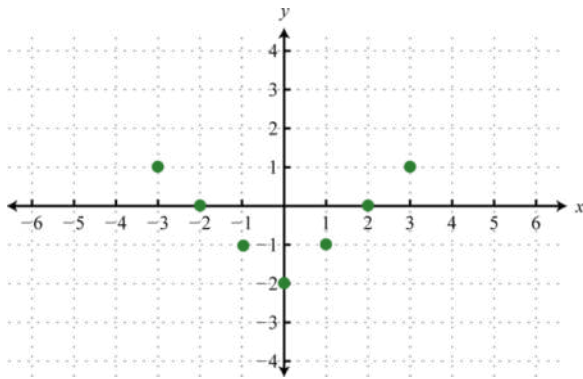
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$y = |x| - 2$  has solutions  $\{(-3, 1), (-2, 0), (-1, -1), (0, -2), (1, -1), (2, 0), (3, 1)\}$   
and

$x = |y| + 1$  has solutions  $\{(4, -3), (3, -2), (2, -1), (1, 0), (2, 1), (3, 2), (4, 3)\}$

We can visually display any relation of this type on a coordinate plane by plotting the points.

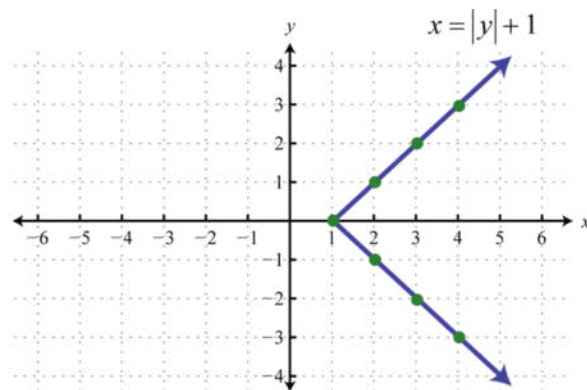
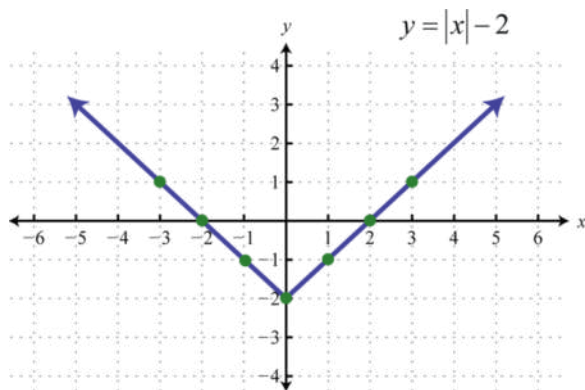


The solution sets of each equation will form a relation consisting of infinitely many ordered pairs. We can use the given ordered pair solutions to estimate all of the other ordered pairs by drawing a line through the given points. Here we put an arrow on the ends of our lines to indicate that this set of ordered pairs continues without bounds.

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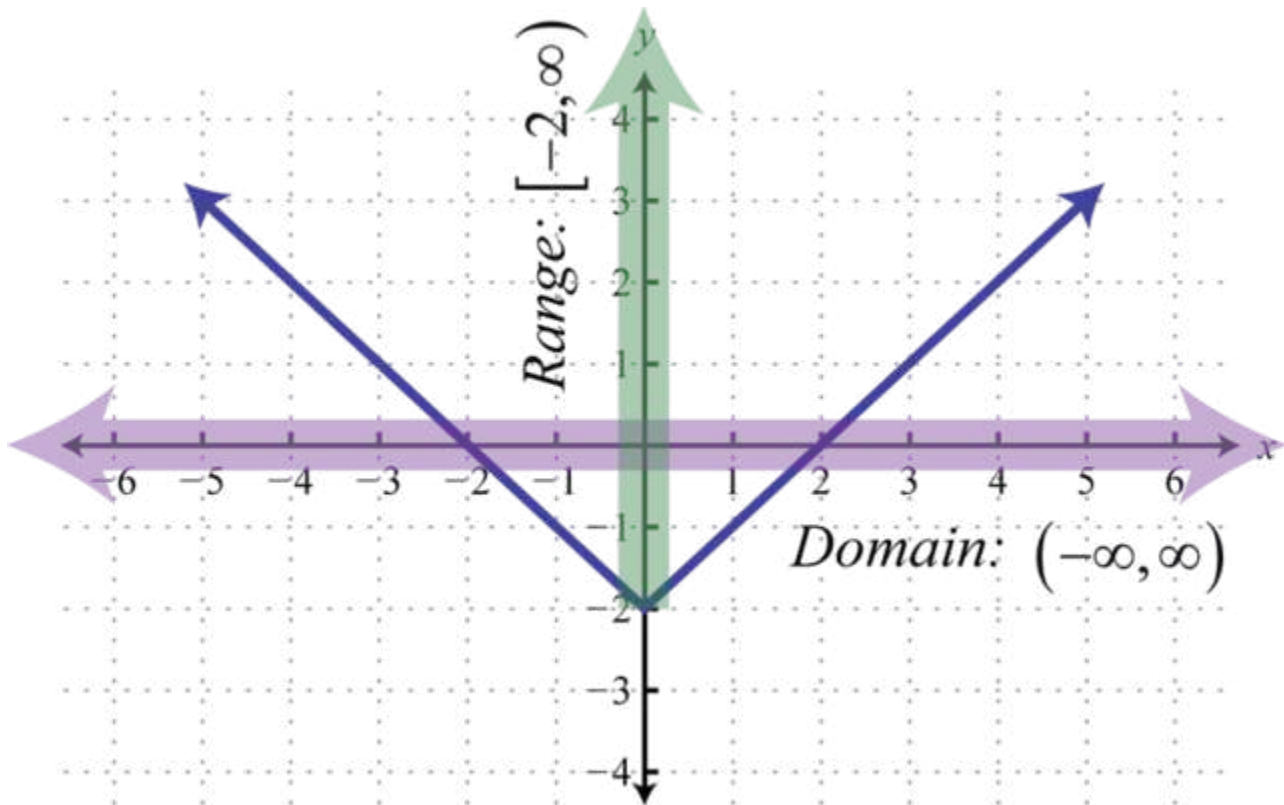




The representation of a relation on a rectangular coordinate plane, as illustrated above, is called a graph. Any curve graphed on a rectangular coordinate plane represents a set of ordered pairs and thus defines a relation.

The set consisting of all of the first components of a relation, in this case the  $x$ -values, is called the **domain**. And the set consisting of all second components of a relation, in this case the  $y$ -values, is called the **range** (or **codomain**). Often, we can determine the domain and range of a relation if we are given its graph.





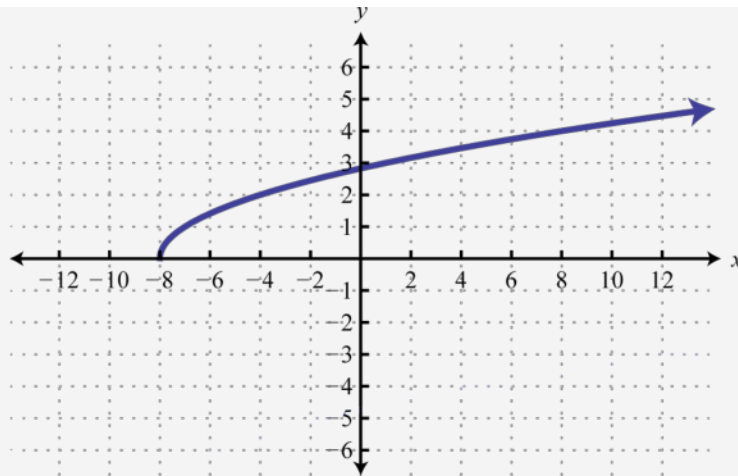
Here we can see that the graph of  $y = |x| - 2$  has a domain consisting of all real numbers,  $\mathbb{R} = (-\infty, \infty)$ , and a range of all  $y$ -values greater than or equal to  $-2$ ,  $[-2, \infty)$ . The domain of the graph of  $x = |y| + 1$  consists of all  $x$ -values greater than or equal to  $1$ ,  $[1, \infty)$ , and the range consists of all real numbers,  $\mathbb{R} = (-\infty, \infty)$ .

### Example 1

Determine the domain and range of the following relation:

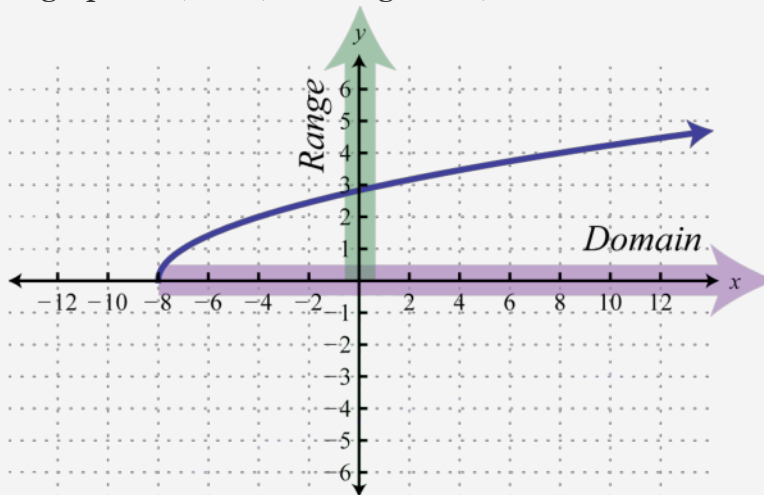
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Solution:

The minimum  $x$ -value represented on the graph is  $-8$  all others are larger. Therefore, the domain consists of all  $x$ -values in the interval  $[-8, \infty)$ . The minimum  $y$ -value represented on the graph is  $0$ ; thus, the range is  $[0, \infty)$ .



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Answer: Domain:  $[-8, \infty)$ ; range:  $[0, \infty)$

## Functions

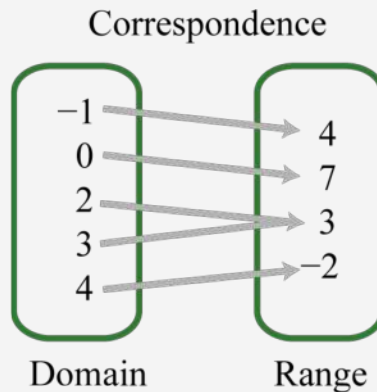
Of special interest are relations where every  $x$ -value corresponds to exactly one  $y$ -value. A relation with this property is called a **function**.

### Example 2

Determine the domain and range of the following relation and state whether it is a function or not:  $\{(-1, 4), (0, 7), (2, 3), (3, 3), (4, -2)\}$

Solution:

Here we separate the domain ( $x$ -values), and the range ( $y$ -values), and depict the correspondence between the values with arrows.



The relation is a function because each  $x$ -value corresponds to exactly one  $y$ -value.

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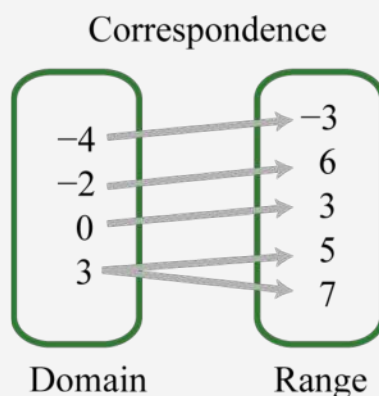


Answer: The domain is  $\{-1, 0, 2, 3, 4\}$  and the range is  $\{-2, 3, 4, 7\}$ . The relation is a function.

### Example 3

Determine the domain and range of the following relation and state whether it is a function or not:  $\{(-4, -3), (-2, 6), (0, 3), (3, 5), (3, 7)\}$

Solution:



The given relation is not a function because the  $x$ -value 3 corresponds to two  $y$ -values. We can also recognize functions as relations where no  $x$ -values are repeated.

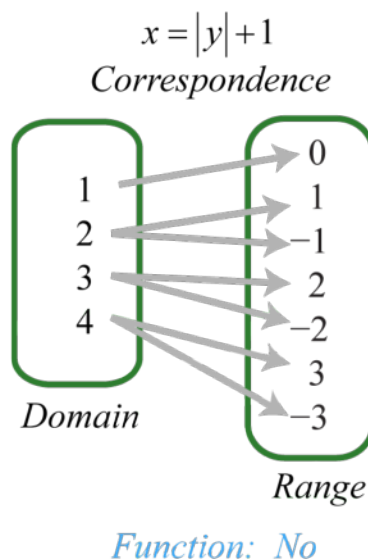
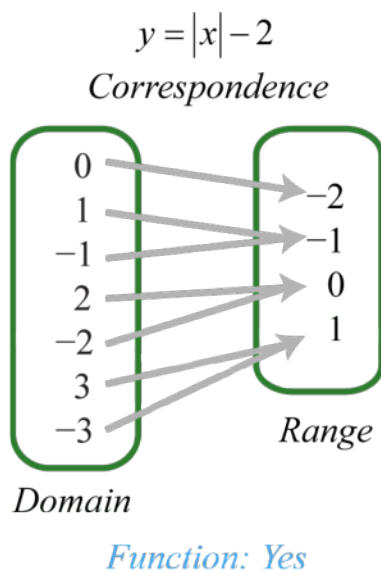
Answer: The domain is  $\{-4, -2, 0, 3\}$  and the range is  $\{-3, 3, 5, 6, 7\}$ . This relation is not a function.

Consider the relations consisting of the seven ordered pair solutions to  $y = |x| - 2$  and  $x = |y| + 1$ . The correspondence between the domain and range of each can be pictured as follows:

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Notice that every element in the domain of the solution set of  $y = |x| - 2$  corresponds to only one element in the range; it is a function. The solutions to  $x = |y| + 1$ , on the other hand, have values in the domain that correspond to two elements in the range. In particular, the  $x$ -value 4 corresponds to two  $y$ -values  $-3$  and  $3$ . Therefore,  $x = |y| + 1$  does not define a function.

We can visually identify functions by their graphs using the **vertical line test**. If any vertical line intersects the graph more than once, then the graph does not represent a function.





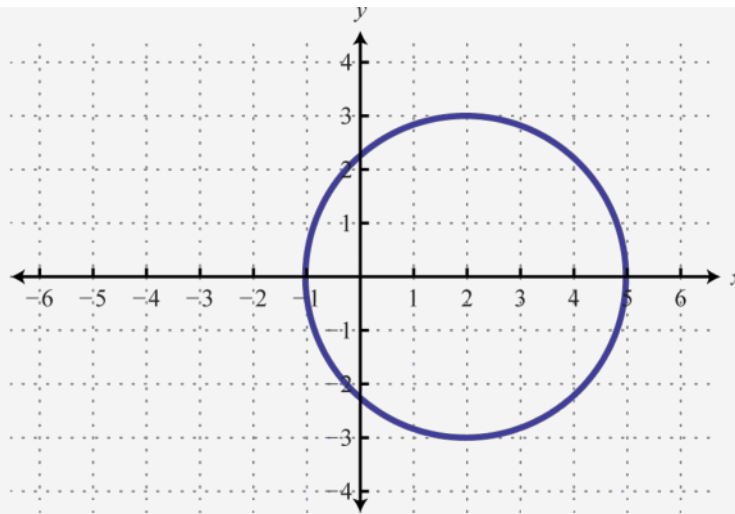
The vertical line represents a value in the domain, and the number of intersections with the graph represent the number of values to which it corresponds. As we can see, any vertical line will intersect the graph of  $y = |x| - 2$  only once; therefore, it is a function. A vertical line can cross the graph of  $x = |y| + 1$  more than once; therefore, it is not a function. As pictured, the  $x$ -value 3 corresponds to more than one  $y$ -value.

## Example 4

Given the graph, state the domain and range and determine whether or not it represents a function:

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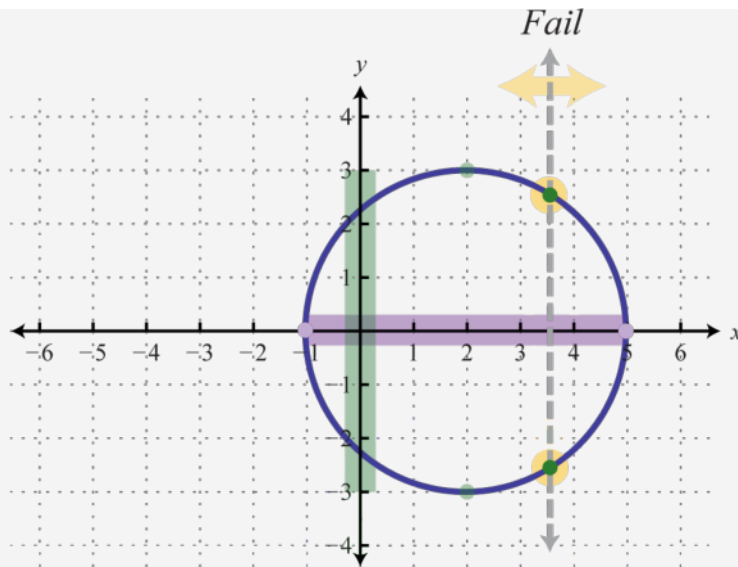
Solution:

From the graph we can see that the minimum  $x$ -value is  $-1$  and the maximum  $x$ -value is  $5$ . Hence, the domain consists of all the real numbers in the set from  $[-1, 5]$ . The maximum  $y$ -value is  $3$  and the minimum is  $-3$ ; hence, the range consists of  $y$ -values in the interval  $[-3, 3]$ .

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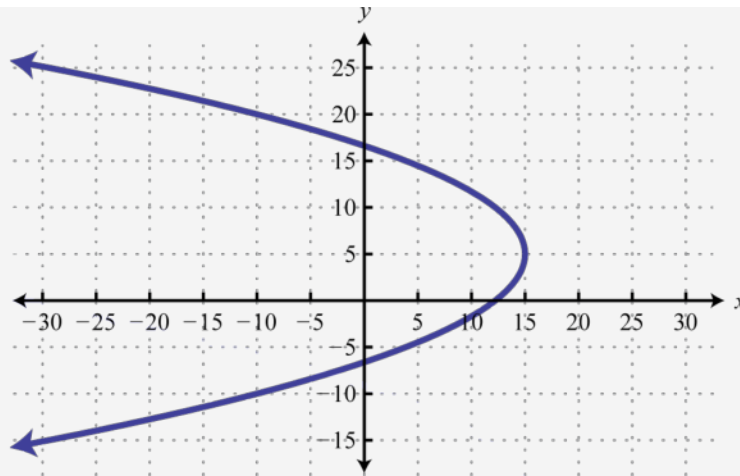
In addition, since we can find a vertical line that intersects the graph more than once, we conclude that the graph is not a function. There are many  $x$ -values in the domain that correspond to two  $y$ -values.

Answer: Domain:  $[-1, 5]$ ; range:  $[-3, 3]$ ; function: no

Try this! Given the graph, determine the domain and range and state whether or not it is a function:

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Answer: Domain:  $(-\infty, 15]$ ; range:  $\mathbb{R}$ ; function: no

## Function Notation

With the definition of a function comes special notation. If we consider each  $x$ -value to be the input that produces exactly one output, then we can use **function notation**:

$$f(x) = y$$

The notation  $f(x)$  reads, “ $f$  of  $x$ ” and should not be confused with multiplication. Algebra frequently involves functions, and so the notation becomes useful when performing common tasks. Here  $f$  is the function name, and  $f(x)$  denotes the value in the range associated with the value  $x$  in the domain. Functions are often named with different letters; some common names for functions are  $f$ ,  $g$ ,  $h$ ,  $C$ , and  $R$ . We have determined

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that the set of solutions to  $y = |x| - 2$  is a function; therefore, using function notation we can write:

It is important to note that  $y$  and  $f(x)$  are used interchangeably. This notation is used as follows:

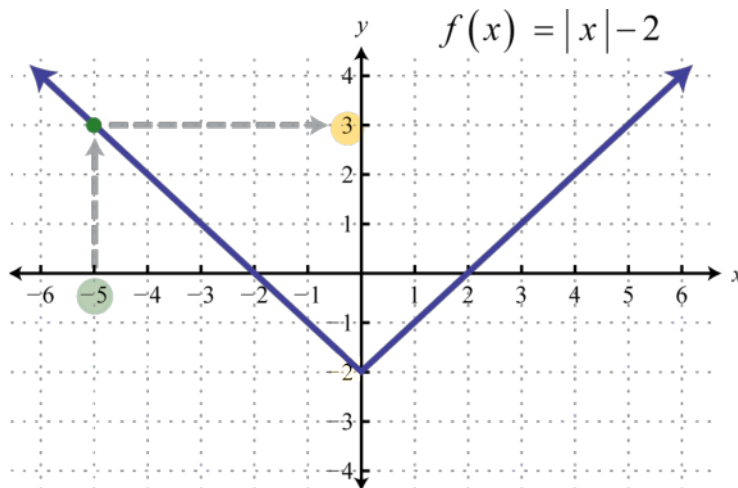
Here the compact notation  $f(-5) = 3$  indicates that where  $x = -5$  (*the input*), the function results in  $y = 3$  (*the output*). In other words, replace the variable with the value given inside the parentheses.

$$\begin{array}{c} \text{Input} \\ \downarrow \\ f(-5) = |-5| - 2 = 3 \\ \uparrow \\ \text{Output} \end{array}$$

Functions are compactly defined by an algebraic equation, such as  $f(x) = |x| - 2$ . Given values for  $x$  in the domain, we can quickly calculate the corresponding values in the range. As we have seen, functions are also expressed using graphs. In this case, we interpret  $f(-5) = 3$  as follows:







Function notation streamlines the task of evaluating. For example, use the function  $h$  defined by to evaluate for  $x$ -values in the set  $\{-2, 0, 7\}$ .

Given any function defined by  $h(x) = y$ , the value  $x$  is called the **argument of the function**. The argument can be any algebraic expression. For example:

## Example 5

Given , find  $g(-2)$ , , and  $g(x + h)$ .

Solution:

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Recall that when evaluating, it is a best practice to begin by replacing the variables with parentheses and then substitute the appropriate values. This helps with the order of operations when simplifying expressions.

Answer:  $g(-2) = 4$ ,  $g(x + h) = x^2 + 2xh + h^2$

At this point, it is important to note that, in general,  $f(x + h) \neq f(x) + f(h)$ . The previous example, where  $f(x) = x^2$ , illustrates this nicely.

## Example 6

Given  $f(x) = x^2 - 2x + 2$ , find  $f(-2)$ ,  $f(0)$ , and

Solution:

Answer:  $f(-2) = 0$ ,  $f(0) = 2$ ,

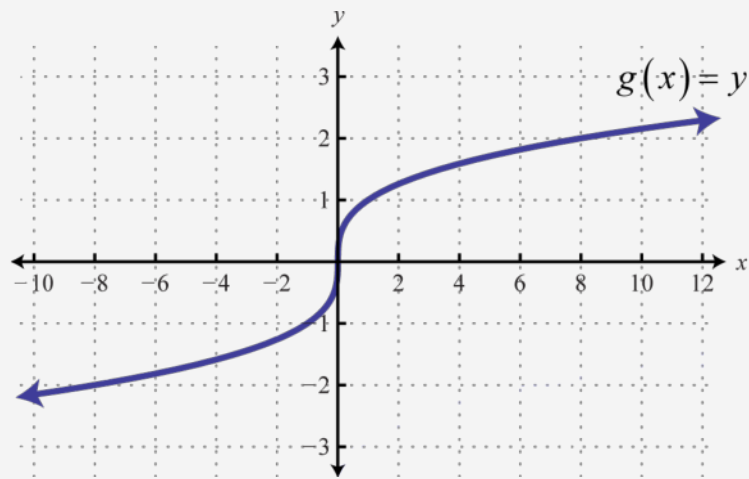
## Example 7

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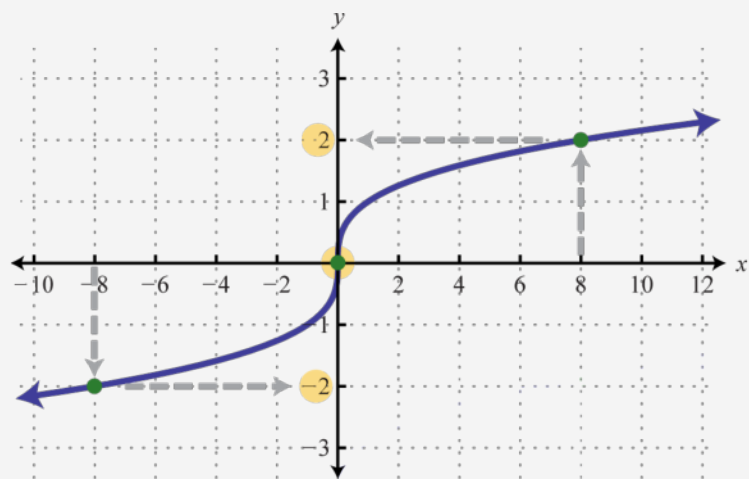


Given the graph of  $g(x)$ , find  $g(-8)$ ,  $g(0)$ , and  $g(8)$ .



Solution:

Use the graph to find the corresponding  $y$ -values where  $x = -8, 0,$  and  $8$ .



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Answer:  $g(-8) = -2$ ,  $g(0) = 0$ ,  $g(8) = 2$

Sometimes the output is given and we are asked to find the input.

## Example 8

Given  $f(x) = 5x + 7$ , find  $x$  where  $f(x) = 27$ .

Solution:

In this example, the output is given and we are asked to find the input.

Substitute  $f(x)$  with 27 and solve.

Therefore,  $f(4) = 27$ . As a check, we can evaluate  $f(4) = 5(4) + 7 = 27$ .

Answer:  $x = 4$

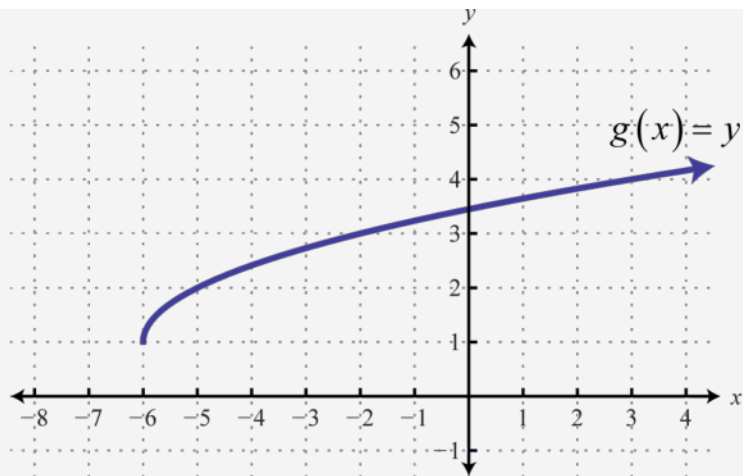
## Example 9

Given the graph of  $g$ , find  $x$  where  $g(x) = 2$ .

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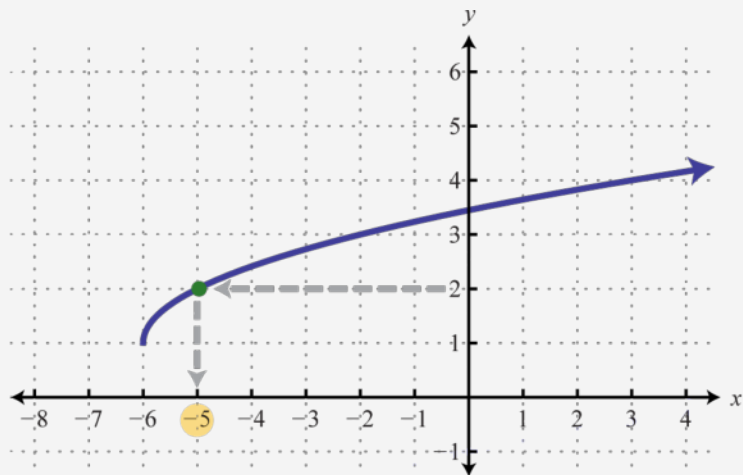
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Solution:

Here we are asked to find the  $x$ -value given a particular  $y$ -value. We begin with 2 on the  $y$ -axis and then read the corresponding  $x$ -value.



We can see that  $g(x) = 2$  where  $x = -5$ ; in other words,  $g(-5) = 2$ .

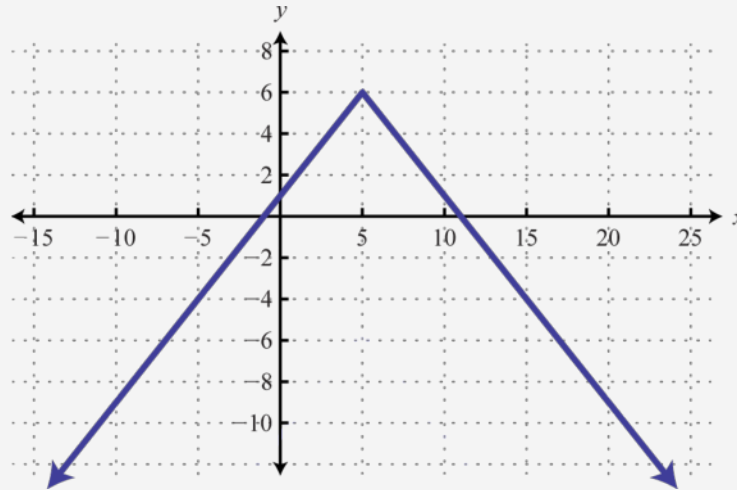
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Answer:  $x = -5$

Try this! Given the graph of  $h$ , find  $x$  where  $h(x) = -4$ .



Answer:  $x = -5$  and  $x = 15$

## KEY TAKEAWAYS

A relation is any set of ordered pairs. However, in this course, we will be working with sets of ordered pairs  $(x, y)$  in the rectangular coordinate system. The set of  $x$ -values defines the domain and the set of  $y$ -values defines the range.

Special relations where every  $x$ -value (input) corresponds to exactly one  $y$ -value (output) are called functions.

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We can easily determine whether or not an equation represents a function by performing the vertical line test on its graph. If any vertical line intersects the graph more than once, then the graph does not represent a function.

If an algebraic equation defines a function, then we can use the notation  $f(x) = y$ . The notation  $f(x)$  is read “ $f$  of  $x$ ” and should not be confused with multiplication. When working with functions, it is important to remember that  $y$  and  $f(x)$  are used interchangeably.

If asked to find  $f(a)$ , we substitute the argument  $a$  in for the variable and then simplify. The argument could be an algebraic expression.

If asked to find  $x$  where  $f(x) = a$ , we set the function equal to  $a$  and then solve for  $x$ .

## TOPIC EXERCISES

### PART A: RELATIONS AND FUNCTIONS

Determine the domain and range and state whether the relation is a function or not.

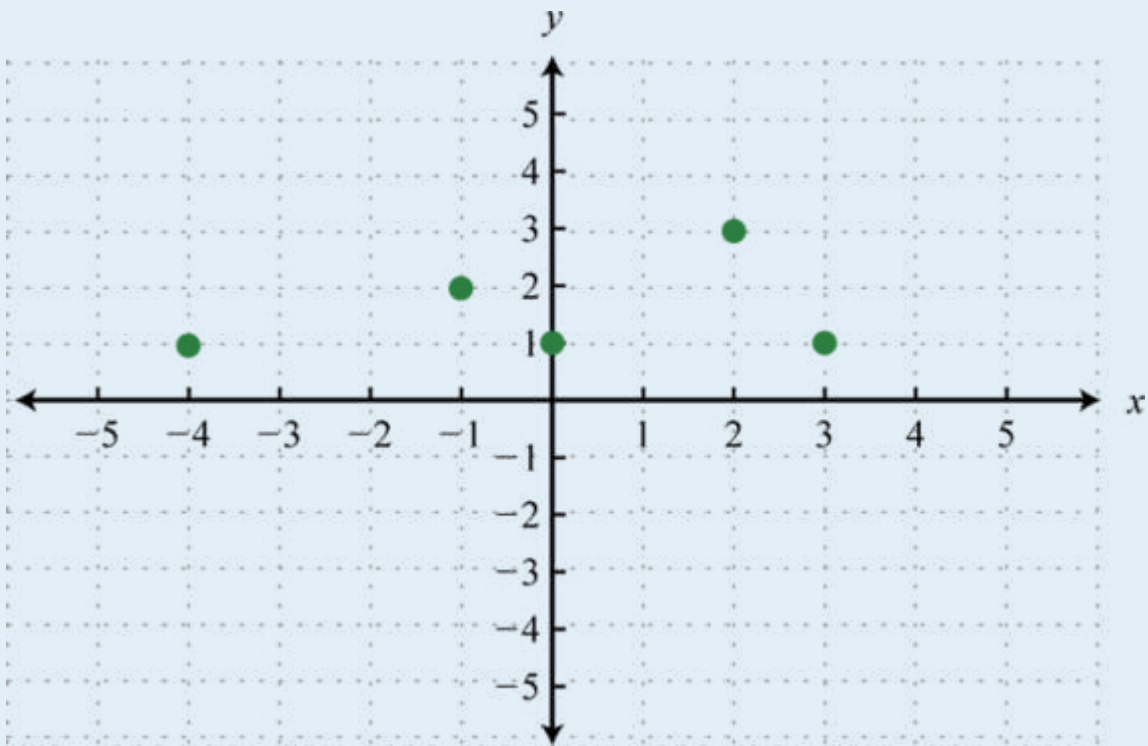
1.  $\{(3, 1), (5, 2), (7, 3), (9, 4), (12, 4)\}$
2.  $\{(2, 0), (4, 3), (6, 6), (8, 6), (10, 9)\}$
3.  $\{(7, 5), (8, 6), (10, 7), (10, 8), (15, 9)\}$
4.  $\{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$
5.  $\{(5, 0), (5, 2), (5, 4), (5, 6), (5, 8)\}$
6.  $\{(-3, 1), (-2, 2), (-1, 3), (0, 4), (0, 5)\}$

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7.

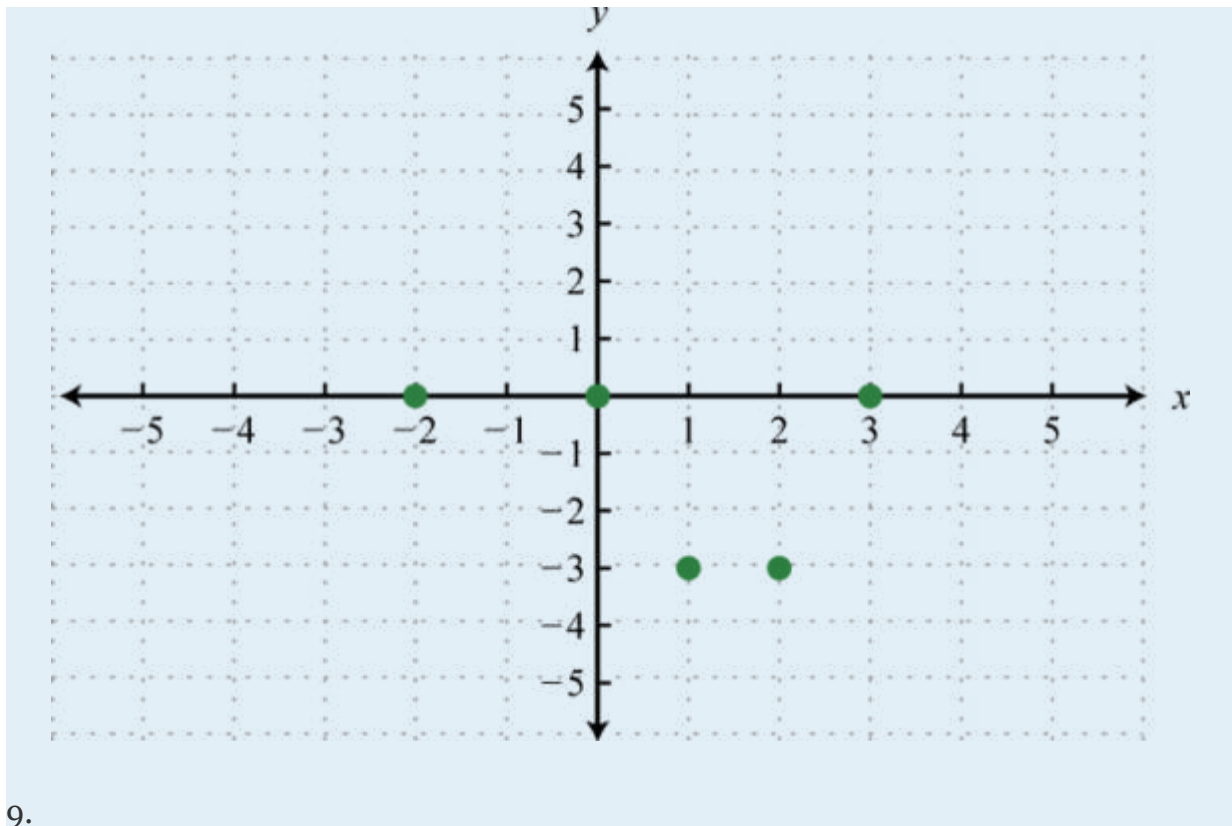


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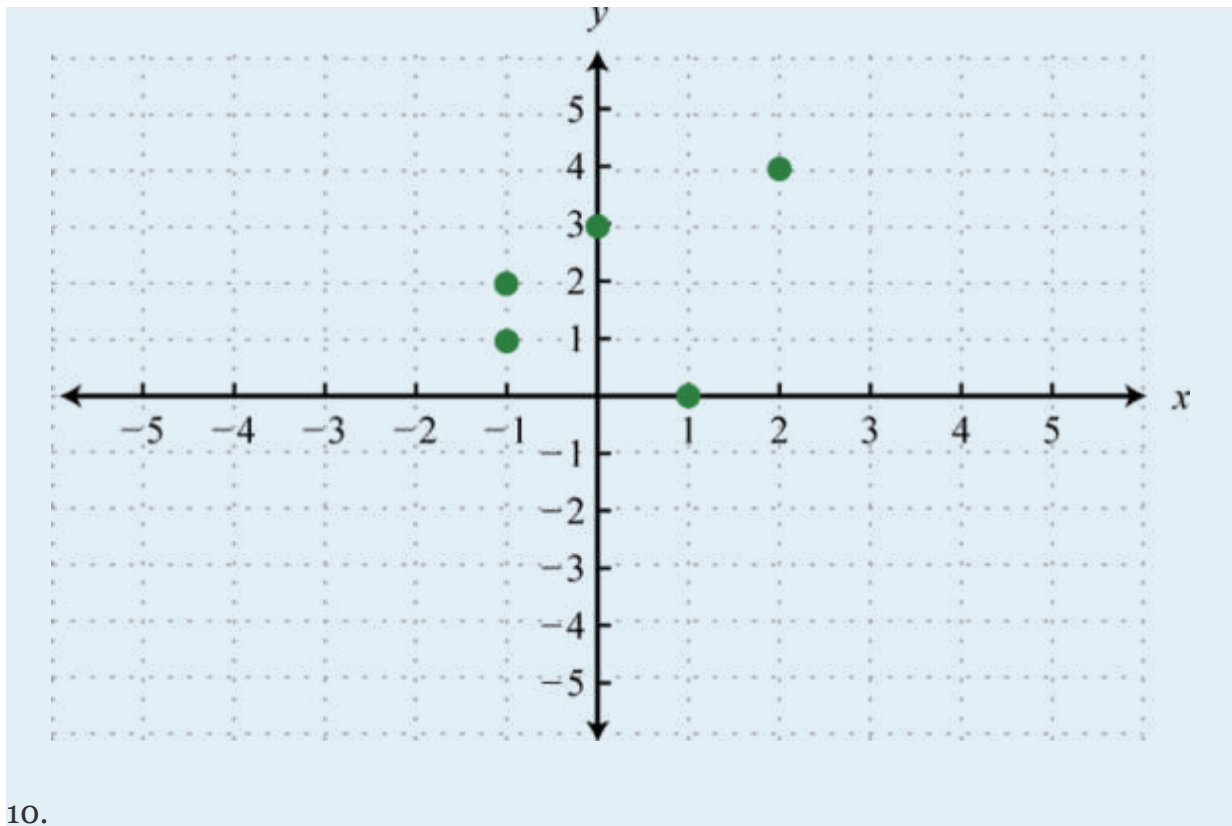


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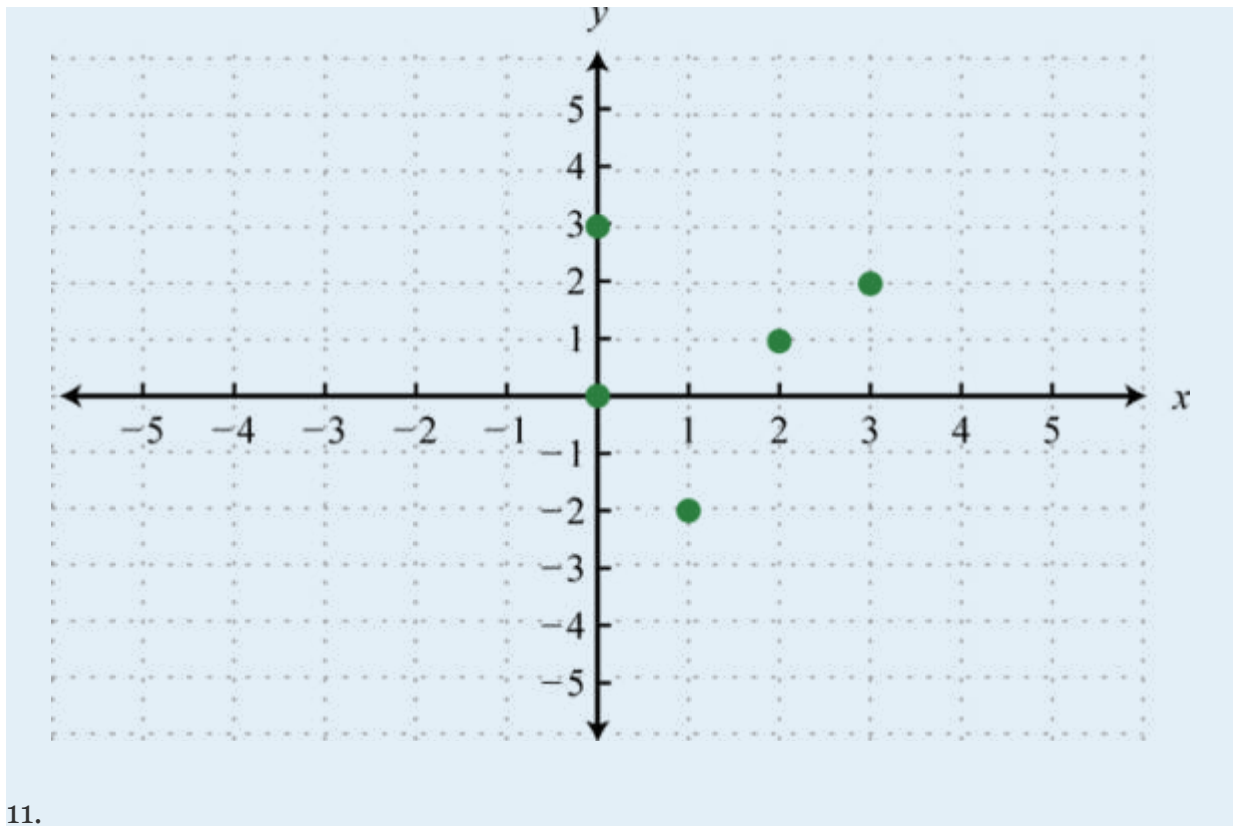




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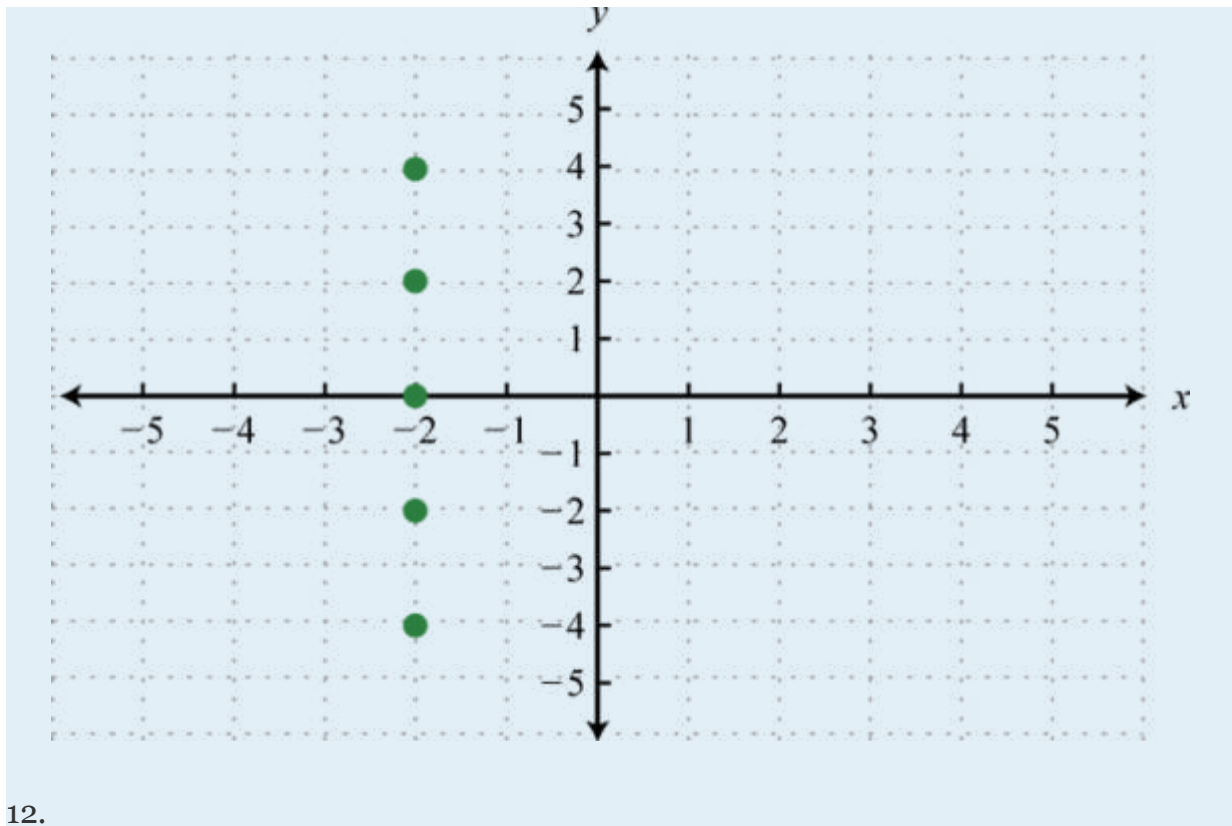




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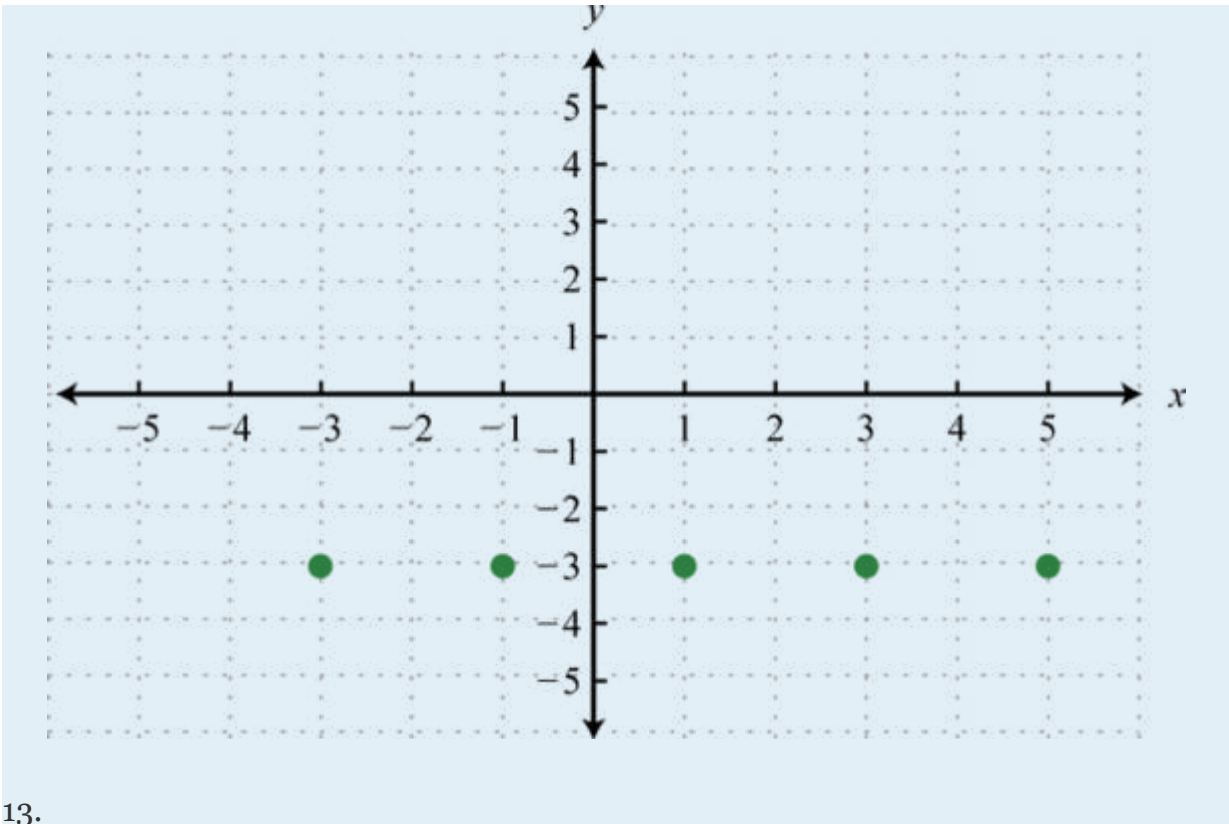


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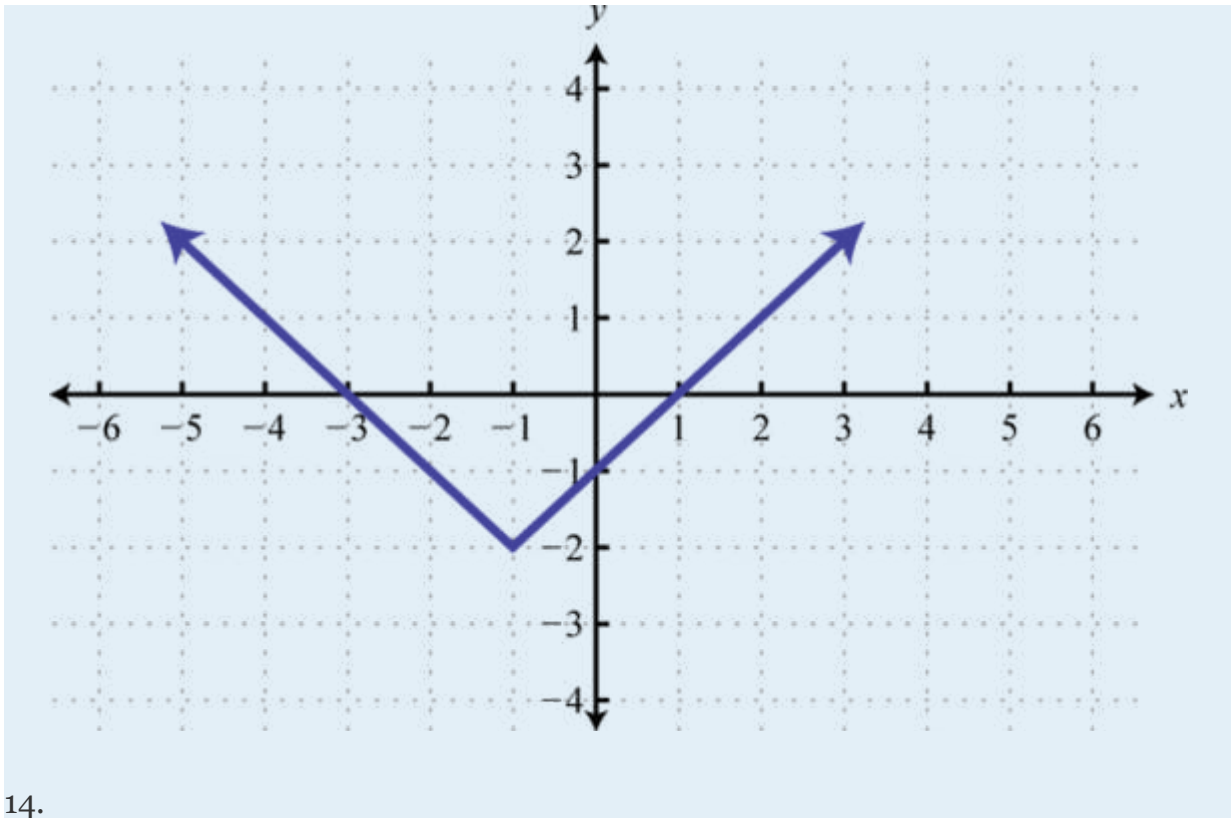




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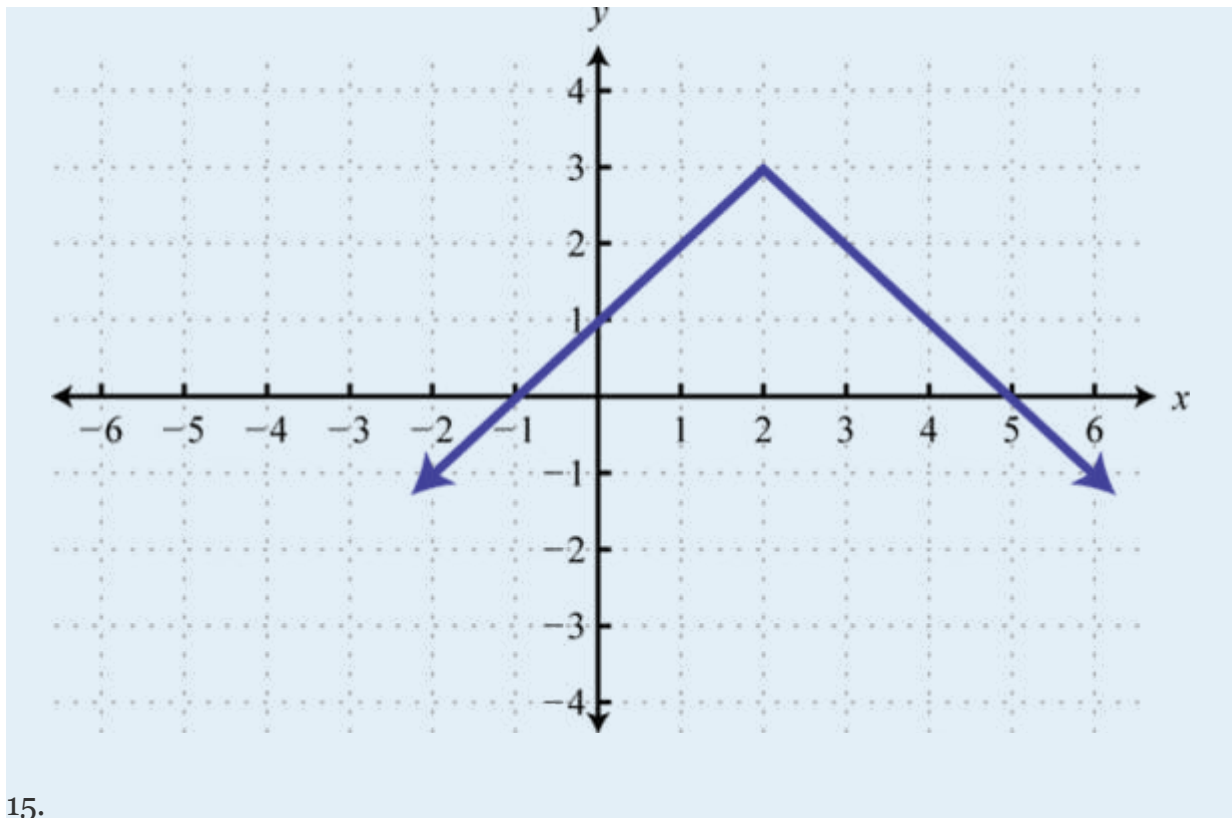




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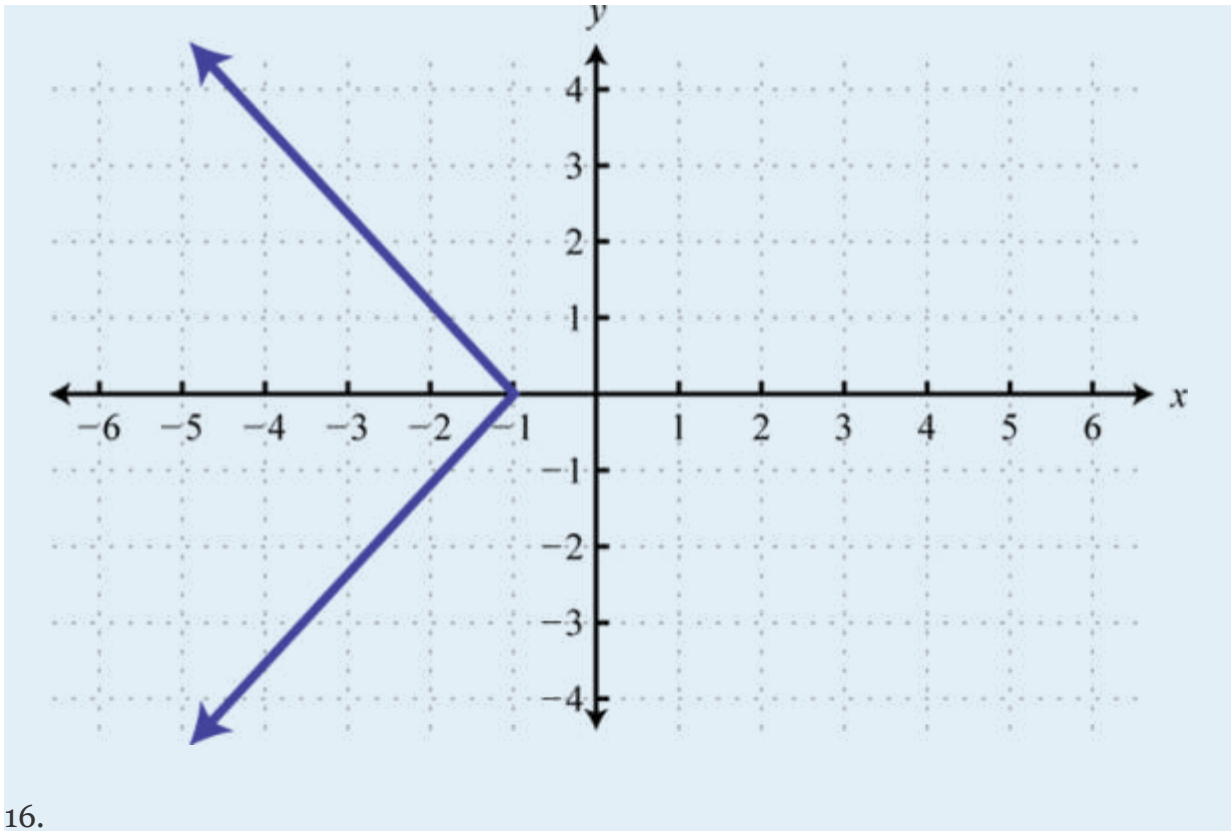




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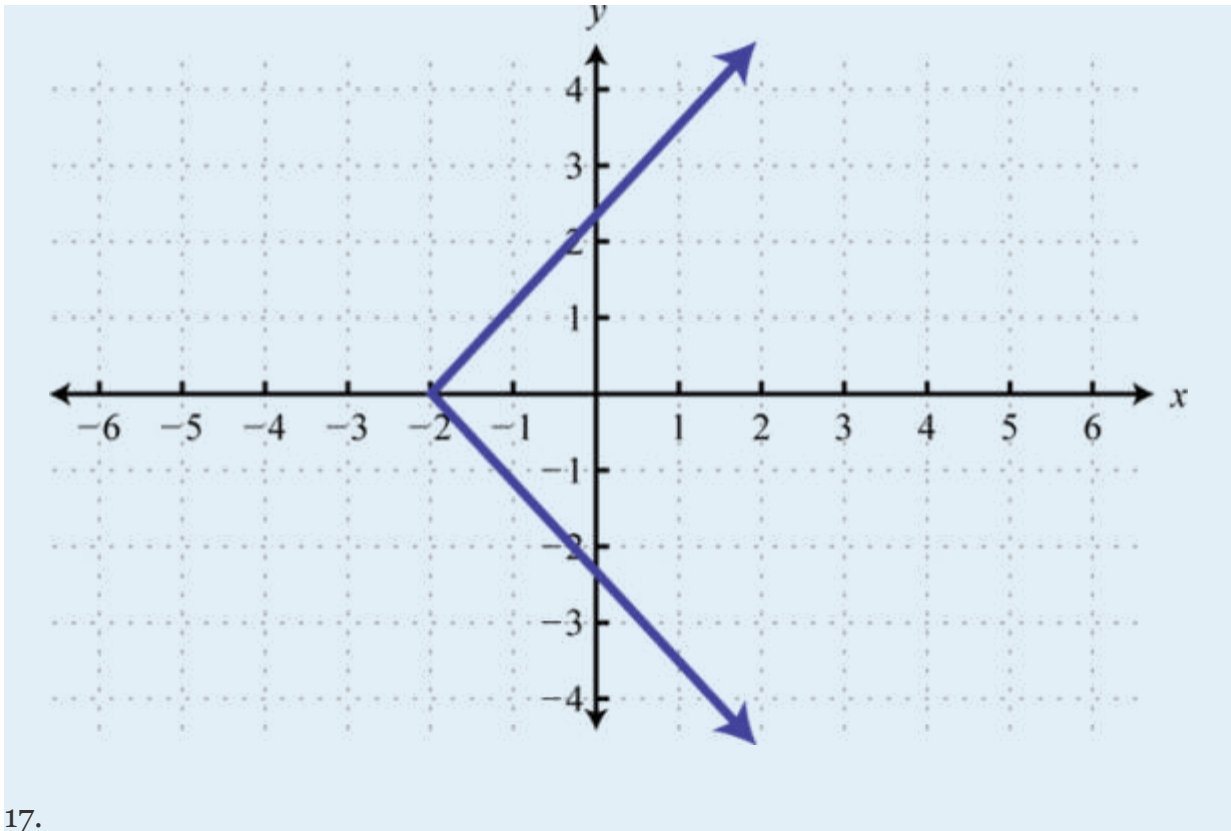


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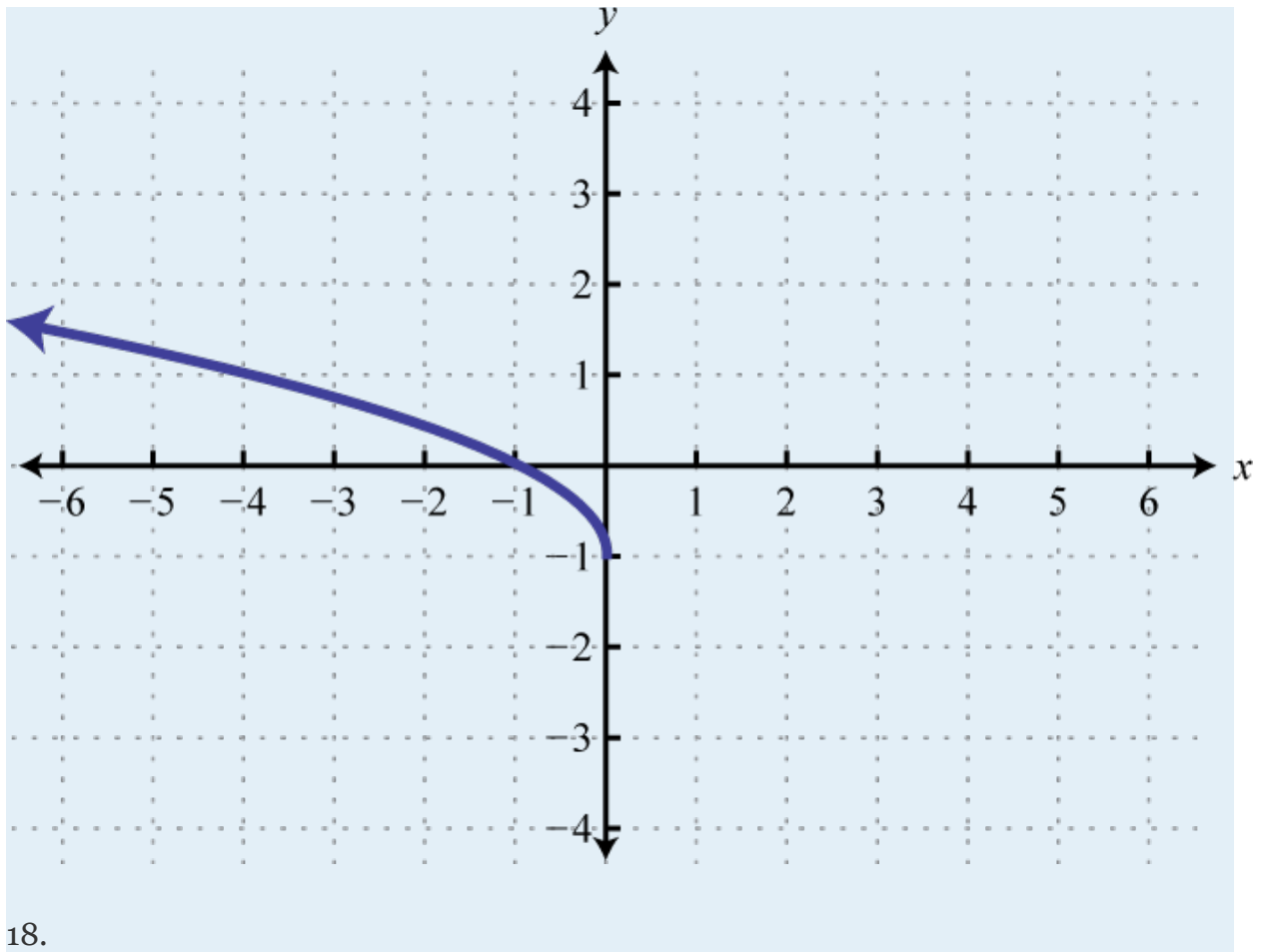




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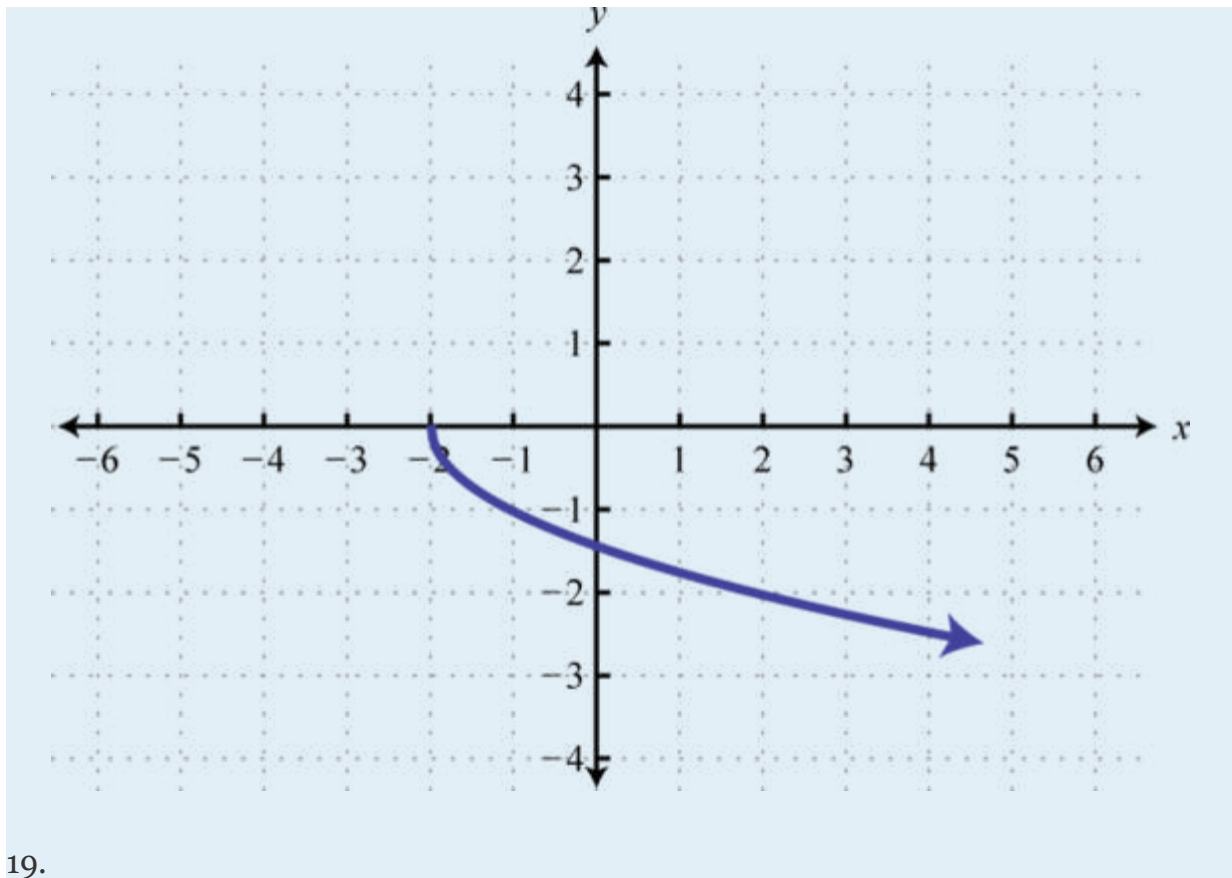
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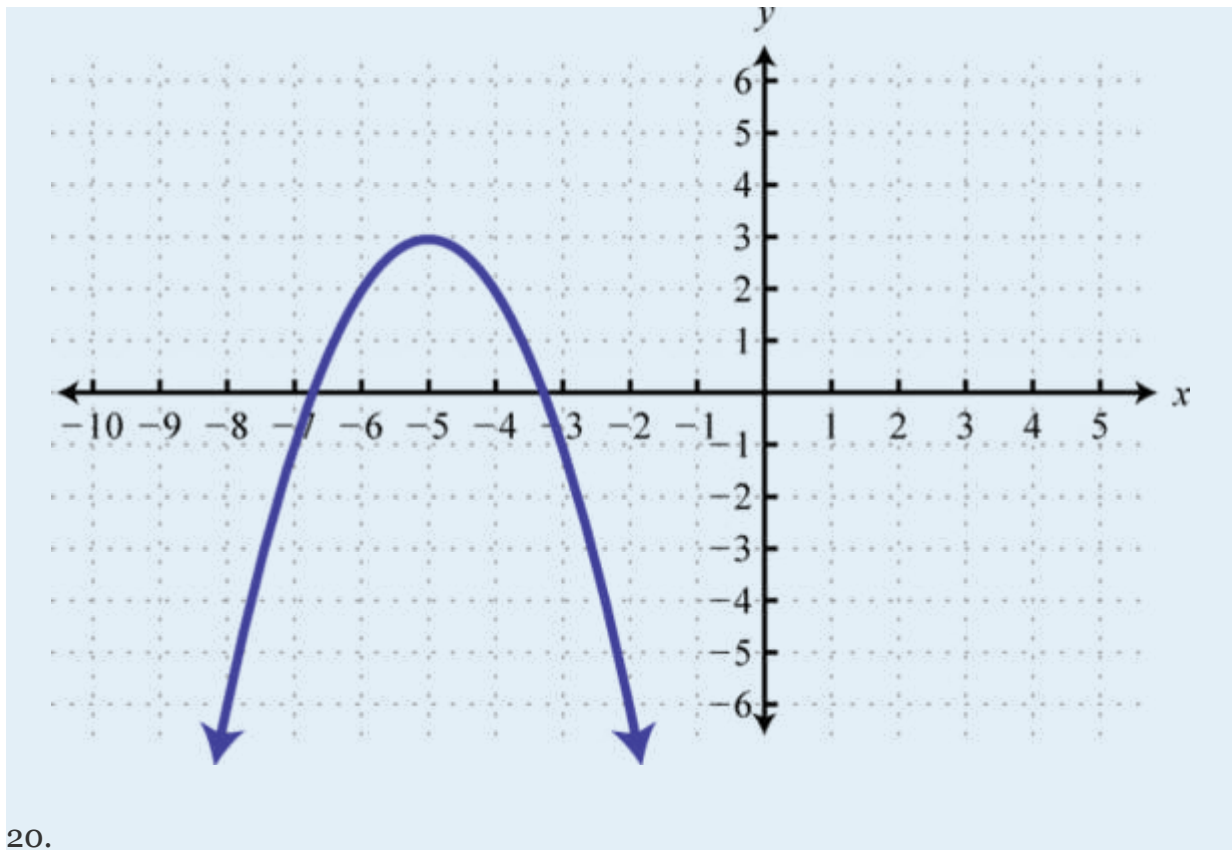




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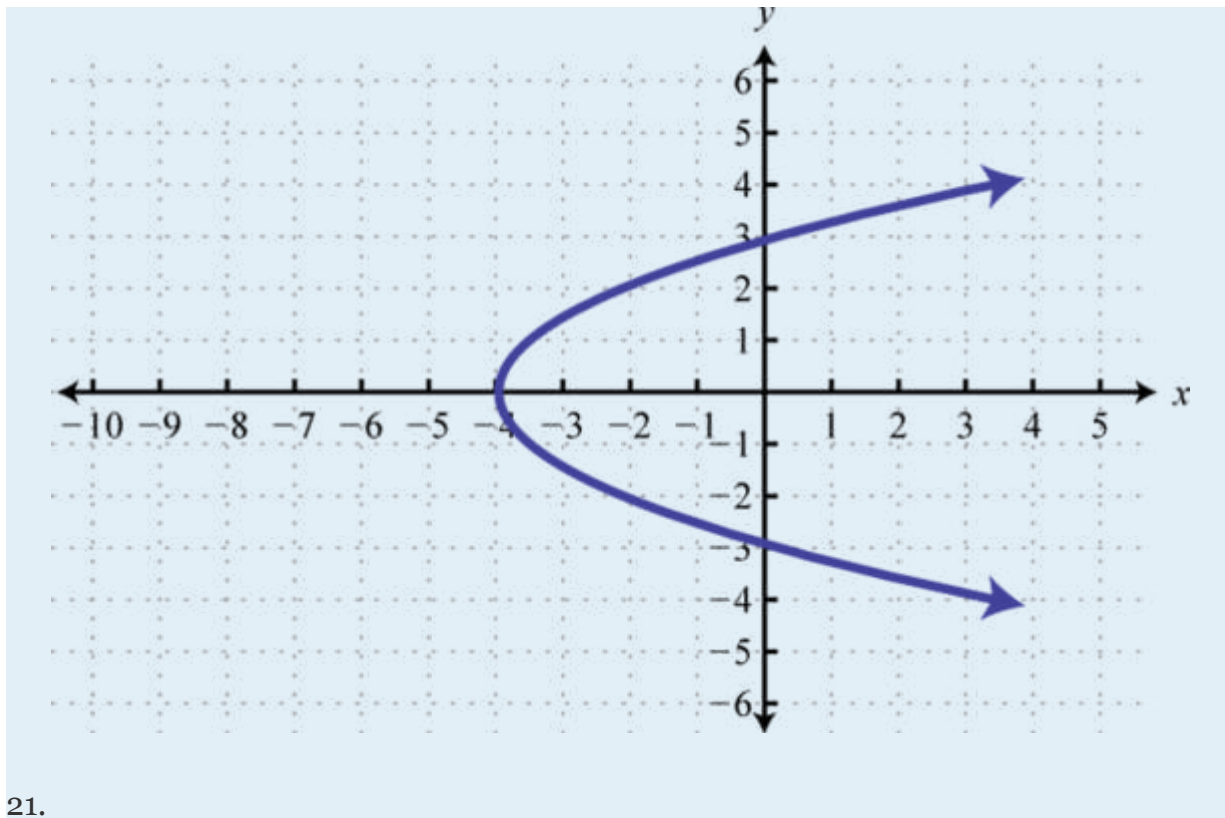




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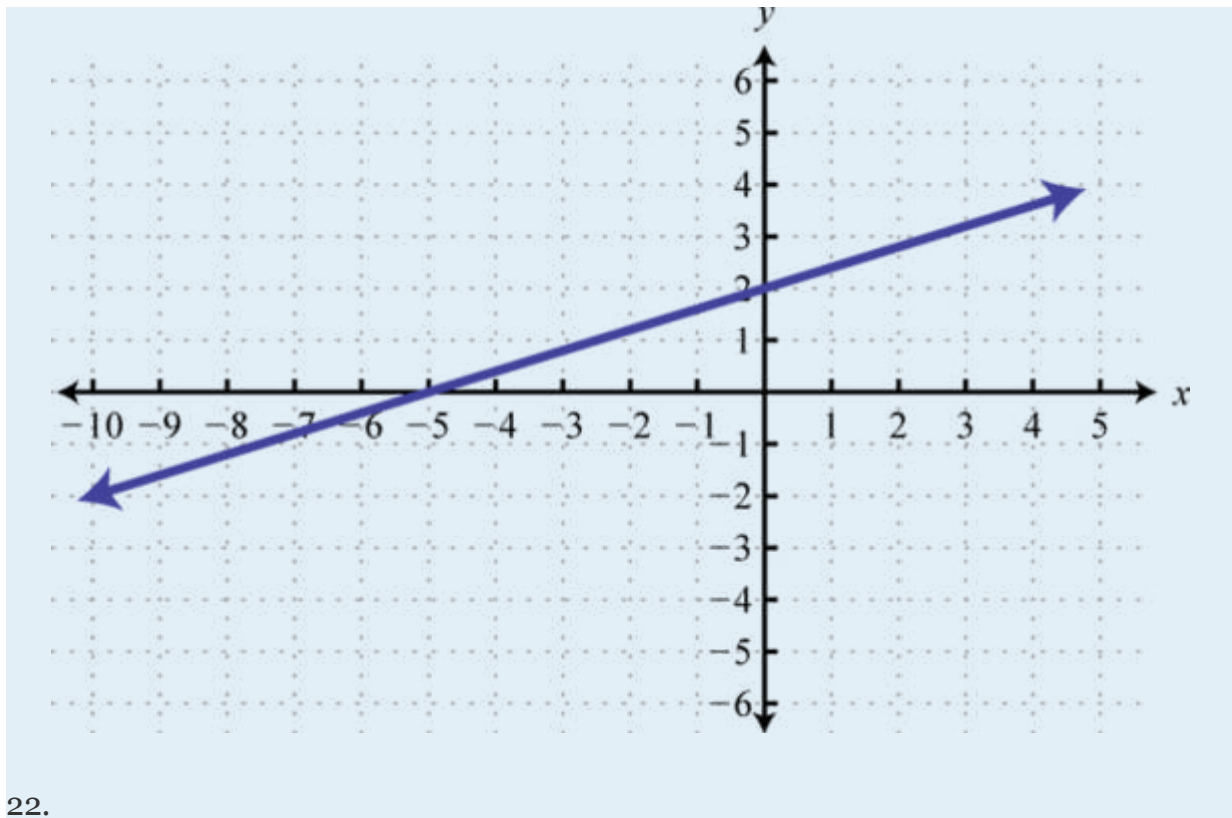




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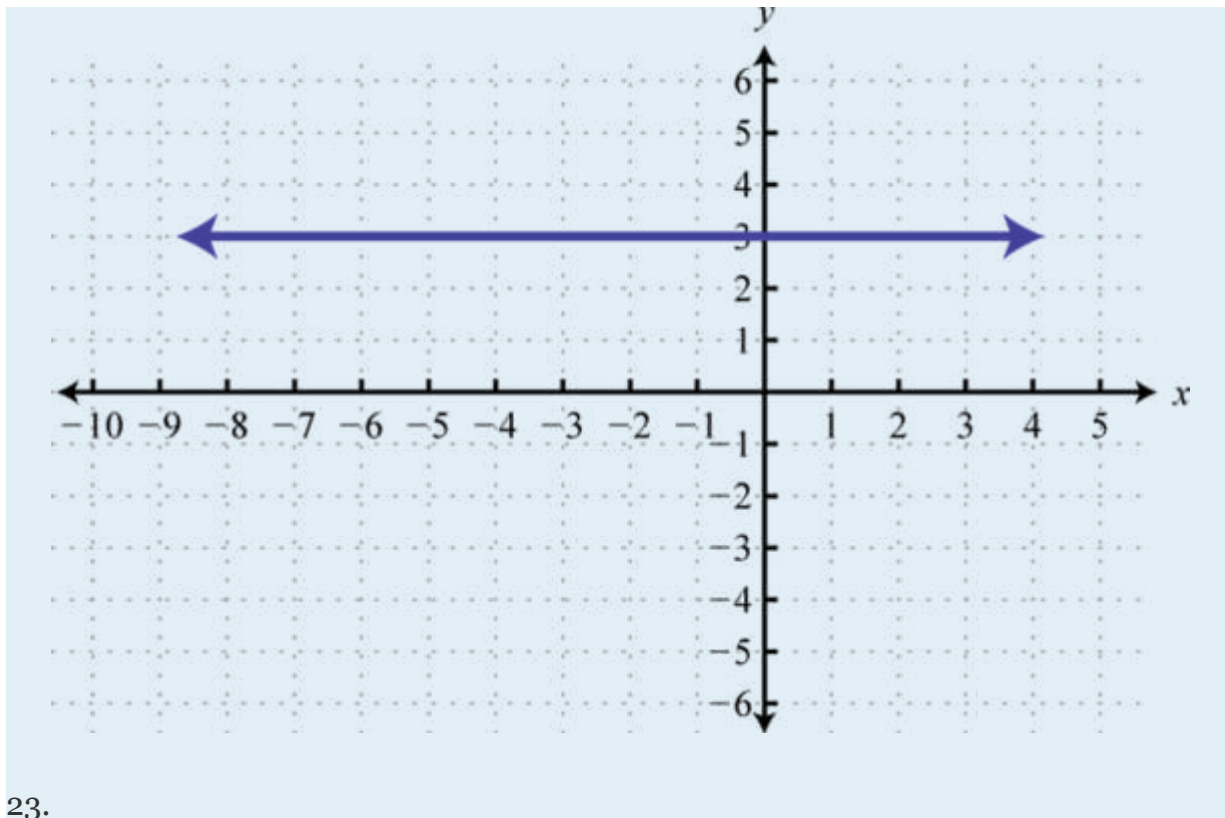


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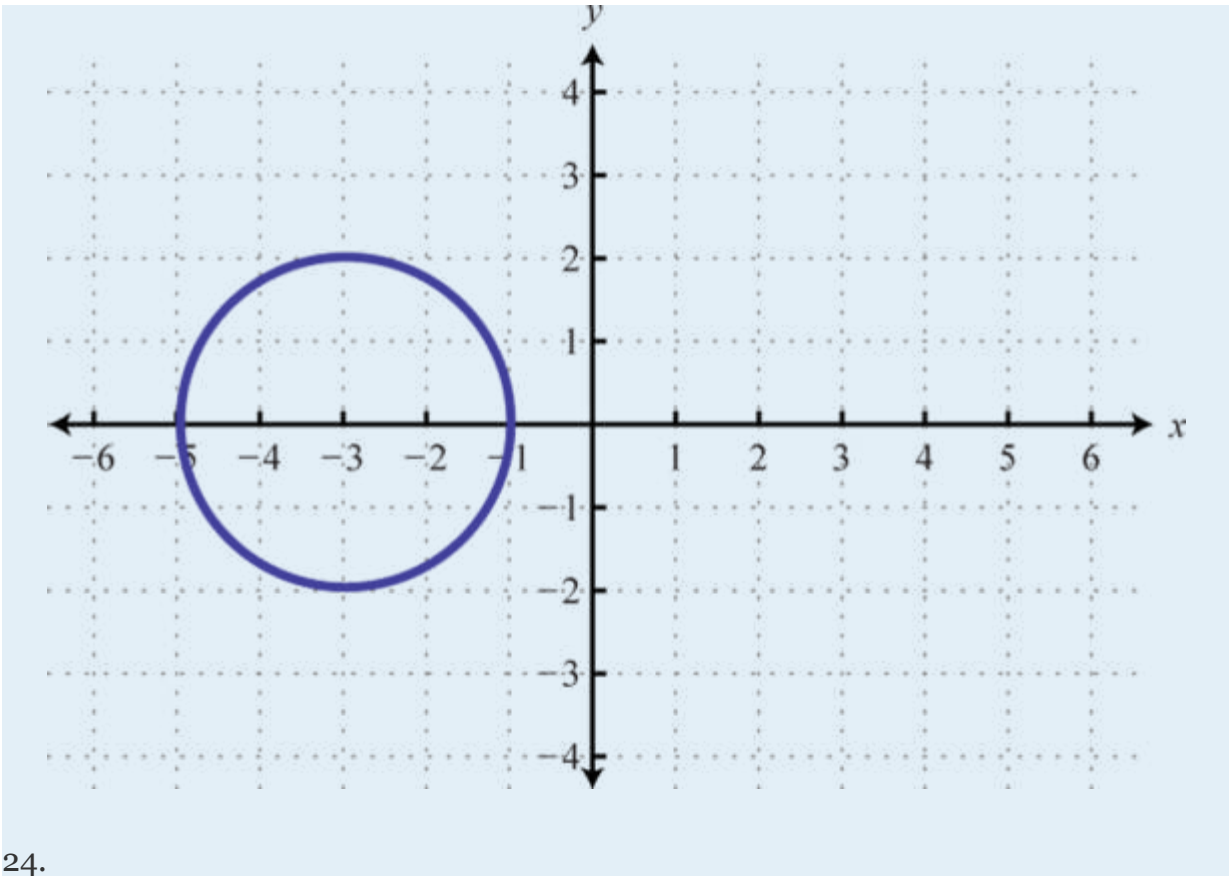




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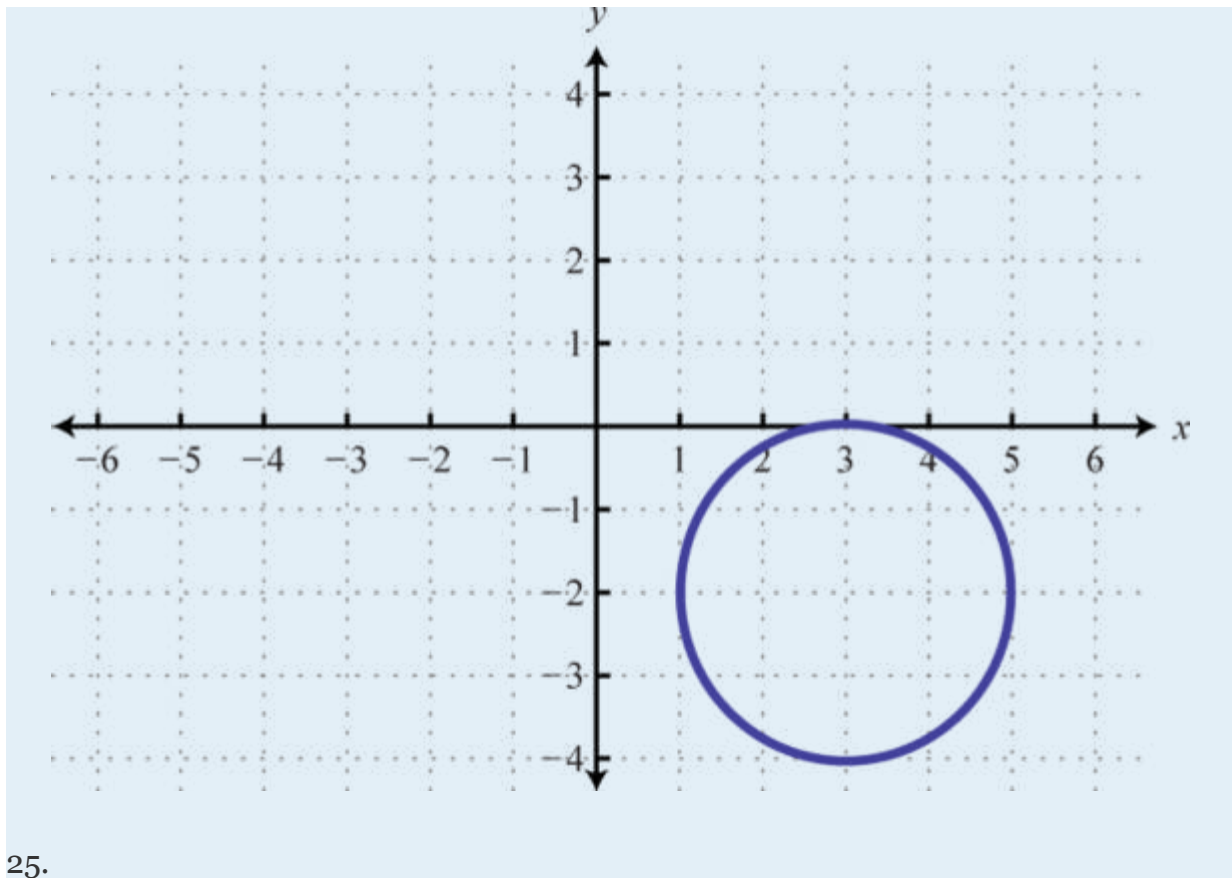


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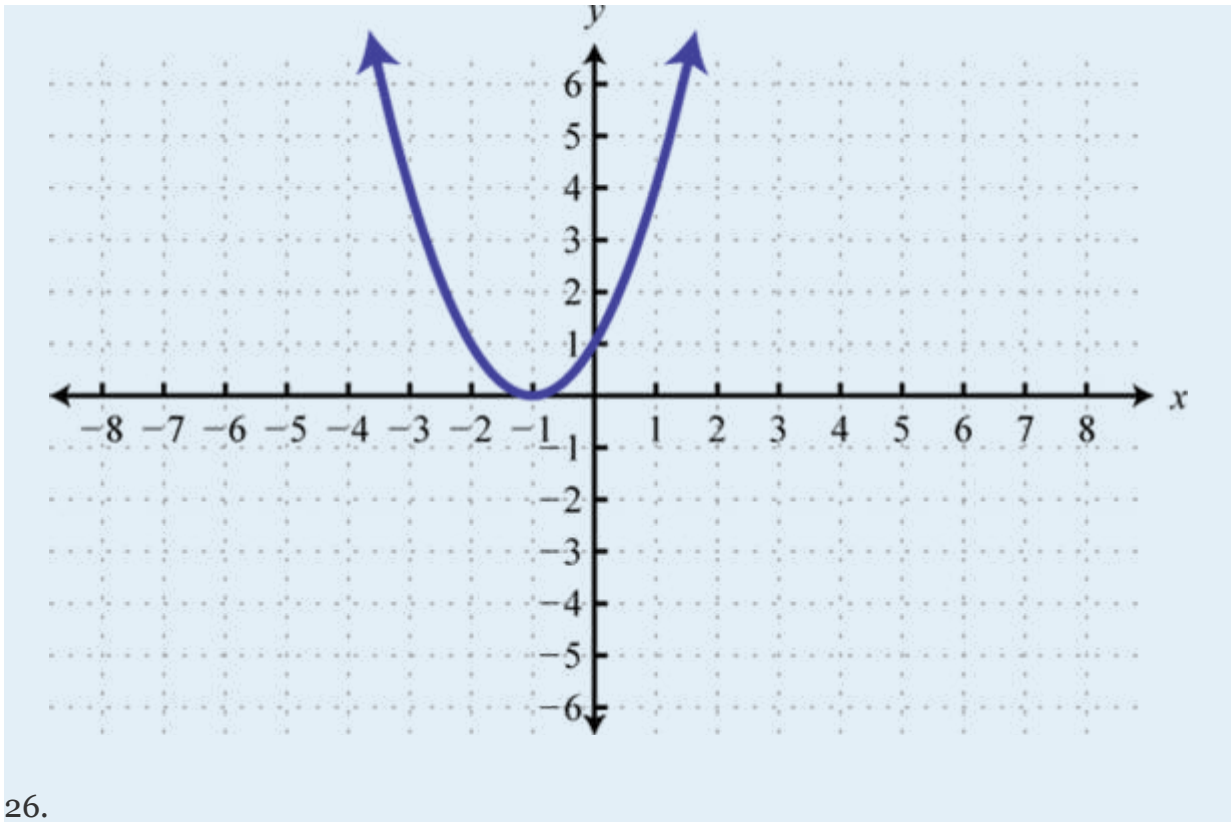




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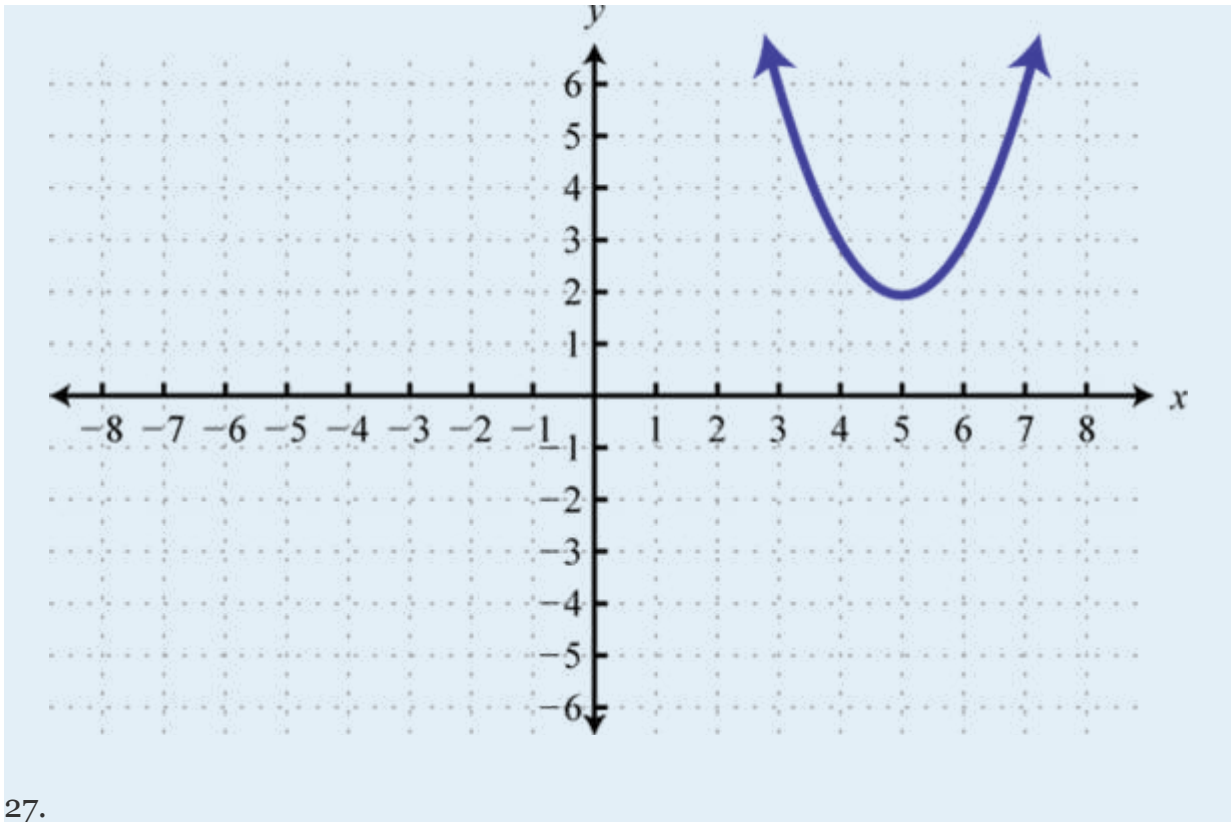




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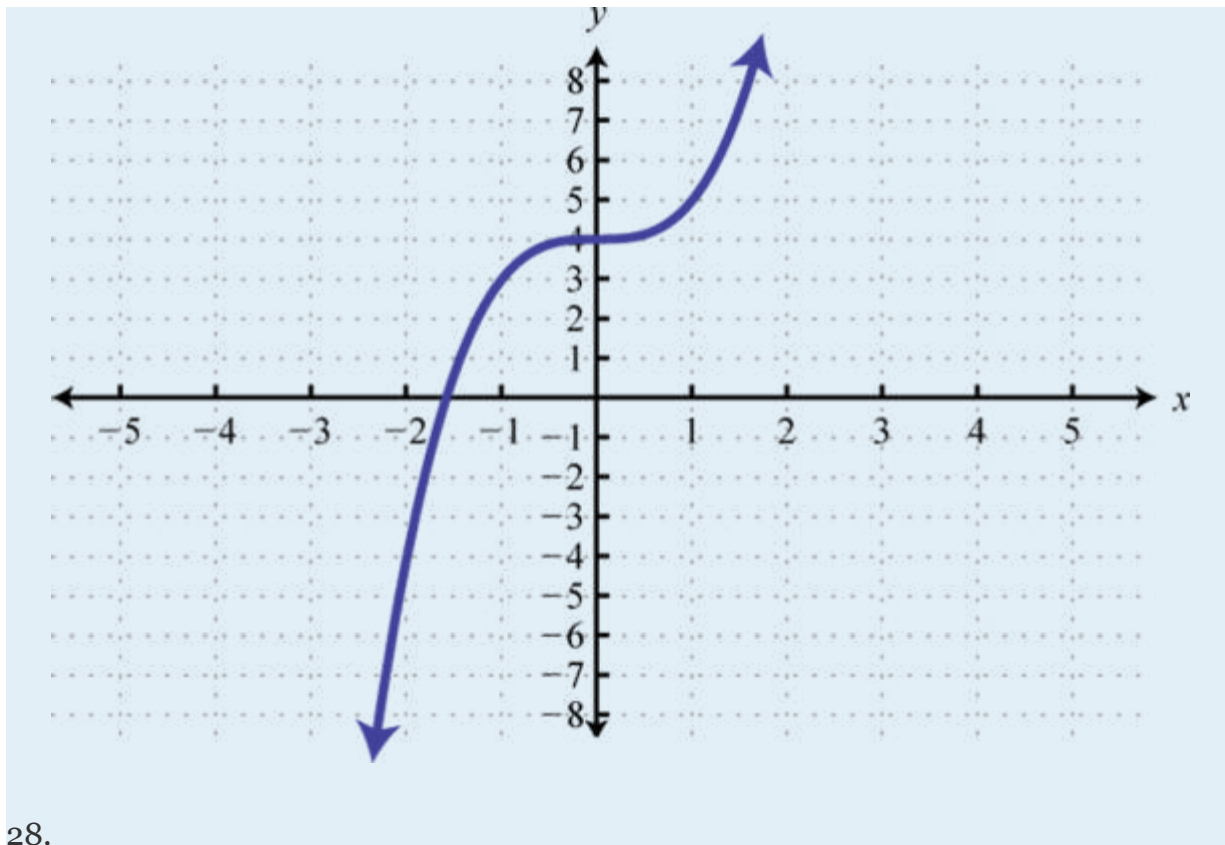




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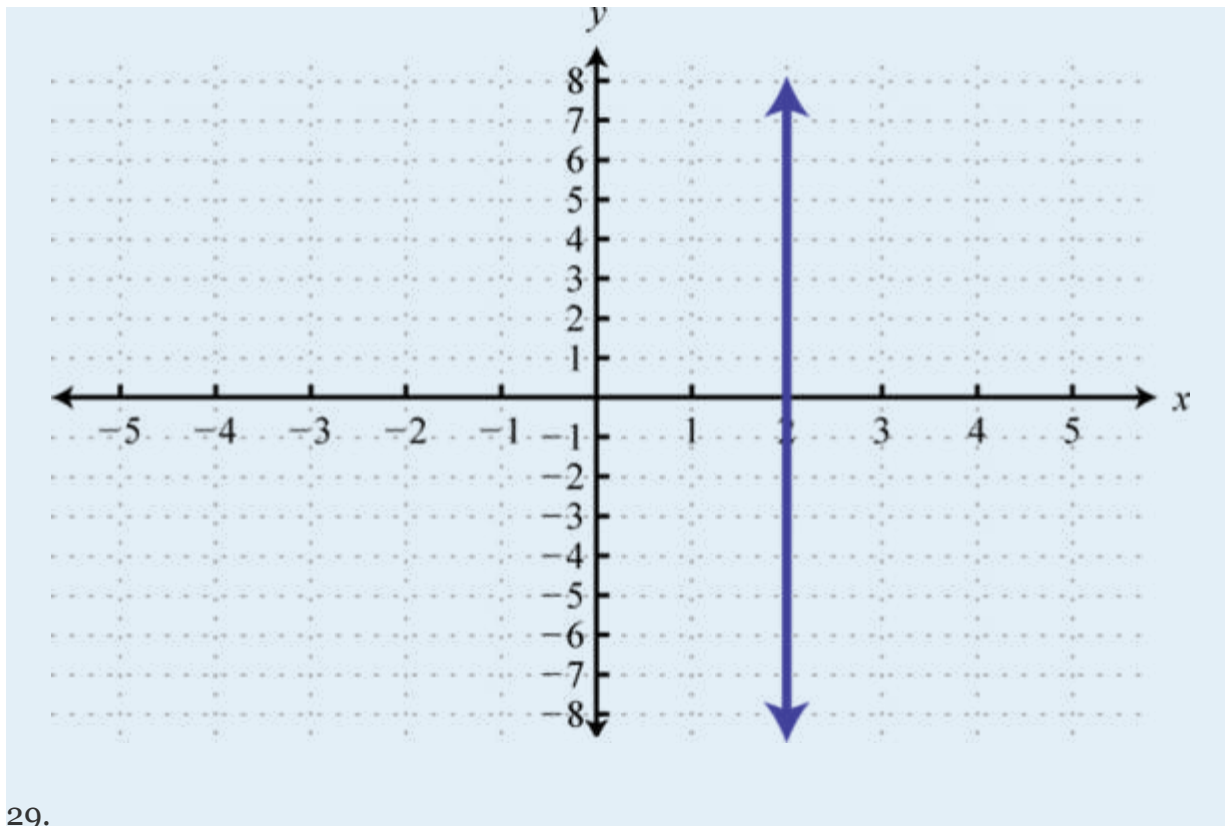




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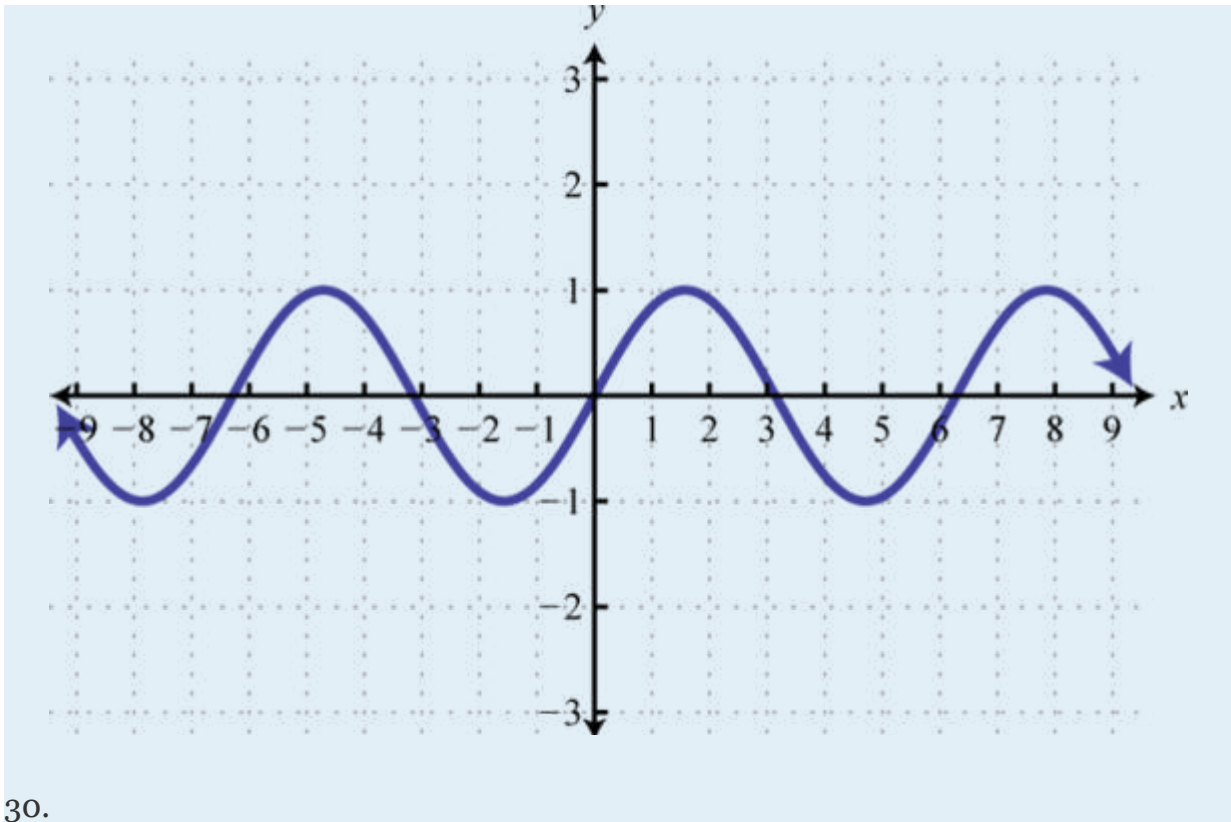




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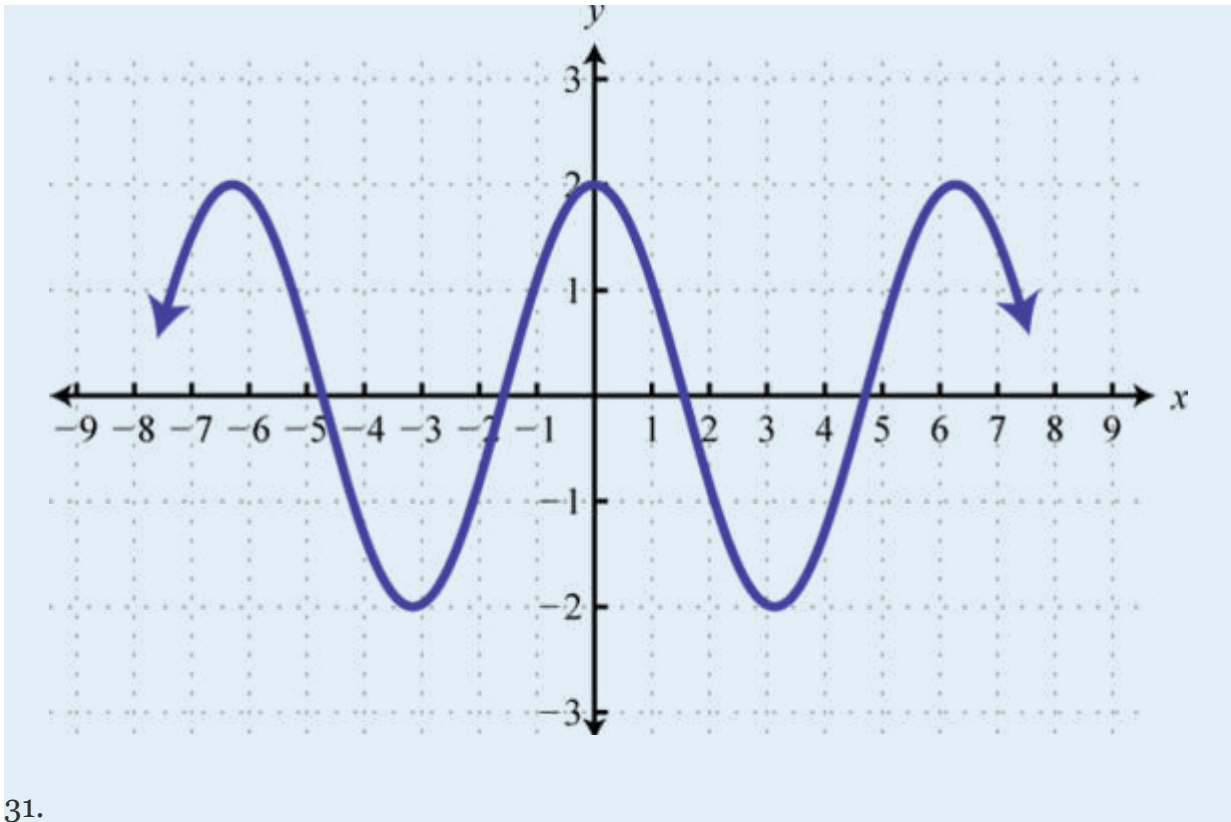




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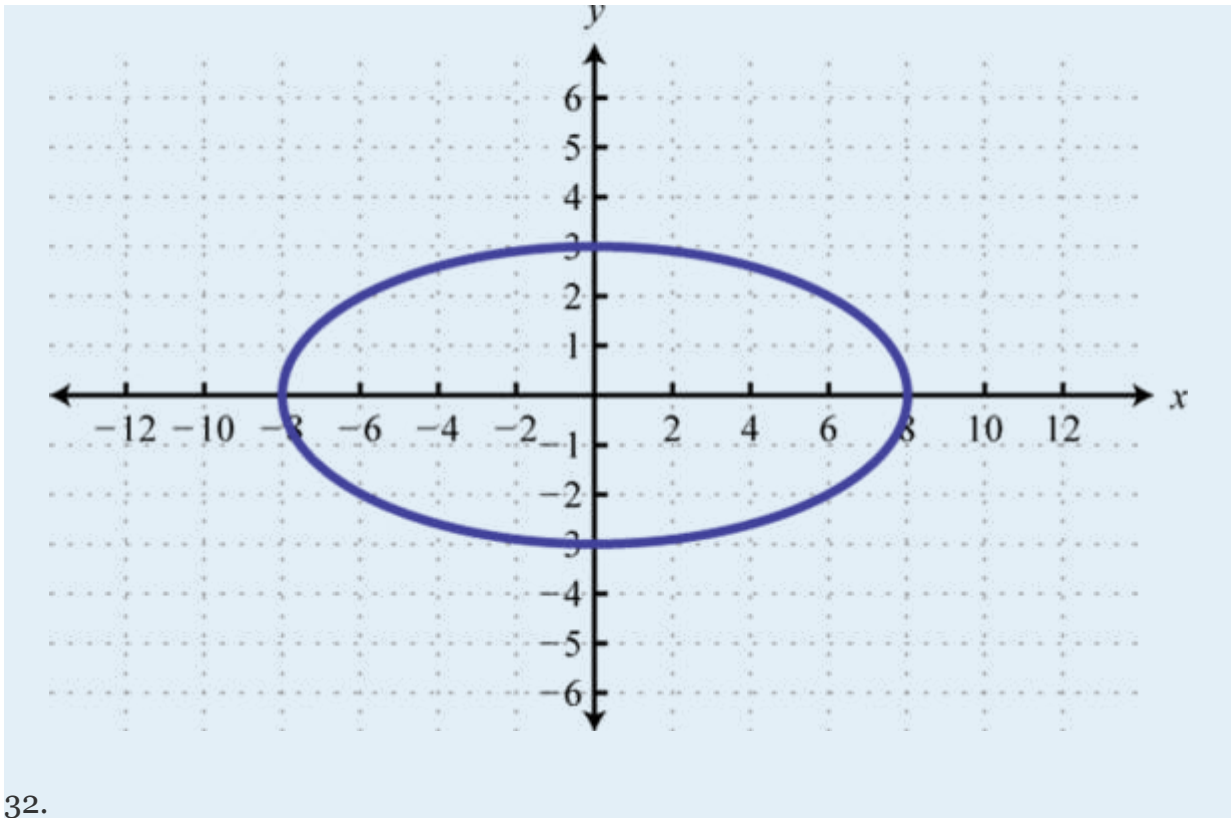




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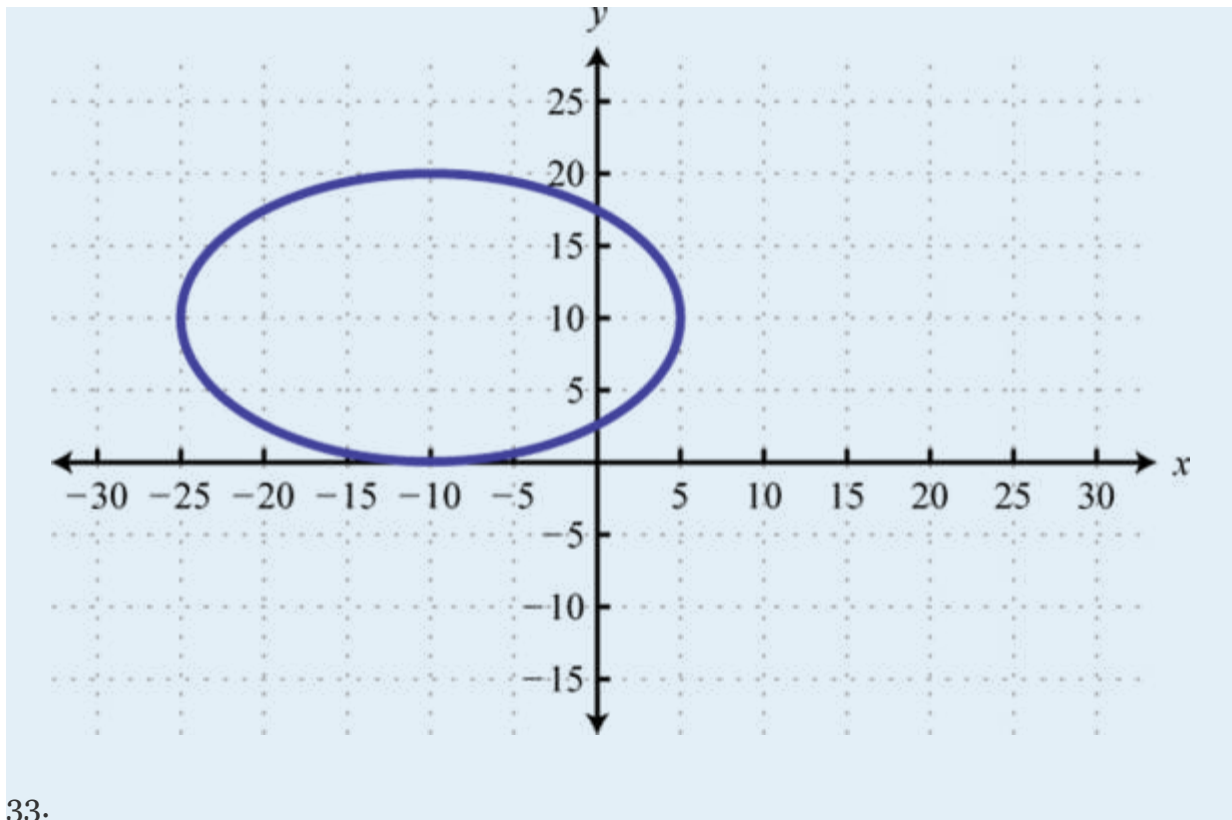


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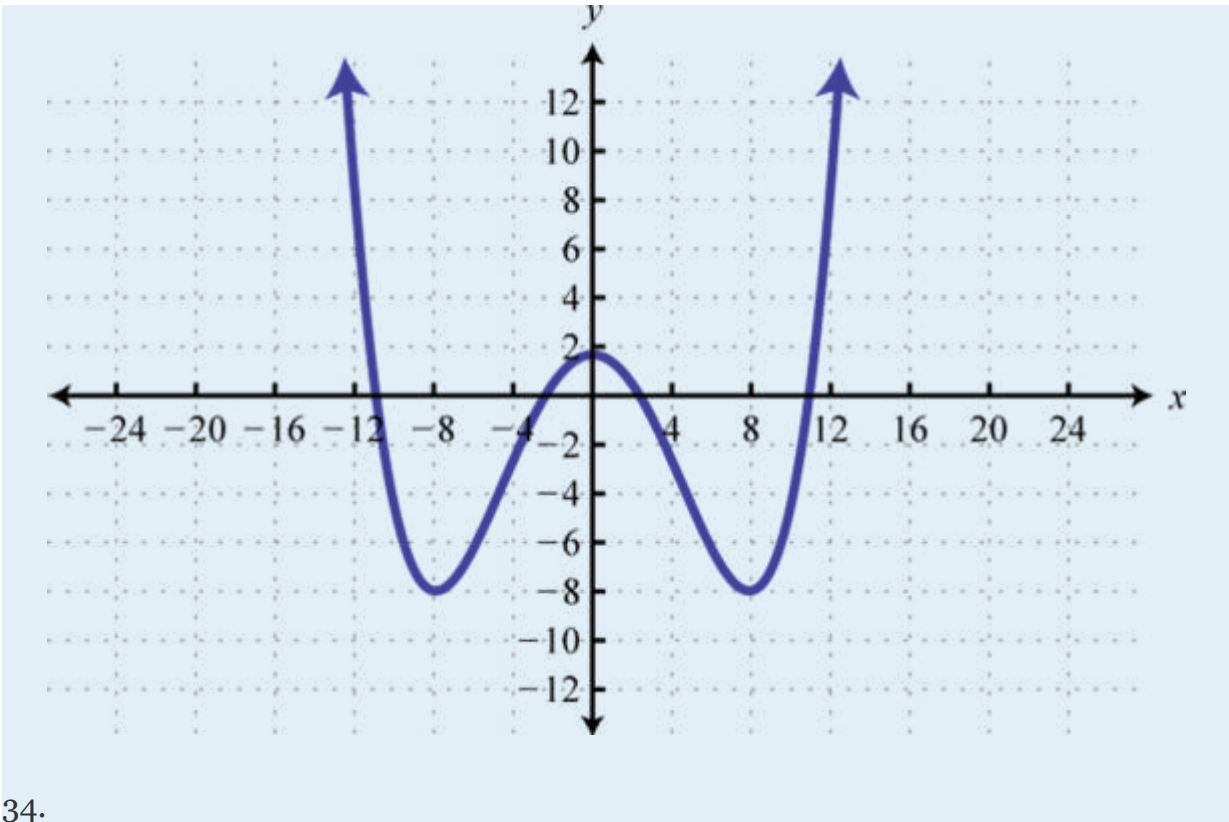




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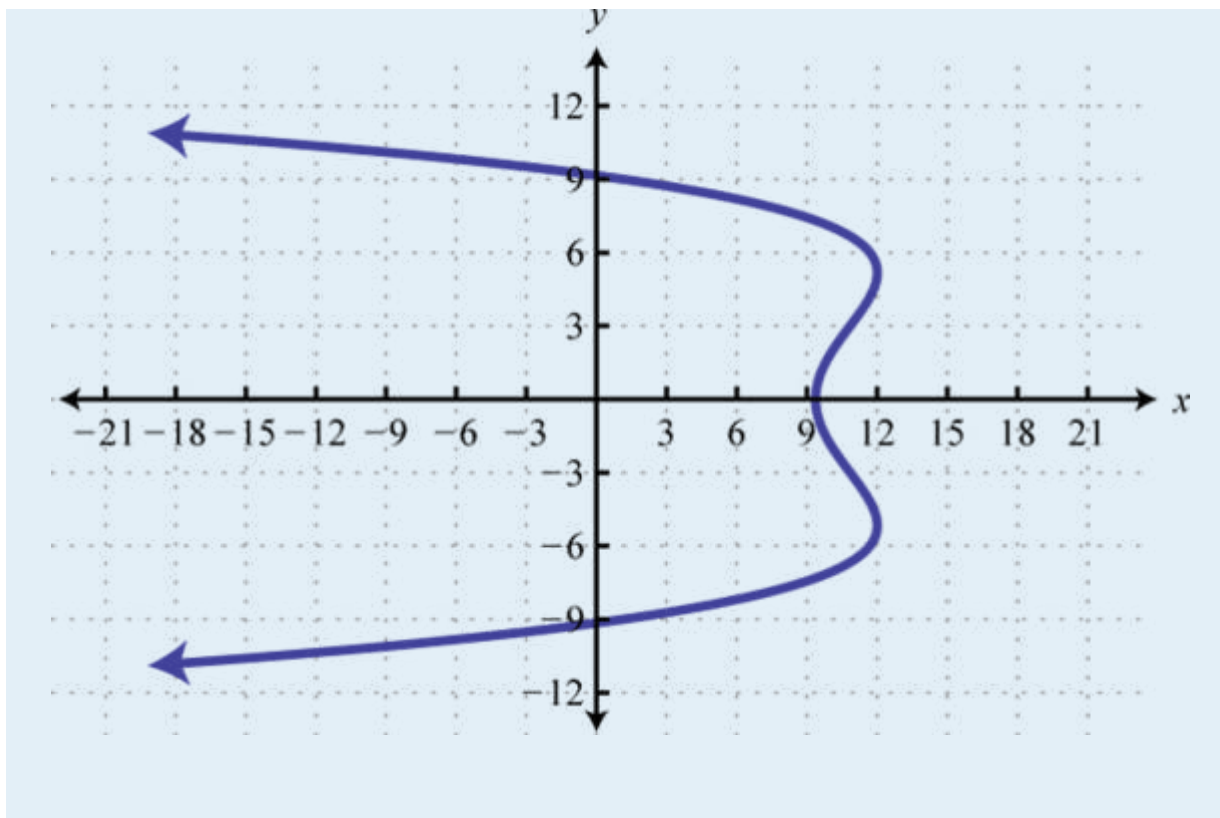




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## PART B: FUNCTION NOTATION

Evaluate.

35.  $g(x) = |x-5|$  find  $g(-5)$ ,  $g(0)$ , and  $g(5)$ .
36.  $g(x) = |x|-5$ ; find  $g(-5)$ ,  $g(0)$ , and  $g(5)$ .
37.  $g(x) = |2x-3|$ ; find  $g(-1)$ ,  $g(0)$ , and  $g(32)$ .
38.  $g(x) = 3-|2x|$ ; find  $g(-3)$ ,  $g(0)$ , and  $g(3)$ .
39.  $f(x) = 2x-3$ ; find  $f(-2)$ ,  $f(0)$ , and  $f(x-3)$ .
40.  $f(x) = 5x-1$ ; find  $f(-2)$ ,  $f(0)$ , and  $f(x+1)$ .
41. ; find  $g(-3)$ ,  $g(0)$ , and  $f(9x+6)$ .

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42. ; find  $g(-4)$ ,  $g(0)$ , and  $g(6x - 2)$ .

43. ; find  $g(-5)$ , , and  $g(x - 5)$ .

44. ; find  $g(-1)$ , , and  $g(2x - 1)$ .

45. ; find  $f(0)$ ,  $f(2)$ , and  $f(x + 2)$ .

46. ; find  $f(-2)$ , , and  $f(x-3)$ .

47. ; find , , and  $h(2a-1)$ .

48. ; find  $h(0)$ , ,  $h(2a + 1)$ .

49. ; find  $f(-1)$ ,  $f(0)$ ,  $f(x-1)$ .

50. ; find  $f(12)$ ,  $f(3)$ ,  $f(x + 3)$ .

51. ; find  $g(0)$ ,  $g(-8)$ , and  $g(x-8)$ .

52. ; find , , and

53. ; find  $f(-1)$ ,  $f(0)$ ,

54. ; find  $f(2)$ ,  $f(0)$ ,

Given the function find  $f(x + h)$ .

35.  $f(x) = 3x-1$

36.  $f(x) = -5x + 2$

37.

38.  $f(x) = x^3$

39.

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Find  $x$  given the function.

35.  $f(x) = 2x - 3$ ; find  $x$  where  $f(x) = 25$ .

36.  $f(x) = 7 - 3x$ ; find  $x$  where  $f(x) = -27$ .

37.  $f(x) = 2x + 5$ ; find  $x$  where  $f(x) = 0$

38.  $f(x) = -2x + 1$ ; find  $x$  where  $f(x) = 0$

39.  $g(x) = 6x + 2$ ; find  $x$  where  $g(x) = 5$ .

40.  $g(x) = 4x + 5$ ; find  $x$  where  $g(x) = 2$ .

41. ; find  $x$  where

42. ; find  $x$  where

43. The value of a new car in dollars is given by the function  $V(t) = -1,800t + 22,000$  where  $t$  represents the age of the car in years. Use the function to determine the value of the car when it is 4 years old. What was the value of the car new?

44. The monthly income in dollars of a commissioned car salesperson is given by the function  $I(n) = 350n + 1,450$  where  $n$  represents the number of cars sold in the month. Use the function to determine the salesperson's income if he sells 3 cars this month. What is his income if he does not sell any cars in one month?

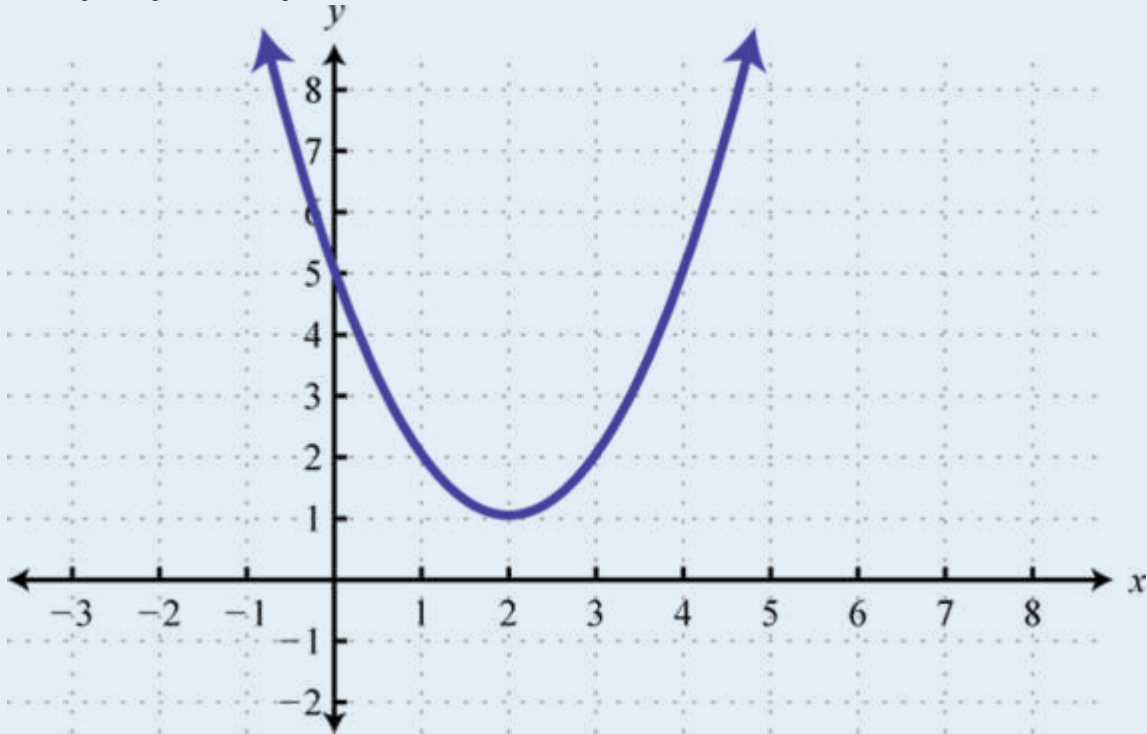
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Given the graph of the function  $f$ , find the function values.

35. Find  $f(0)$ ,  $f(2)$ , and  $f(4)$ .

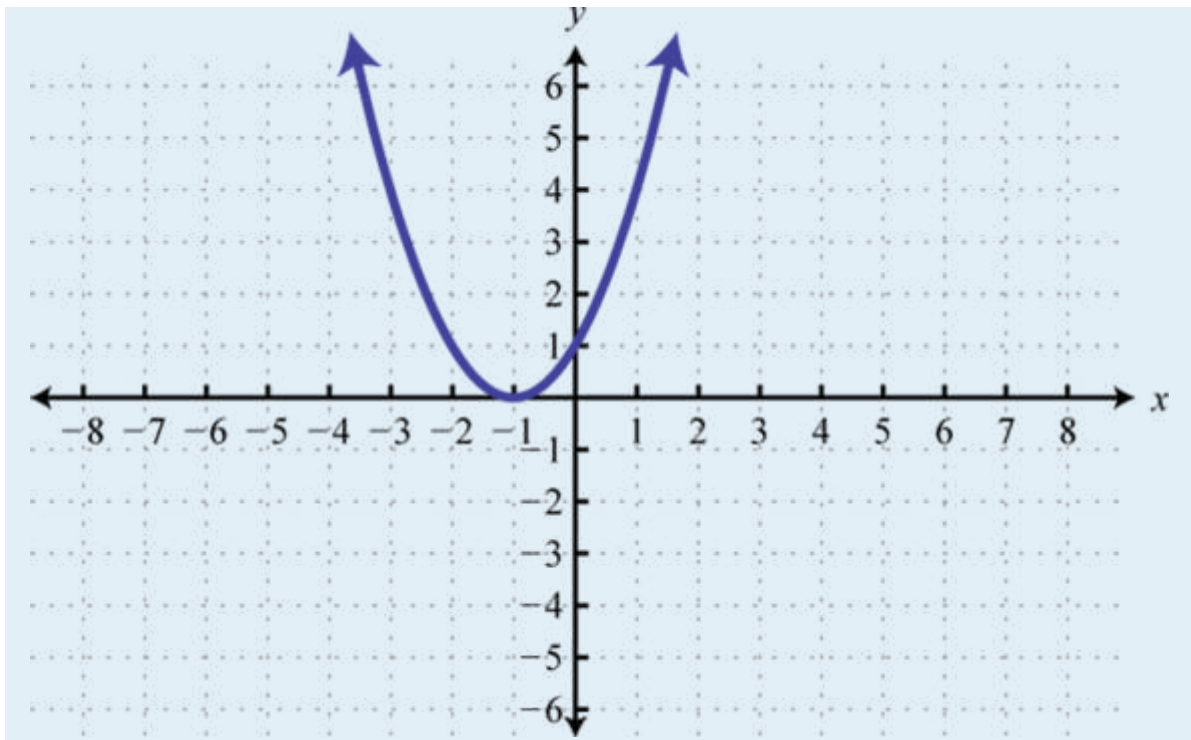


36. Find  $f(-1)$ ,  $f(0)$ , and  $f(1)$ .

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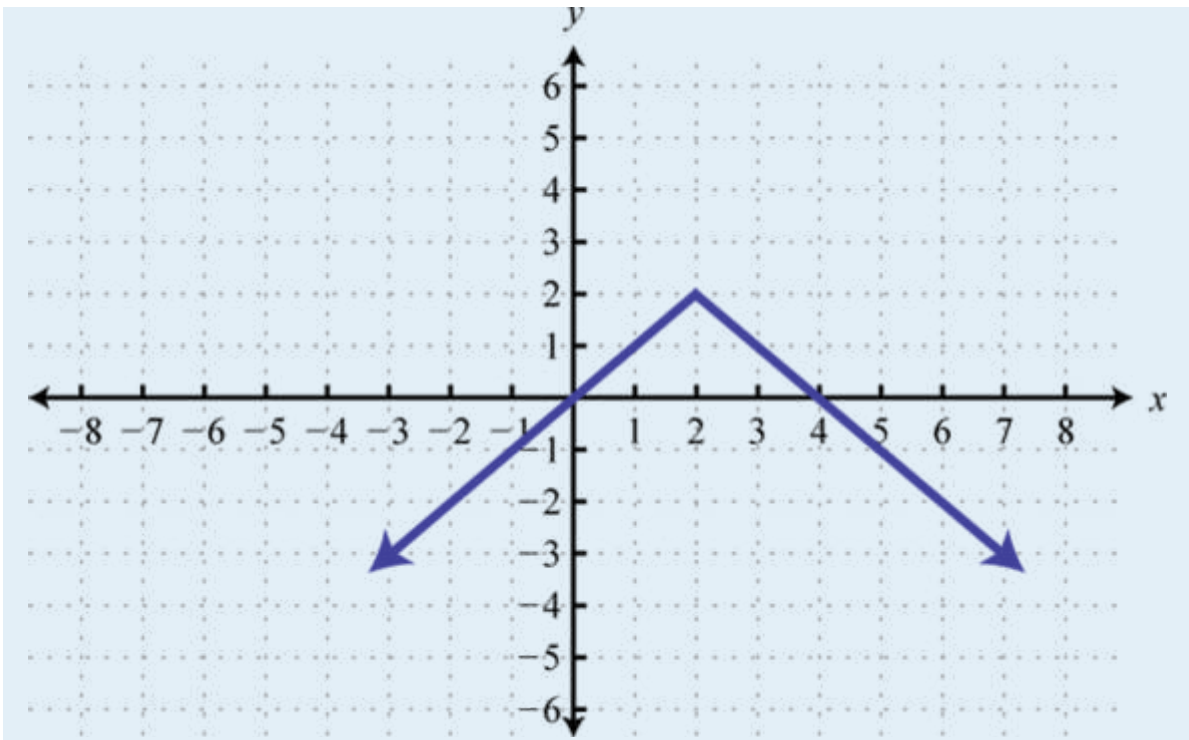
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37. Find  $f(0)$ ,  $f(2)$ , and  $f(4)$ .

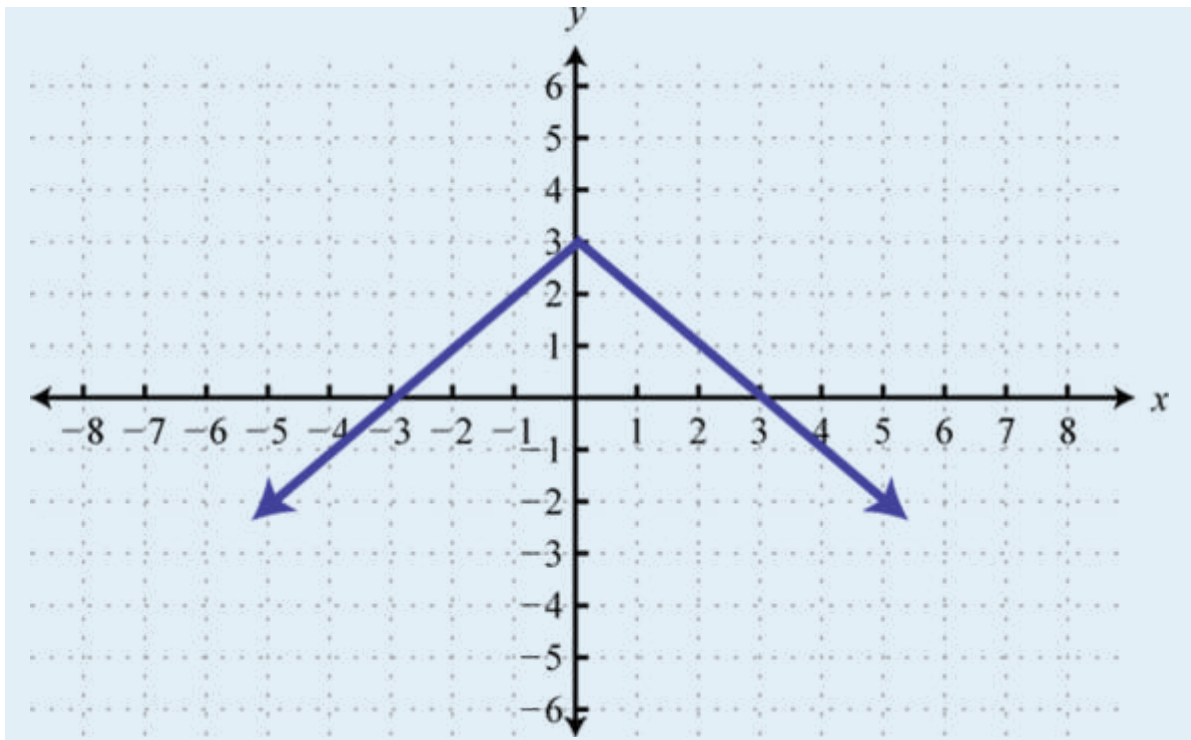




38. Find  $f(-3)$ ,  $f(0)$ , and  $f(3)$ .

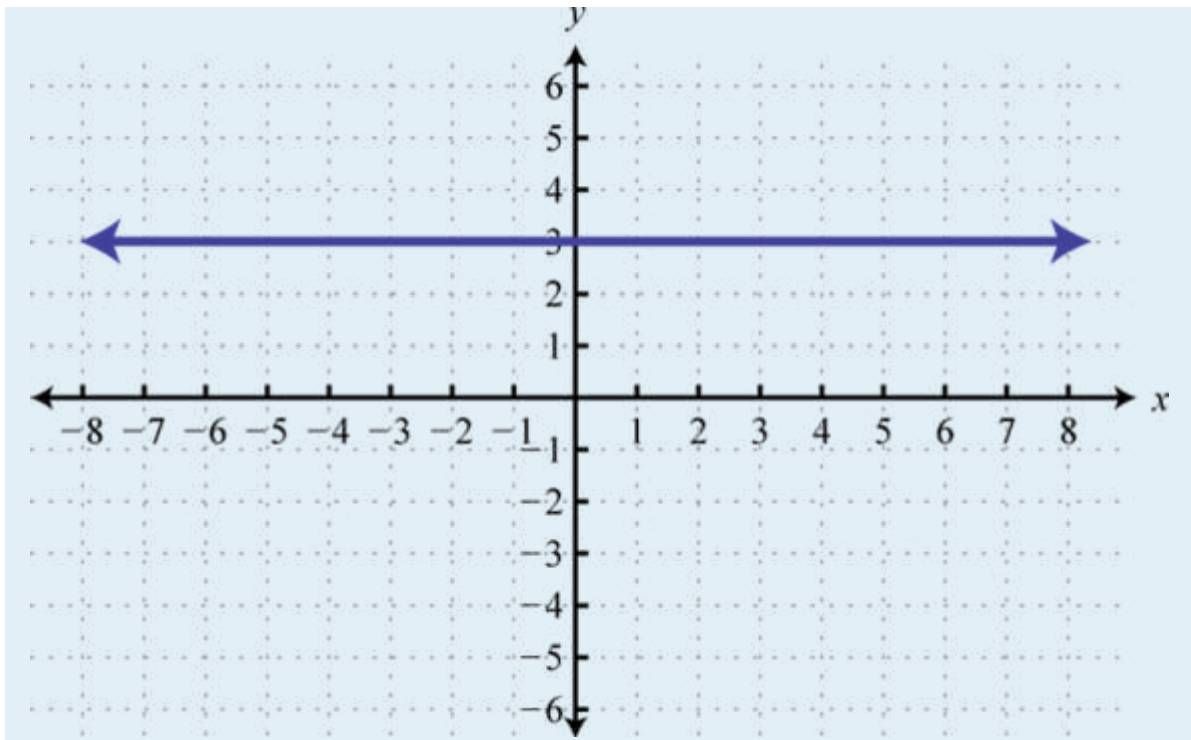






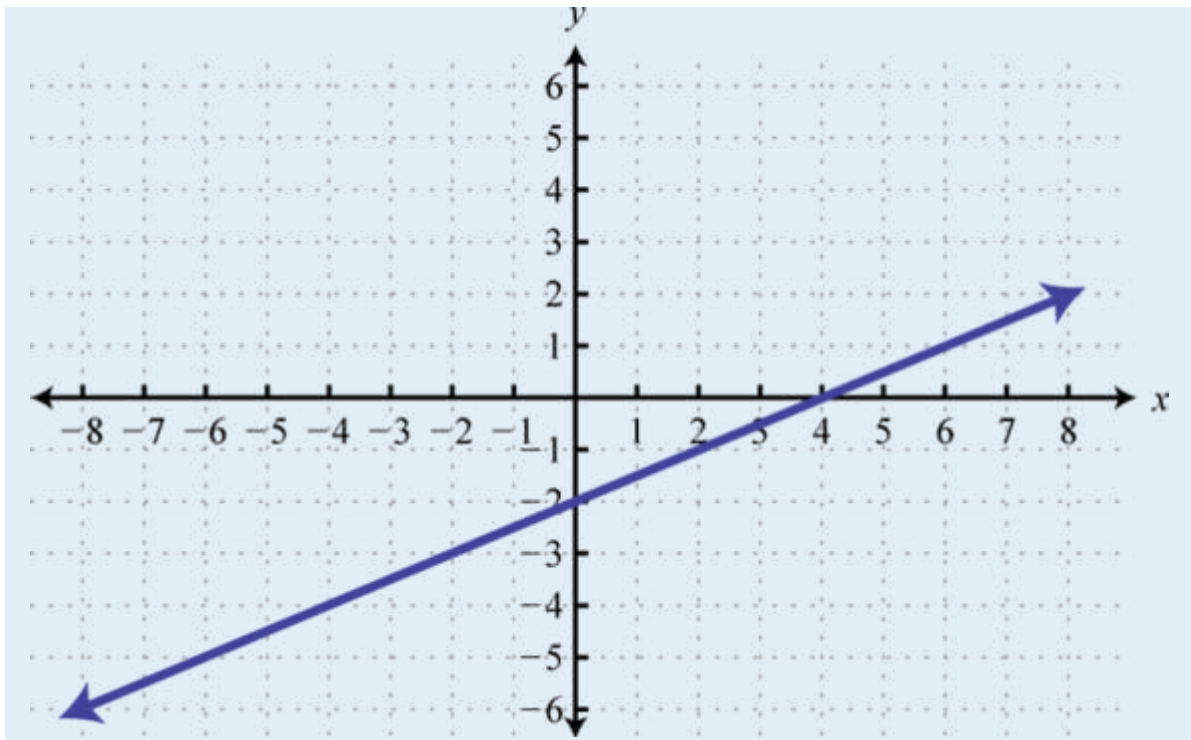
39. Find  $f(-4)$ ,  $f(0)$ , and  $f(2)$ .





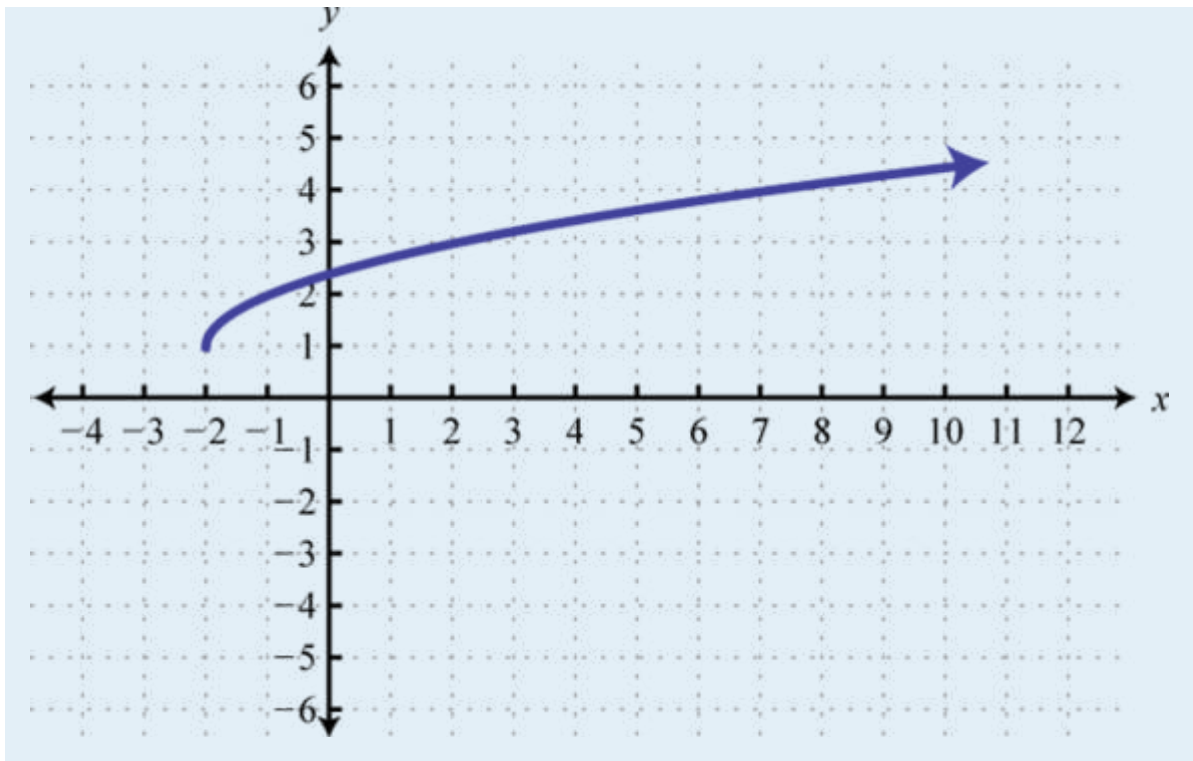
40. Find  $f(-6)$ ,  $f(0)$ , and  $f(6)$ .





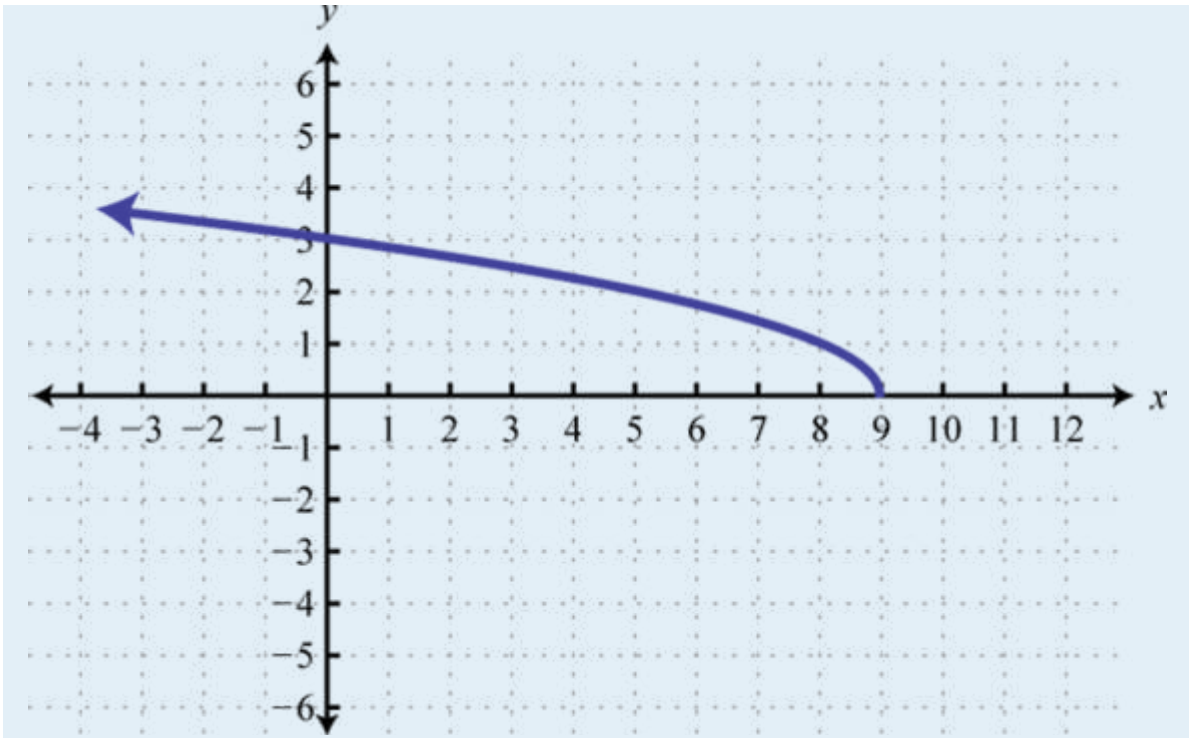
41. Find  $f(-2)$ ,  $f(2)$ , and  $f(7)$ .





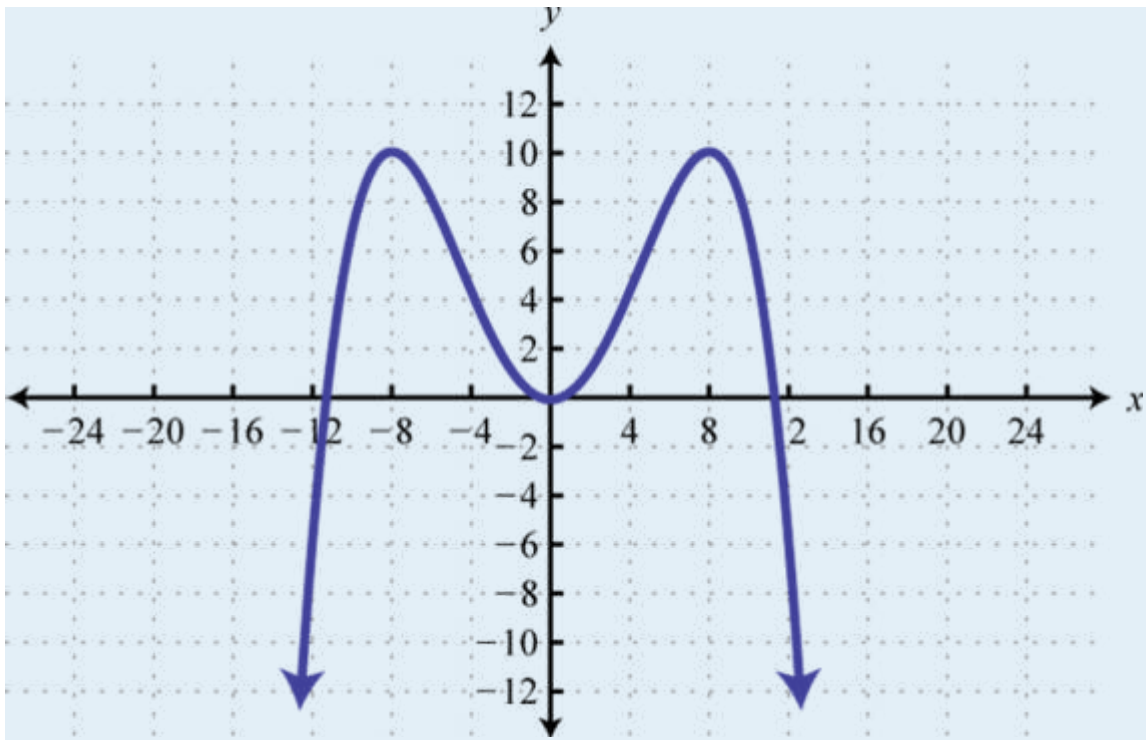
42. Find  $f(0)$ ,  $f(5)$ , and  $f(9)$ .





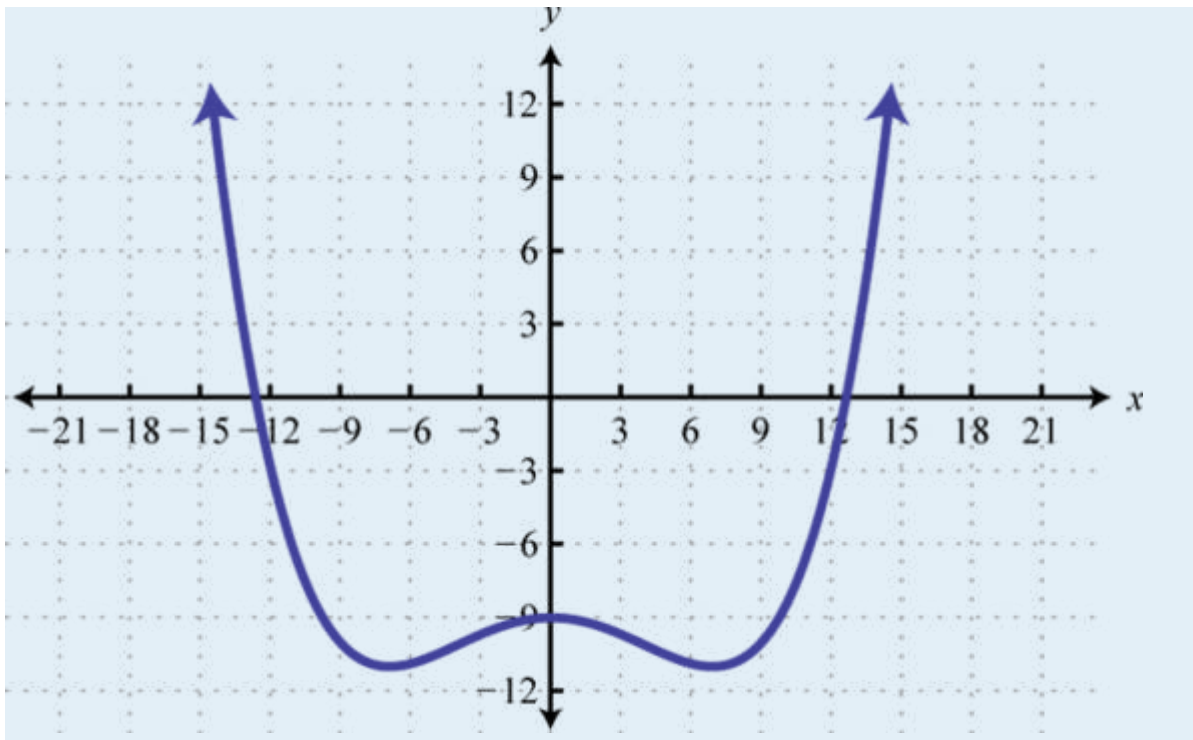
43. Find  $f(-8)$ ,  $f(0)$ , and  $f(8)$ .





44. Find  $f(-12)$ ,  $f(0)$ , and  $f(12)$ .

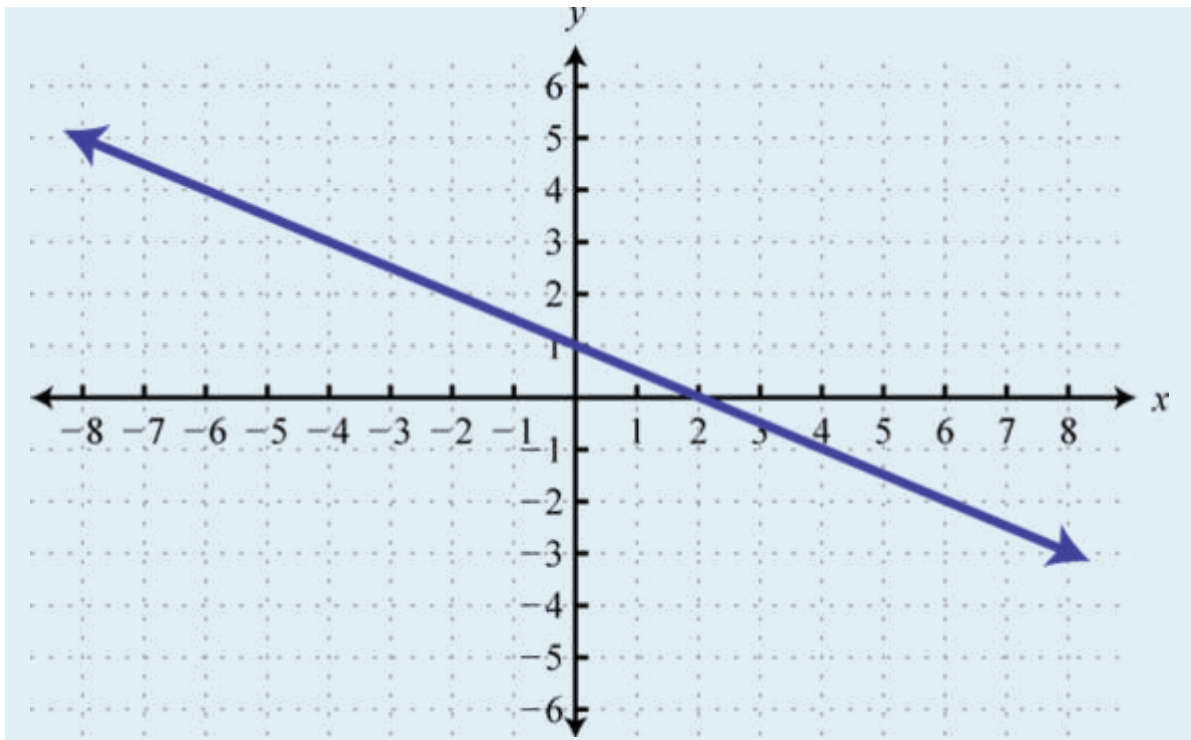




Given the graph of a function  $g$ , find the  $x$ -values.

35. Find  $x$  where  $g(x) = 3$ ,  $g(x) = 0$ , and  $g(x) = -2$ .

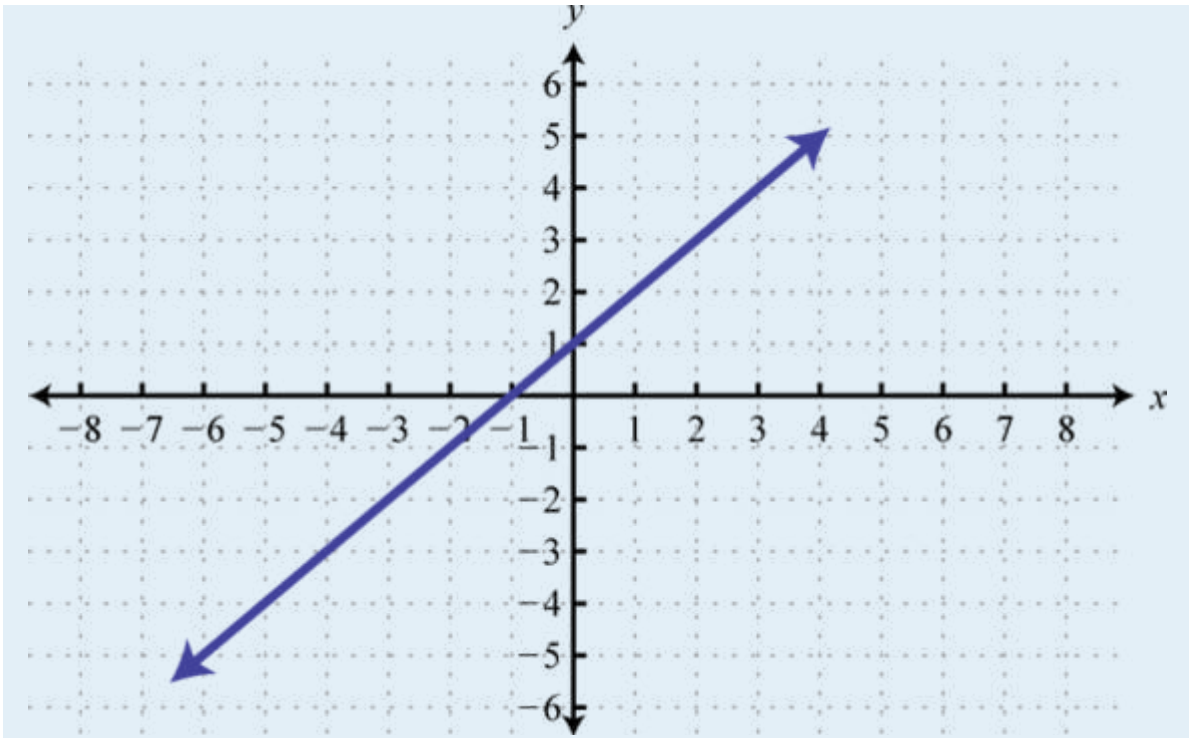




36. Find  $x$  where  $g(x) = 0$ ,  $g(x) = 1$ , and  $g(x) = 4$ .





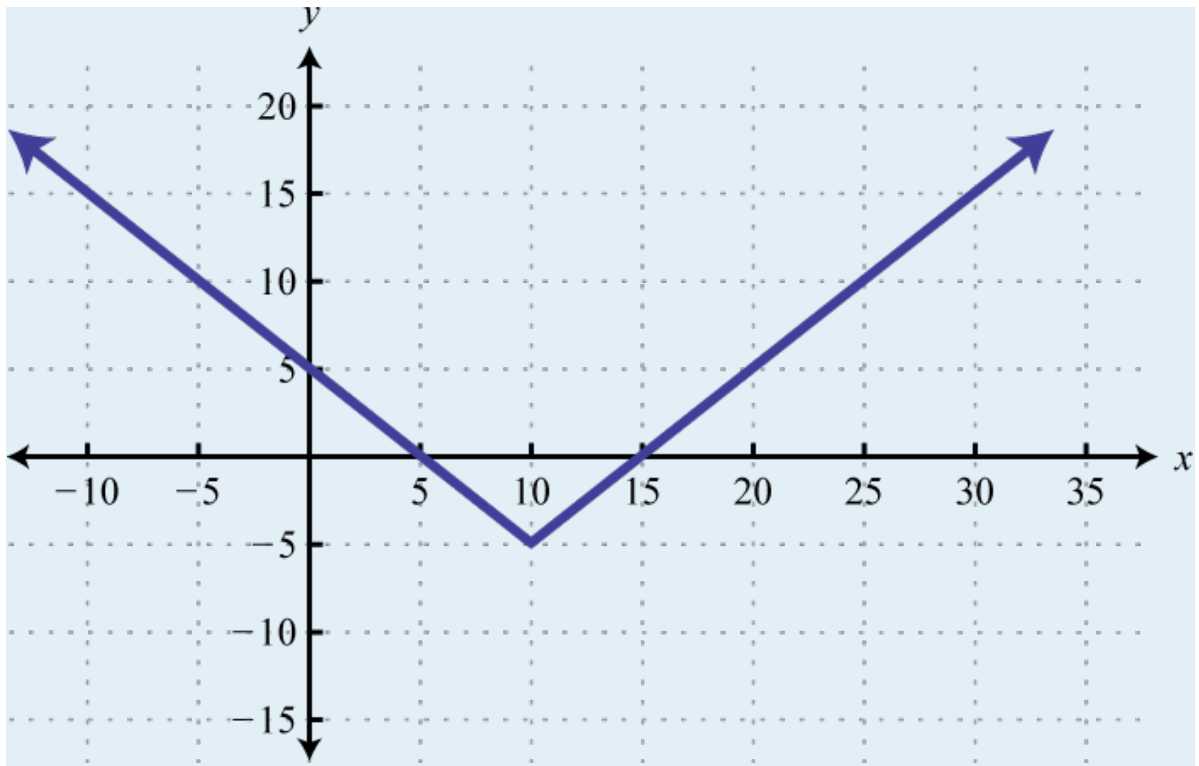


37. Find  $x$  where  $g(x) = -5$ ,  $g(x) = 0$ , and  $g(x) = 10$ .

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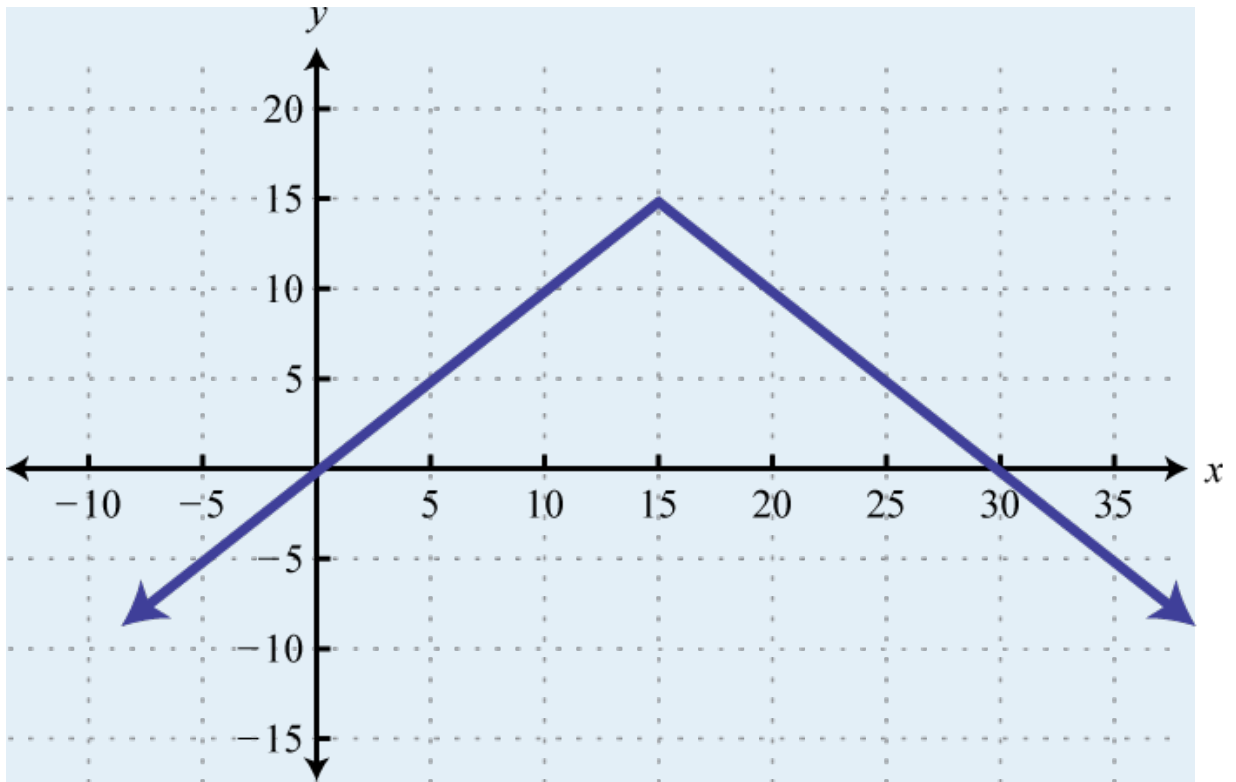
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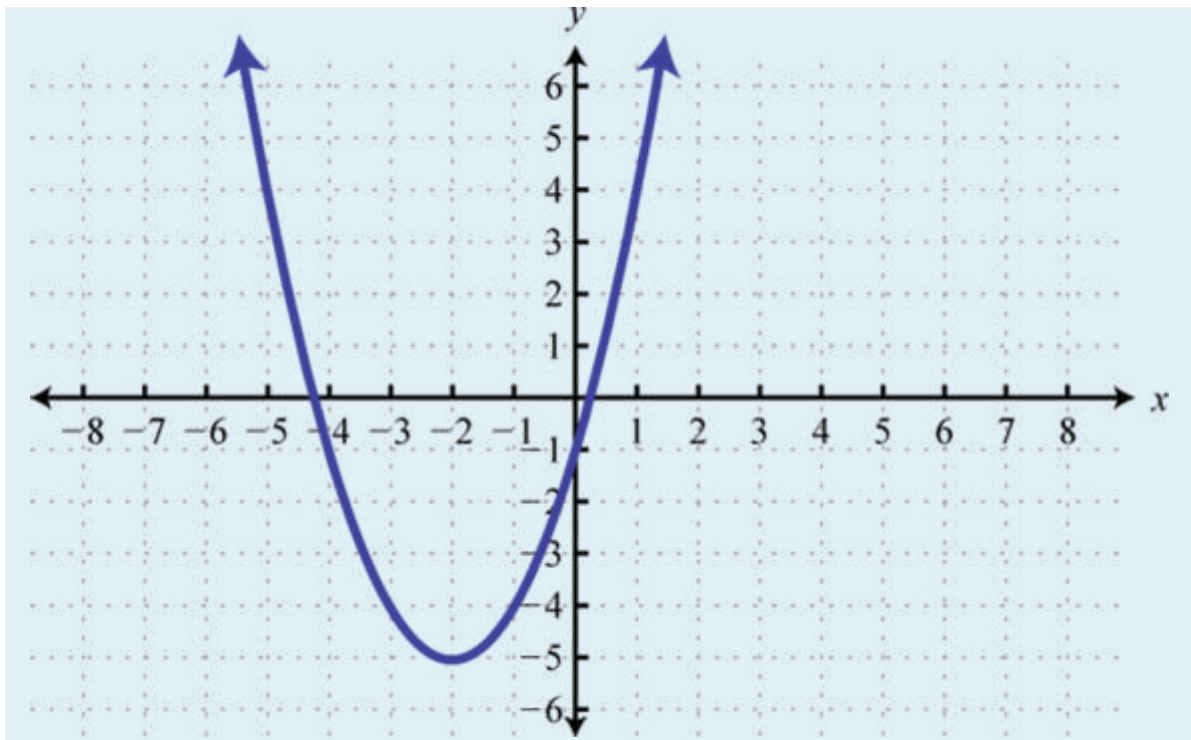
38. Find  $x$  where  $g(x) = 0$ ,  $g(x) = 10$ , and  $g(x) = 15$ .





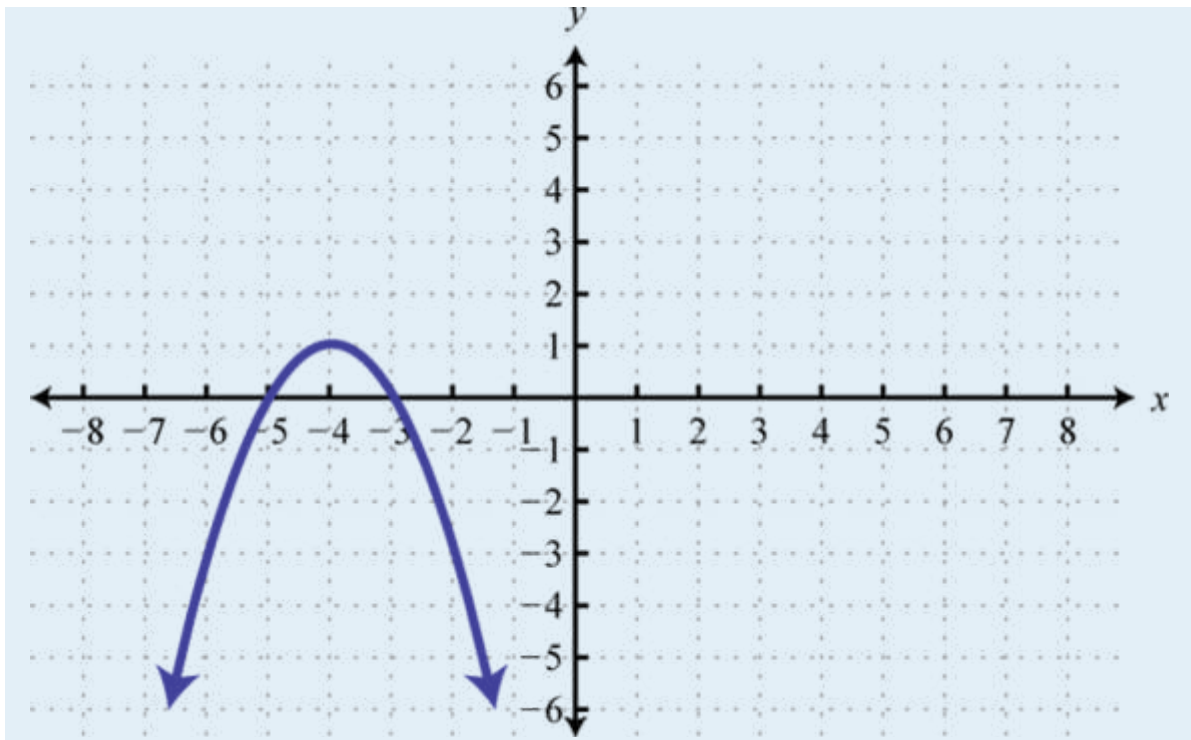
39. Find  $x$  where  $g(x) = -5$ ,  $g(x) = -4$ , and  $g(x) = 4$ .





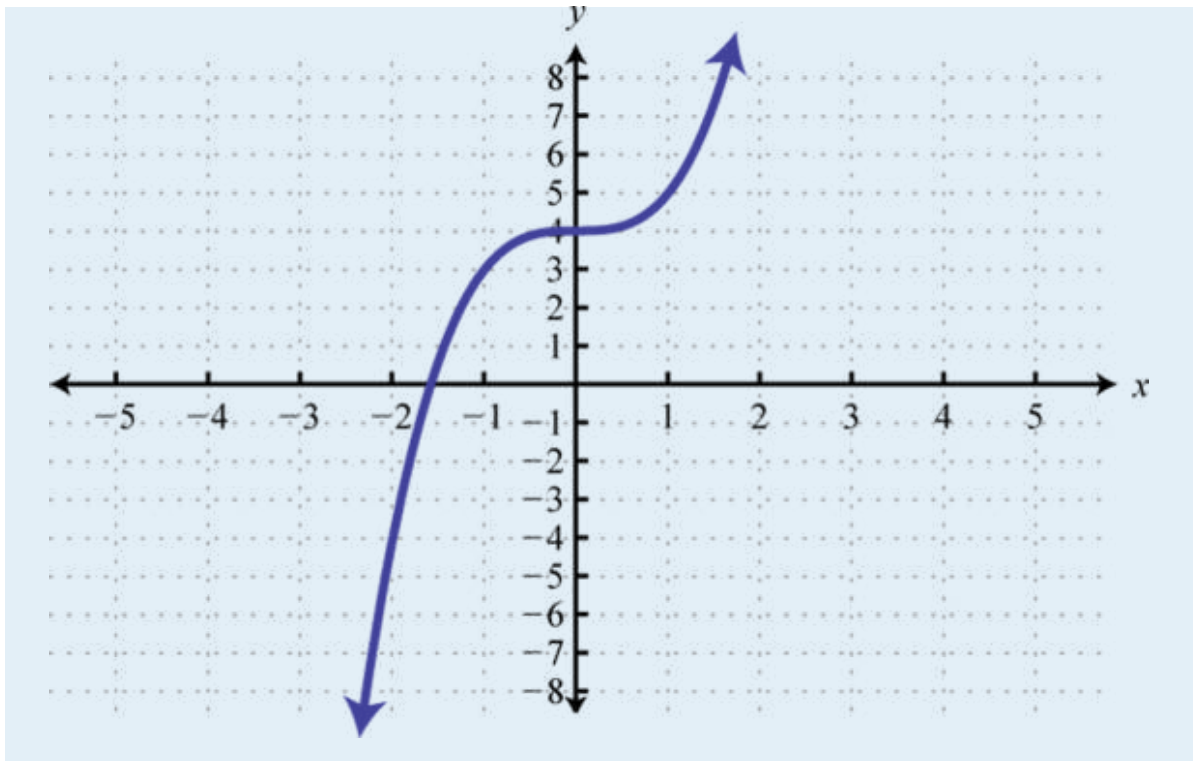
40. Find  $x$  where  $g(x) = 1$ ,  $g(x) = 0$ , and  $g(x) = -3$ .





41. Find  $x$  where  $g(x) = -4$ ,  $g(x) = 3$ , and  $g(x) = 4$ .



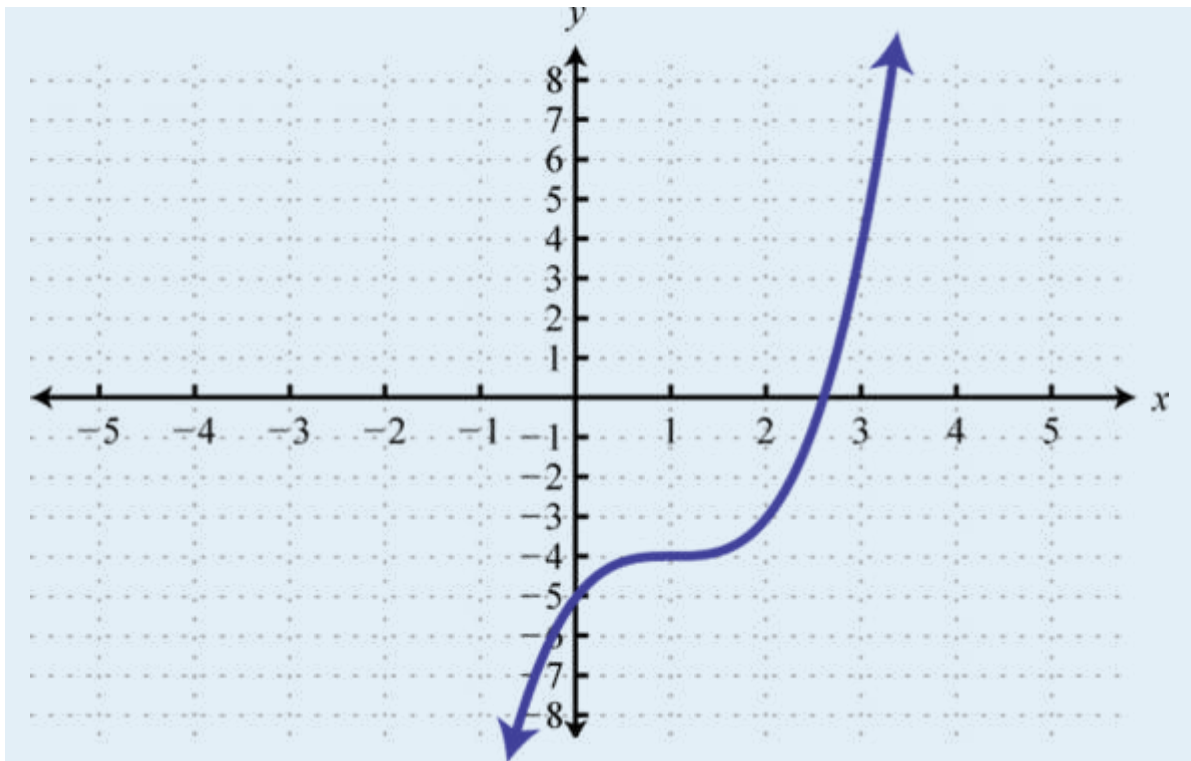


42. Find  $x$  where  $g(x) = -5$ ,  $g(x) = -4$ , and  $g(x) = 4$ .

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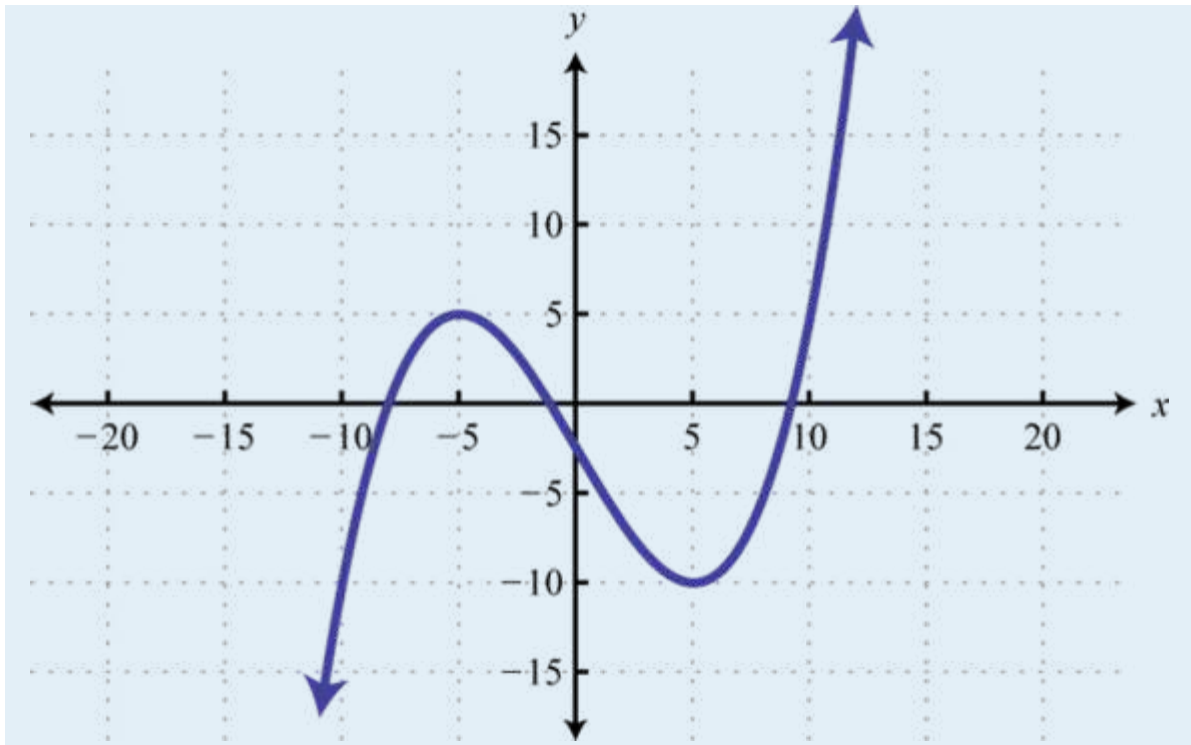
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43. Find  $x$  where  $g(x) = -10$  and  $g(x) = 5$ .

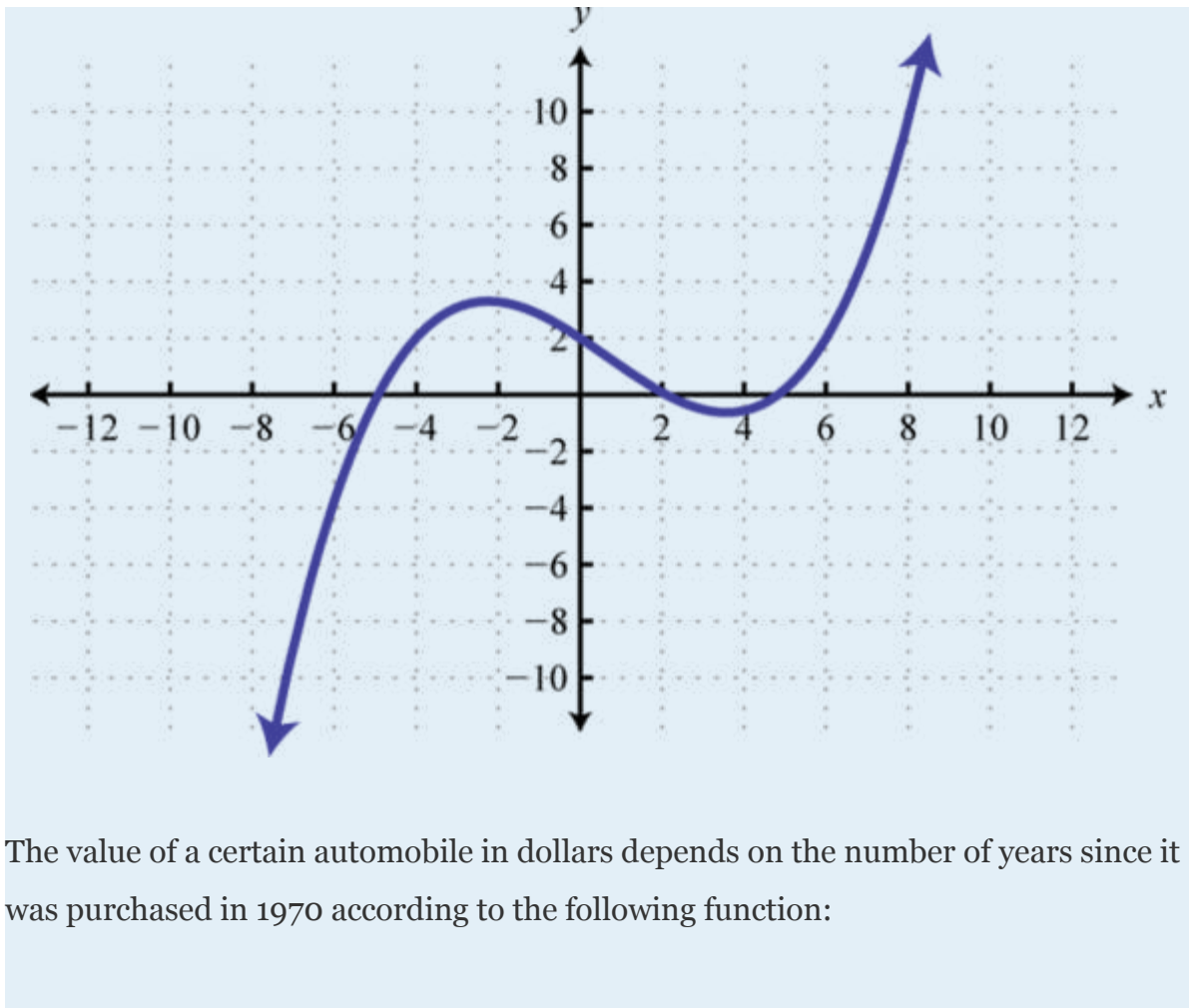




44. Find  $x$  where  $g(x) = 2$ .



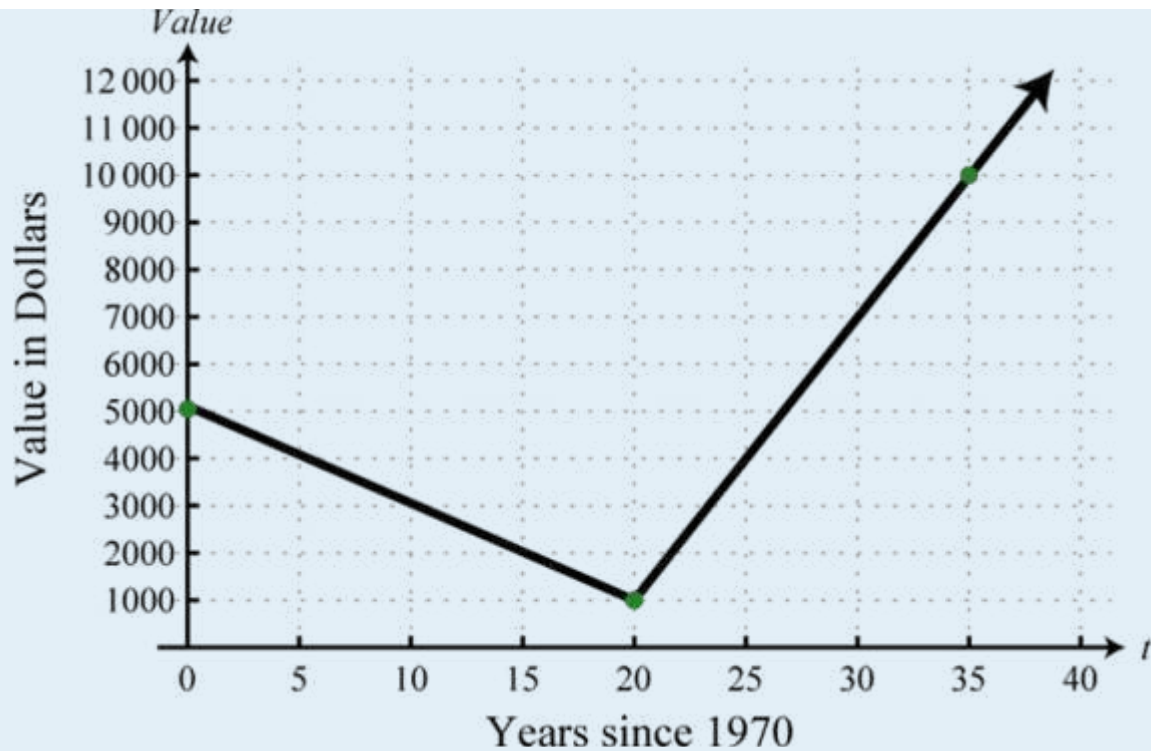




The value of a certain automobile in dollars depends on the number of years since it was purchased in 1970 according to the following function:

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35. What was the value of the car when it was new in 1970?

36. In what year was the value of the car at a minimum?

37. What was the value of the car in 2005?

38. In what years was the car valued at \$4,000?

Given the linear function defined by  $f(x) = 2x - 5$ , simplify the following.

35.  $f(5) - f(3)$

36.  $f(0) - f(7)$

37.  $f(x + 2) - f(2)$

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38.  $f(x + 7) - f(7)$

39.

40.

41. Simplify  $h$  given  $c(x) = 3x + 1$ .

42. Simplify given  $p(x) = 7x - 3$ .

43. Simplify given  $g(x) = mx + b$ .

44. Simplify given  $q(x) = ax$ .

## PART C: DISCUSSION BOARD

105. Who is credited with the introduction of the notation  $y = f(x)$ ? Provide a brief summary of his life and accomplishments.

106. Explain to a beginning algebra student what the vertical line test is and why it works.

107. Research and discuss the life and contributions of René Descartes.

108. Conduct an Internet search for the vertical line test, functions, and evaluating functions.

Share a link to a page that you think others may find useful.

## ANSWERS

1. Domain:  $\{3, 5, 7, 9, 12\}$ ; range:  $\{1, 2, 3, 4\}$ ; function: yes

2.

3. Domain:  $\{7, 8, 10, 15\}$ ; range:  $\{5, 6, 7, 8, 9\}$ ; function: no

4.

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5. Domain:  $\{5\}$ ; range:  $\{0, 2, 4, 6, 8\}$ ; function: no
- 6.
7. Domain:  $\{-4, -1, 0, 2, 3\}$ ; range:  $\{1, 2, 3\}$ ; function: yes
- 8.
9. Domain:  $\{-1, 0, 1, 2\}$ ; range:  $\{0, 1, 2, 3, 4\}$ ; function: no
- 10.
11. Domain:  $\{-2\}$ ; range:  $\{-4, -2, 0, 2, 4\}$ ; function: no
- 12.
13. Domain: ; range:  $[-2, \infty)$ ; function: yes
- 14.
15. Domain:  $(-\infty, -1]$ ; range: ; function: no
- 16.
17. Domain:  $(-\infty, 0]$ ; range:  $[-1, \infty)$ ; function: yes
- 18.
19. Domain: ; range:  $(-\infty, 3]$ ; function: yes
- 20.
21. Domain: ; range: ; function: yes
- 22.
23. Domain:  $[-5, -1]$ ; range:  $[-2, 2]$ ; function: no
- 24.
25. Domain: ; range:  $[0, \infty]$ ; function: yes
- 26.
27. Domain: ; range: ; function: yes
- 28.

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29. Domain: ; range:  $[-1, 1]$ ; function: yes
- 30.
31. Domain:  $[-8, 8]$ ; range:  $[-3, 3]$ ; function: no
- 32.
33. Domain: ; range:  $[-8, \infty]$ ; function: yes
- 34.
35.  $g(-5) = 10, g(0) = 5, g(5) = 0$
- 36.
37.  $g(-1) = 5, g(0) = 3,$
- 38.
39.  $f(-2) = -7, f(0) = -3, f(x-3) = 2x-9$
- 40.
41.  $g(-3) = -1, g(0) = 1, g(9x + 6) = 6x + 5$
- 42.
43.  $g(-5) = 25, 3,$
- 44.
45.  $f(0) = -2, f(2) = 0,$
- 46.
47. , ,
- 48.
49.  $f(-1) = -2, f(0) = -1,$
- 50.
51. ,  $g(-8) = 0,$
- 52.

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53.  $f(-1) = 0, f(0) = 1,$
- 54.
55.  $f(x + h) = 3x + 3h - 1$
- 56.
- 57.
- 58.
- 59.
60.  $x = 14$
- 61.
- 62.
- 63.
64.  $x = 1$
- 65.
66. New: \$22,000; 4 yrs old: \$14, 800
- 67.
68.  $f(0) = 5, f(2) = 1, f(4) = 5$
- 69.
70.  $f(0) = 0, f(2) = 2, f(4) = 0$
- 71.
72.  $f(-4) = 3, f(0) = 3, f(2) = 3$
- 73.
74.  $f(-2) = 1, f(2) = 3, f(7) = 4$
- 75.
76.  $f(-8) = 10, f(0) = 0, f(8) = 10$
- 77.

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78.  $g(-4) = 3, g(2) = 0,$  and  $g(6) = -2.$
- 79.
80.  $g(10) = -5, g(5) = 0$  and  $g(15) = 0, g(-5) = 10$  and  $g(25) = 10$
- 81.
82.  $g(-2) = -5, g(-3) = -4$  and  $g(-1) = -4, g(-5) = 4$  and  $g(1) = 4$
- 83.
84.  $g(-2) = -4, g(-1) = 3, g(0) = 4$
- 85.
86.  $g(-10) = -10$  and  $g(5) = -10; g(-5) = 5$  and  $g(10) = 5$
- 87.
88. \$5,000
- 89.
90. \$10,000
- 91.
92. 4
- 93.
94.  $2x$
- 95.
96.  $2h$
- 97.
98. 3
- 99.
100.  $m$
- 101.

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105. Answer may vary

106.

107. Answer may vary

## 2.2 Linear Functions and Their Graphs

### LEARNING OBJECTIVES

1. Graph a line by plotting points.
2. Determine the slope of a line.
3. Identify and graph a linear function using the slope and  $y$ -intercept.

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4. Interpret solutions to linear equations and inequalities graphically.

## A Review of Graphing Lines

Recall that the set of all solutions to a linear equation can be represented on a rectangular coordinate plane using a straight line through at least two points; this line is called its graph. For example, to graph the linear equation  $8x + 4y = 12$  we would first solve for  $y$ .

Written in this form, we can see that  $y$  depends on  $x$ ; in other words,  $x$  is the **independent variable** and  $y$  is the **dependent variable**. Choose at least two  $x$ -values and find the corresponding  $y$ -values. It is a good practice to choose zero, some negative numbers, as well as some positive numbers. Here we will choose five  $x$  values, determine the corresponding  $y$ -values, and then form a representative set of ordered pair solutions.

$x$	$y$	$y = -2x + 3$	<i>Solutions</i>
-2	7	$y = -2(-2) + 3 = 4 + 3 = 7$	$(-2, 7)$
-1	5	$y = -2(-1) + 3 = 2 + 3 = 5$	$(-1, 5)$
0	3	$y = -2(0) + 3 = 0 + 3 = 3$	$(0, 3)$
4	-5	$y = -2(4) + 3 = -8 + 3 = -5$	$(4, -5)$

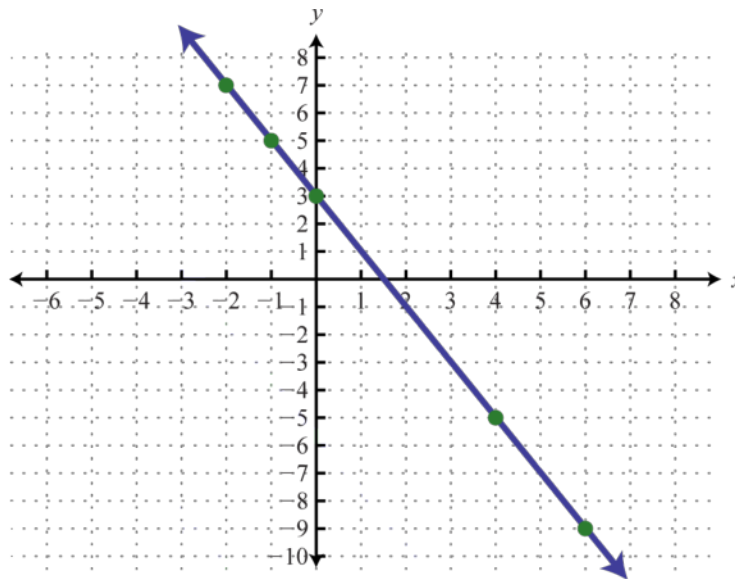
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$x$	$y$	$y = -2x + 3$	<i>Solutions</i>
6	-9	$y = -2(6) + 3 = -12 + 3 = -9$	(6, -9)

Plot the points and draw a line through the points with a straightedge. Be sure to add arrows on either end to indicate that the graph extends indefinitely.

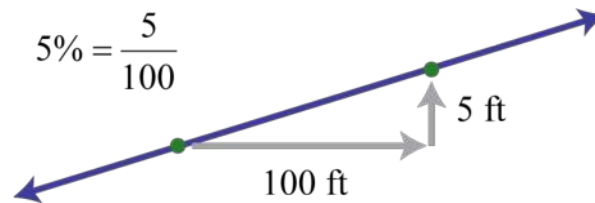


The resulting line represents all solutions to  $8x + 4y = 12$ , of which there are infinitely many. The above process describes the technique for graphing known as **plotting points**. This technique will be used to graph more complicated functions as we progress in this course.

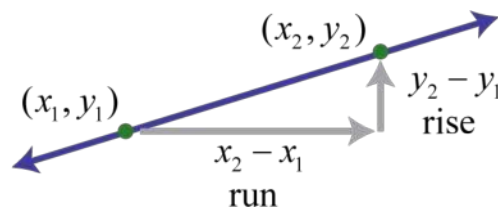
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The steepness of any incline can be measured as the ratio of the vertical change to the horizontal change. For example, a 5% incline can be written as  $\frac{5}{100}$ , which means that for every 100 feet forward, the height increases 5 feet.



In mathematics, we call the incline of a line the **slope**, denoted by the letter  $m$ . The vertical change is called the **rise** and the horizontal change is called the **run**. Given any two points  $(x_1, y_1)$  and  $(x_2, y_2)$ , we can obtain the rise and run by subtracting the corresponding coordinates.



This leads us to the **slope formula**. Given any two points  $(x_1, y_1)$  and  $(x_2, y_2)$ , the slope is given by:

The Greek letter delta ( $\Delta$ ) is often used to describe the change in a quantity. Therefore, the slope is sometimes described using the notation  $\frac{\Delta y}{\Delta x}$ , which represents the change in  $y$  divided by the change in  $x$ .



## Example 1

Find the slope of the line passing through  $(-3, -5)$  and  $(2, 1)$ .

Solution:

Given  $(-3, -5)$  and  $(2, 1)$ , calculate the difference of the  $y$ -values divided by the difference of the  $x$ -values. Take care to be consistent when subtracting the coordinates:

It does not matter which point you consider to be the first and second. However, because subtraction is not commutative, you must take care to subtract the coordinates of the first point from the coordinates of the second point in the same order. For example, we obtain the same result if we apply the slope formula with the points switched:

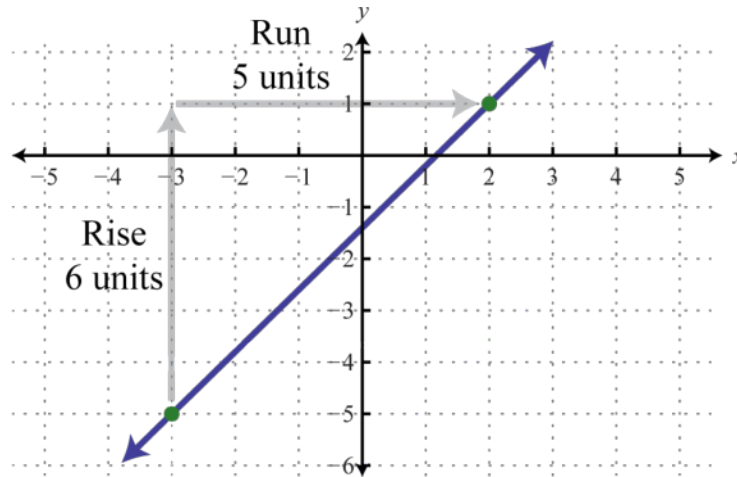
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Answer:

Verify that the slope is by graphing the line described in the previous example.



Certainly the graph is optional; the beauty of the slope formula is that, given any two points, we can obtain the slope using only algebra.

## Example 2

Find the  $y$ -value for which the slope of the line passing through  $(6, -3)$  and  $(-9, y)$  is

Solution:

Substitute the given information into the slope formula.

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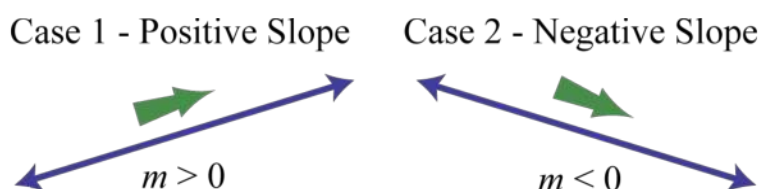
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After substituting in the given information, the only variable left is  $y$ . Solve.

Answer:  $y = 7$

There are four geometric cases for the value of the slope.



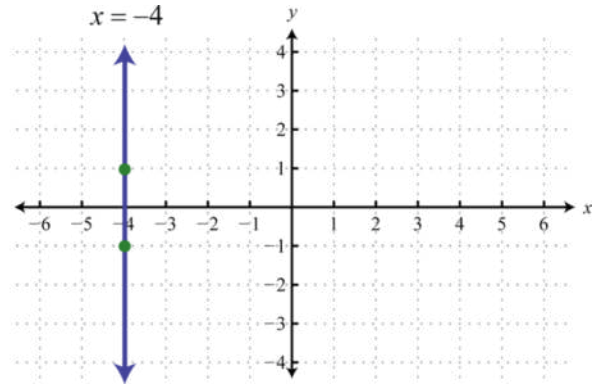
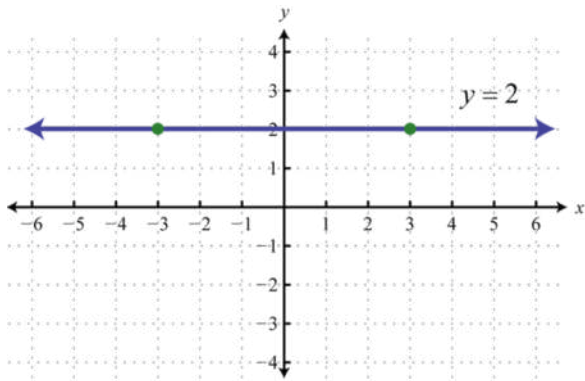
Reading the graph from left to right, lines with an upward incline have positive slopes and lines with a downward incline have negative slopes. The other two cases involve horizontal and vertical lines. Recall that if  $k$  is a real number we have

For example, if we graph  $y = 2$  we obtain a horizontal line, and if we graph  $x = -4$  we obtain a vertical line.

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From the graphs we can determine two points and calculate the slope using the slope formula.

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<i>Horizontal Line</i>	<i>Vertical Line</i>

Notice that the points on the horizontal line share the same  $y$ -values. Therefore, the rise is zero and hence the slope is zero. The points on the vertical line share the same  $x$ -values. Consequently, the run is zero, leading to an undefined slope. In general,

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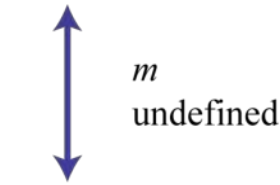
Case 3 - Zero Slope



$$m = 0$$

Horizontal Line

Case 4 - Undefined Slope



$m$

undefined

Vertical Line

## Linear Functions

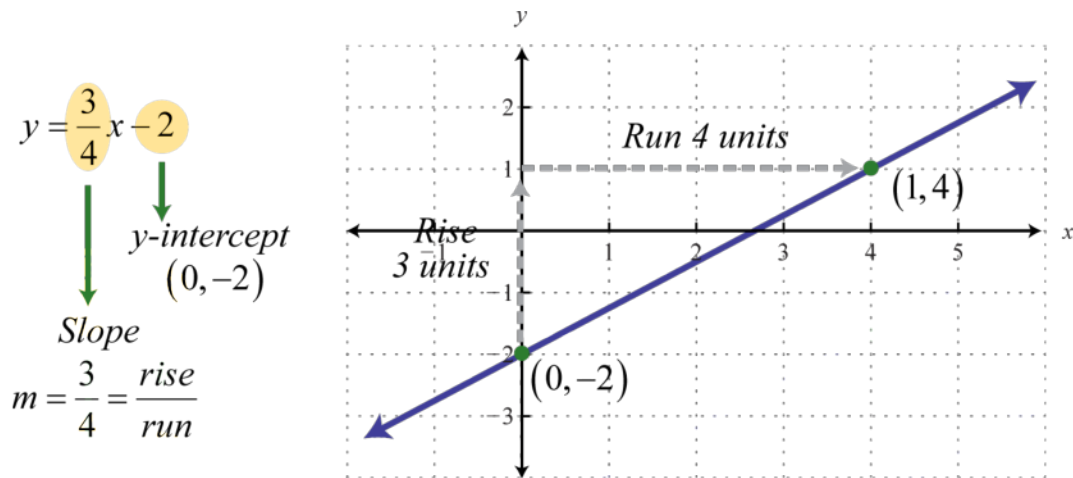
Given any linear equation in **standard form**,  $ax + by = c$ , we can solve for  $y$  to obtain **slope-intercept form**,  $y = mx + b$ . For example,

Where  $x = 0$ , we can see that  $y = -2$  and thus  $(0, -2)$  is an ordered pair solution. This is the point where the graph intersects the  $y$ -axis and is called the  $y$ -intercept. We can use this point and the slope as a means to quickly graph a line. For example, to graph , start at the  $y$ -intercept  $(0, -2)$  and mark off the slope to find a second point. Then use these points to graph the line as follows:

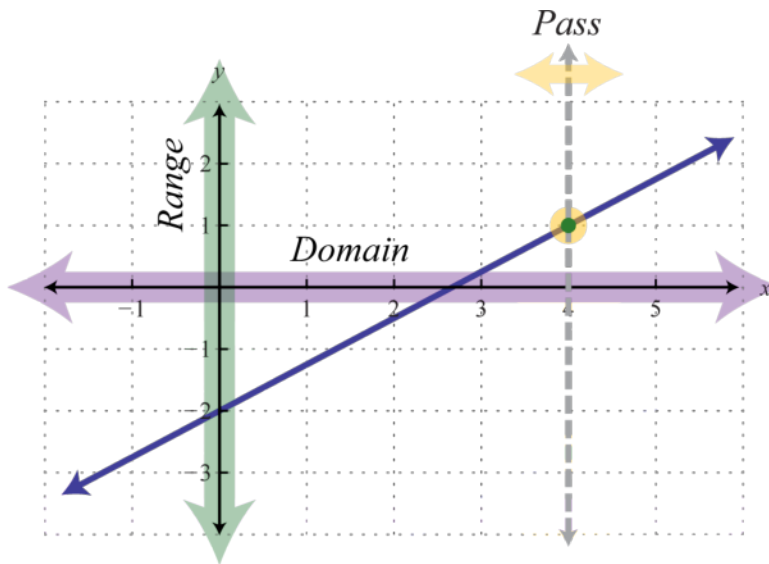
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The vertical line test indicates that this graph represents a function. Furthermore, the domain and range consists of all real numbers.



In general, a **linear function** is a function that can be written in the form

$$f(x) = mx + b \text{ Linear Function}$$

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where the slope  $m$  and  $b$  represent any real numbers. Because  $y = f(x)$ , we can use  $y$  and  $f(x)$  interchangeably, and ordered pair solutions on the graph  $(x, y)$  can be written in the form  $(x, f(x))$ .

$$(x, y) \Leftrightarrow (x, f(x))$$

We know that any  $y$ -intercept will have an  $x$ -value equal to zero. Therefore, the  $y$ -intercept can be expressed as the ordered pair  $(0, f(0))$ . For linear functions,

$$\begin{aligned} f(0) &= m(0) + b \\ &= b \end{aligned}$$

Hence, the  $y$ -intercept of any linear function is  $(0, b)$ . To find the **x-intercept**, the point where the function intersects the  $x$ -axis, we find  $x$  where  $y = 0$  or  $f(x) = 0$ .

### Example 3

Graph the linear function and label the  $x$ -intercept.

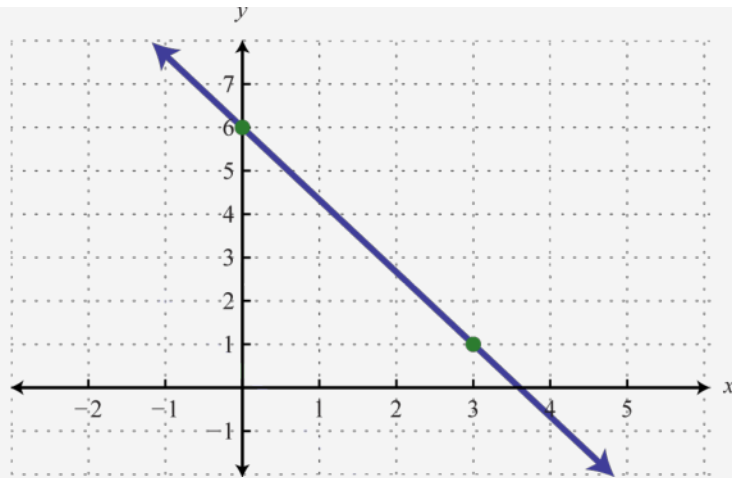
Solution:

From the function, we see that  $f(0) = 6$  (or  $b = 6$ ) and thus the  $y$ -intercept is  $(0, 6)$ . Also, we can see that the slope is  $\frac{5}{3}$ . Starting from the  $y$ -intercept, mark a second point down 5 units and right 3 units. Draw the line passing through these two points with a straightedge.

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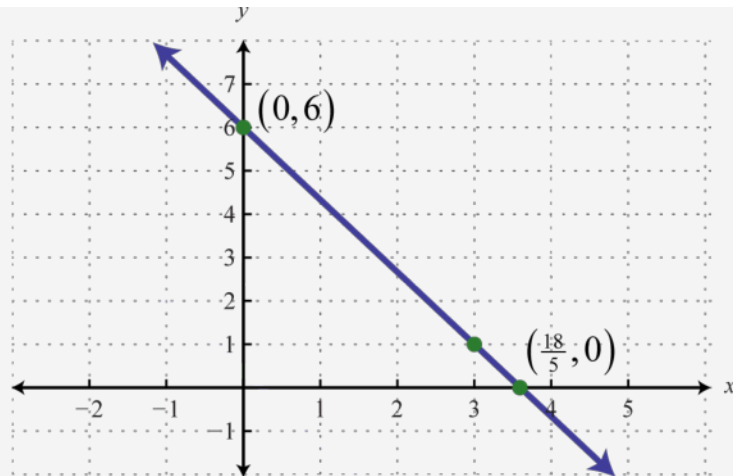


To determine the  $x$ -intercept, find the  $x$ -value where the function is equal to zero. In other words, determine  $x$  where  $f(x) = 0$ .

Therefore, the  $x$ -intercept is The general rule is to label all important points that cannot be clearly read from the graph.

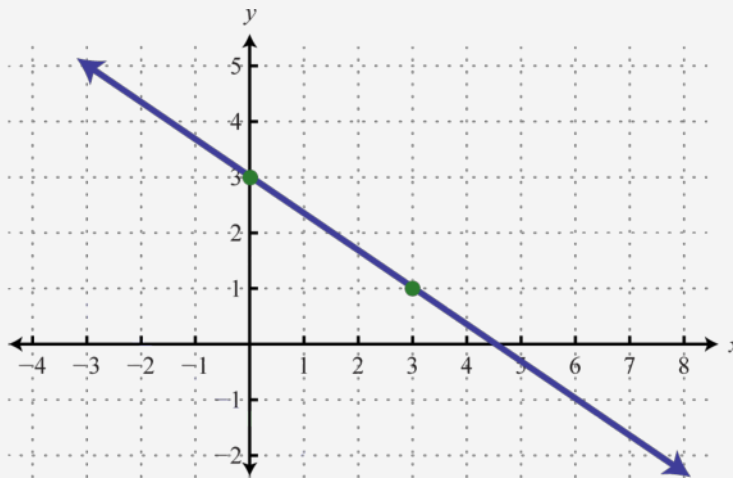
Answer:





## Example 4

Determine a linear function that defines the given graph and find the x-intercept.



Solution:

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We begin by reading the slope from the graph. In this case, two points are given and we can see that,

In addition, the  $y$ -intercept is  $(0, 3)$  and thus  $b = 3$ . We can substitute into the equation for any linear function.

To find the  $x$ -intercept, we set  $g(x) = 0$  and solve for  $x$ .

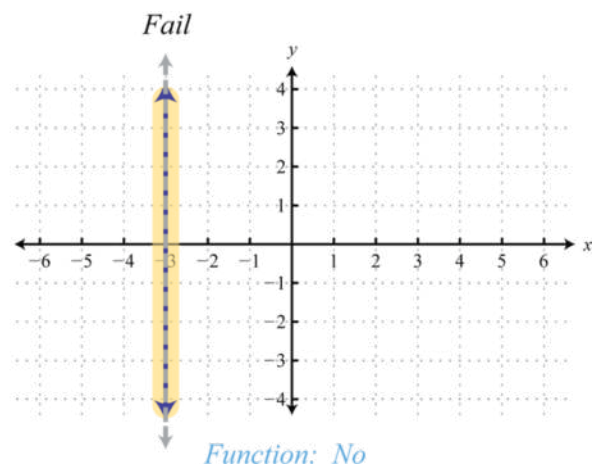
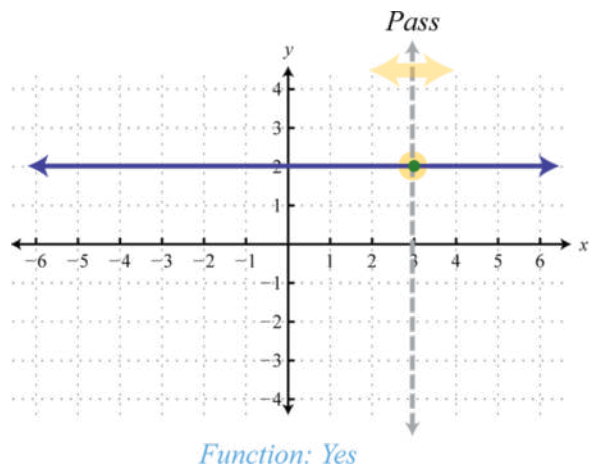
Answer: ;  $x$ -intercept:

Next, consider horizontal and vertical lines. Use the vertical line test to see that any horizontal line represents a function, and that a vertical line does not.

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Given any horizontal line, the vertical line test shows that every  $x$ -value in the domain corresponds to exactly one  $y$ -value in the range; it is a function. A vertical line, on the other hand, fails the vertical line test; it is not a function. A vertical line represents a set of ordered pairs where all of the elements in the domain are the same. This violates the requirement that functions must associate exactly one element in the range to each element in the domain. We summarize as follows:



	<i>Horizontal Line</i>	<i>Vertical Line</i>
<i>Equation:</i>	$y = 2$	$x = -3$
<i>x-intercept:</i>	None	$(-3, 0)$

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	<i>Horizontal Line</i>	<i>Vertical Line</i>
<i>y-intercept:</i>	$(0, 2)$	None
<i>Domain:</i>	$(-\infty, \infty)$	$\{-3\}$

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	<i>Horizontal Line</i>	<i>Vertical Line</i>
<i>Range:</i>	$\{2\}$	$(-\infty, \infty)$
<i>Function:</i>	Yes	No

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A horizontal line is often called a constant function. Given any real number  $c$ ,

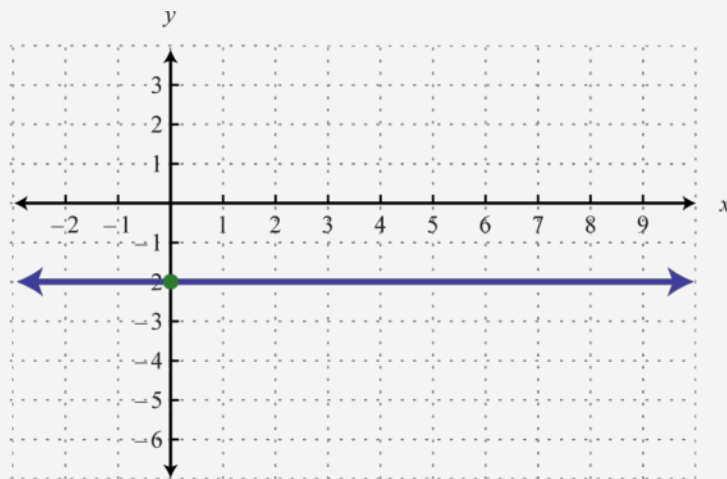
$$f(x) = c \text{ Constant Function}$$

## Example 5

Graph the constant function  $g(x) = -2$  and state the domain and range.

Solution:

Here we are given a constant function that is equivalent to  $y = -2$ . This defines a horizontal line through  $(0, -2)$ .



Answer: Domain:  $\mathbb{R}$ ; range:  $\{-2\}$

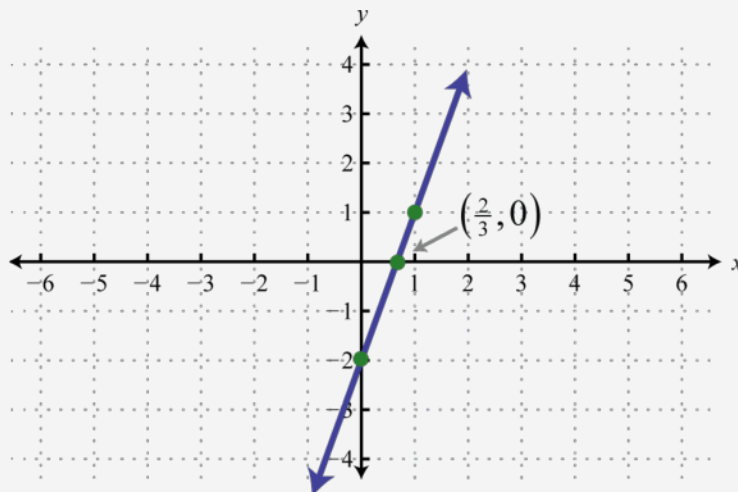
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Try this! Graph  $f(x) = 3x - 2$  and label the  $x$ -intercept.

Answer:



## Linear Equations and Inequalities: A Graphical Interpretation

We can use the ideas in this section to develop a geometric understanding of what it means to solve equations of the form  $f(x) = g(x)$ , where  $f$  and  $g$  are linear functions. Using algebra, we can solve the linear equation as follows:

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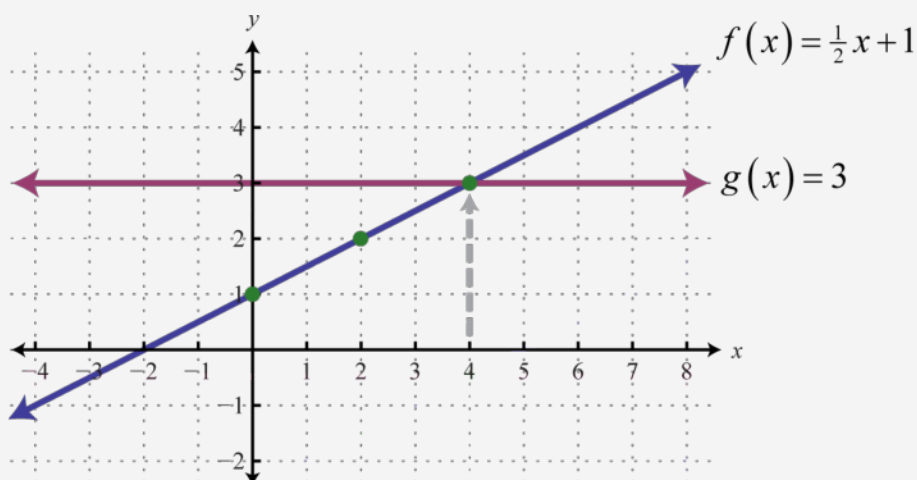
The solution to this equation is  $x = 4$ . Geometrically, this is the  $x$ -value of the intersection of the two graphs and  $g(x) = 3$ . The idea is to graph the linear functions on either side of the equation and determine where the graphs coincide.

## Example 6

Graph and  $g(x) = 3$  on the same set of axes and determine where  $f(x) = g(x)$ .

Solution:

Here  $f$  is a linear function with slope 12 and  $y$ -intercept  $(0, 1)$ . The function  $g$  is a constant function and represents a horizontal line. Graph both of these functions on the same set of axes.



From the graph we can see that  $f(x) = g(x)$  where  $x = 4$ . In other words, where  $x = 4$ .

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Answer:  $x = 4$

We can extend the geometric interpretation a bit further to solve inequalities. For example, we can solve the linear inequality , using algebra, as follows:

The solution set consists of all real numbers greater than or equal to 4. Geometrically, these are the  $x$ -values for which the graph lies above the graph of  $g(x) = 3$ .

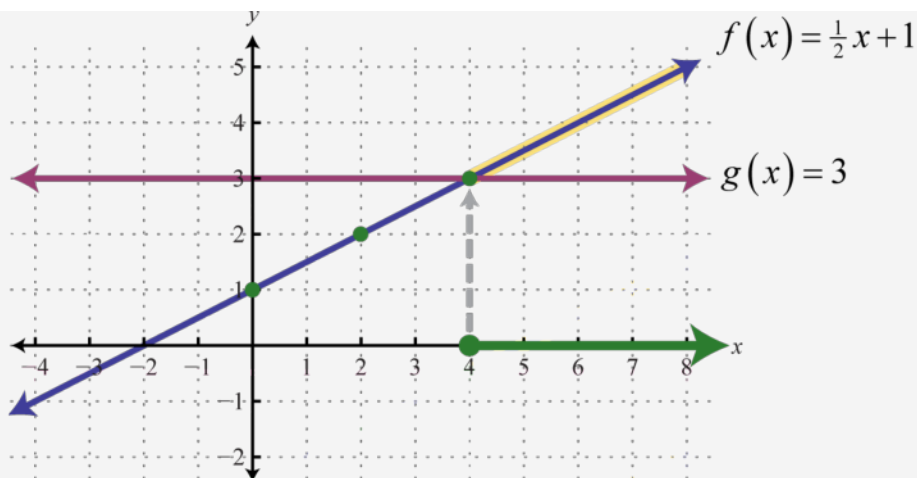
## Example 7

Graph and  $g(x) = 3$  on the same set of axes and determine where  $f(x) \geq g(x)$ .

Solution:

On the graph we can see this shaded.





From the graph we can see that  $f(x) \geq g(x)$  or where  $x \geq 4$ .

Answer: The  $x$ -values that solve the inequality, in interval notation, are  $[4, \infty)$ .

### KEY TAKEAWAYS

We can graph lines by plotting points. Choose a few values for  $x$ , find the corresponding  $y$ -values, and then plot the resulting ordered pair solutions. Draw a line through the points with a straightedge to complete the graph.

Given any two points on a line, we can calculate the slope algebraically using the slope formula,

Use slope-intercept form  $y = mx + b$  to quickly sketch the graph of a line. From the  $y$ -intercept  $(0, b)$ , mark off the slope to determine a second point. Since two points determine a line, draw a line through these two points with a straightedge to complete the graph.

Linear functions have the form  $f(x) = mx + b$ , where the slope  $m$  and  $b$  are real numbers. To find the  $x$ -intercept, if one exists, set  $f(x) = 0$  and solve for  $x$ .

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Since  $y = f(x)$  we can use  $y$  and  $f(x)$  interchangeably. Any point on the graph of a function can be expressed using function notation  $(x, f(x))$ .

## TOPIC EXERCISES

### PART A: GRAPHING LINES BY PLOTTING POINTS

Find five ordered pair solutions and graph.

1.  $y = 3x - 6$

2.  $y = 2x - 4$

3.  $y = -5x + 15$

4.  $y = -3x + 18$

5.

6.

7.

8.

9.

10.

11.  $y = 10$

12.  $x = -1$

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13.  $6x + 3y = 18$

14.  $8x - 2y = 16$

15.  $-2x + 4y = 8$

16.  $-x + 3y = 18$

17.

18.

19.  $x + y = 0$

20.  $-x + y = 0$

Find the slope of the line passing through the given points.

1.  $(-2, -4)$  and  $(1, -1)$

2.  $(-3, 0)$  and  $(3, -4)$

3. and

4.  $(-4, -3)$  and  $(-2, -3)$

5.  $(9, -5)$  and  $(9, -6)$

6. and

Find the  $y$ -value for which the slope of the line passing through given points has the given slope.

1. ;  $(6, 10), (-4, y)$

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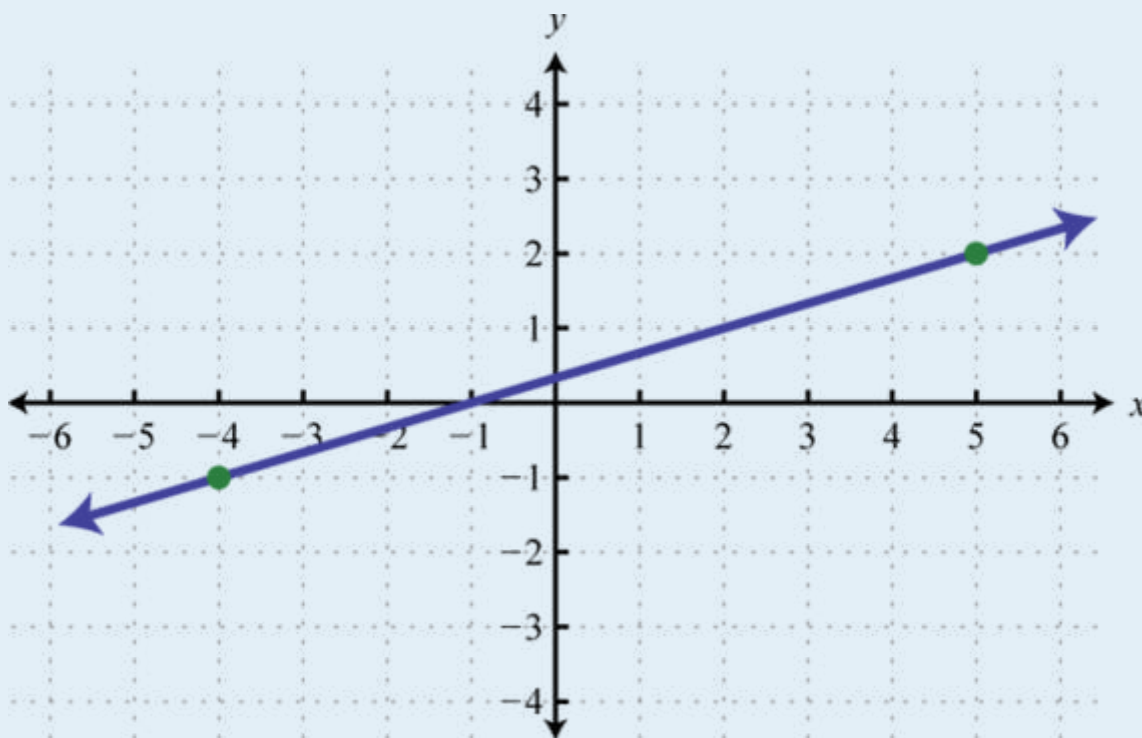
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2. ;  $(-6, 4), (9, y)$
3.  $m = -4$ ;  $(-2, 5), (-1, y)$
4.  $m = 3$ ;  $(1, -2), (-2, y)$
5. ;  $(1, y), (6, 15)$
6. ;  $(-1, y), (-4, 5)$

Given the graph, determine the slope.

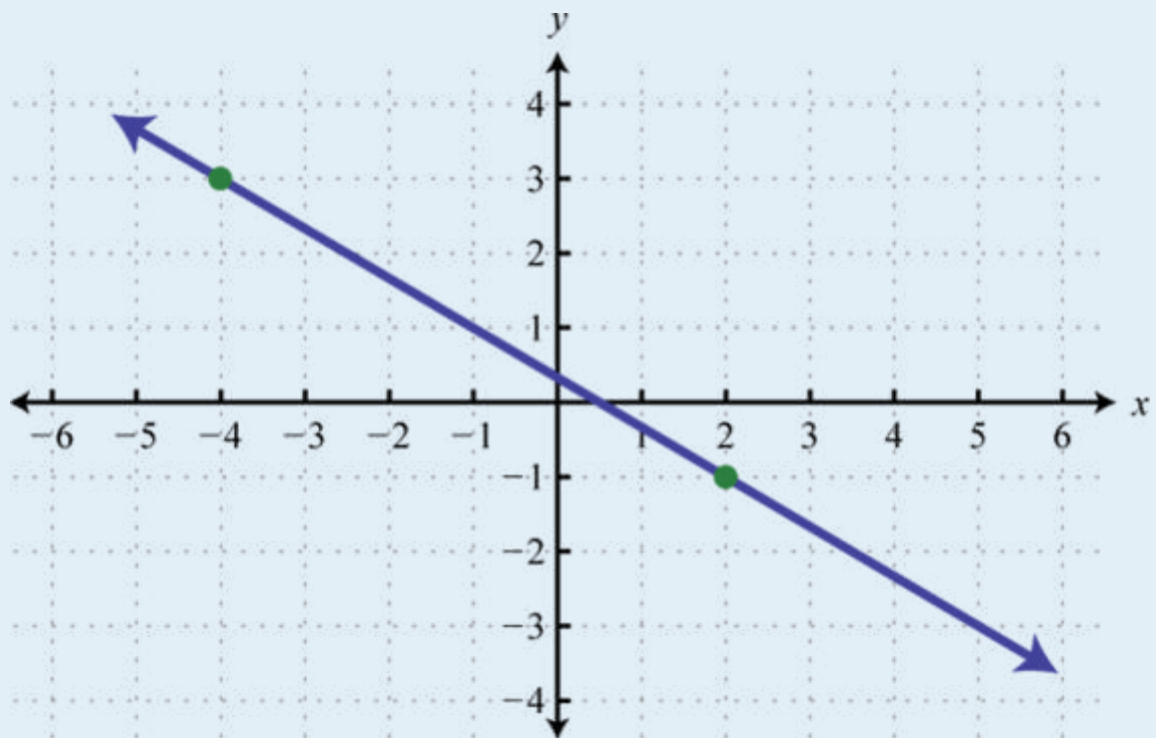
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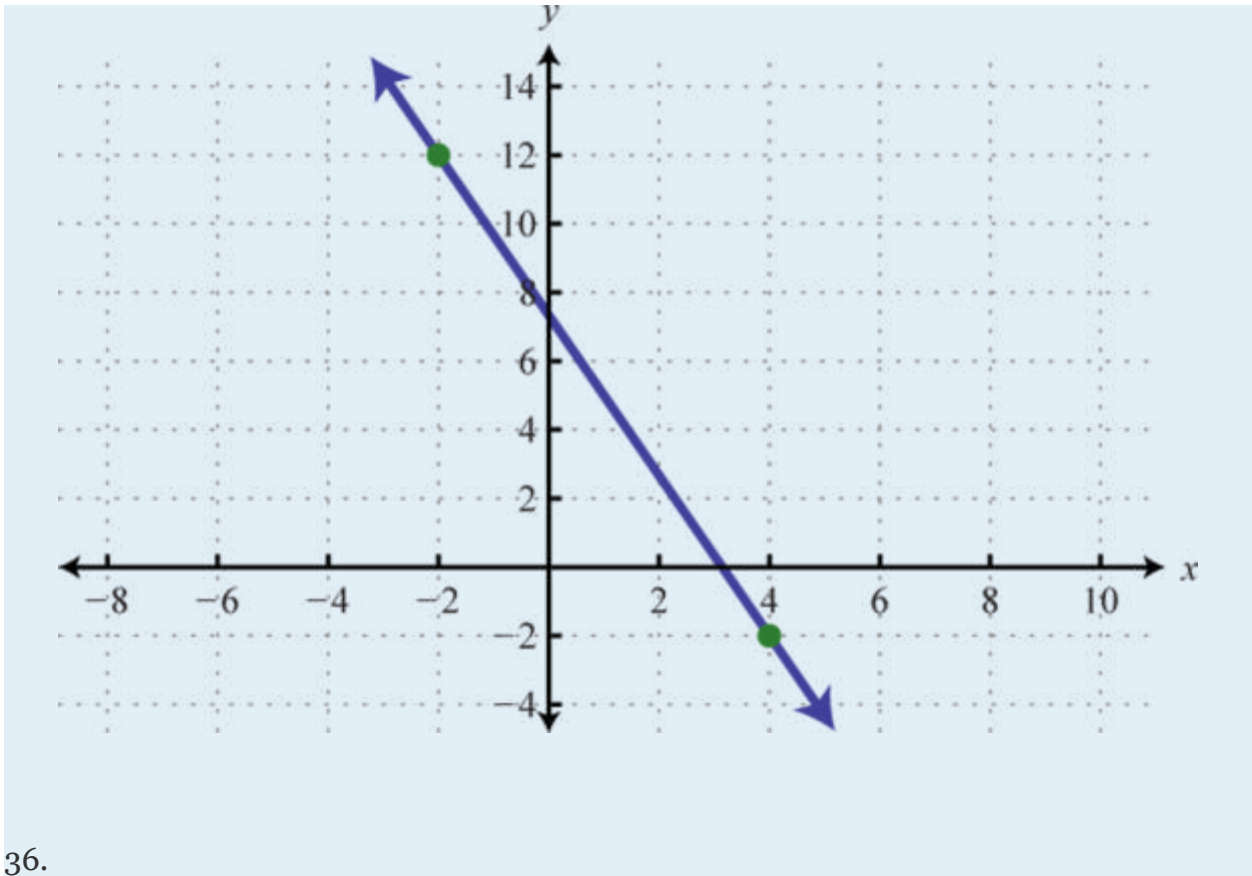
34.



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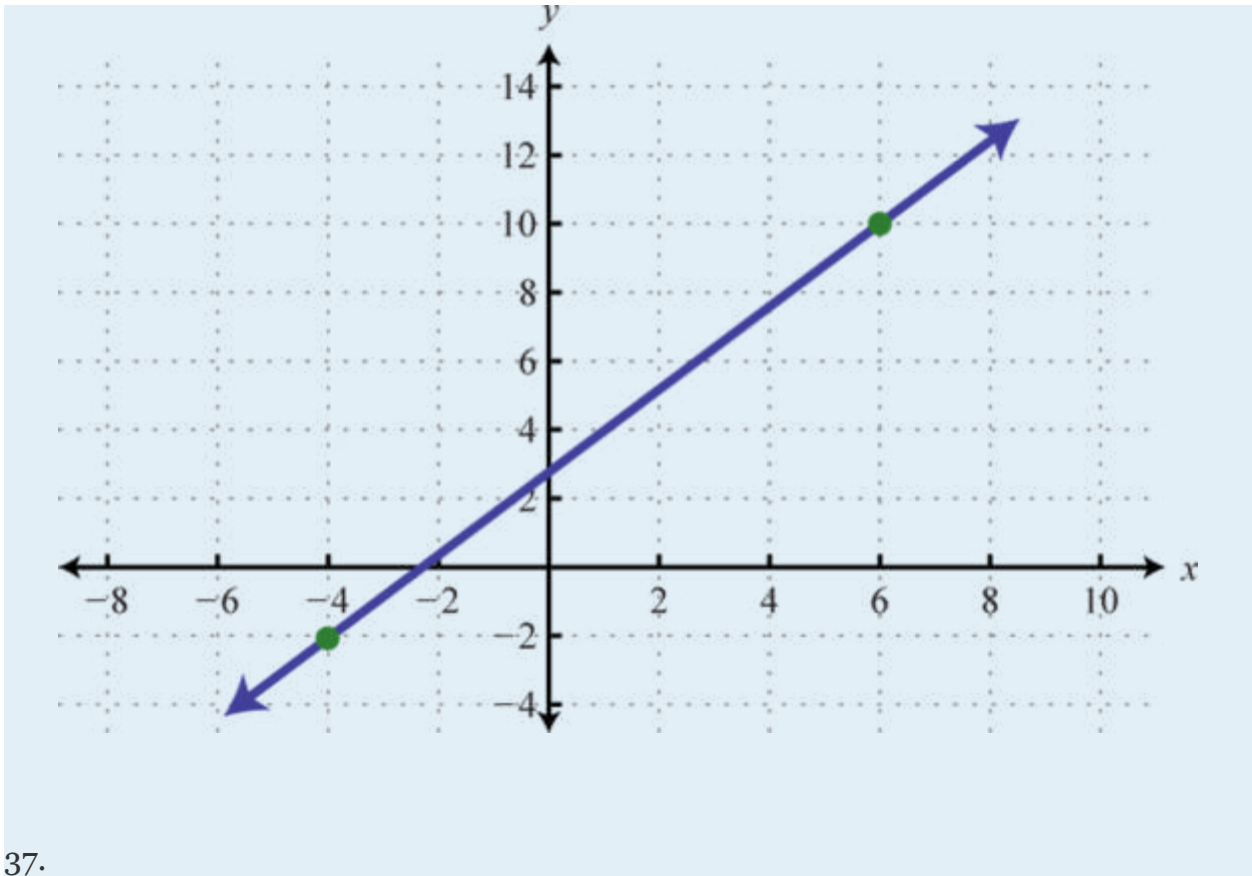
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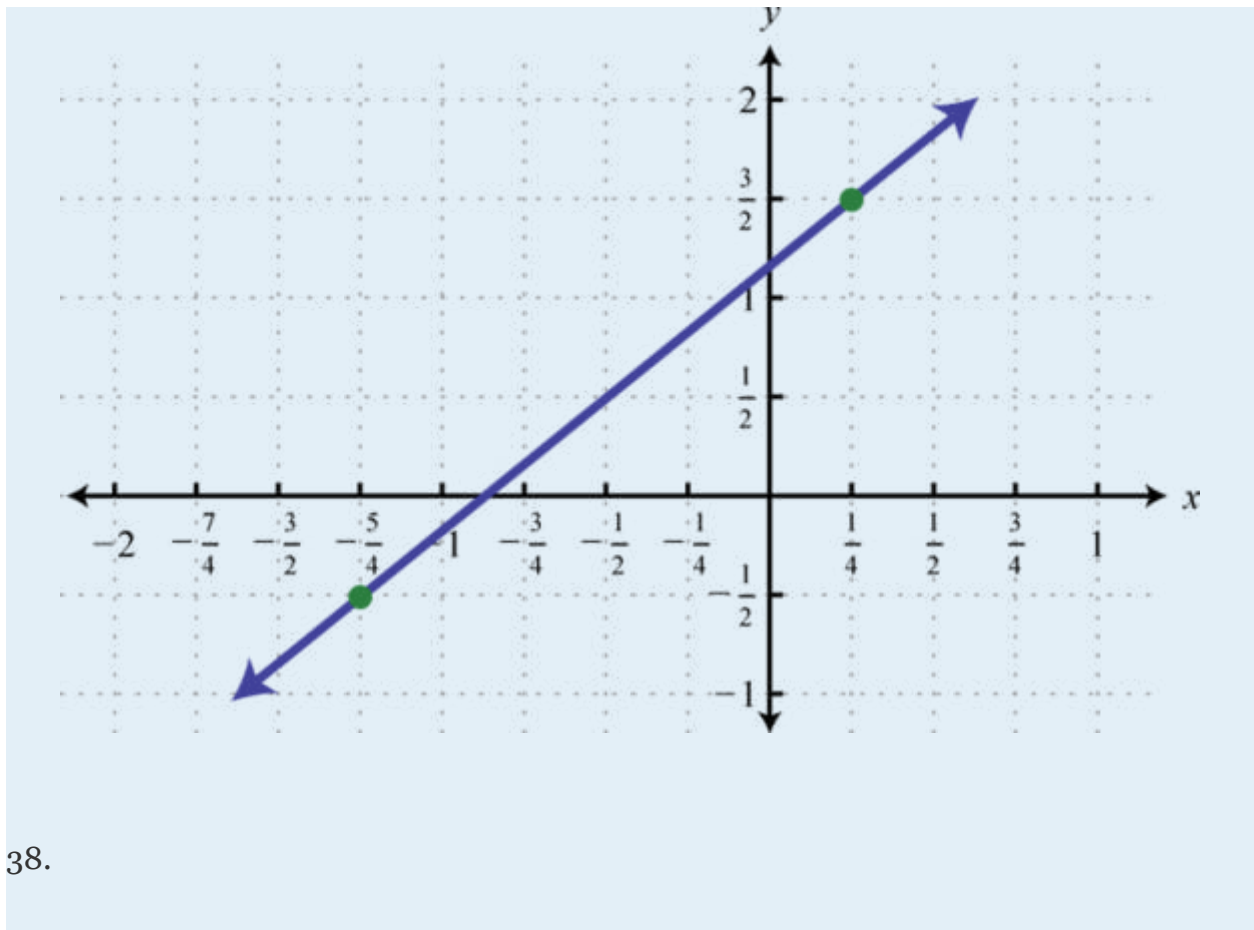




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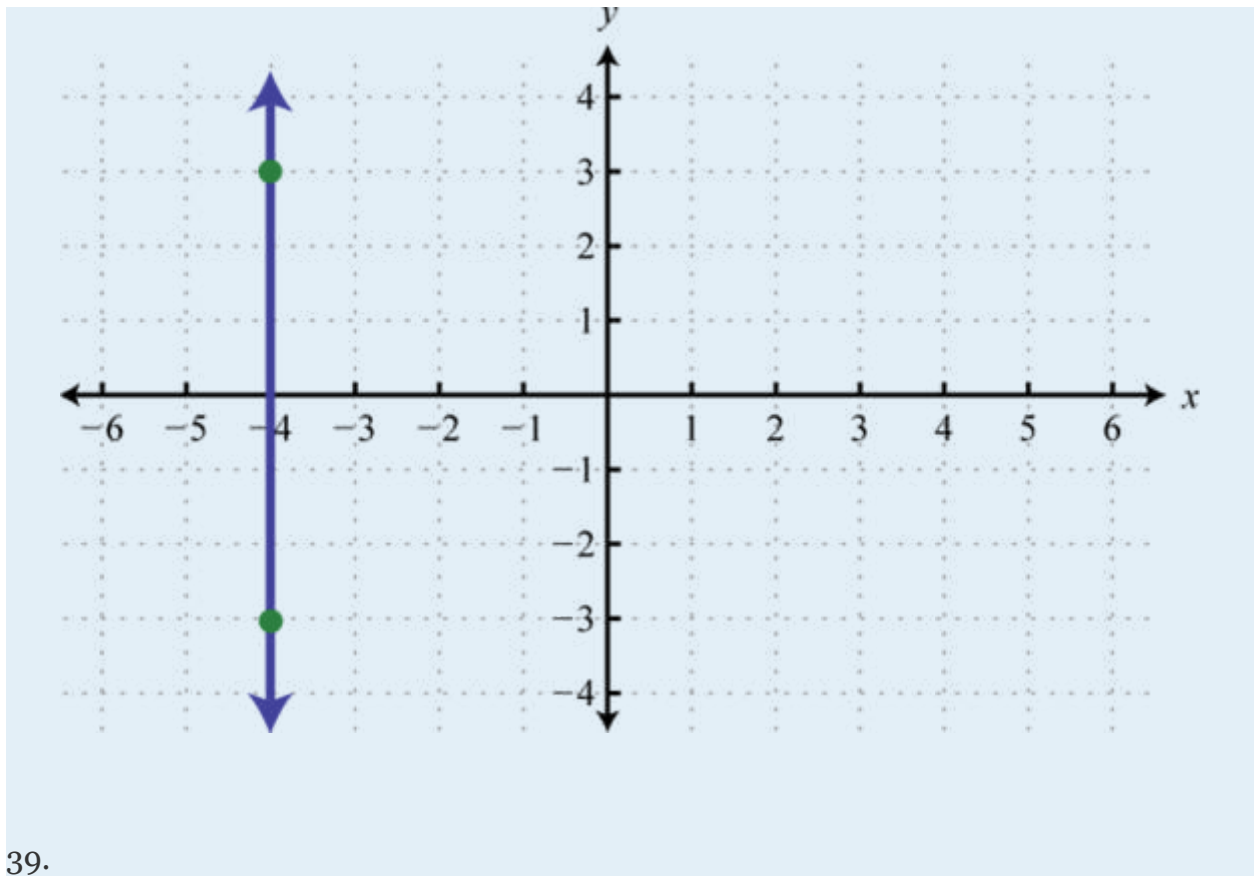




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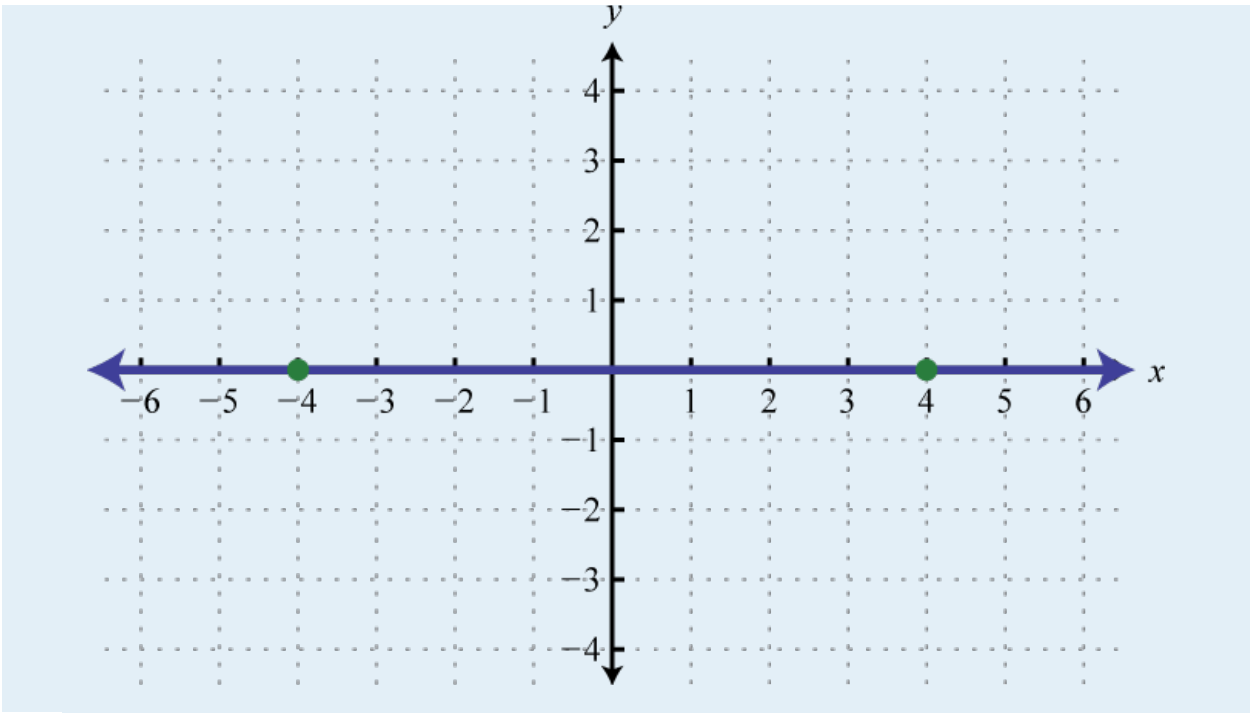




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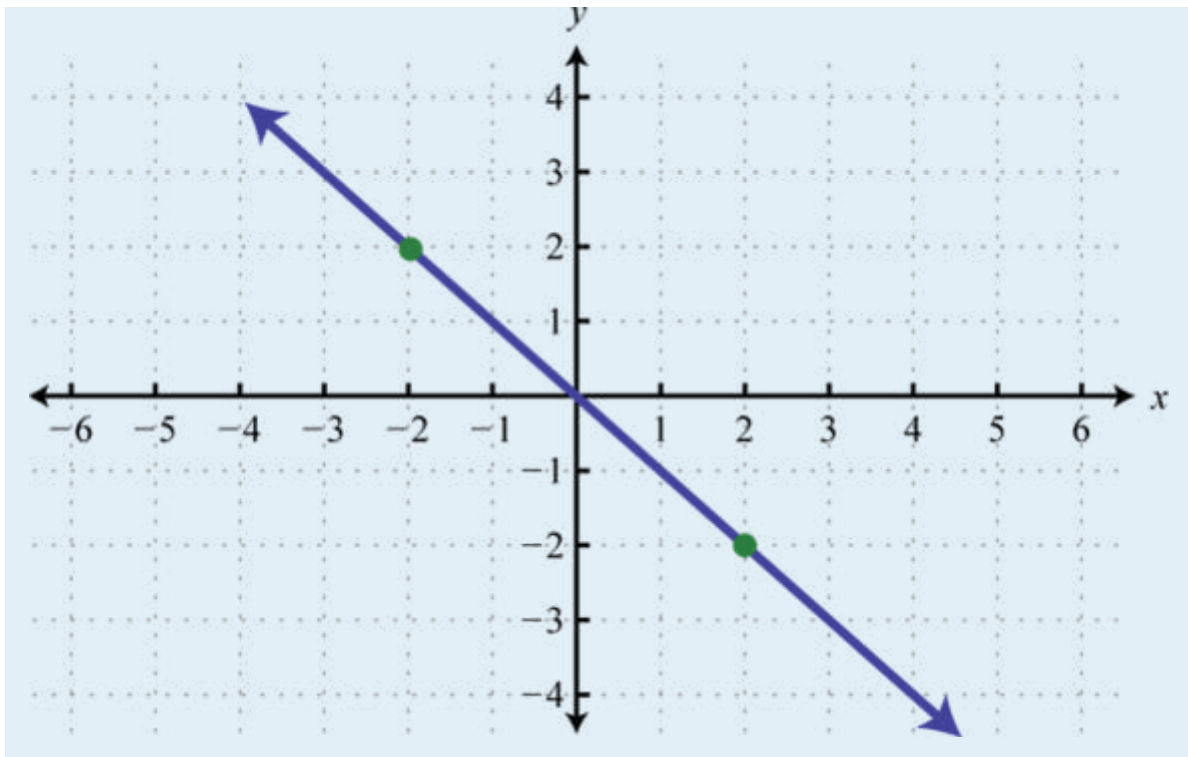


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## PART B: LINEAR FUNCTIONS

Find the  $x$ - and  $y$ -intercepts and use them to graph the following functions.

41.  $6x - 3y = 18$

42.  $8x - 2y = 8$

43.  $-x + 12y = 6$

44.  $-2x - 6y = 8$

45.  $x - 2y = 5$

46.  $-x + 3y = 1$

47.  $2x + 3y = 2$

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48.  $5x - 4y = 2$

49.  $9x - 4y = 30$

50.  $-8x + 3y = 28$

51.

52.

53.

54.

55.

56.

57.

58.

59.

60.

Graph the linear function and label the  $x$ -intercept.

41.  $f(x) = -5x + 15$

42.  $f(x) = -2x + 6$

43.  $f(x) = -x - 2$

44.  $f(x) = x + 3$

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46.

47.

48.  $f$

49.

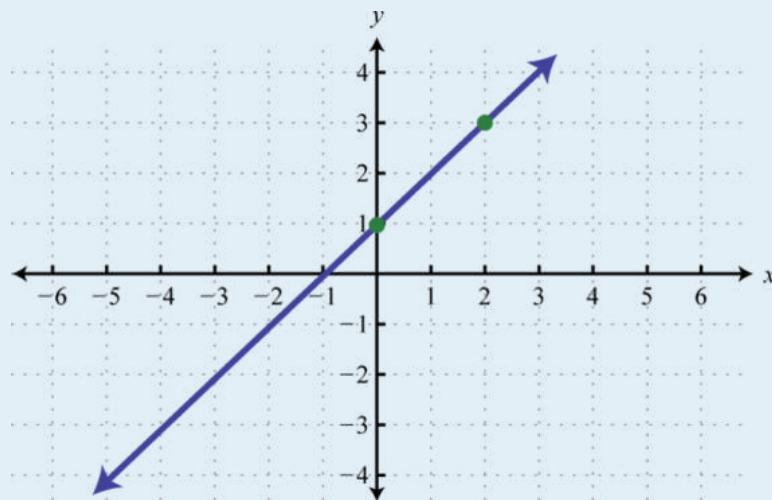
50.

51.  $f(x) = 2x$

52.  $f(x) = 3$

Determine the linear function that defines the given graph and find the  $x$ -intercept.

73.

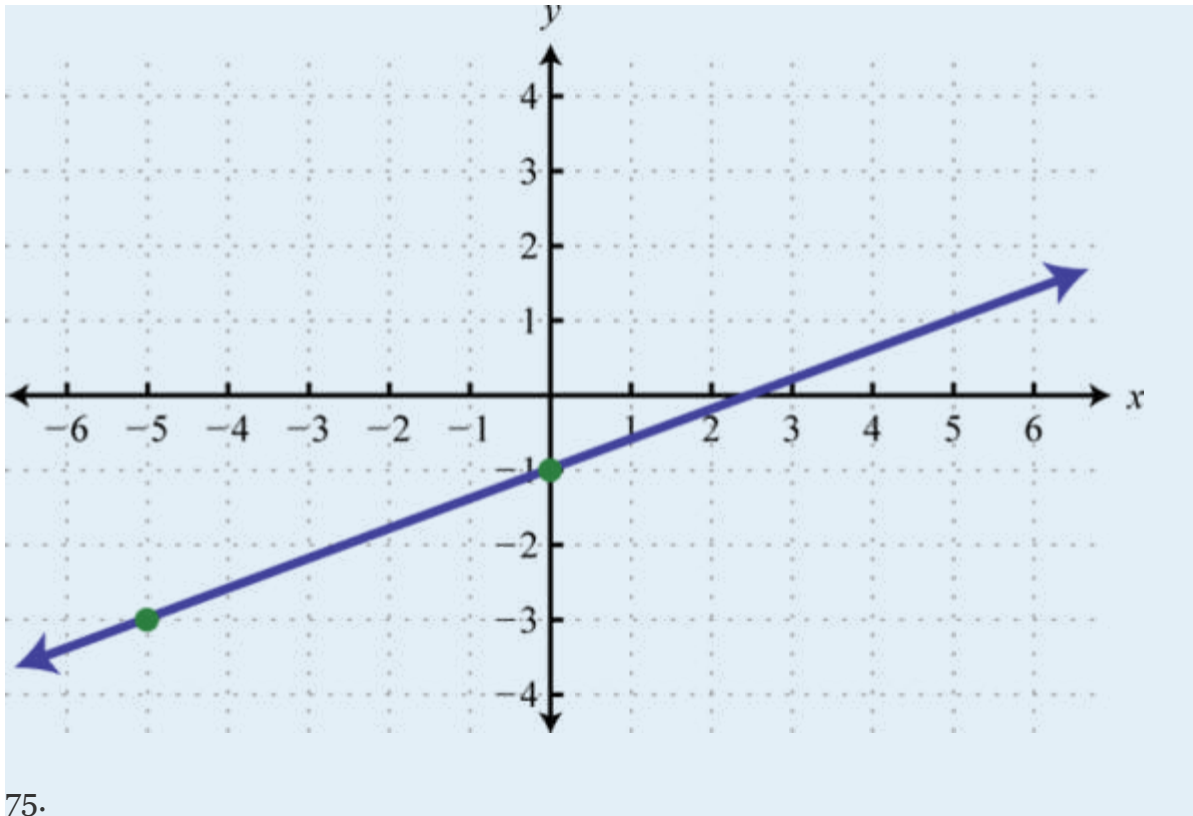


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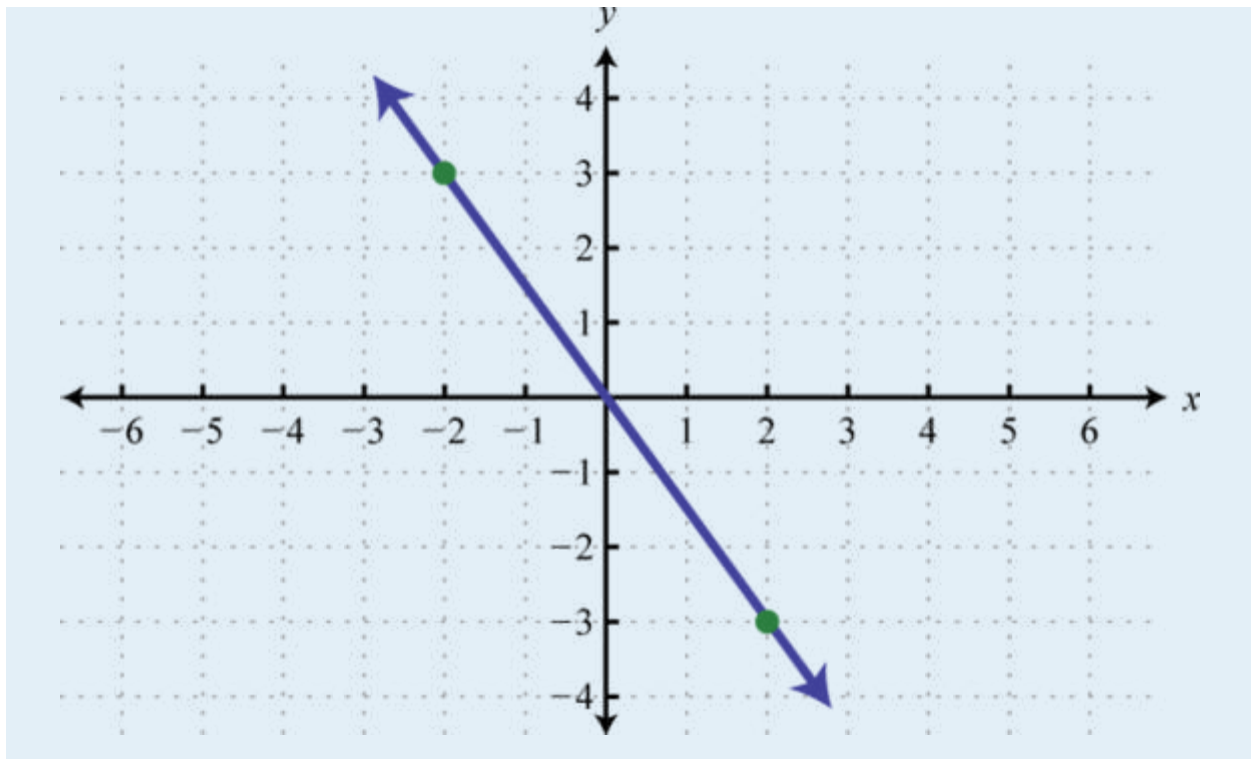




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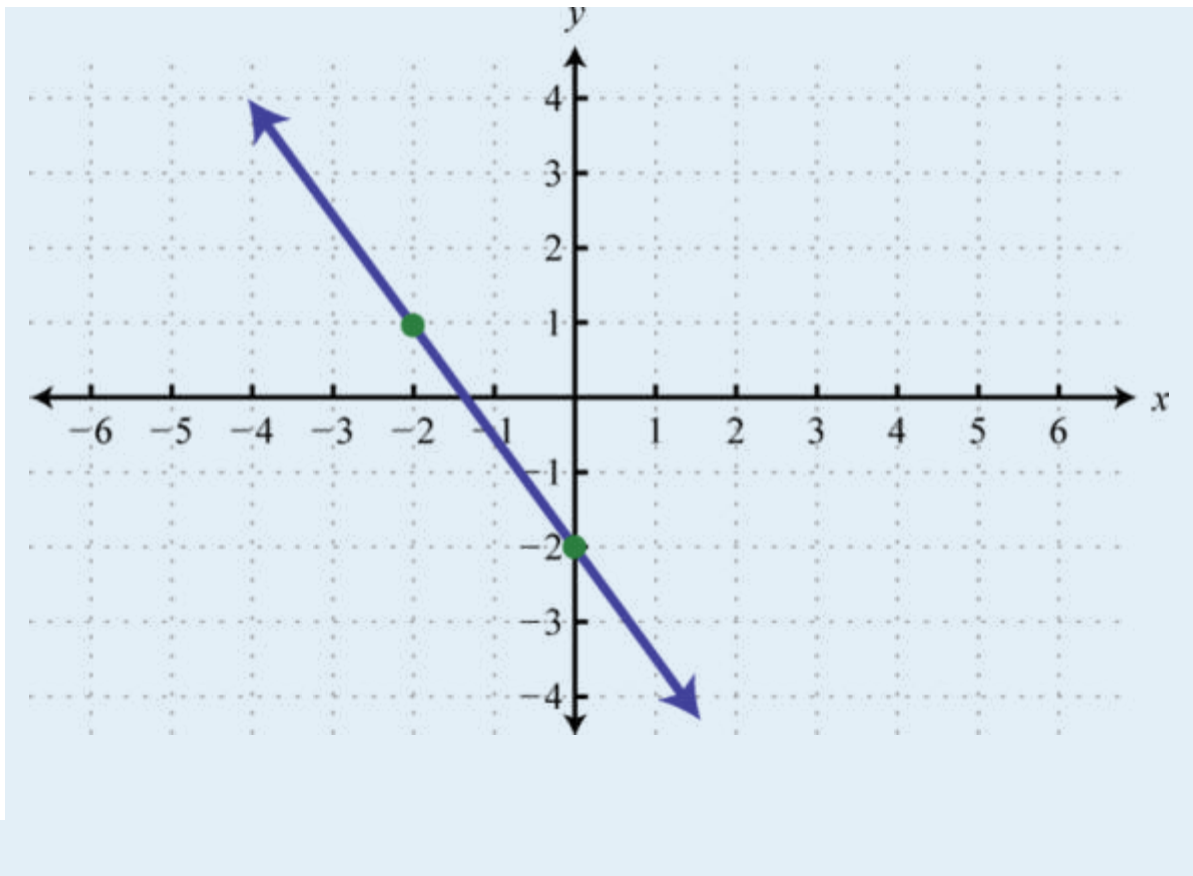




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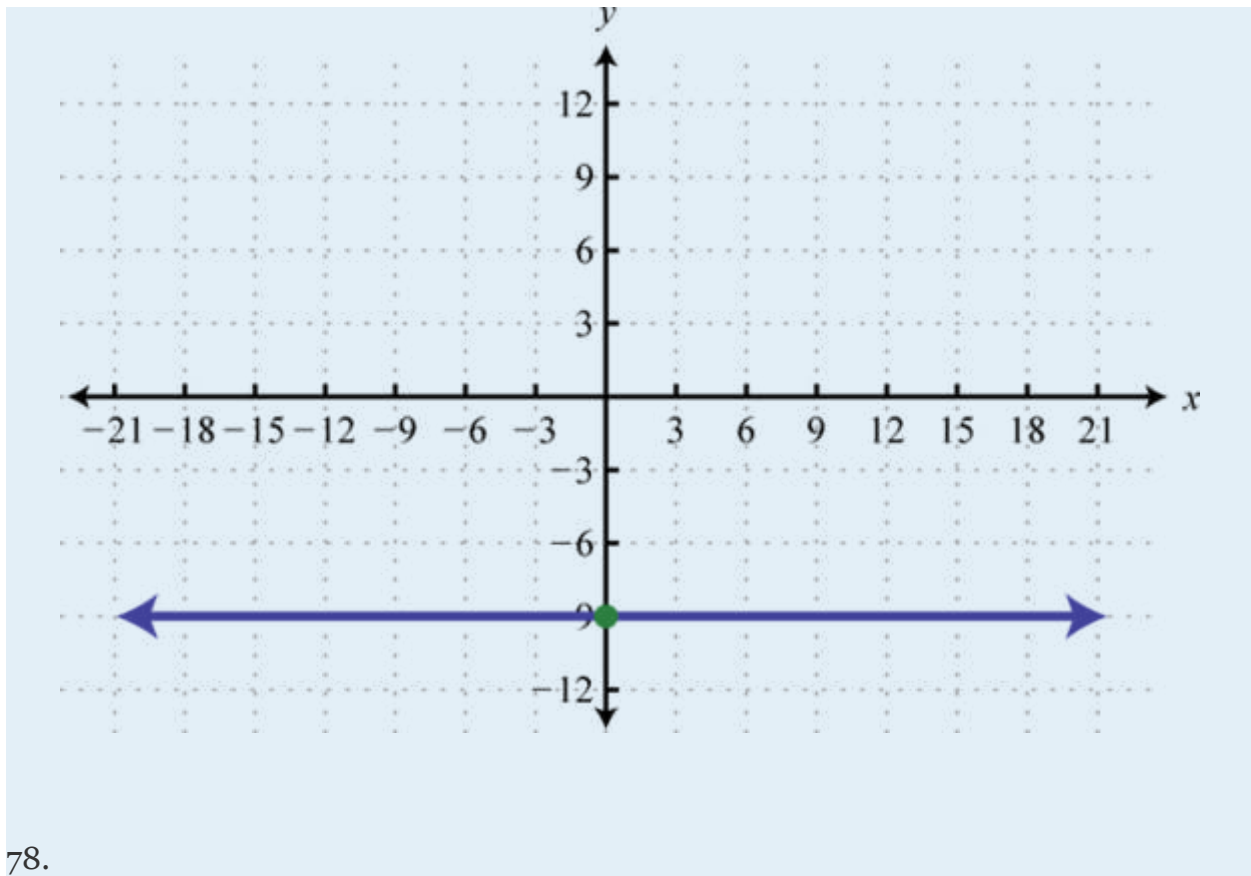




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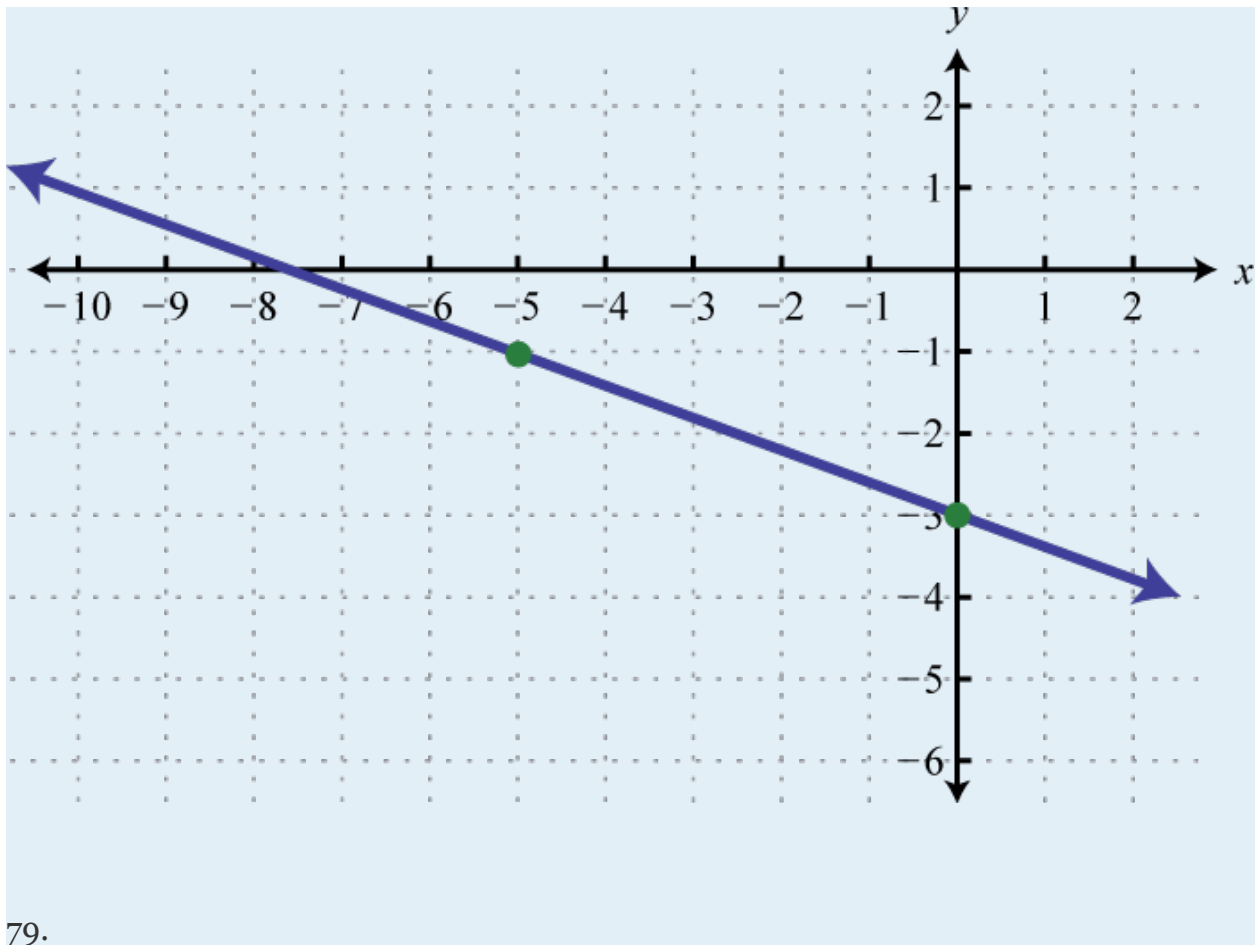




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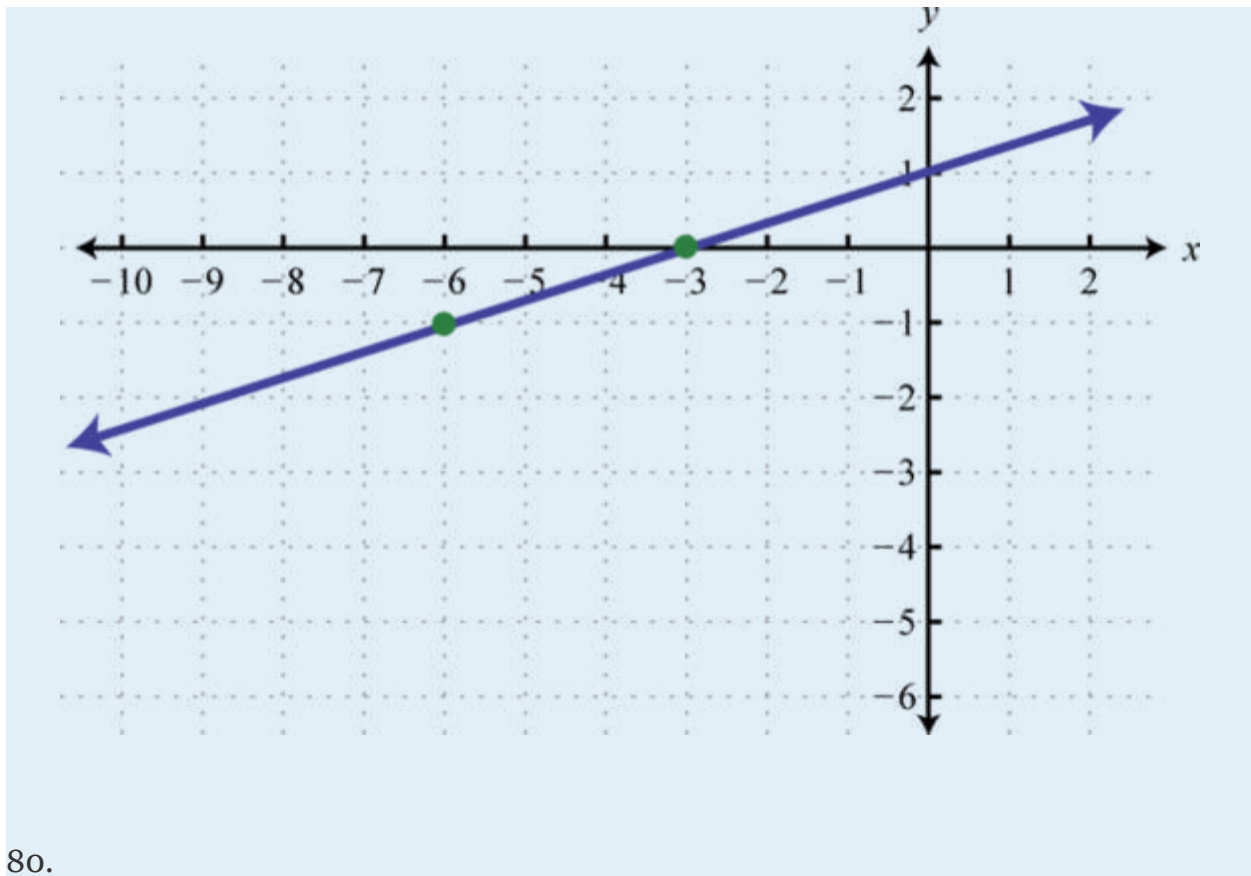


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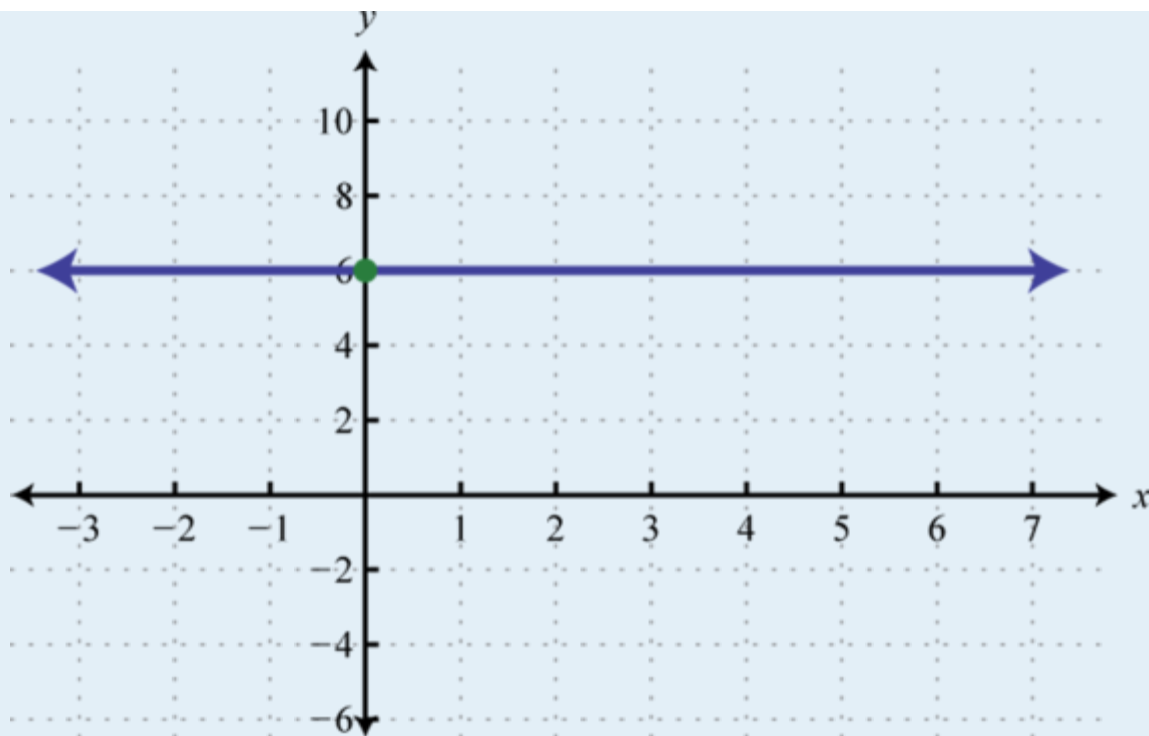




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## PART C: A GRAPHICAL INTERPRETATION OF LINEAR EQUATIONS AND INEQUALITIES

Graph the functions  $f$  and  $g$  on the same set of axes and determine where  $f(x) = g(x)$ . Verify your answer algebraically.

81.       ,  $g(x) = 1$
82.       ,  $g(x) = -1$
83.        $f(x) = 3x - 2$ ,  $g(x) = -5$
84.        $f(x) = x + 2$ ,  $g(x) = -3$



85. ,  $g(x) = 2$
86. ,  $g(x) = 1$
87.  $f(x) = 3x - 2$ ,  $g(x) = -2x + 3$
88.  $f(x) = -x + 6$ ,  $g(x) = x + 2$
89. ,
90. ,

Graph the functions  $f$  and  $g$  on the same set of axes and determine where  $f(x) \geq g(x)$ . Verify your answer algebraically.

81.  $f(x) = 3x + 7$ ,  $g(x) = 1$
82.  $f(x) = 5x - 3$ ,  $g(x) = 2$
83. ,  $g(x) = -3$
84. ,  $g(x) = -1$
85.  $f(x) = -x + 1$ ,  $g(x) = -3$
86.  $f(x) = -4x + 4$ ,  $g(x) = 8$
87.  $f(x) = x - 2$ ,  $g(x) = -x + 4$
88.  $f(x) = 4x - 5$ ,  $g(x) = x + 1$

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Graph the functions  $f$  and  $g$  on the same set of axes and determine where  $f(x) < g(x)$ . Verify your answer algebraically.

81.  $f(x) = x + 5, g(x) = -1$

82.  $f(x) = 3x - 3, g(x) = 6$

83.  $f(x) = 2x + 1, g(x) = -8$

84.  $f(x) = x - 2, g(x) = -3$

85.  $f(x) = x + 1, g(x) = 0$

86.  $f(x) = x - 1, g(x) = 0$

87.  $f(x) = x + 1, g(x) = x - 1$

88.  $f(x) = 32x + 3, g(x) = 32x + 3$

## PART D: DISCUSSION BOARD

107. Do all linear functions have  $y$ -intercepts? Do all linear functions have  $x$ -intercepts? Explain.

108. Can a function have more than one  $y$ -intercept? Explain.

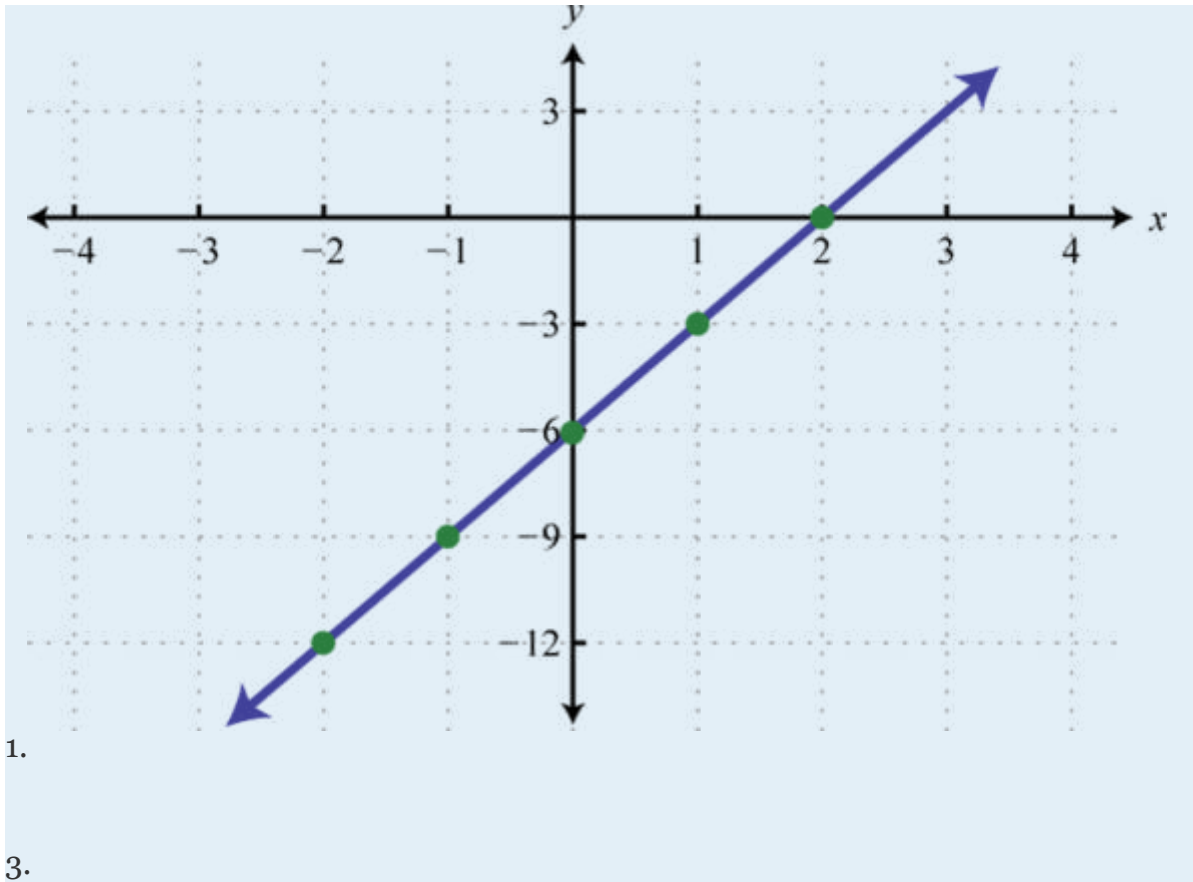
109. How does the vertical line test show that a vertical line is not a function?

## ANSWERS

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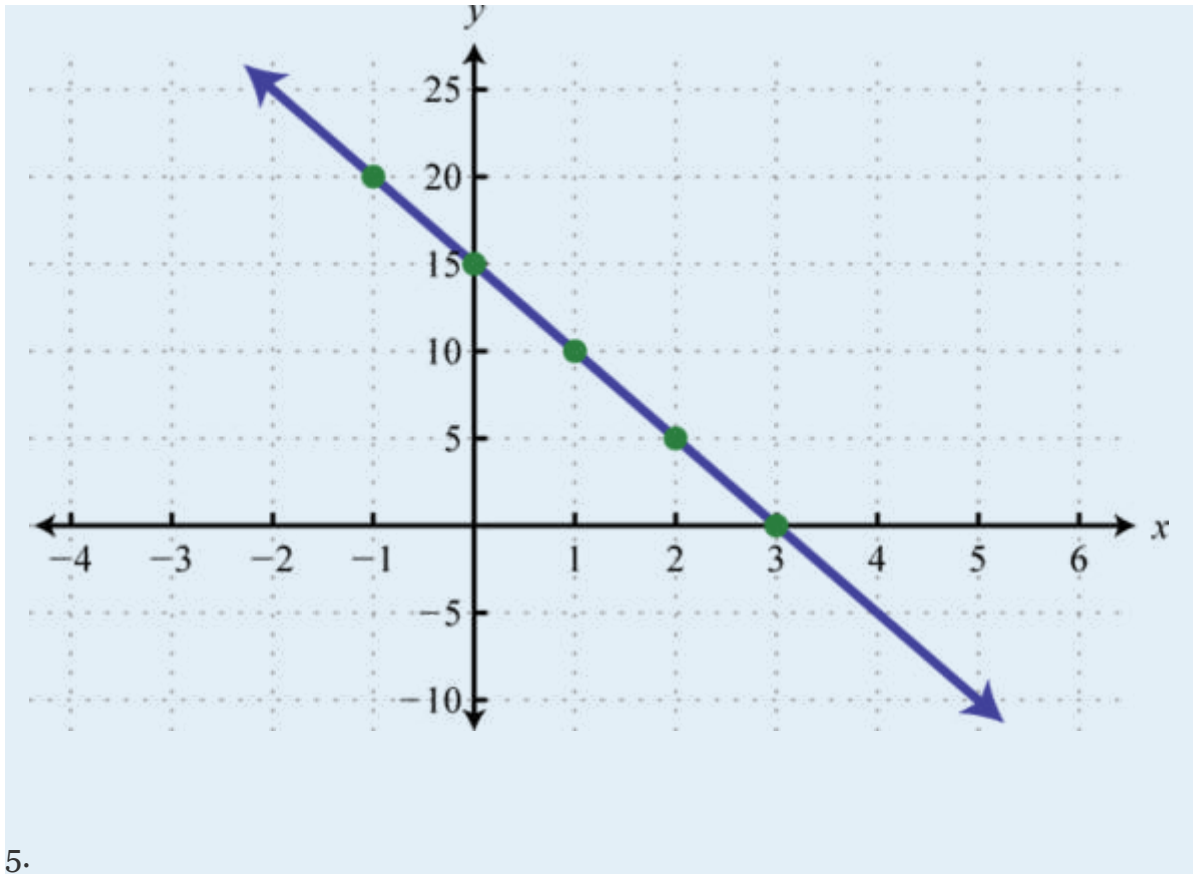
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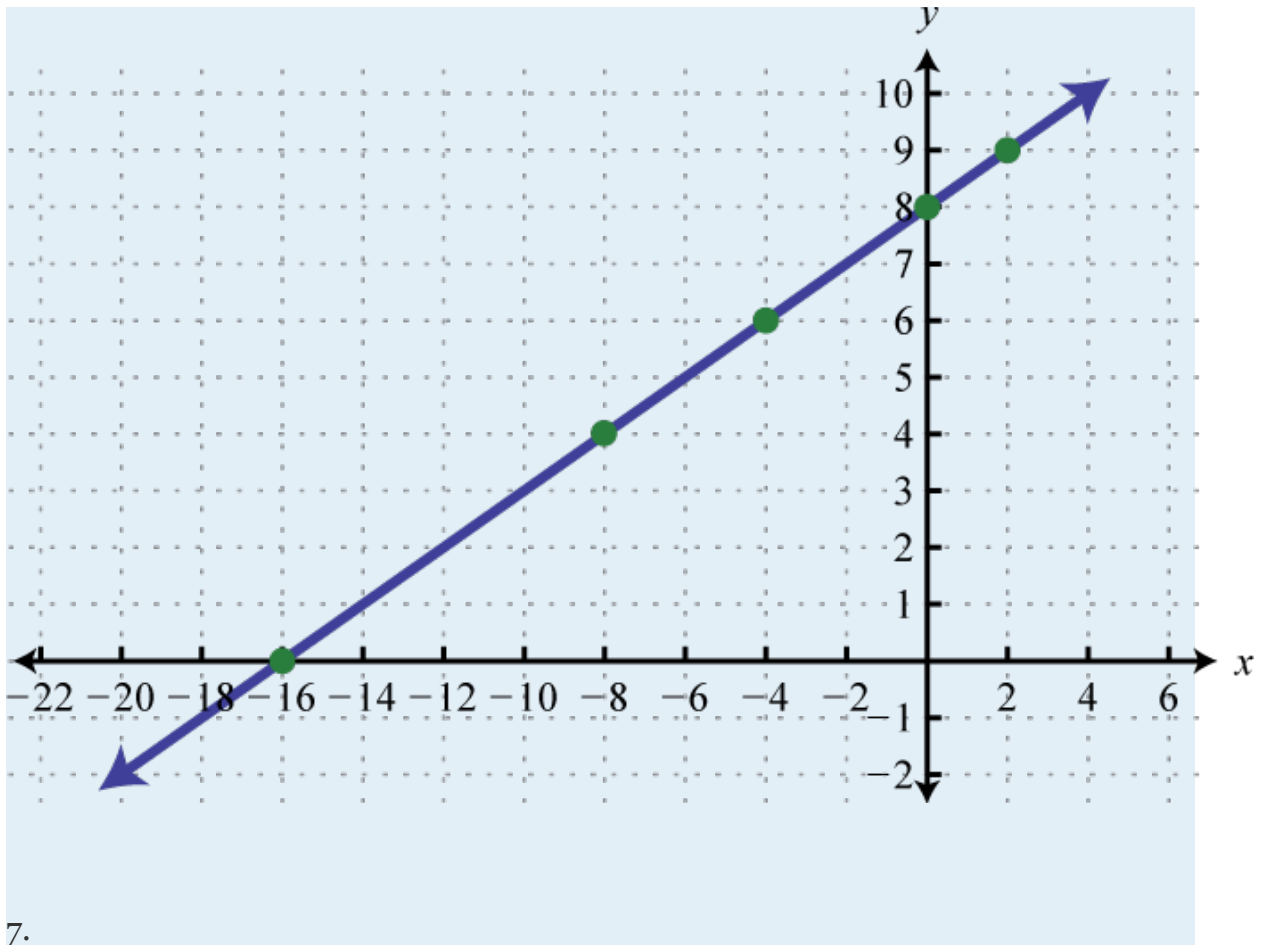




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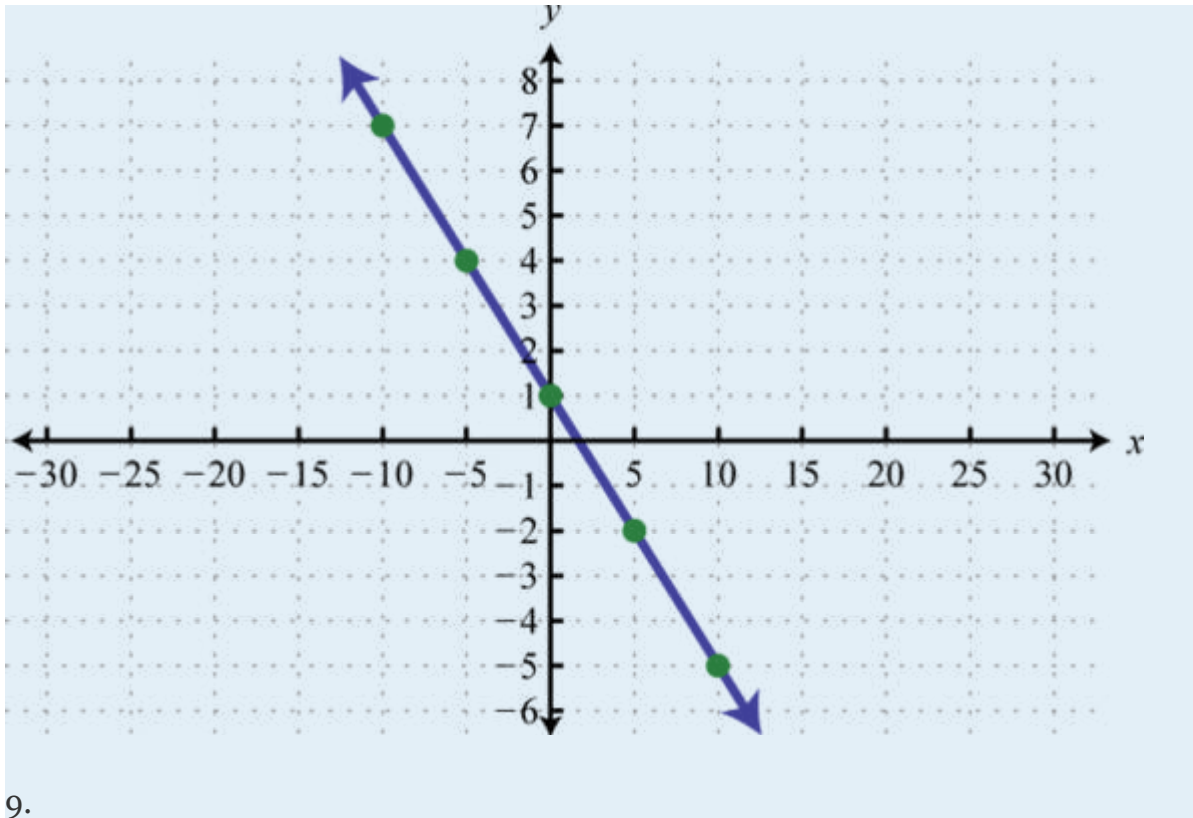
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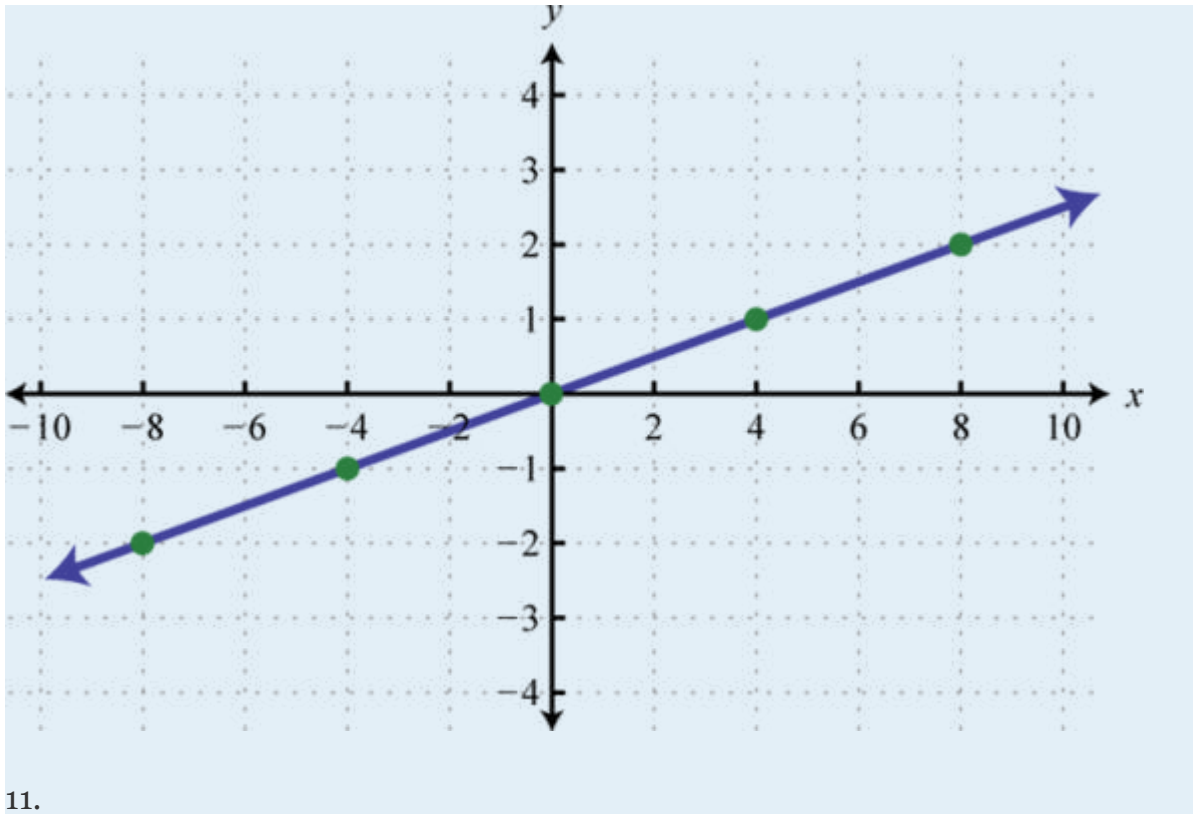


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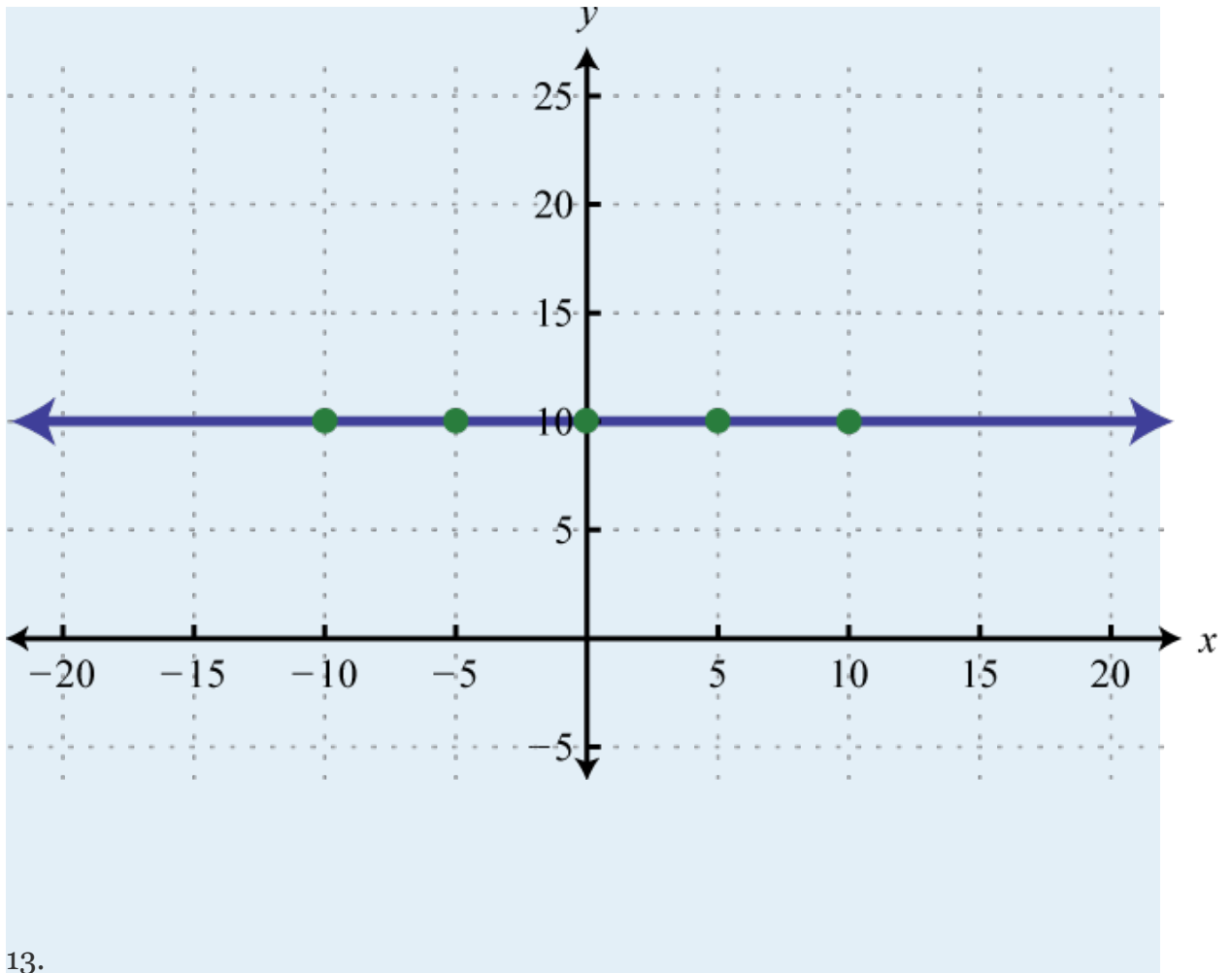




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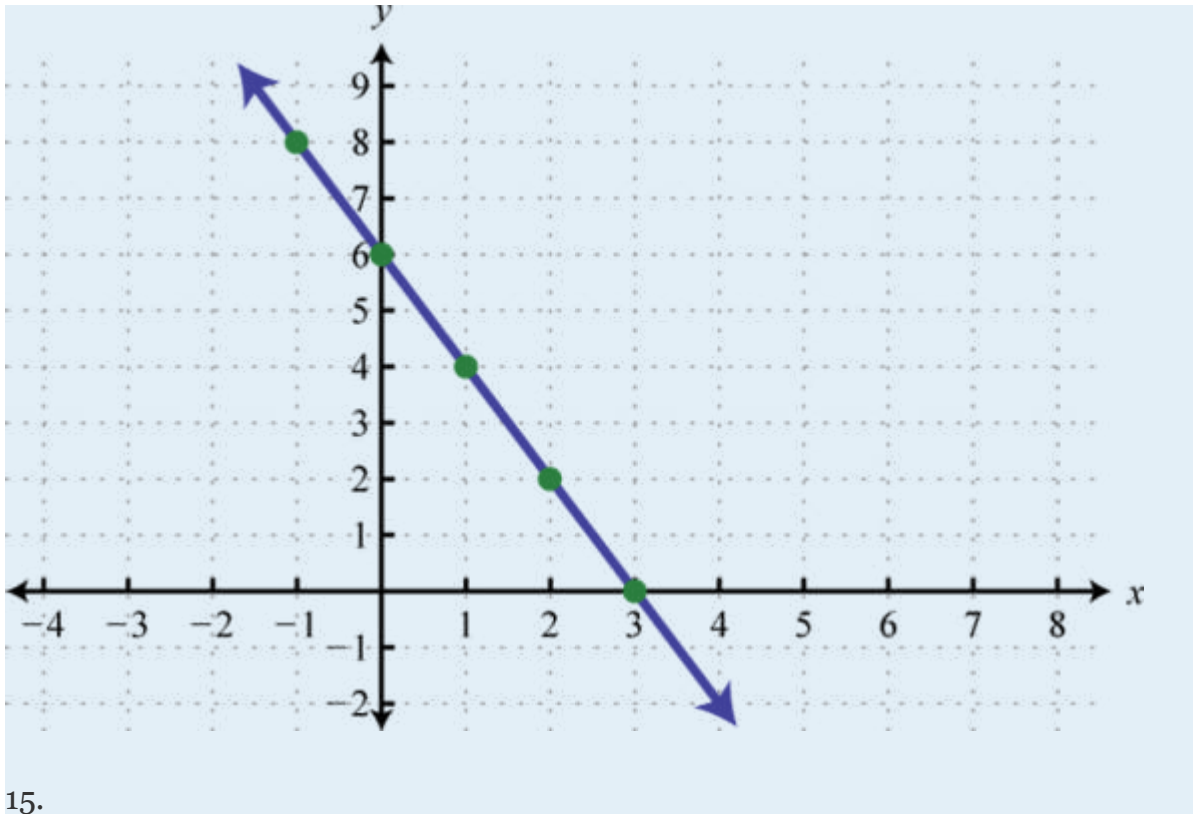




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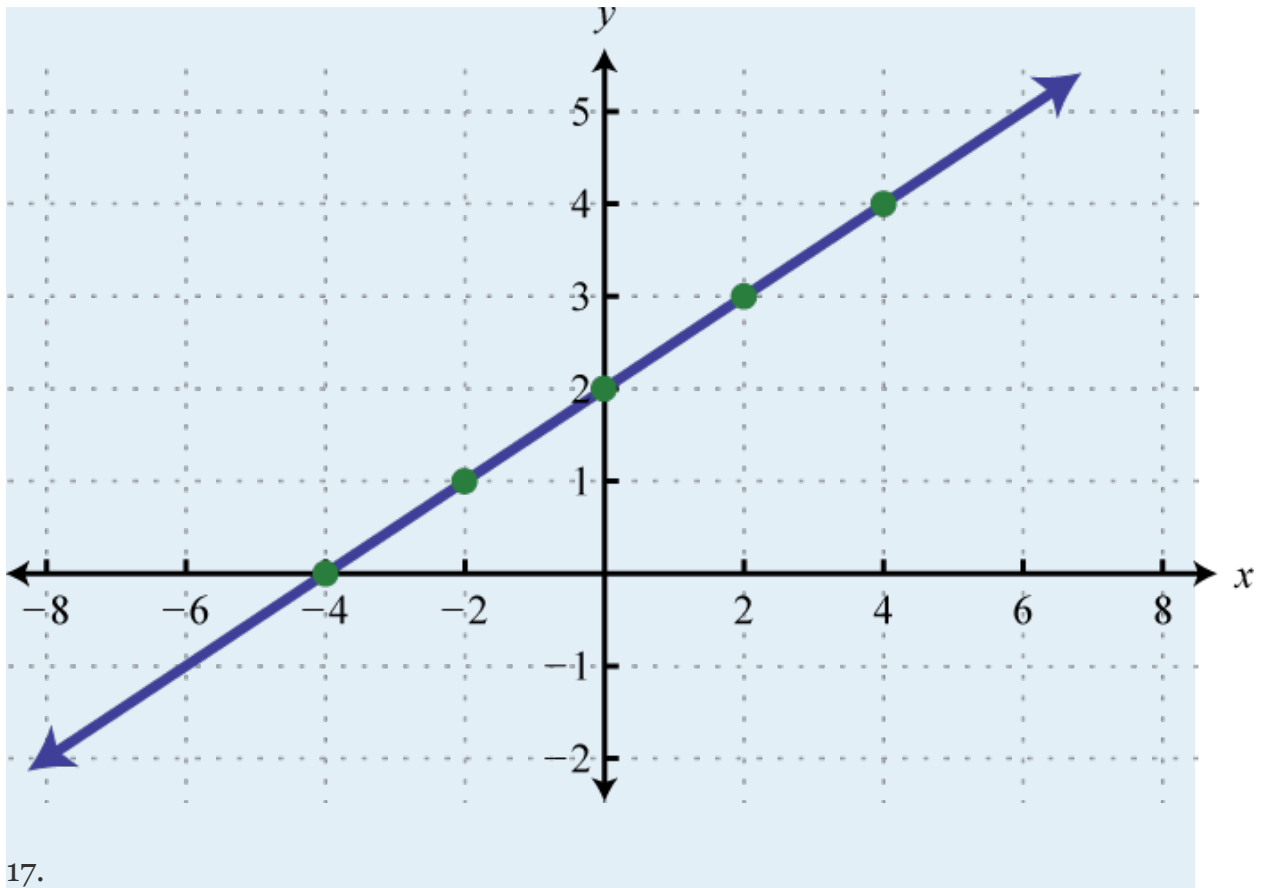




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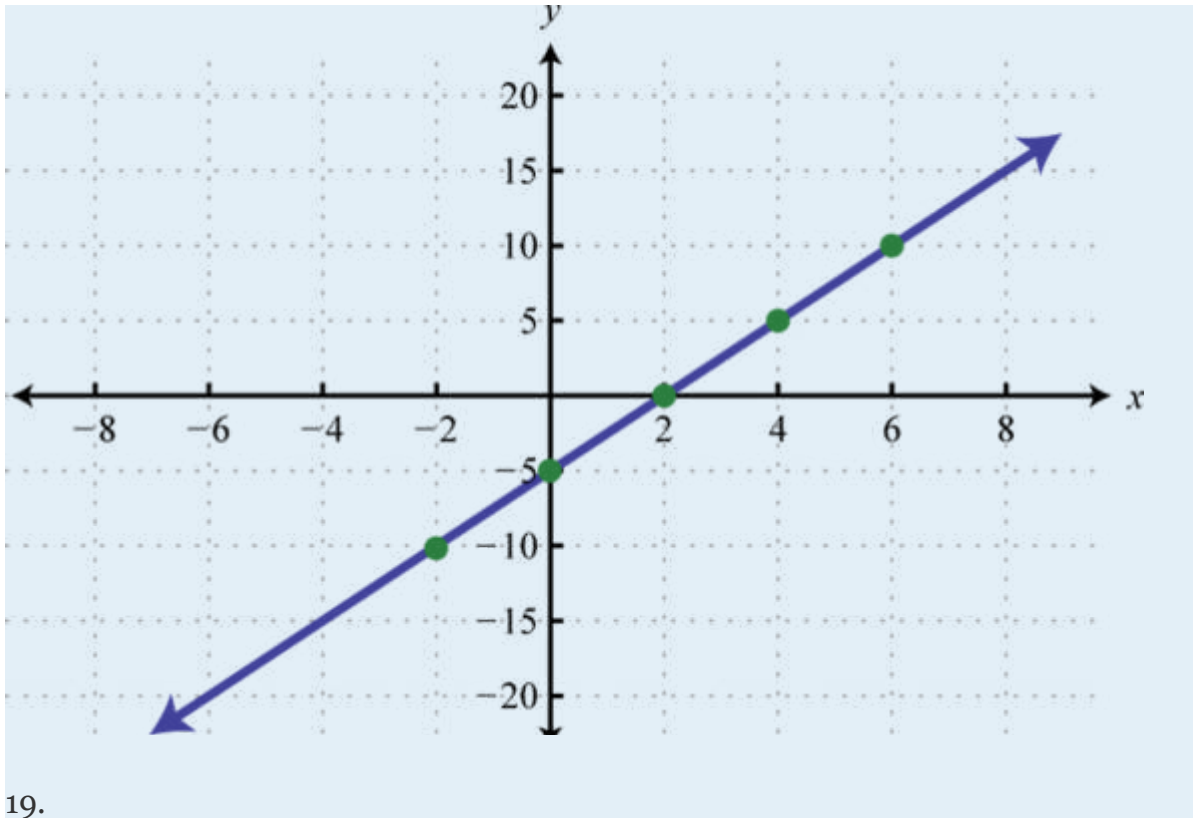




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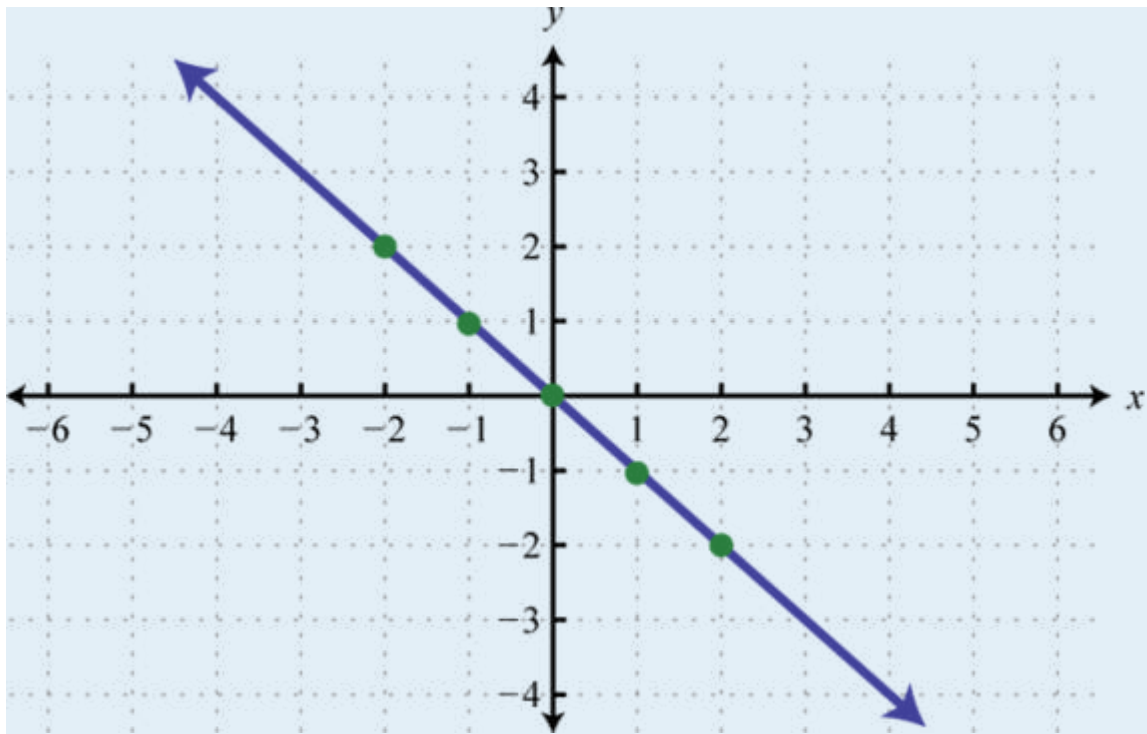




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21. 1

23.

25. Undefined

27.  $y = -5$

29.  $y = 1$

31.

33.

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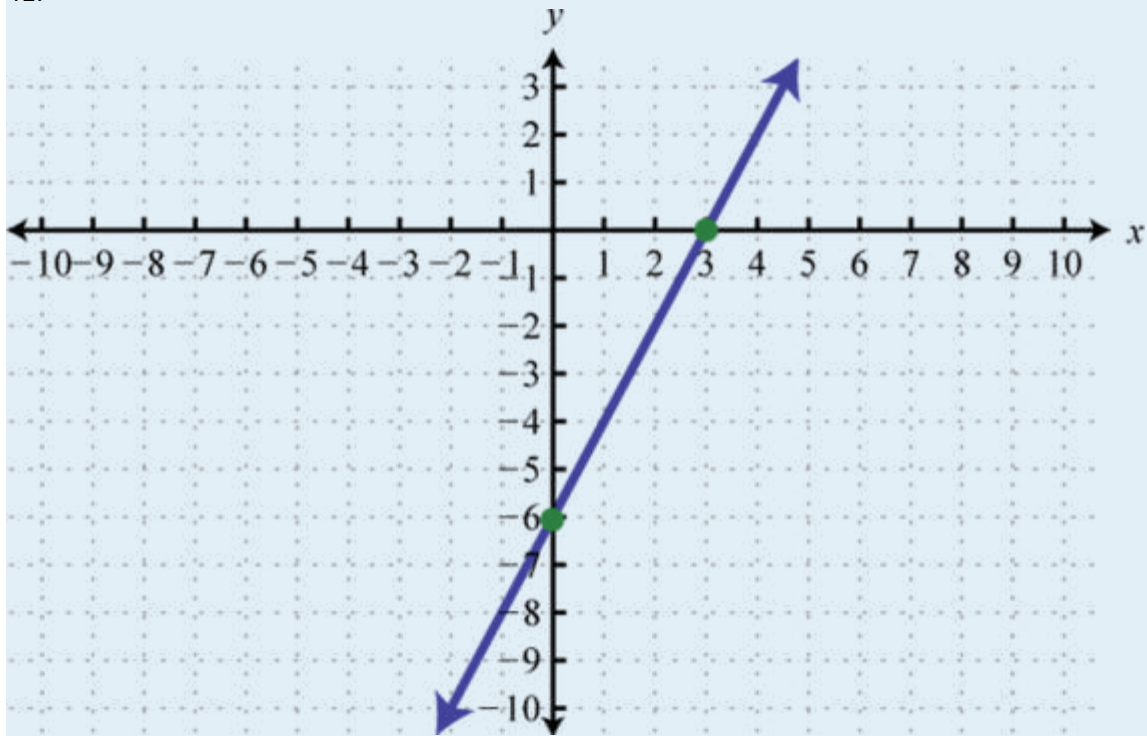


35.

37.

39.  $m = 0$

41.

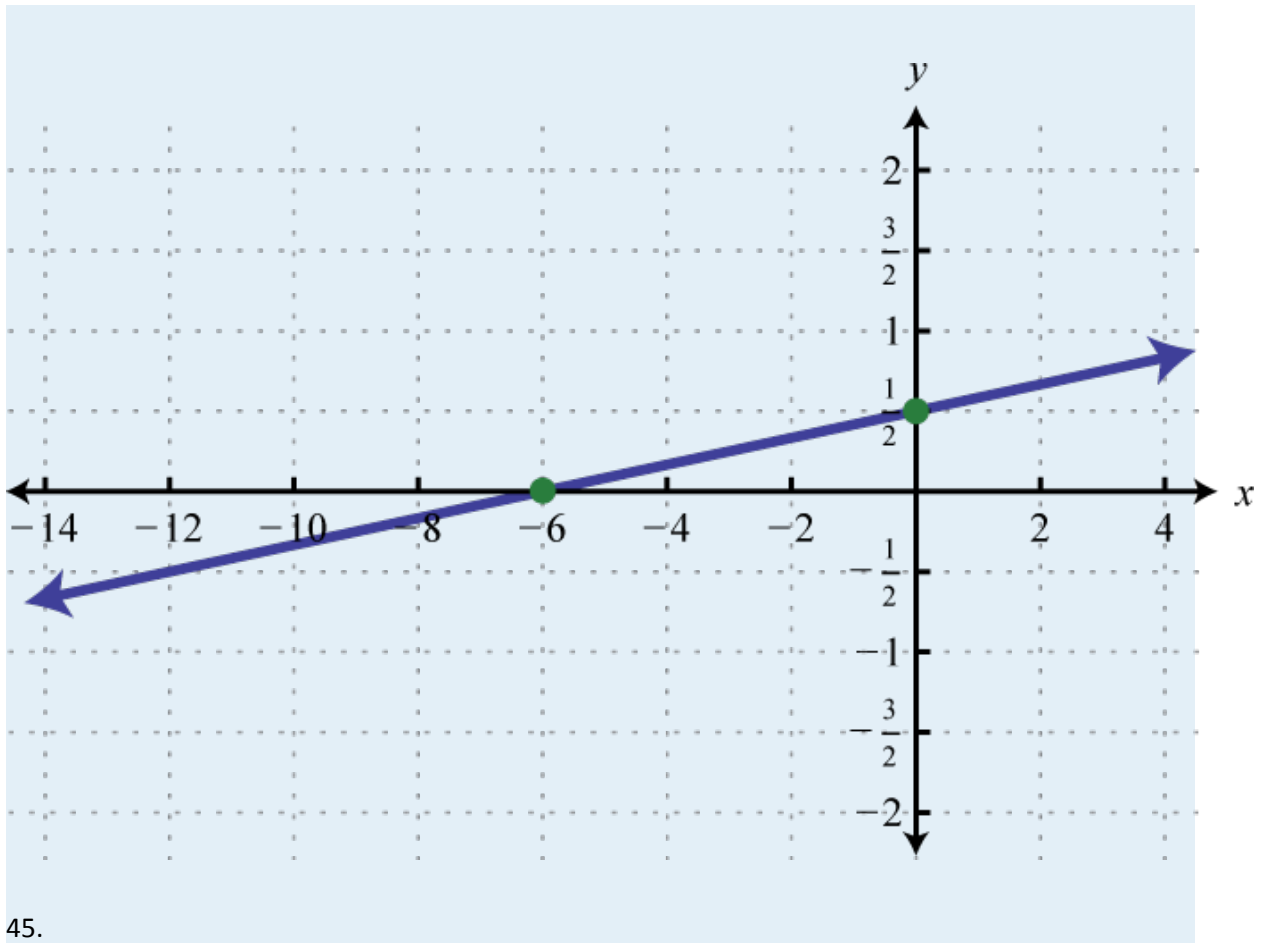


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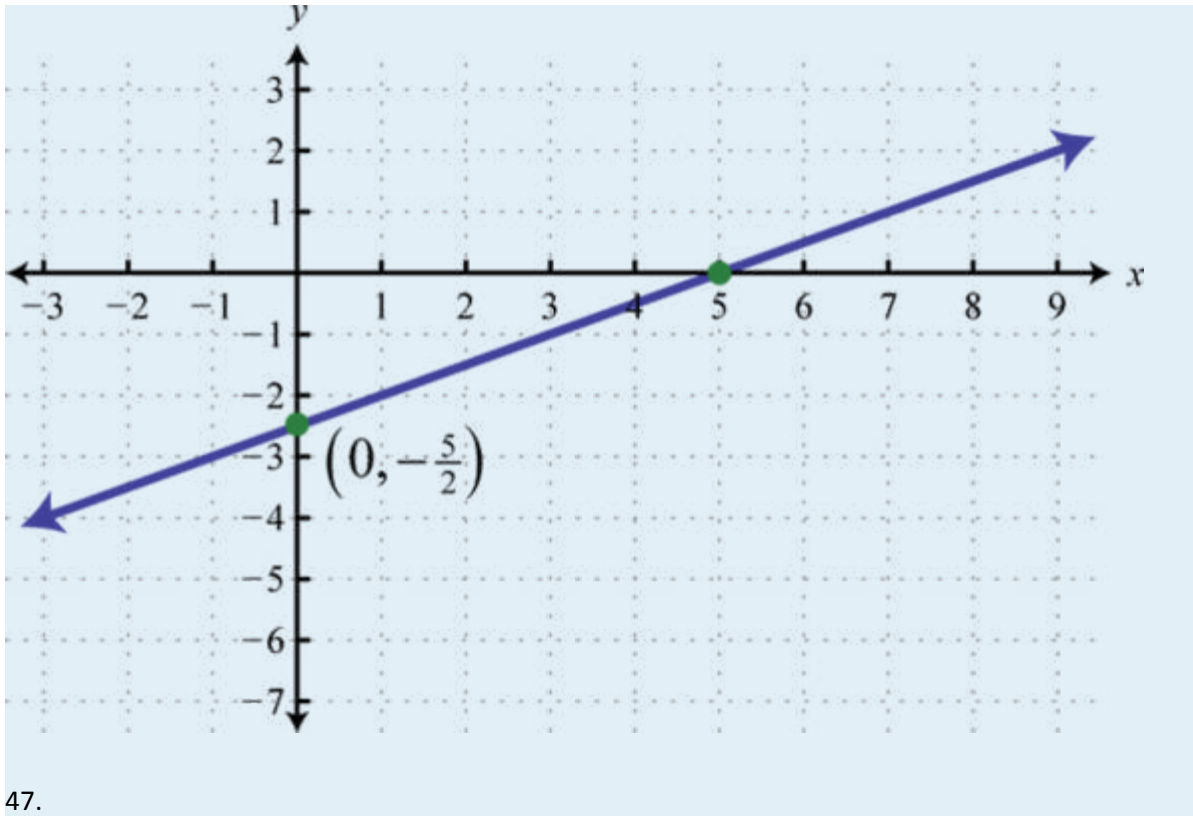


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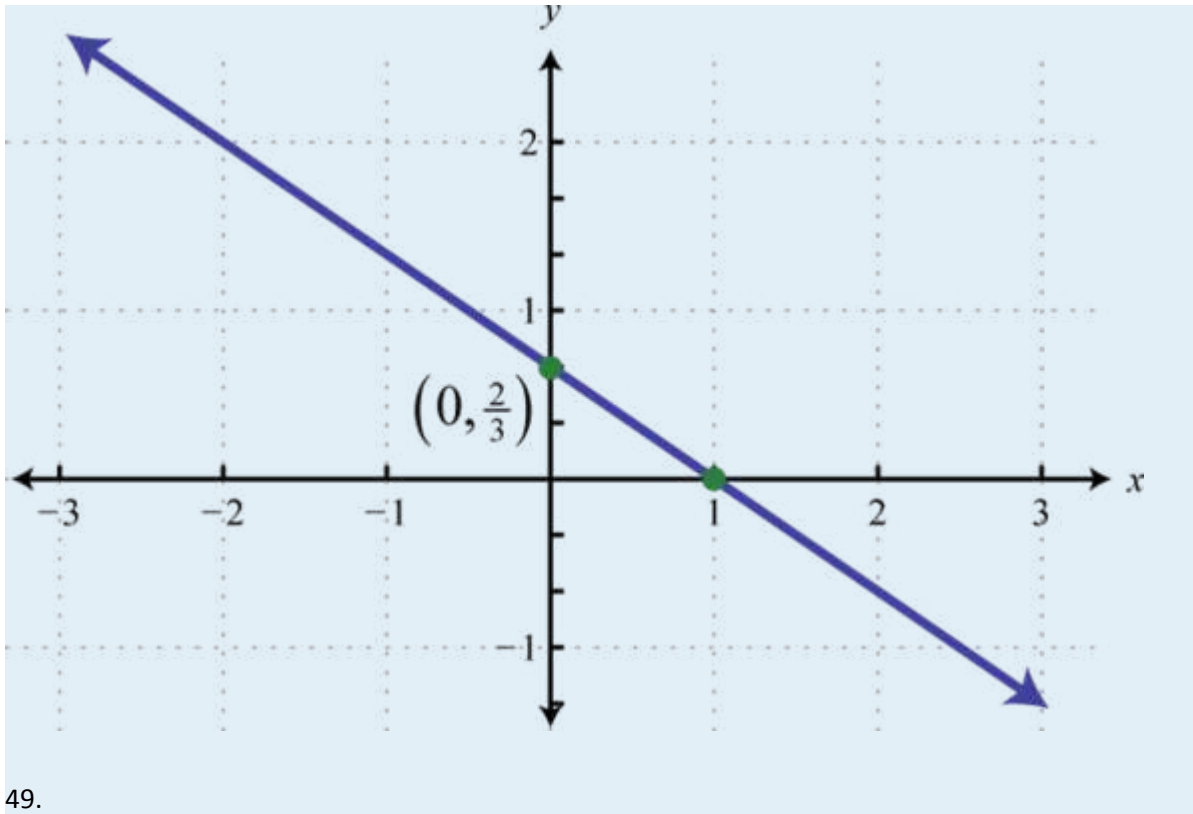




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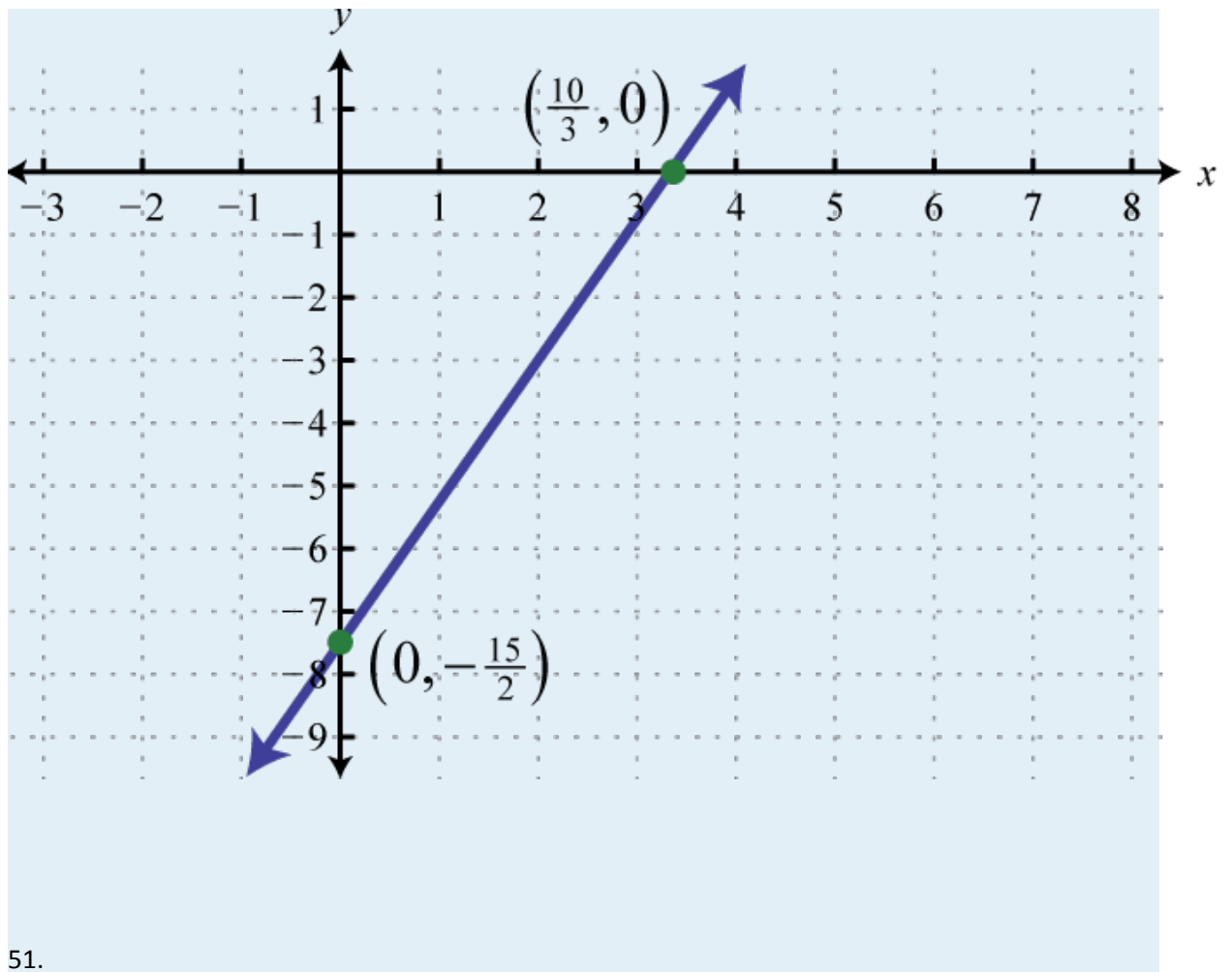




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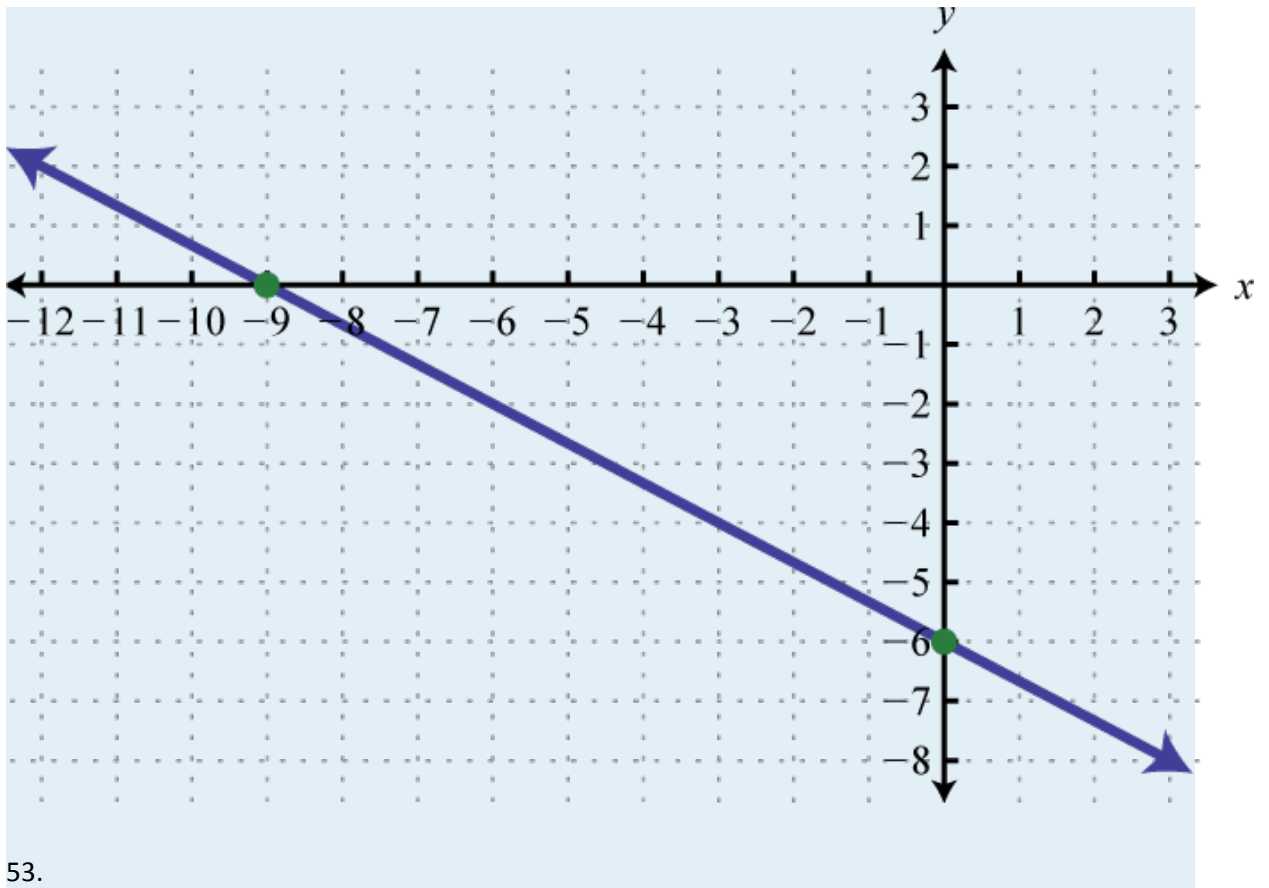
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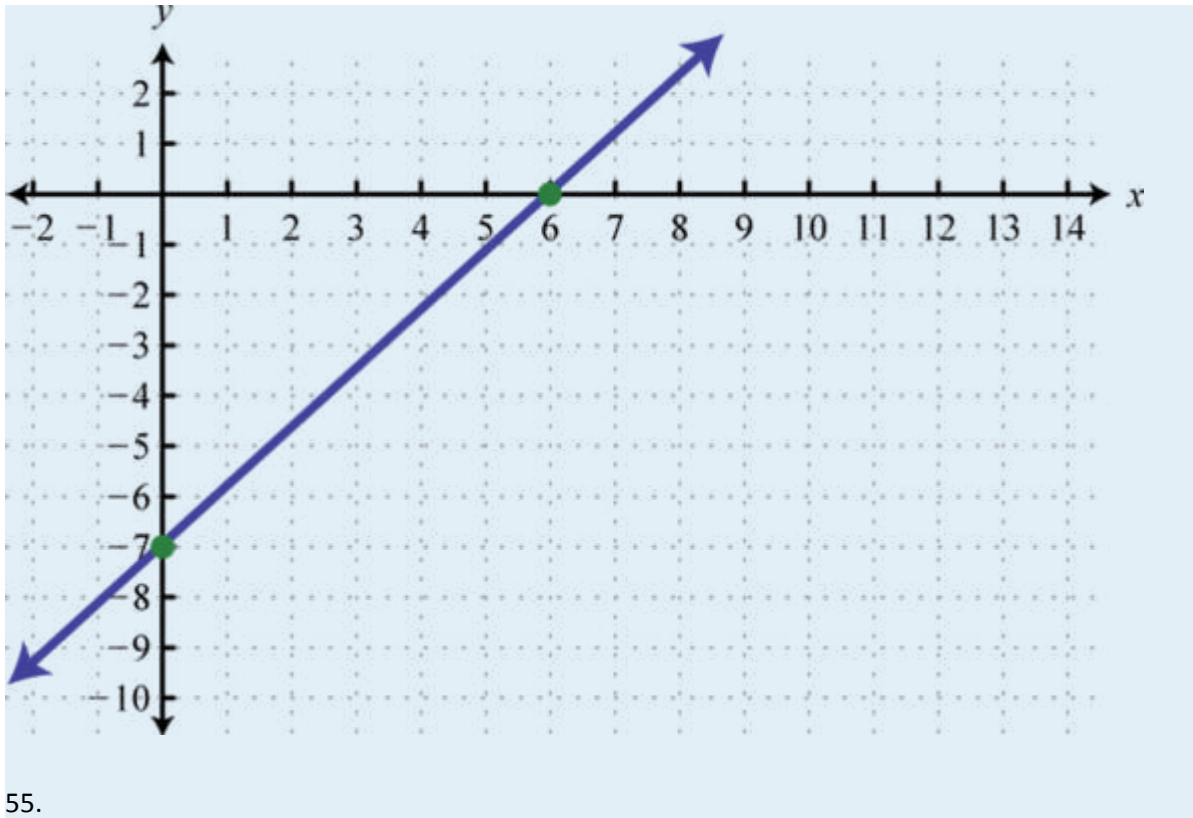




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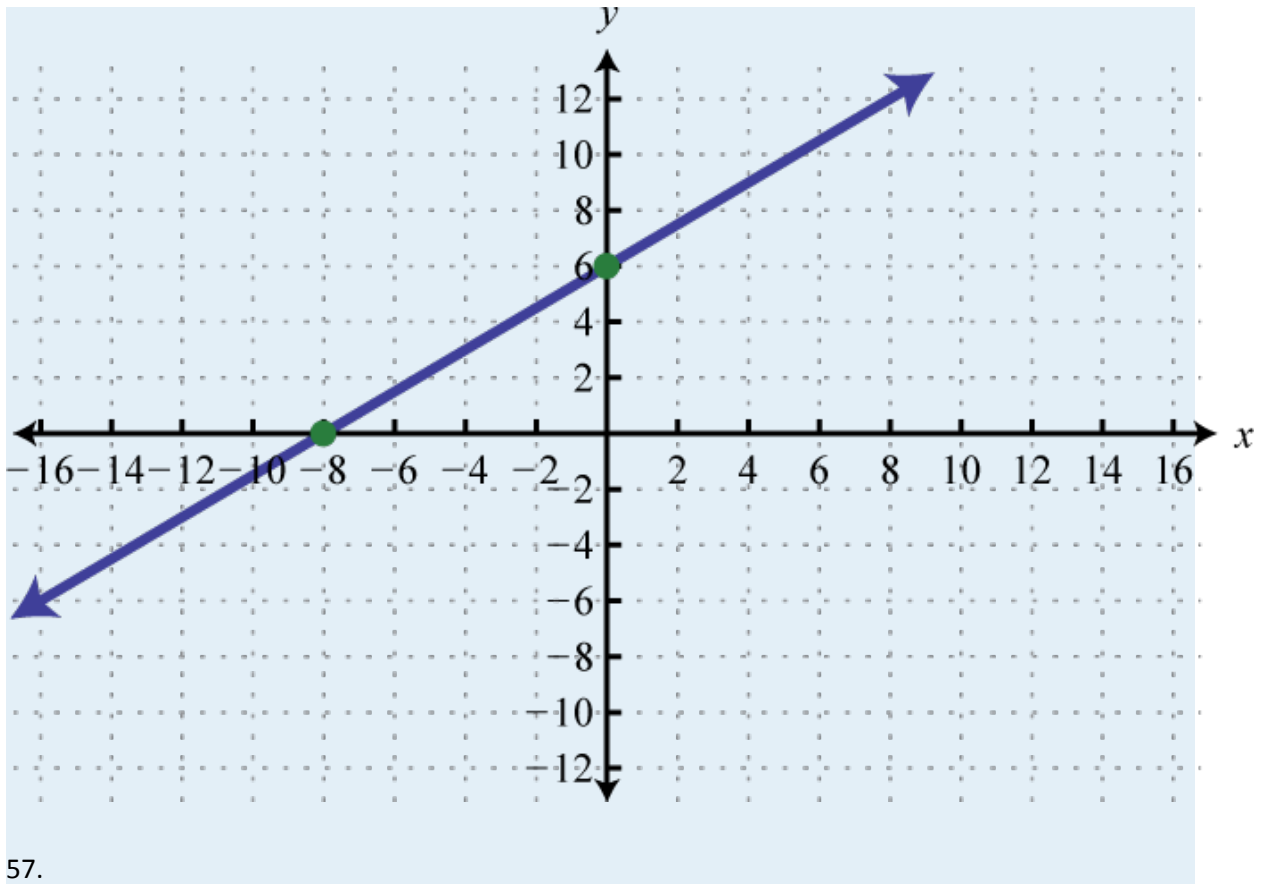




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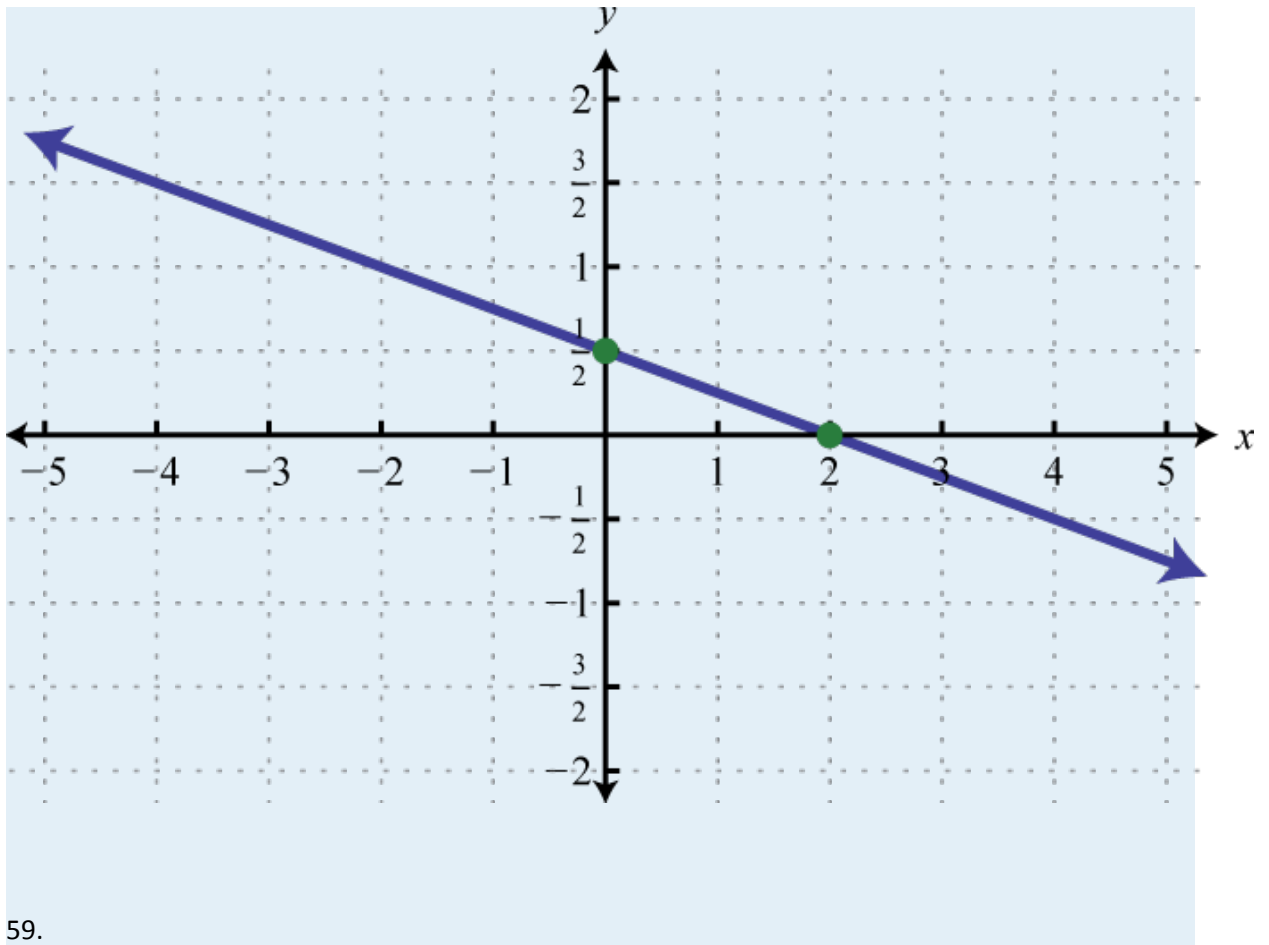
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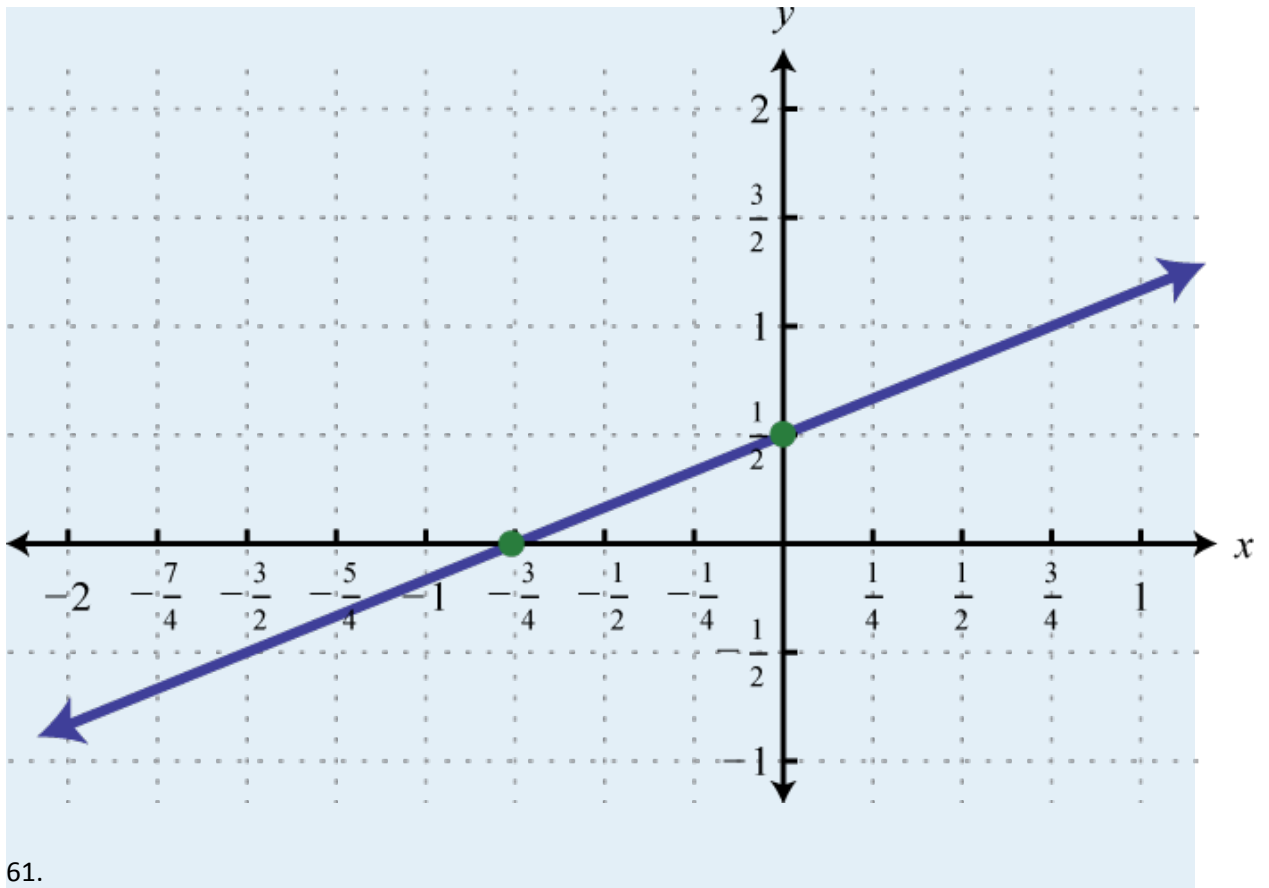




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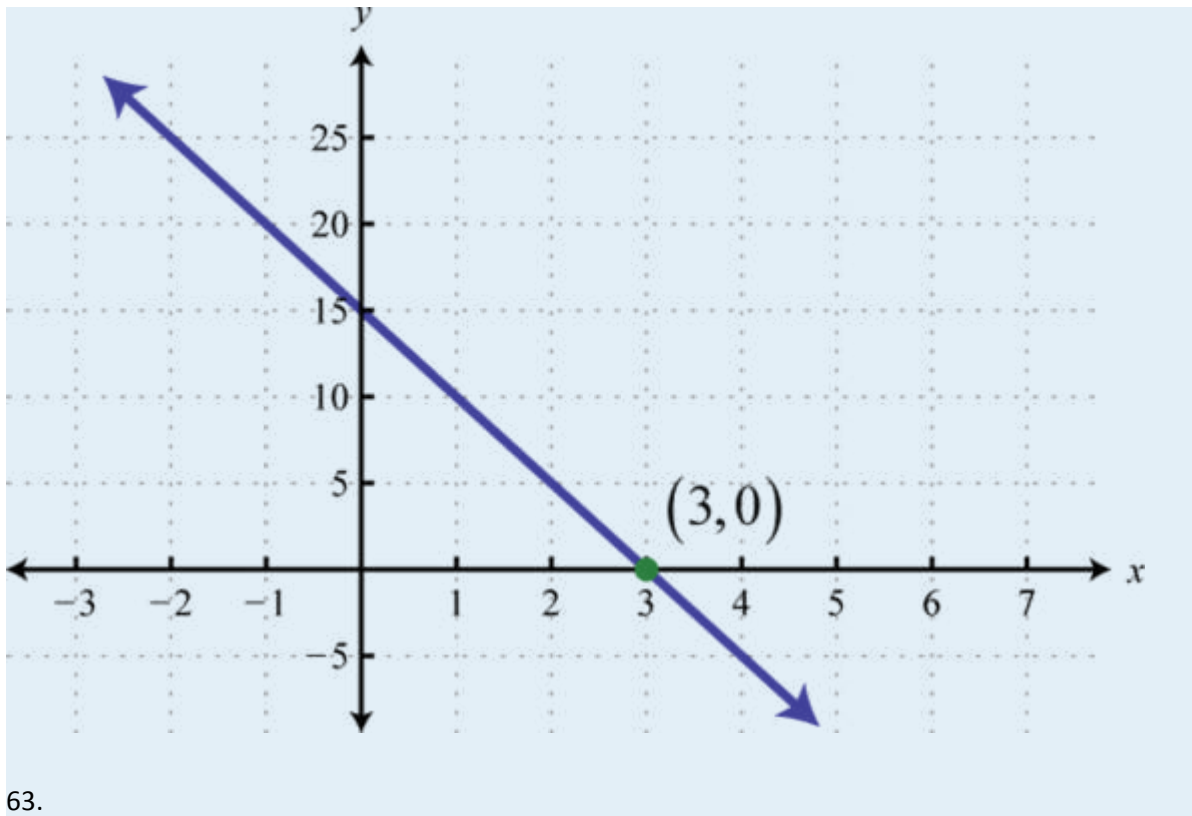




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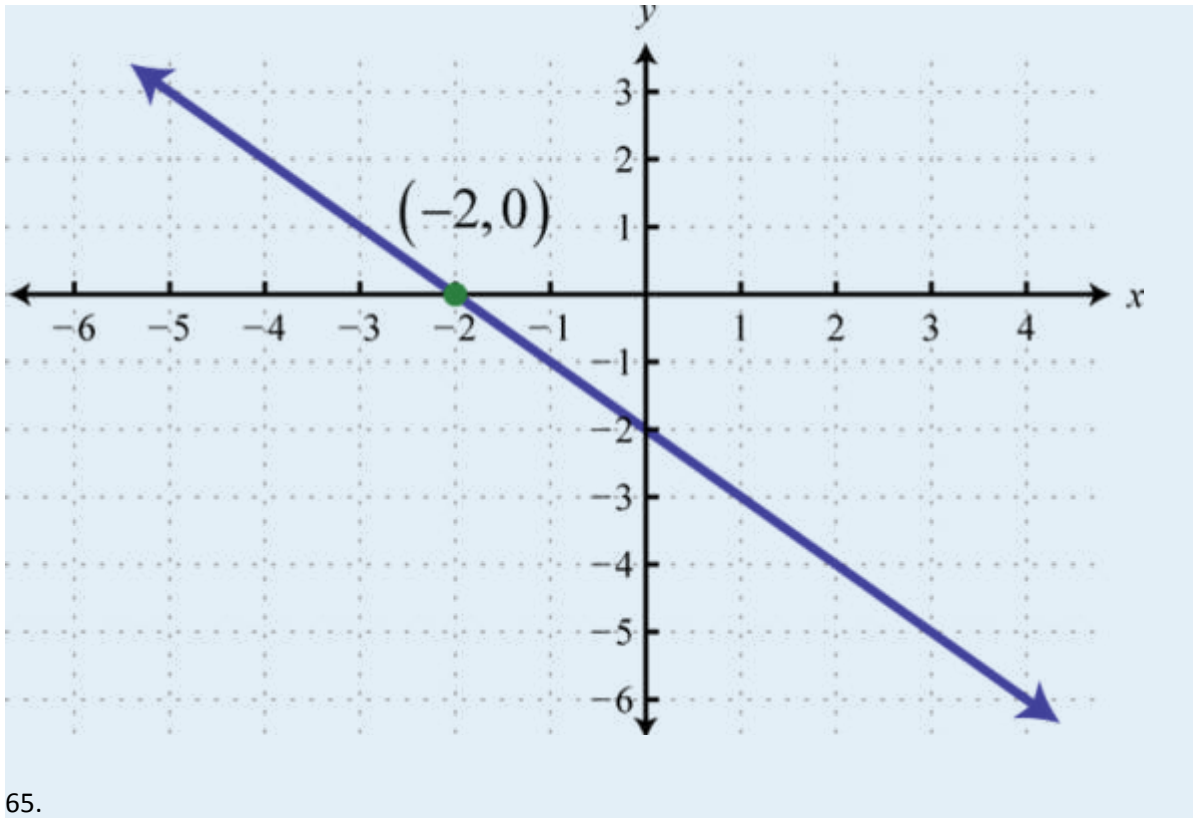




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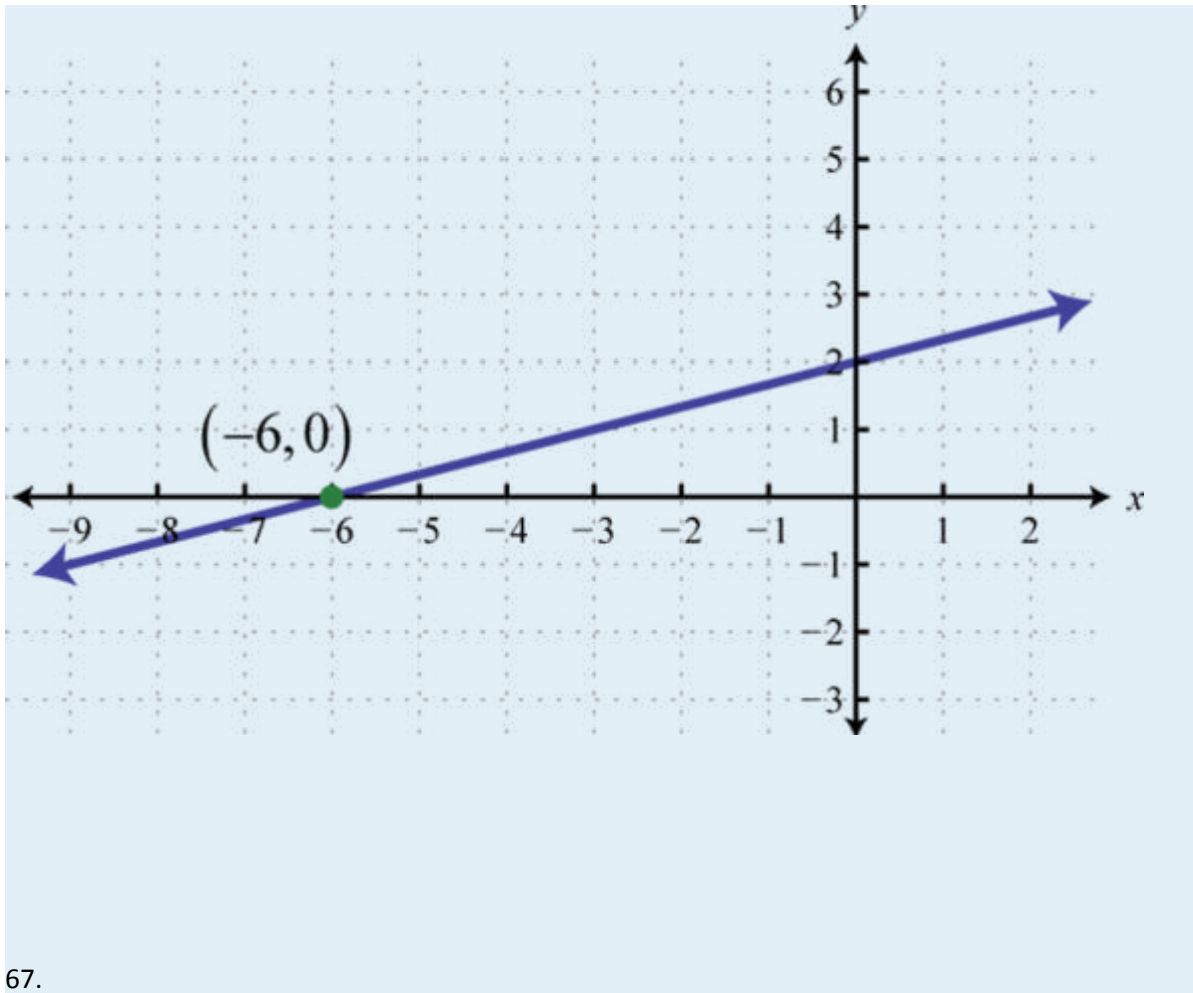




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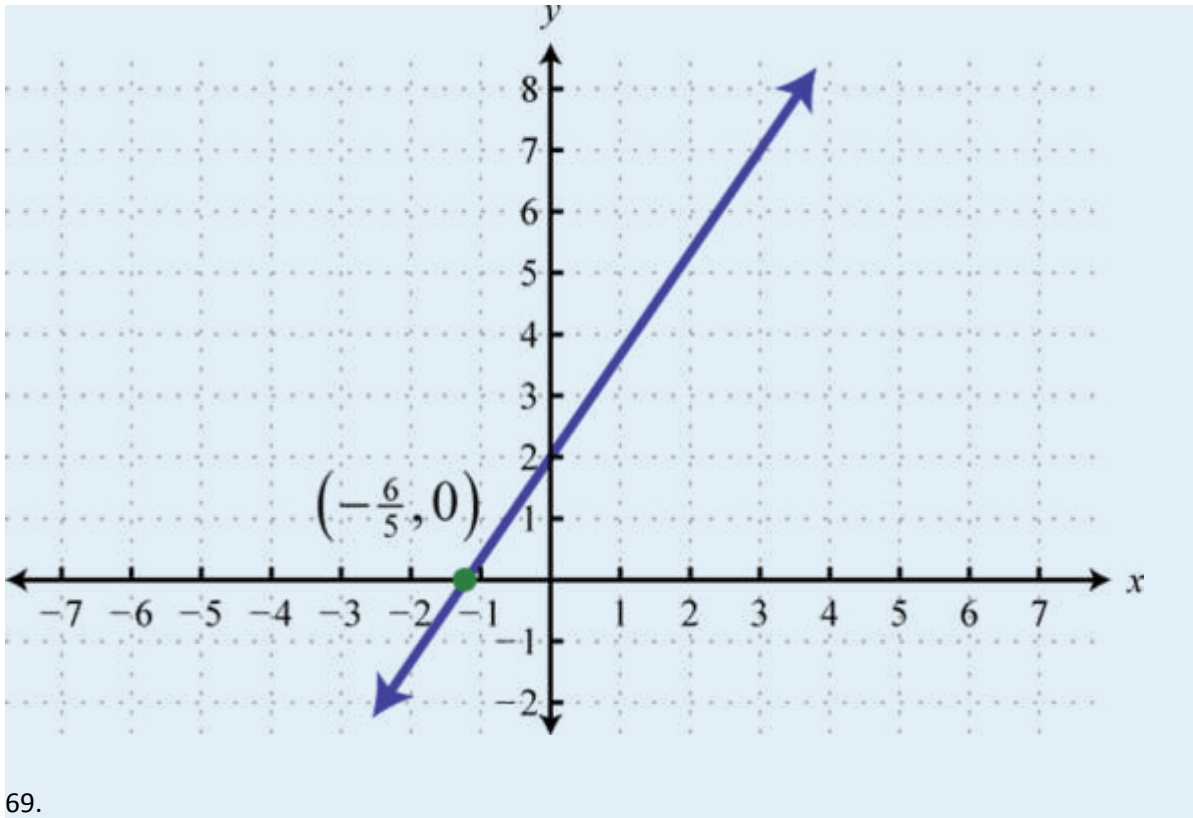




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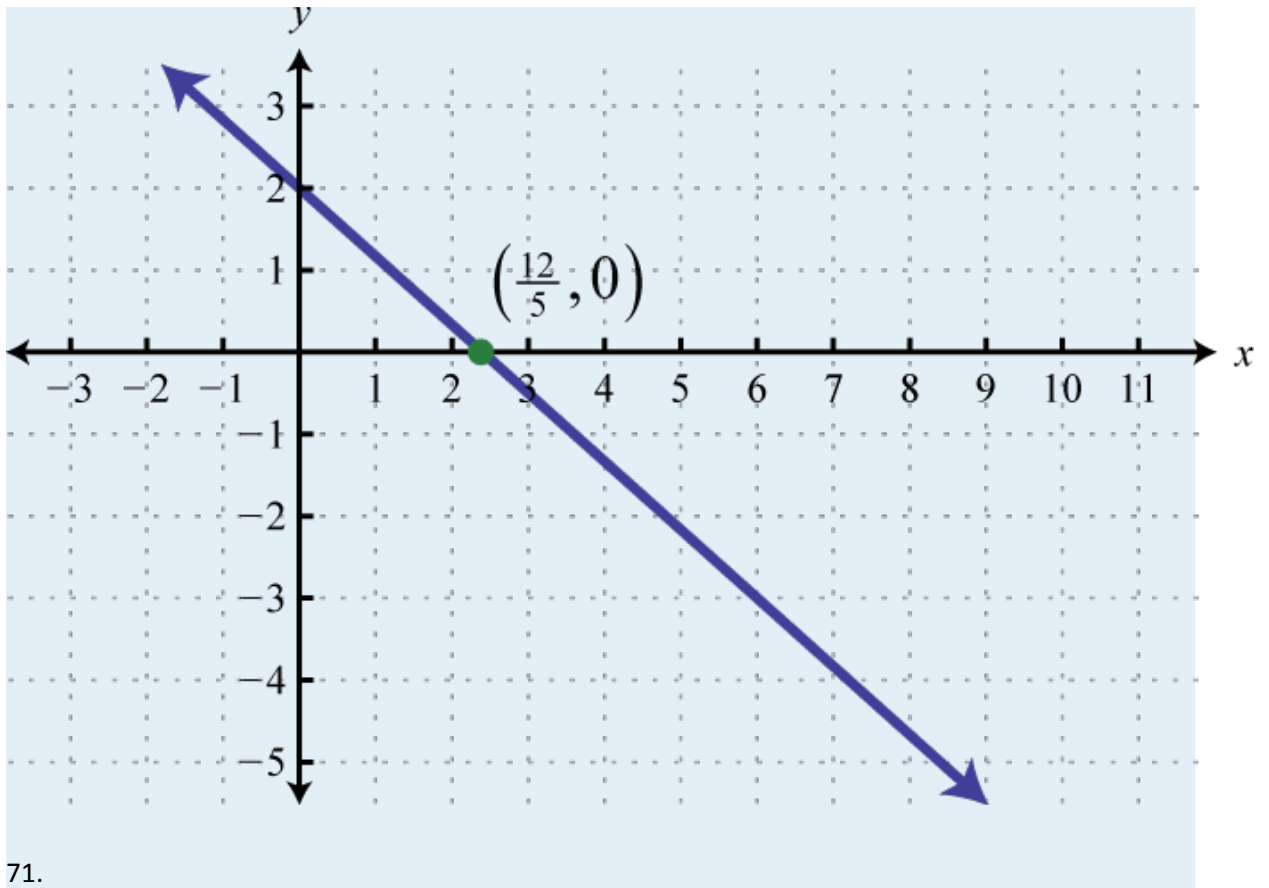




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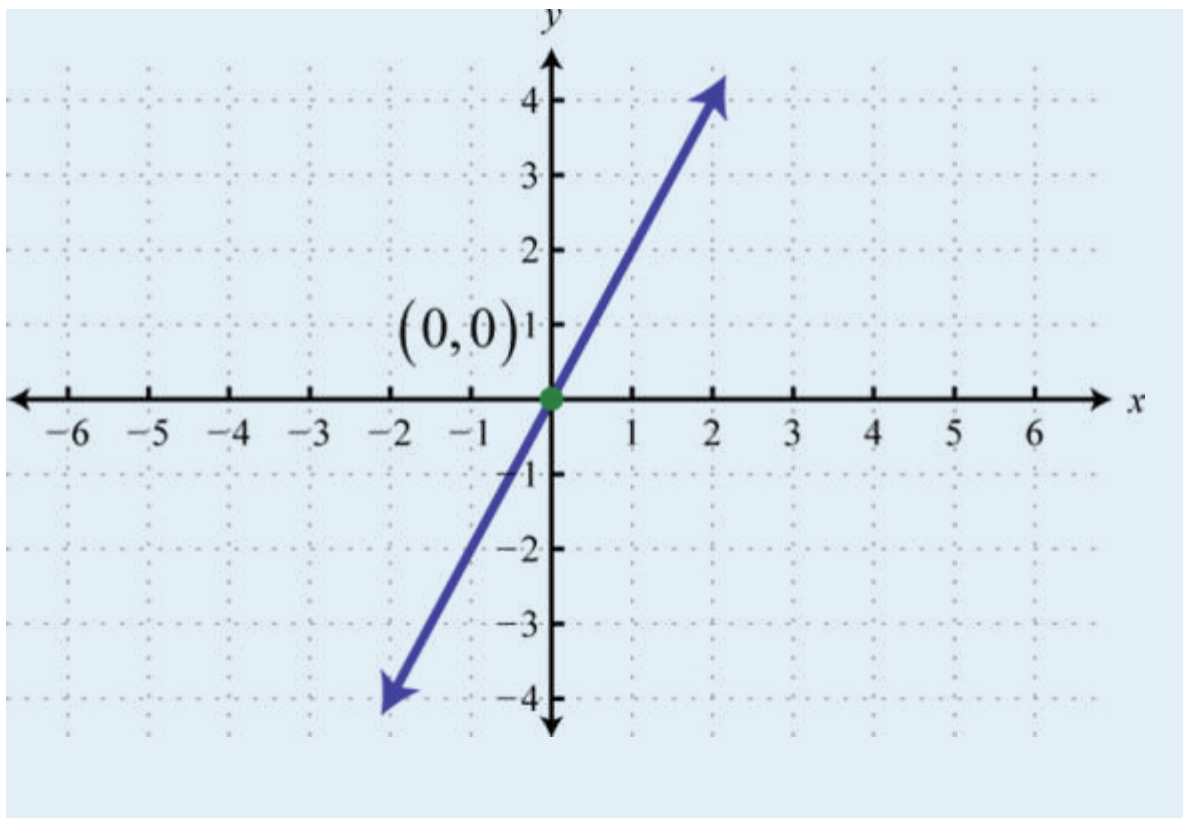
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71.





53.  $f(x) = x + 1$ ;  $(-1, 0)$

75.  $f(x) = -32x$ ;  $(0, 0)$

77.  $f(x) = -9$ ; none

79.  $f(x) = 13x + 1$ ;  $(-3, 0)$

81.  $x = 8$

83.  $x = -1$



85.  $x = 3$

86.

87.  $x = 1$

88.

89.  $x = 3$

90.

91.  $[-2, \infty)$

92.

93.  $[0, \infty)$

94.

95.  $(-\infty, 4]$

96.

97.  $[3, \infty)$

98.

99.  $(-\infty, -6)$

100.

101.  $(10, \infty)$

102.

103.  $(-\infty, -4)$

104.

105.  $(-\infty, -3)$

106.

107. Answer may vary

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108.

109. Answer may vary

## 2.3 Modeling Linear Functions

### LEARNING OBJECTIVES

1. Determine the equation of a line given two points.
2. Determine the equation of a line given the slope and  $y$ -intercept.
3. Find linear functions that model common applications.

## Equations of Lines

Given the algebraic equation of a line, we can graph it in a number of ways. In this section, we will be given a geometric description of a line and find the algebraic equation. Finding the equation of a line can be accomplished in a number of ways. The following example makes use of slope-intercept form,  $y = mx + b$ , or using function

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notation,  $f(x) = mx + b$ . If we can determine the slope,  $m$ , and the  $y$ -intercept,  $(0, b)$ , we can then construct the equation.

## Example 1

Find the equation of the line passing through  $(-3, 6)$  and  $(5, -4)$ .

Solution:

We begin by finding the slope. Given two points, we can find the slope using the slope formula.

Here and we have

To find  $b$ , substitute either one of the given points through which the line passes. Here we will use  $(-3, 6)$ , but  $(5, -4)$  would work just as well:

Therefore, the equation of the line passing through the two given points is:

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Answer:

Next, we outline an alternative method for finding equations of lines. Begin by applying the slope formula with a given point  $(x_1, y_1)$  and a variable point  $(x, y)$ .

Therefore, the equation of a nonvertical line can be written in **point-slope form**:

Point-slope form is particularly useful for finding the equation of a line given the slope and any ordered pair solution. After finding the slope, in the previous example, we could use this form to find the equation.

Substitute as follows.

Notice that we obtain the same linear function

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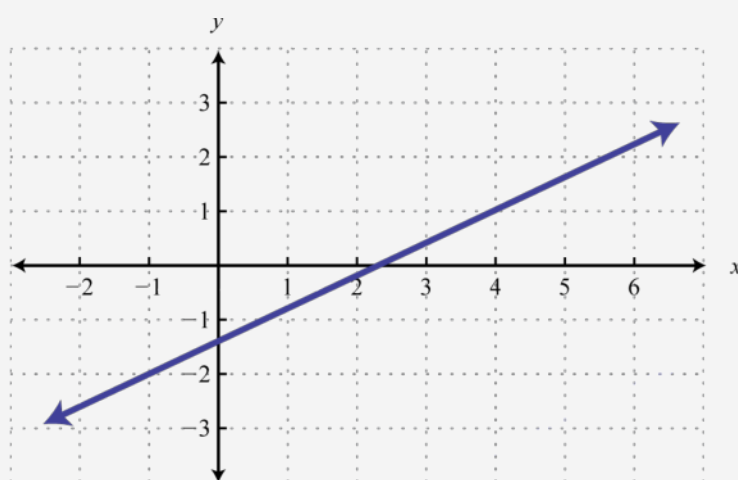
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Note: Sometimes a variable is not expressed explicitly in terms of another; however, it is still assumed that one variable is dependent on the other. For example, the equation  $2x + 3y = 6$  implicitly represents the function. You should become comfortable with working with functions in either form.

## Example 2

Find the equation of the following linear function:



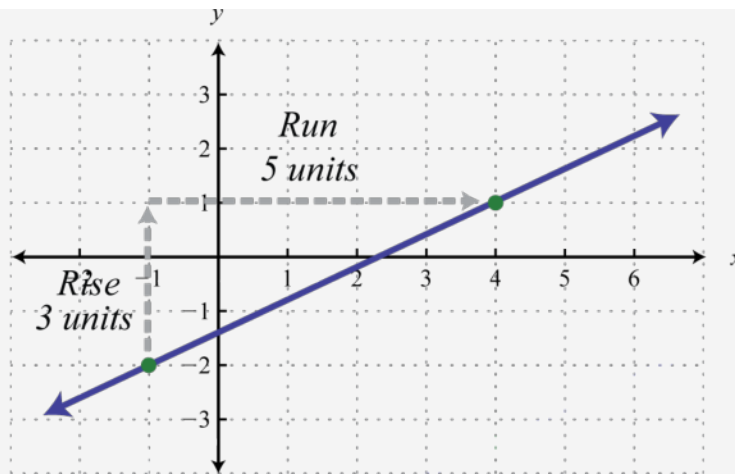
Solution:

From the graph we can determine two points  $(-1, -2)$  and  $(4, 1)$ . Use these points to read the slope from the graph. The rise is 3 units and the run is 5 units.

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Therefore, we have the slope and a point. (It does not matter which of the given points we use, the result will be the same.)

Use point-slope form to determine the equation of the line.

Answer:

Recall that **parallel lines** are lines in the same plane that never intersect. Two non-vertical lines in the same plane with slopes  $m_1$  and  $m_2$  are parallel if their slopes are the same,

## Example 3

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Find the equation of the line passing through  $(3, -2)$  and parallel to  $x - 2y = -2$ .

Solution:

To find the slope of the given line, solve for  $y$ .

Here the given line has slope  $\frac{1}{2}$  and thus the slope of a parallel line is  $\frac{1}{2}$ . The notation  $m$  reads “ $m$  parallel.” Since we are given a point and we now have the slope, we will choose to use point-slope form of a line to determine the equation.

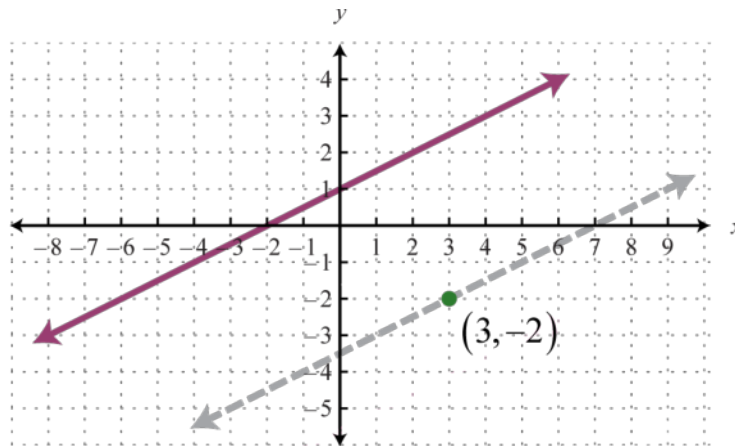
Answer:

It is important to have a geometric understanding of this question. We were asked to find the equation of a line parallel to another line passing through a certain point.

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Through the point  $(3, -2)$  we found a parallel line, shown as a dashed line. Notice that the slope is the same as the given line, but the  $y$ -intercept is different.

Recall that **perpendicular lines** are lines in the same plane that intersect at right angles (90 degrees). Two nonvertical lines, in the same plane with slopes  $m_1$  and  $m_2$ , are perpendicular if the product of their slopes is  $-1$ ,  $m_1 \cdot m_2 = -1$ . We can solve for  $m_2$  and obtain  $m_2 = -\frac{1}{m_1}$ . In this form, we see that perpendicular lines have slopes that are **negative reciprocals**, or **opposite reciprocals**. In general, given real numbers  $a$  and  $b$ ,

The mathematical notation  $m \perp n$  reads “ $m$  perpendicular”. For example, the opposite reciprocal of  $2$  is  $-\frac{1}{2}$ . We can verify that two slopes produce perpendicular lines if their product is  $-1$ .



## Example 4

Find the equation of the line passing through  $(-5, -2)$  and perpendicular to  $x + 4y = 4$ .

Solution:

To find the slope of the given line, solve for  $y$ .

The given line has slope  $-\frac{1}{4}$ , and thus, the perpendicular line has slope  $4$ . Substitute this slope and the given point into point-slope form.

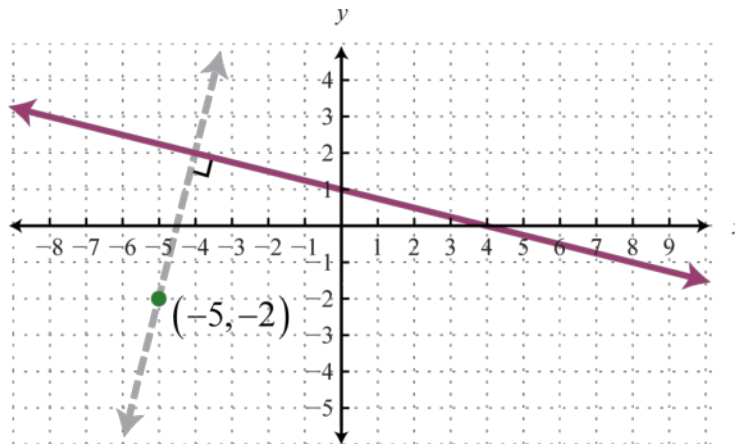
Answer:  $f(x) = 4x + 18$

Geometrically, we see that the line  $y = 4x + 18$ , shown as a dashed line in the graph, passes through  $(-5, -2)$  and is perpendicular to the given line  $y = -\frac{1}{4}x + 1$ .

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Try this! Find the equation of the line passing through  $(-5, -2)$  and perpendicular to

Answer:

## Modeling Linear Applications

Data can be used to construct functions that model real-world applications. Once an equation that fits given data is determined, we can use the equation to make certain predictions; this is called **mathematical modeling**.

### Example 5

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The cost of a daily truck rental is \$48.00, plus an additional \$0.45 for every mile driven. Write a function that gives the cost of the daily truck rental and use it to determine the total cost of renting the truck for a day and driving it 60 miles.

Solution:

The total cost of the truck rental depends on the number of miles driven. If we let  $x$  represent the number of miles driven, then  $0.45x$  represents the variable cost of renting the truck. Use this and the fixed cost, \$48.00, to write a function that models the total cost,

$$C(x) = 0.45x + 48$$

Use this function to calculate the cost of the rental when  $x = 60$  miles.

$$\begin{aligned} C(60) &= 0.45(60) + 48 \\ &= 27 + 48 \\ &= 75 \end{aligned}$$

Answer: The total cost of renting the truck for the day and driving it 60 miles would be \$75.

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We can use the model  $C(x) = 0.45x + 48$  to answer many more questions. For example, how many miles can be driven to keep the cost of the rental at most \$66? To answer this question, set up an inequality that expresses the cost less than or equal to \$66.

Solve for  $x$  to determine the number of miles that can be driven.

To limit the rental cost to \$66, the truck can be driven 40 miles or less.

## Example 6

A company purchased a new piece of equipment for \$12,000. Four years later it was valued at \$9,000 dollars. Use this data to construct a linear function that models the value of the piece of equipment over time.

Solution:

The value of the item depends on the number of years after it was purchased. Therefore, the age of the piece of equipment is the independent variable. Use ordered pairs where the  $x$ -values represent the age and the  $y$ -values represent the corresponding value.

*(age, value)*

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From the problem, we can determine two ordered pairs. Purchased new ( $age = 0$ ), the item cost \$12,000, and 4 years later the item was valued at \$9,000. Therefore, we can write the following two ( $age, value$ ) ordered pairs:

$$(0, 12,000) \text{ and } (4, 9,000)$$

Use these two ordered pairs to construct a linear model. Begin by finding the slope  $m$ .

Here we have  $m = -750$ . The ordered pair  $(0, 12,000)$  gives the  $y$ -intercept; therefore,  $b = 12,000$ .

Lastly, write this model as a function which gives the value of the piece of equipment over time. Choose the function name  $V$ , for value, and the variable  $t$  instead of  $x$  to represent time in years.

$$V(t) = -750t + 12,000$$

Answer:  $V(t) = -750t + 12,000$

The function  $V(t) = -750t + 12,000$  called a **linear depreciation model**. It uses a linear equation to express the declining value of an item over time. Using this function to determine the value of the item between the given data points is

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called **interpolation**. For example, we can use the function to determine the value of the item where  $t = 2$ ,

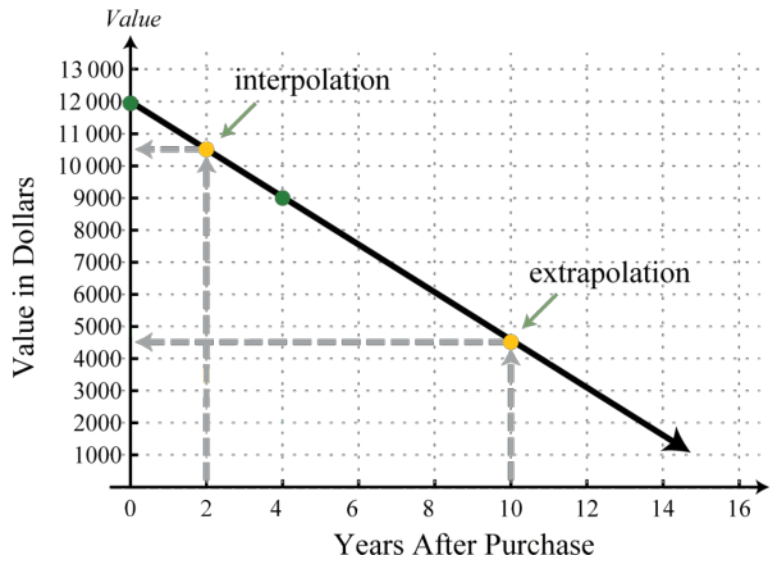
$$\begin{aligned}V(2) &= -750(2) + 12,000 \\ &= 10,500\end{aligned}$$

The function shows that the item was worth \$10,500 two years after it was purchased. Using this model to predict the value outside the given data points is called **extrapolation**. For example, we can use the function to determine the value of the item when  $t = 10$ :

$$\begin{aligned}V(10) &= -750(10) + 12,000 \\ &= -7,500 + 12,000 \\ &= 4,500\end{aligned}$$

The model predicts that the piece of equipment will be worth \$4,500 ten years after it is purchased.





In a business application, revenue results from the sale of a number of items. For example, if an item can be sold for \$150 and we let  $n$  represent the number of units sold, then we can form the following **revenue function**:

$$R(n) = 150n$$

Use this function to determine the revenue generated from selling  $n = 100$  units,

$$R(100) = 150(100) = 15,000$$

The function shows that the revenue generated from selling 100 items is \$15,000. Typically, selling items does not represent the entire story. There are a number of costs associated with the generation of revenue. For example, if there is a one-time set up fee of \$5,280 and each item cost \$62 to produce, then we can form the following **cost function**:

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$$C(n) = 62n + 5,280$$

Here  $n$  represents the number of items produced. Use this function to determine the cost associated with producing  $n = 100$  units:

$$C(100) = 62(100) + 5,280 = 11,480$$

The function shows that the cost associated with producing 100 items is \$11,480. Profit is revenue less costs:

Therefore, the profit generated by producing and selling 100 items is \$3,520. In general, given a revenue function  $R$  and a cost function  $C$ , we can form a **profit function** by subtracting as follows:

$$P(n) = R(n) - C(n)$$

## Example 7

The cost in dollars of producing  $n$  items is given by the formula  $C(n) = 62n + 5,280$ . The revenue in dollars is given by  $R(n) = 150n$ , where  $n$  represents the number items sold. Write a function that gives the profit generated by producing and selling  $n$  items. Use the function to determine how many items must be produced and sold in order to earn a profit of at least \$7,000.

Solution:

Obtain the profit function by subtracting the cost function from the revenue function.

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Therefore,  $P(n) = 88n + 5,280$  models the profit. To determine the number of items that must be produced and sold to profit at least \$7,000, solve the following:

Round up because the number of units produced and sold must be an integer. To see this, calculate the profit where  $n$  is 139 and 140 units.

Answer: 140 or more items must be produced and sold in order to earn a profit of at least \$7,000.

Sometimes the costs exceed the revenue, in which case, the profit will be negative. For example, use the profit function of the previous example,  $P(n) = 88n - 5,280$ , to calculate the profit generated where  $n = 50$ .

$$P(50) = 88(50) - 5,280 = -880$$

This indicates that when 50 units are produced and sold the corresponding profit is a loss of \$880.

It is often important to determine how many items must be produced and sold to break even. To break even means to neither have a gain nor a loss; in this case, the profit will

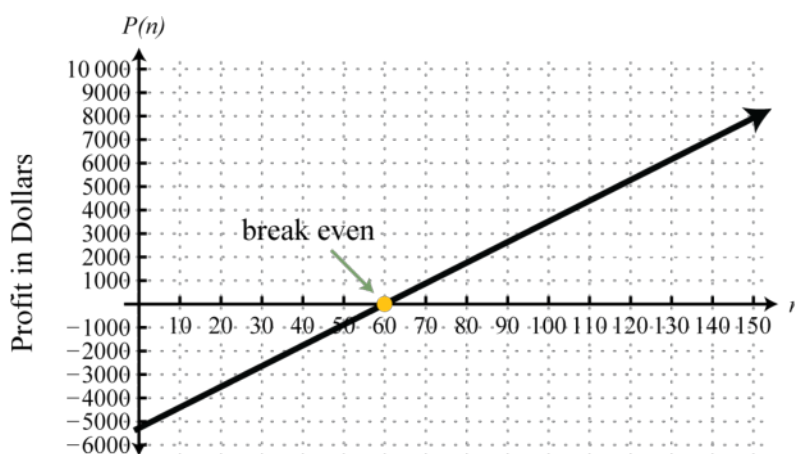
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be equal to zero. To determine the **breakeven point**, set the profit function equal to zero and solve:

Therefore, 60 items must be produced and sold to break even.



Try this! Custom t-shirts can be sold for \$6.50 each. In addition to an initial set-up fee of \$120, each t-shirt cost \$3.50 to produce. a. Write a function that models the revenue and a function that models the cost. b. Determine a function that models the profit and use it to determine the profit from producing and selling 150 t-shirts. c. Calculate the number of t-shirts that must be sold to break even.

Answer: a. Revenue:  $R(x) = 6.50x$ ; cost:  $C(x) = 3.50x + 120$ ; b. profit:  $P(x) = 3x + 120$ ; \$330 c. 40

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## KEY TAKEAWAYS

Given two points we can find the equation of a line.

Parallel lines have the same slope.

Perpendicular lines have slopes that are opposite reciprocals. In other words, if  $m$  is the slope of a line, then the slope of a line perpendicular to it is  $-\frac{1}{m}$ .

To find an equation of a line, first use the given information to determine the slope. Then use the slope and a point on the line to find the equation using point-slope form.

To construct a linear function that models a real-world application, first identify the dependent and independent variables. Next, find two ordered pairs that describe the given situation. Use these two ordered pairs to construct a linear function by finding the slope and  $y$ -intercept.

## TOPIC EXERCISES

### PART A: EQUATIONS OF LINES

Find the linear function  $f$  passing through the given points.

1.  $(-1, 2)$  and  $(3, -4)$
2.  $(3, -2)$  and  $(-1, -4)$
3.  $(-5, -6)$  and  $(-4, 2)$
4.  $(2, -7)$  and  $(3, -5)$
5.  $(10, -15)$  and  $(7, -6)$
6.  $(-9, 13)$  and  $(-8, 12)$
7.  $(-12, 22)$  and  $(6, -20)$

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8.  $(6, -12)$  and  $(-4, 13)$

9. and

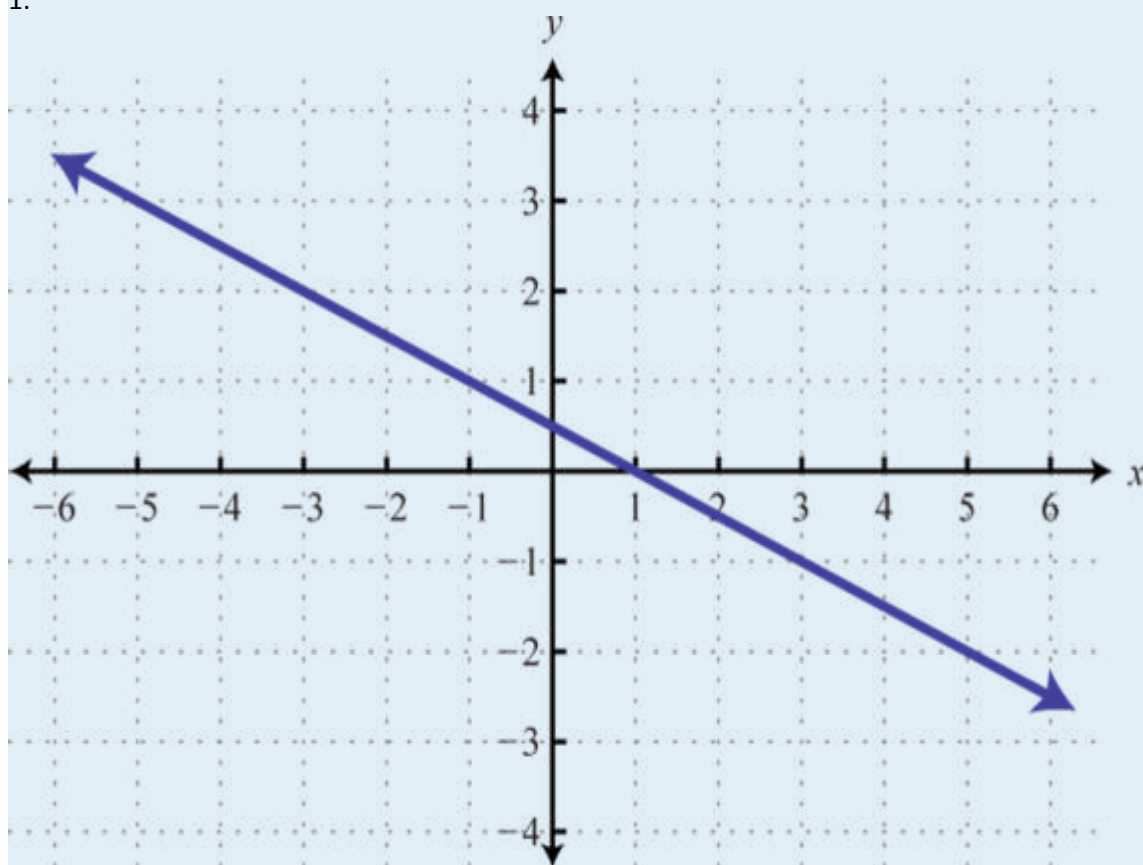
10. and

11.  $(-5, 10)$  and  $(-1, 10)$

12.  $(4, 0)$  and  $(-7, 0)$

Find the equation of the given linear function.

1.

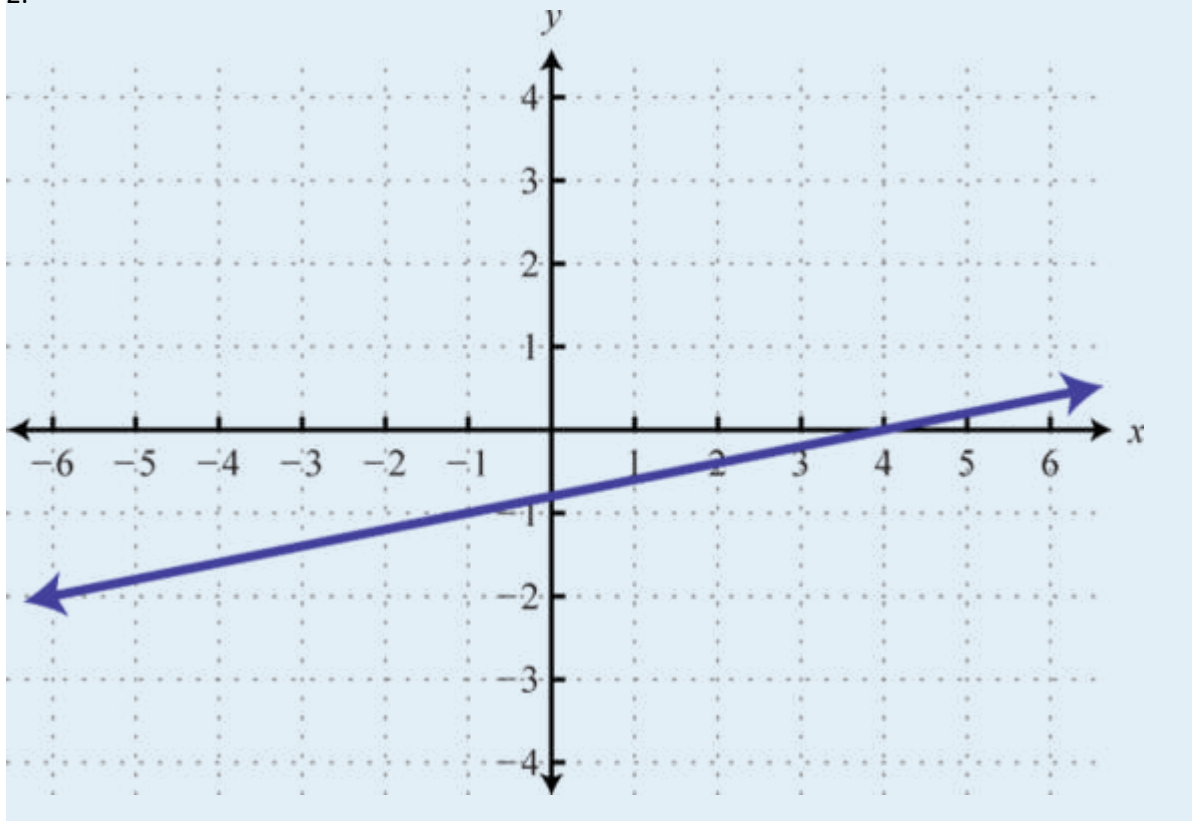


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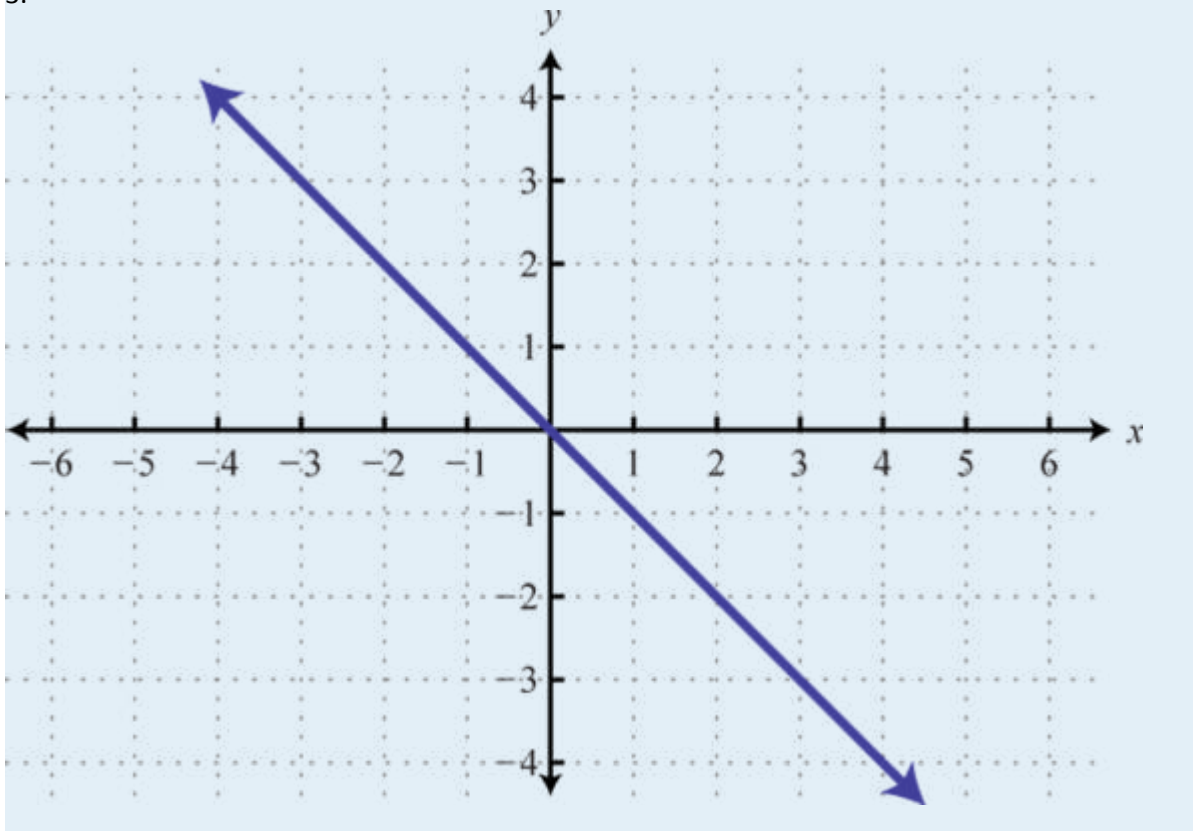


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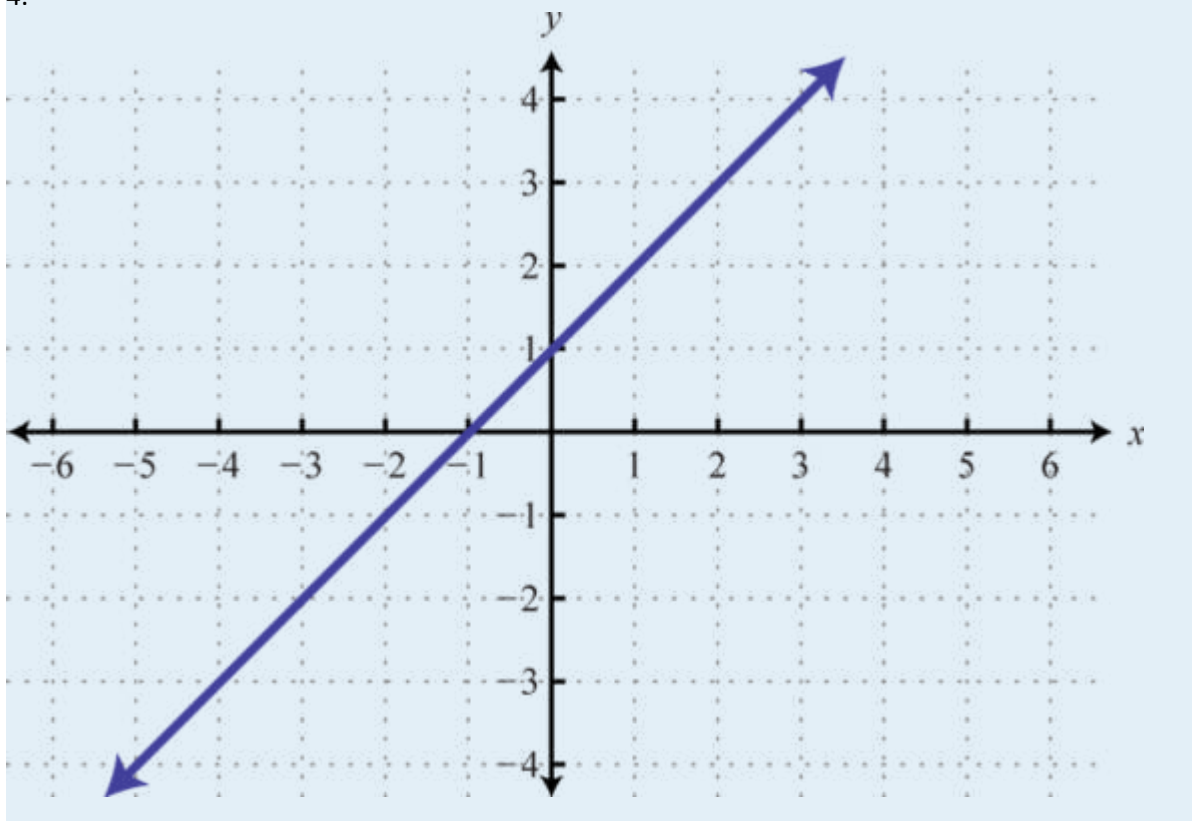


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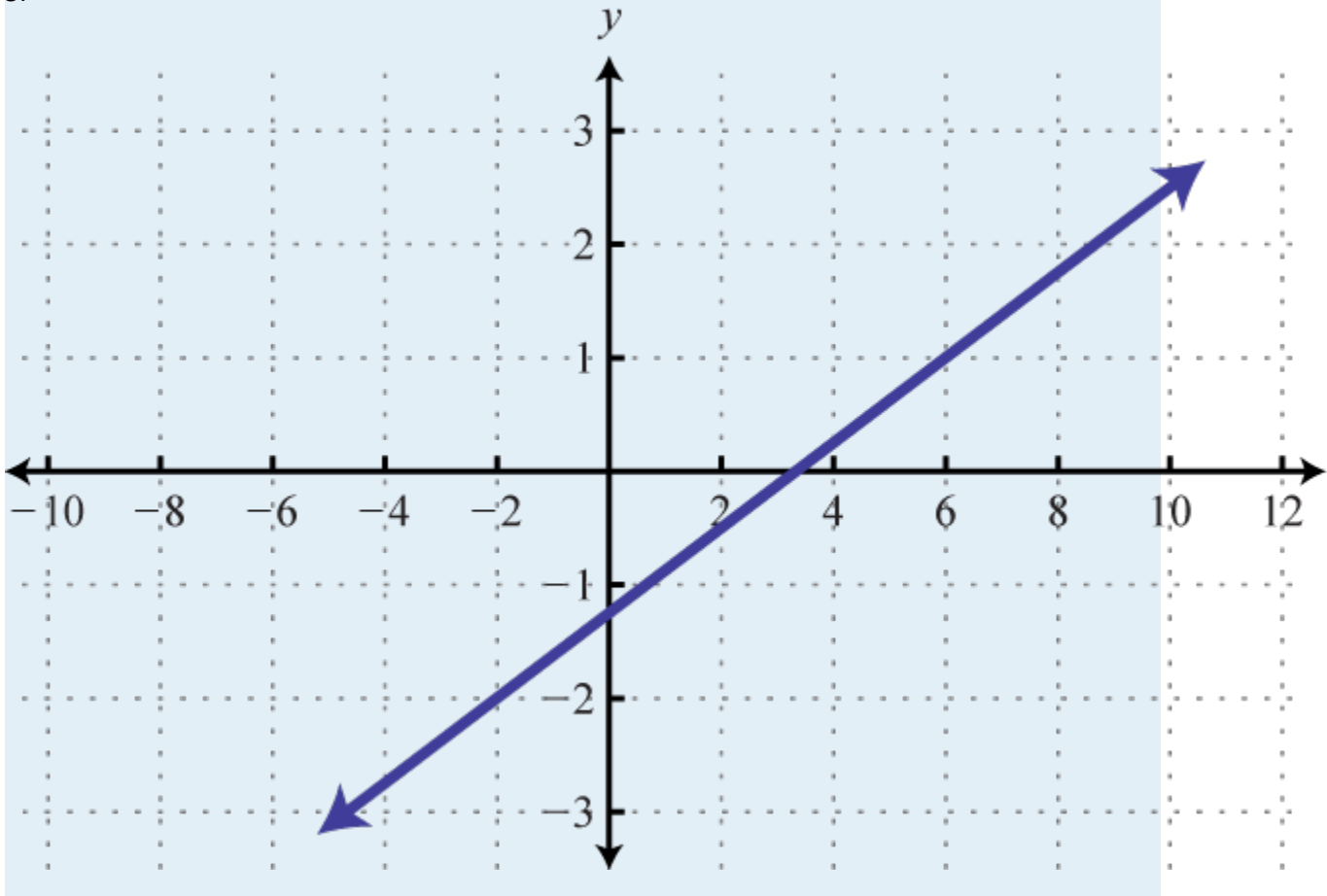


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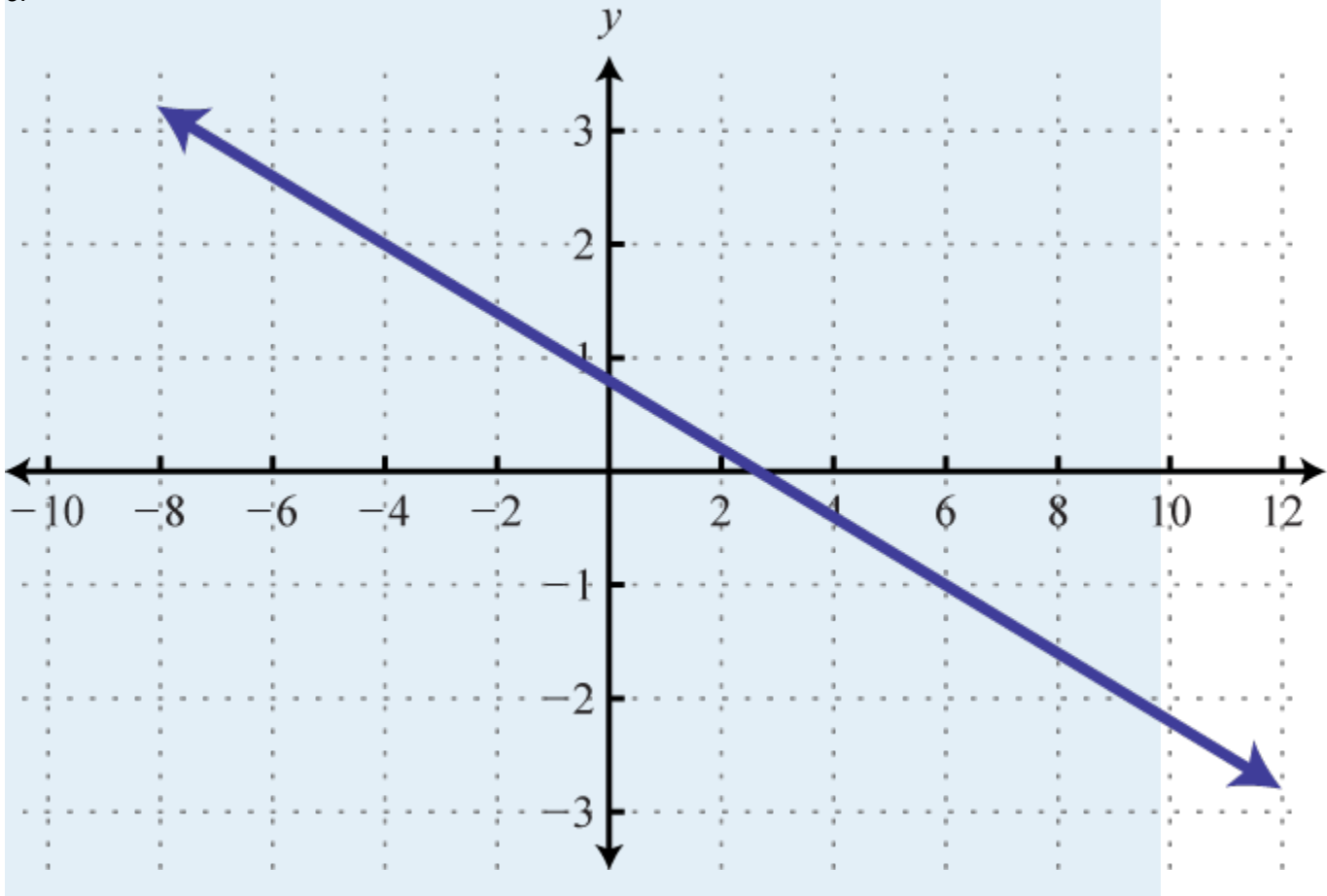


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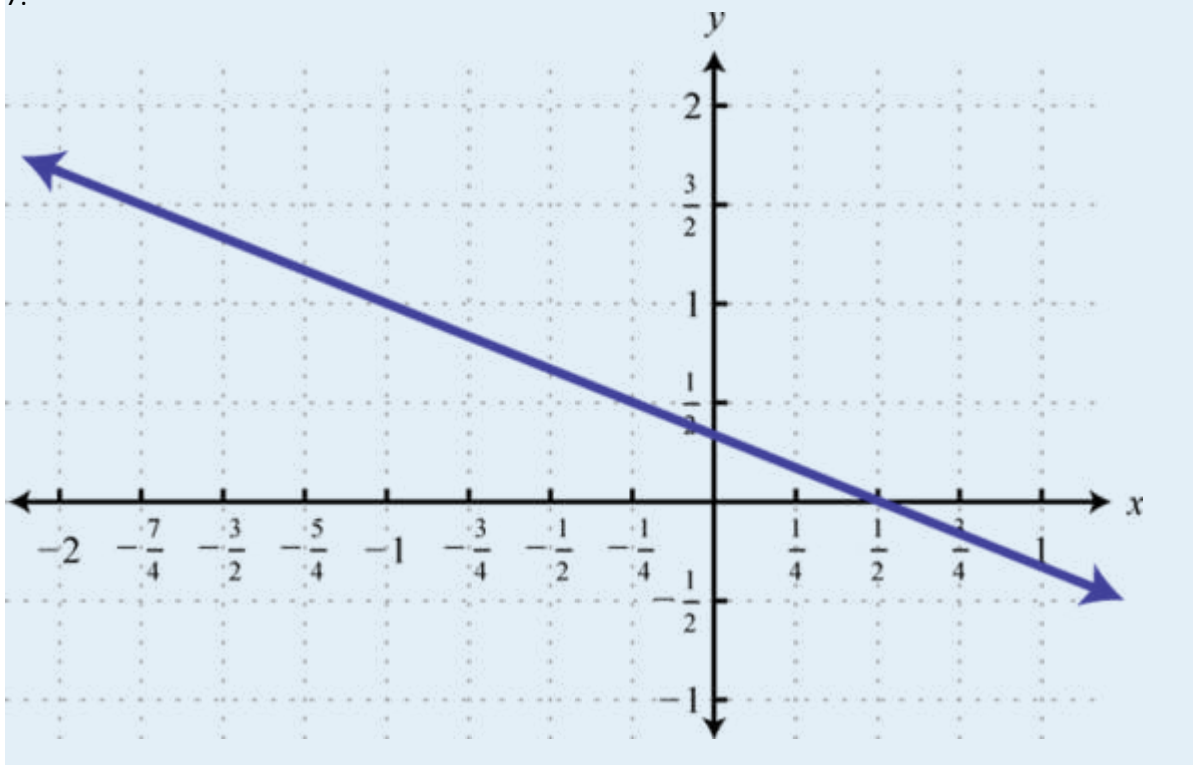


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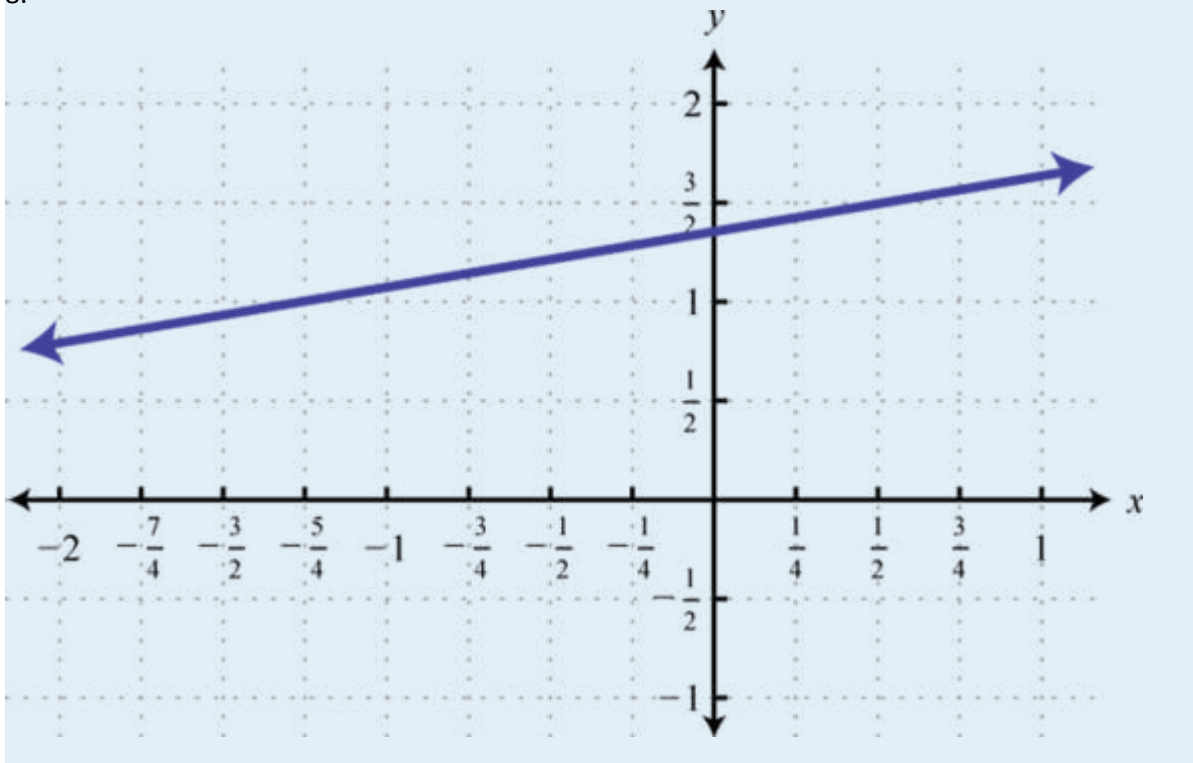
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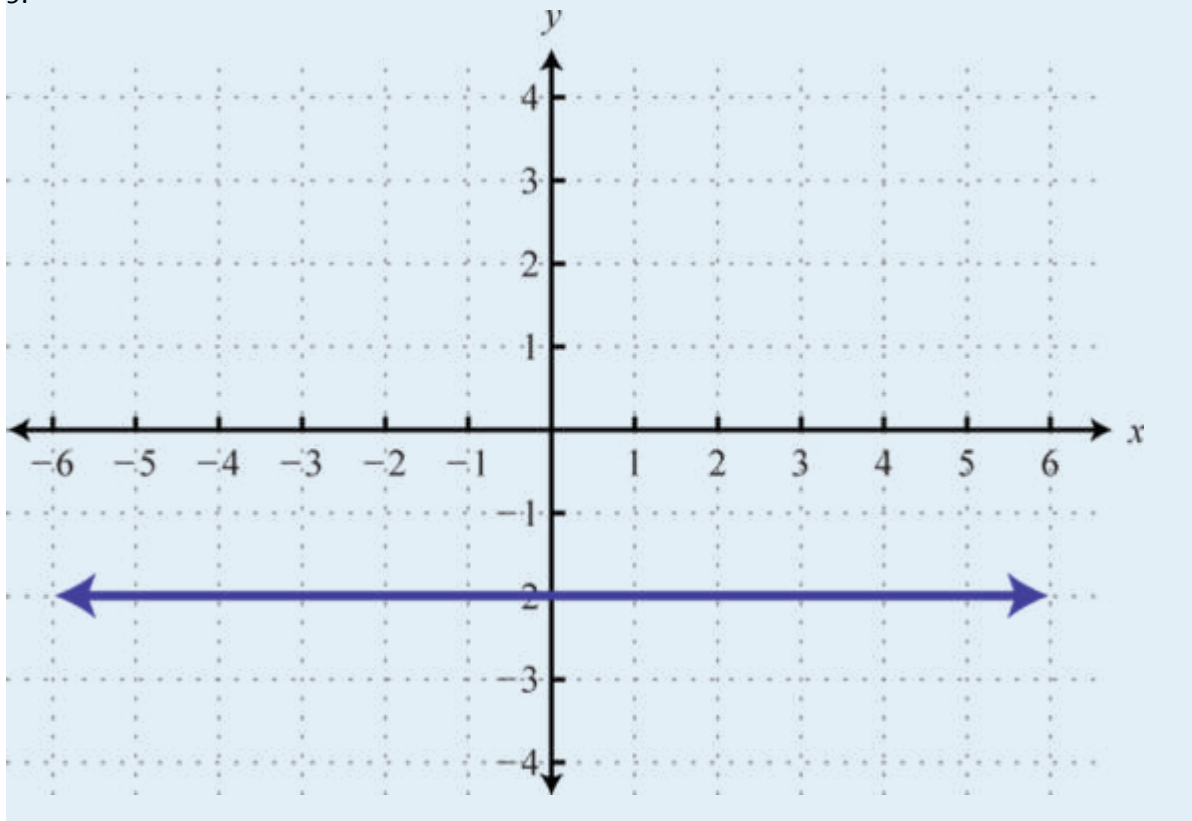


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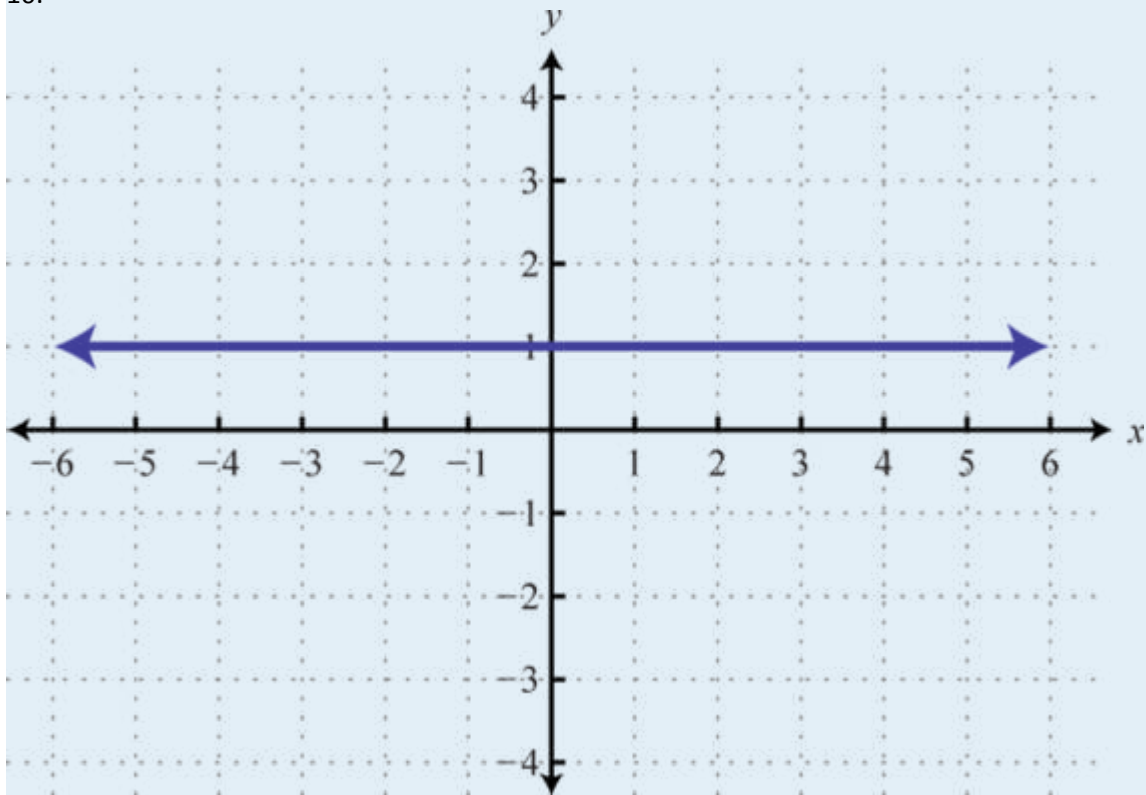


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10.



Find the equation of the line:

1. Parallel to and passing through  $(-3, 4)$ .
2. Parallel to and passing through  $(-8, -1)$ .
3. Parallel to and passing through  $(2, -5)$ .
4. Parallel to and passing through  $(5, 6)$ .
5. Parallel to  $4x - 5y = 15$  and passing through  $(-1, -2)$ .
6. Parallel to  $3x - 4y = 2$  and passing through  $(-6, 8)$ .
7. Parallel to  $2x + 12y = 9$  and passing through  $(10, -9)$ .

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8. Parallel to  $9x + 24y = 2$  and passing through  $(-12, -4)$ .
9. Parallel to  $9x + 24y = 2$  and passing through  $(-15, 4)$ .
10. Parallel to  $9x + 24y = 2$  and passing through  $(12, -11)$ .
11. Perpendicular to  $y = 5x + 2$  and passing through  $(10, -5)$ .
12. Perpendicular to  $y = -2x + 1$  and passing through  $(-8, -11)$ .
13. Perpendicular to  $y = 5x + 2$  and passing through  $(5, -3)$ .
14. Perpendicular to  $y = -2x + 1$  and passing through  $(-6, -4)$ .
15. Perpendicular to  $12x + 15y = 3$  and passing through  $(12, 15)$ .
16. Perpendicular to  $24x + 15y = 12$  and passing through  $(2, -1)$ .
17. Perpendicular to  $14x - y = 3$  and passing through  $(7, 3)$ .
18. Perpendicular to  $x - y = 4$  and passing through  $(6, -2)$ .
19. Perpendicular to  $9x + 24y = 2$  and passing through  $(1, -1)$ .
20. Perpendicular to  $9x + 24y = 2$  and passing through  $(-3, 6)$ .
21. Give the equation of the line that coincides with the  $x$ -axis.
22. Give the equation of the line that coincides with the  $y$ -axis.
23. Given any line in standard form,  $ax + by = c$ , determine the slope of any perpendicular line.
24. Given any line in standard form,  $ax + by = c$ , determine the slope of any parallel line.

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## PART B: MODELING LINEAR APPLICATIONS

Use algebra to solve the following.

47. A company wishes to purchase pens stamped with the company logo. In addition to an initial set-up fee of \$90, each pen cost \$1.35 to produce. Write a function that gives the cost in terms of the number of pens produced. Use the function to determine the cost of producing 500 pens with the company logo stamped on it.
48. A rental car company charges a daily rate of \$42.00 plus \$0.51 per mile driven. Write a function that gives the cost of renting the car for a day in terms of the number of miles driven. Use the function to determine the cost of renting the car for a day and driving it 76 miles.
49. A certain cellular phone plan charges \$16 per month and \$0.15 per minute of usage. Write a function that gives the cost of the phone per month based on the number of minutes of usage. Use the function to determine the number of minutes of usage if the bill for the first month was \$46.
50. A web-services company charges \$2.50 a month plus \$0.14 per gigabyte of storage on their system. Write a function that gives the cost of storage per month in terms of the number of gigabytes stored. How many gigabytes are stored if the bill for this month was \$6.00?

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51. Mary has been keeping track of her cellular phone bills for the last two months. The bill for the first month was \$45.00 for 150 minutes of usage. The bill for the second month was \$25.00 for 50 minutes of usage. Find a linear function that gives the total monthly bill based on the minutes of usage.
52. A company in its first year of business produced 1, 200 brochures for a total cost of \$5, 050. The following year, the company produced 500 more brochures at a cost of \$2, 250. Use this information to find a linear function that gives the total cost of producing brochures from the number of brochures produced.
53. A Webmaster has noticed that the number of registered users has been steadily increasing since beginning an advertising campaign. Before starting to advertise, he had 2, 200 registered users, and after 4 months of advertising he now has 5, 480 registered users. Use this data to write a linear function that gives the total number of registered users, given the number of months after starting to advertise. Use the function to predict the number of users 8 months into the advertising campaign.
54. A corn farmer in California was able to produce 154 bushels of corn per acre 2 years after starting his operation. Currently, after 7 years of operation, he has increased his yield to 164 bushels per acre. Use this information to write a linear function that gives the total yield per acre based on the number of years of operation, and use it to predict the yield for next year.
55. A commercial van was purchased new for \$22, 500 and is expected to be worthless in 12 years. Use this information to write a linear depreciation function for the value of the van. Use the function to determine the value of the van after 8 years of use.

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56. The average lifespan of an industrial welding robot is 10 years, after which it is considered to have no value. If an industrial welding robot was purchased new for \$58,000, write a function that gives the value of the robot in terms of the number of years of operation. Use the function to value the robot after 3 years of operation.
57. A business purchased a piece of equipment new for \$2,400. After 5 years of use the equipment is valued at \$1,650. Find a linear function that gives the value of the equipment in terms of years of usage. Use the function to determine the number of years after which the piece of equipment will have no value.
58. A salesman earns a base salary of \$2,400 a month plus a 5% commission on all sales. Write a function that gives the salesman's monthly salary in terms of sales. Use the function to determine the monthly sales required to earn at least \$3,600 a month.
59. When a certain professor was hired in 2005, the enrollment at a college was 8,500 students. Five years later, in 2010, the enrollment grew to 11,200 students. Determine a linear growth function that models the student population in years since 2005. Use the model to predict the year in which enrollment will exceed 13,000 students.
60. In 1980, the population of California was about 24 million people. Twenty years later, in the year 2000, the population was about 34 million. Use this data to

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construct a linear function to model the population growth in years since 1980. Use the function to predict the year in which the population will reach 40 million.

61. A classic car is purchased for \$24,500 and is expected to increase in value each year by \$672. Write a linear function that models the appreciation of the car in terms of the number of years after purchase. Use the function to predict the value of the car in 7 years.
62. A company reported first and second quarter sales of \$52,000 and \$64,500, respectively.
- 47.a) Write a linear function that models the sales for the year in terms of the quarter  $n$ .
- 47.b) Use the model to predict the sales in the third and fourth quarters.
63. A particular search engine assigns a ranking to a webpage based on the number of links that direct users to the webpage. If no links are found, the webpage is assigned a ranking of 1. If 20 links are found directing users to the webpage, the search engine assigns a page ranking of 3.5. a. Find a linear function that gives the webpage ranking based on the number of links that direct users to it. b. How many links will be needed to obtain a page ranking of 5?
64. Online sales of a particular product are related to the number of clicks on its advertisement. It was found that 1,520 clicks in a month results in \$2,748 of online sales, and that 1,840 clicks results in \$2,956 of online sales. Write a linear function that models the online sales of the product based on the number of clicks on its advertisement. How many clicks would we need to expect \$3,385 in monthly online sales from this particular product?

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65. A bicycle manufacturing business can produce  $x$  bicycles at a cost, in dollars, given by the formula  $C(x) = 85x + 2,400$ . The company sells each bicycle at a wholesale price of \$145. The revenue, in dollars, is given by  $R(x) = 145x$ , where  $x$  represents the number of bicycles sold. Write a function that gives profit in terms of the number of bicycles produced and sold. Use the function to determine the number of bicycles that need to be produced and sold to break even.
66. The cost, in dollars, of producing  $n$  custom lamps is given by the formula  $C(n) = 28n + 360$ . Each lamp can be sold online for \$79. The revenue in dollars, is given by  $R(n) = 79n$ , where  $n$  represents the number of lamps sold. Write a function that gives the profit from producing and selling  $n$  custom lamps. Use the function to determine how many lamps must be produced and sold to earn at least \$1,000 in profit.
67. A manufacturer can produce a board game at a cost of \$12 per unit after an initial fixed retooling investment of \$12,500. The games can be sold for \$22 each to retailers.
- 47.a) Write a function that gives the manufacturing costs when  $n$  games are produced.
- 47.b) Write a function that gives the revenue from selling  $n$  games to retailers.
- 47.c) Write a function that gives the profit from producing and selling  $n$  units.
- 47.d) How many units must be sold to earn a profit of at least \$37,500?
68. A vending machine can be leased at a cost of \$90 per month. The items used to stock the machine can be purchased for \$0.50 each and sold for \$1.25 each.

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- 68.a. Write a function that gives the monthly cost of leasing and stocking the vending machine with  $n$  items.
- 68.b. Write a function that gives the revenue generated by selling  $n$  items.
- 68.c. Write a function that gives the profit from stocking and selling  $n$  items per month.
- 68.d. How many items must be sold each month to break even?

## PART C: DISCUSSION BOARD

69. Research and discuss linear depreciation. In a linear depreciation model, what do the slope and  $y$ -intercept represent?
70. Write down your own steps for finding the equation of a line. Post your steps on the discussion board.

## ANSWERS

- 1.
- 2.
3.  $f(x) = 8x + 34$
- 4.
5.  $f(x) = -3x + 15$
- 6.
- 7.
- 8.
- 9.
- 10.
11.  $f(x) = 10$
- 12.



13.

14.

15.  $f(x) = -x$

16.

17.

18.

19.

20.

21.  $f(x) = -2$

22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.

35.

36.

37.

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38.

39.

40.

41.

42.

43.  $y = 0$

44.

45.

46.

47.  $C(x) = 1.35x + 90$ ; \$765

48.

49.  $C(x) = 0.15x + 16$ ; 200 minutes

50.

51.  $C(x) = 0.20x + 15$

52.

53.  $U(x) = 820x + 2,200$ ; 8,760 users

54.

55.  $V(t) = -1,875t + 22,500$ ; \$7,500

56.

57.  $V(t) = -150t + 2,400$ ; 16 years

58.

59.  $P(x) = 540x + 8,500$ ; 2013

60.

61.  $V(t) = 672t + 24,500$ ; \$29,204

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62.

63.

a)  $r(n) = 0.125n + 1$ ;

b) 32 links

64.

65.  $P(x) = 60x - 2,400$ ; 40 bicycles

66.

67.

a.  $C(n) = 12n + 12,500$ ;

b.  $R(n) = 22n$ ;

c.  $P(n) = 10n - 12,500$ ;

d. at least 5,000 units

68.

69. Answer may vary

## 2.4 Graphing the Basic Functions

### LEARNING OBJECTIVES

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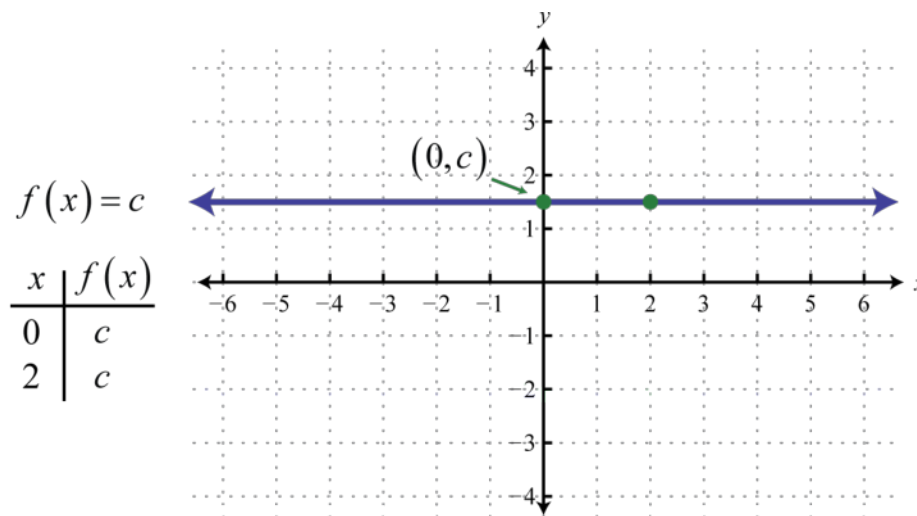


1. Define and graph seven basic functions.
2. Define and graph piecewise functions.
3. Evaluate piecewise defined functions.
4. Define the greatest integer function.

## Basic Functions

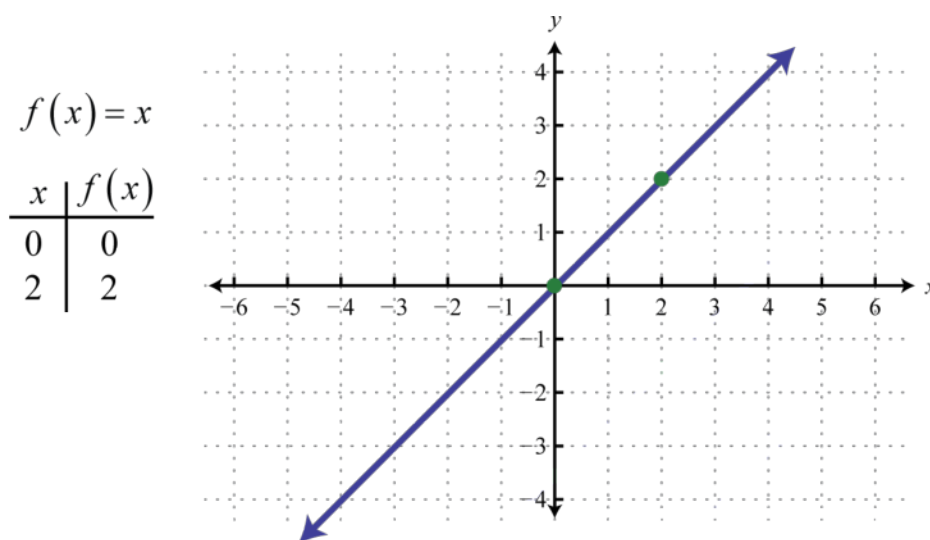
In this section we graph seven basic functions that will be used throughout this course. Each function is graphed by plotting points. Remember that  $f(x) = y$  and thus  $f(x)$  and  $y$  can be used interchangeably.

Any function of the form  $f(x) = c$ , where  $c$  is any real number, is called a **constant function**. Constant functions are linear and can be written  $f(x) = 0x + c$ . In this form, it is clear that the slope is 0 and the  $y$ -intercept is  $(0, c)$ . Evaluating any value for  $x$ , such as  $x = 2$ , will result in  $c$ .



The graph of a constant function is a horizontal line. The domain consists of all real numbers and the range consists of the single value  $\{c\}$ .

We next define the **identity function**  $f(x) = x$ . Evaluating any value for  $x$  will result in that same value. For example,  $f(0) = 0$  and  $f(2) = 2$ . The identity function is linear,  $f(x) = 1x + 0$ , with slope  $m = 1$  and  $y$ -intercept  $(0, 0)$ .



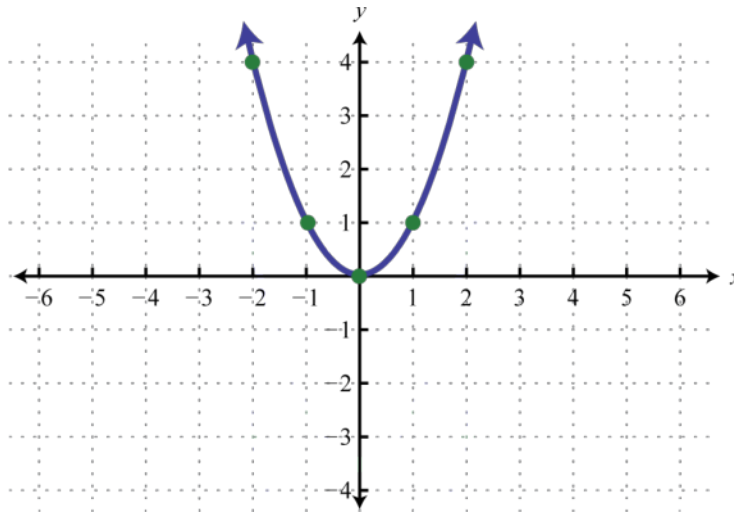
The domain and range both consist of all real numbers.

The **squaring function**, defined by  $f(x) = x^2$ , is the function obtained by squaring the values in the domain. For example,  $f(3) = 9$  and  $f(-3) = 9$ . The result of squaring nonzero values in the domain will always be positive.



$$f(x) = x^2$$

$x$	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

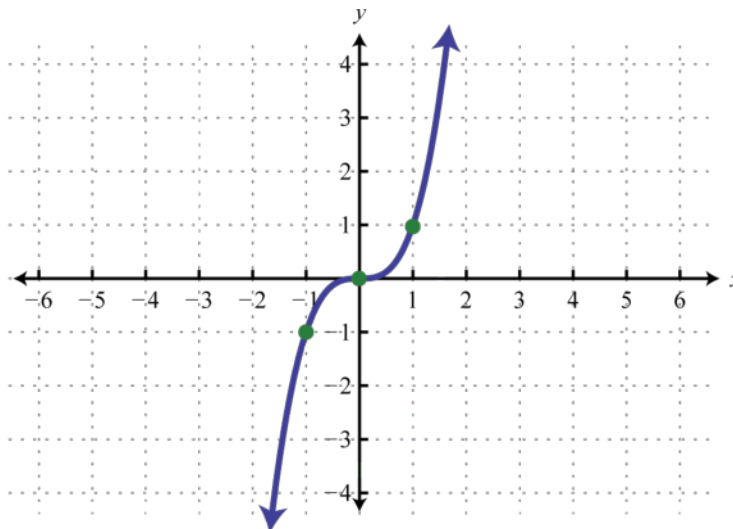


The resulting curved graph is called a **parabola**. The domain consists of all real numbers and the range consists of all  $y$ -values greater than or equal to zero  $[0, \infty)$ .

The **cubing function**, defined by  $f(x) = x^3$ , raises all of the values in the domain to the third power. The results can be either positive, zero, or negative. For example,  $2^3 = 8$ ,  $0^3 = 0$ , and  $(-2)^3 = -8$ .

$$f(x) = x^3$$

$x$	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8



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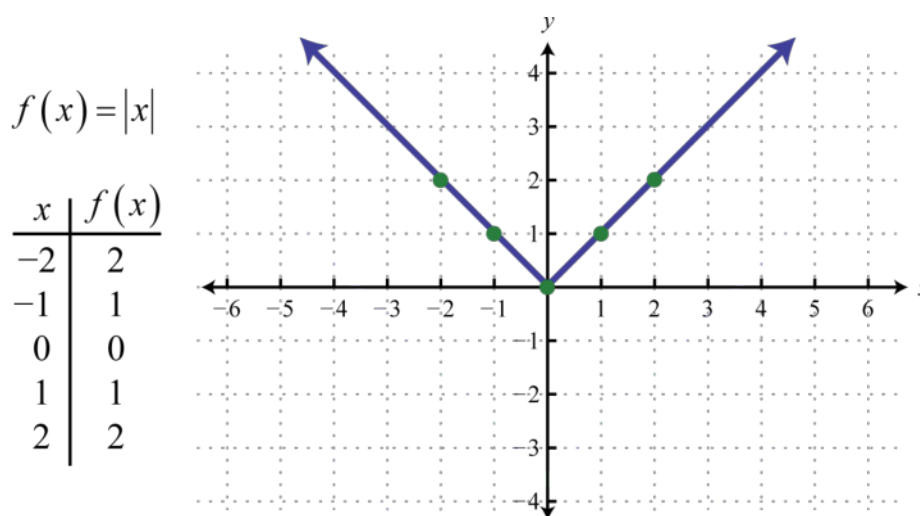




The domain and range both consist of all real numbers .

Note that the constant, identity, squaring, and cubing functions are all examples of basic polynomial functions. The next three basic functions are not polynomials.

The **absolute value function**, defined by  $f(x) = |x|$ , is a function where the output represents the distance to the origin on a number line. The result of evaluating the absolute value function for any nonzero value of  $x$  will always be positive. For example,  $f(-2) = |-2| = 2$  and  $f(2) = |2| = 2$ .



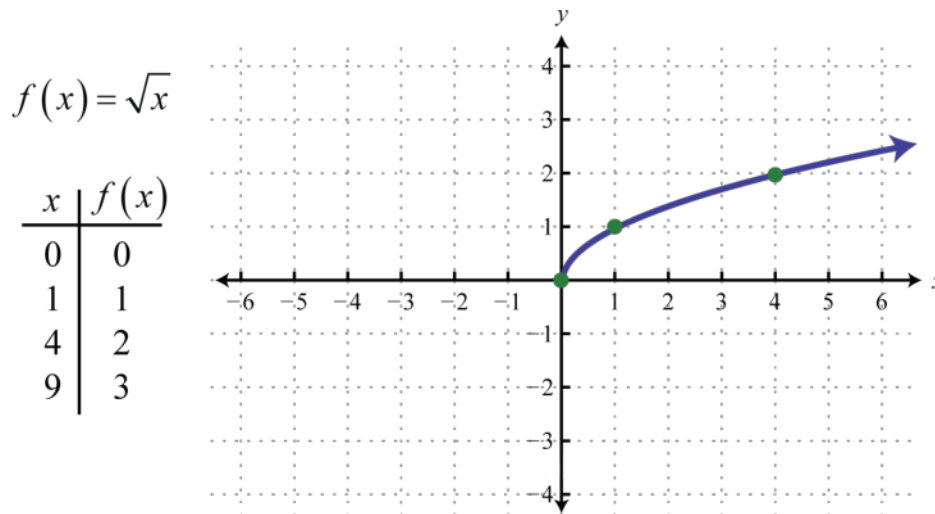
The domain of the absolute value function consists of all real numbers and the range consists of all  $y$ -values greater than or equal to zero  $[0, \infty)$ .

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The **square root function**, defined by  $f(x) = \sqrt{x}$ , is not defined to be a real number if the  $x$ -values are negative. Therefore, the smallest value in the domain is zero. For example,  $f(1) = 1$  and  $f(4) = 2$ .



The domain and range both consist of real numbers greater than or equal to zero  $[0, \infty)$ .

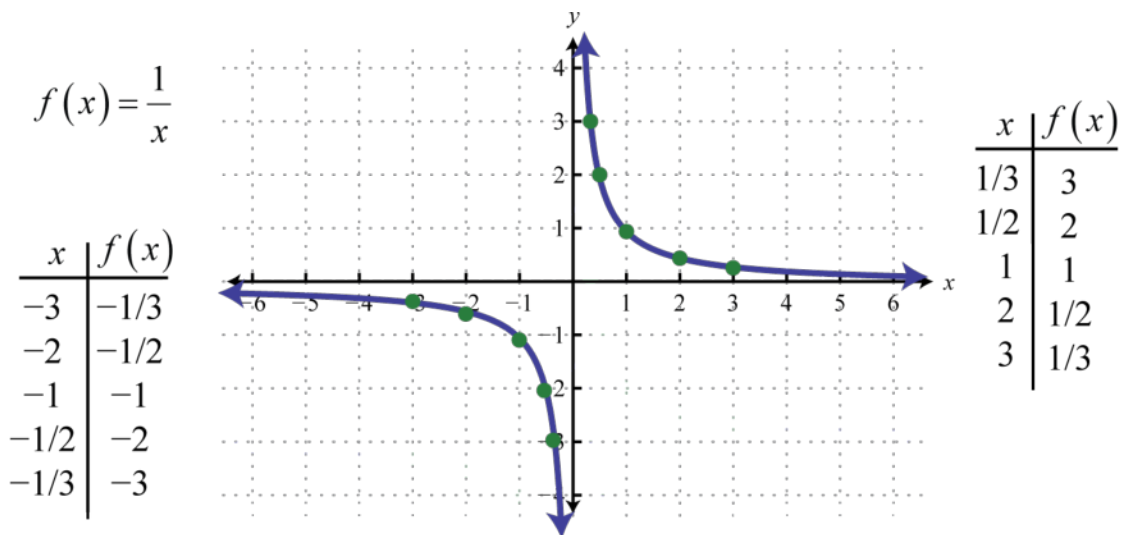
The **reciprocal function**, defined by  $f(x) = \frac{1}{x}$ , is a rational function with one restriction on the domain, namely  $x \neq 0$ . The reciprocal of an  $x$ -value very close to zero is very large. For example,  $f(0.001) = 1000$ .

In other words, as the  $x$ -values approach zero their reciprocals will tend toward either positive or negative infinity. This describes a **vertical asymptote** at the  $y$ -axis.



Furthermore, where the  $x$ -values are very large the result of the reciprocal function is very small.

In other words, as the  $x$ -values become very large the resulting  $y$ -values tend toward zero. This describes a **horizontal asymptote** at the  $x$ -axis. After plotting a number of points the general shape of the reciprocal function can be determined.



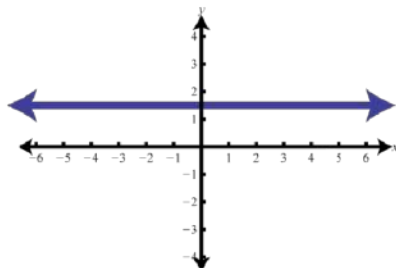
Both the domain and range of the reciprocal function consists of all real numbers except 0, which can be expressed using interval notation as follows:  $(-\infty, 0) \cup (0, \infty)$ .

In summary, the basic polynomial functions are:



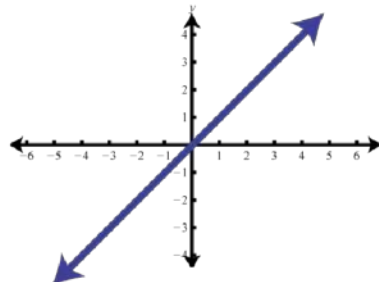
Constant Function

$$f(x) = c$$



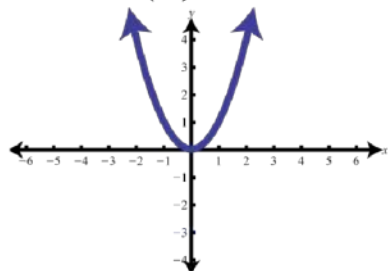
Identity Function

$$f(x) = x$$



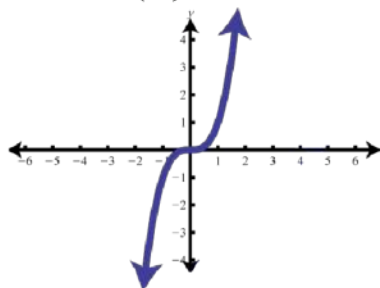
Squaring Function

$$f(x) = x^2$$



Cubing Function

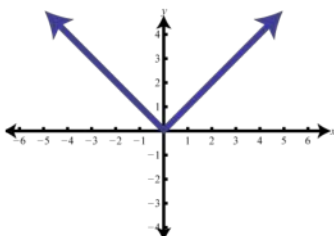
$$f(x) = x^3$$



The basic nonpolynomial functions are:

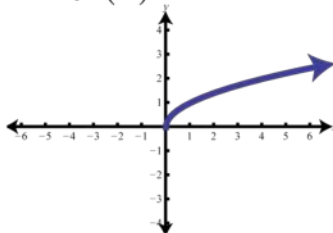
Absolute Value Function

$$f(x) = |x|$$



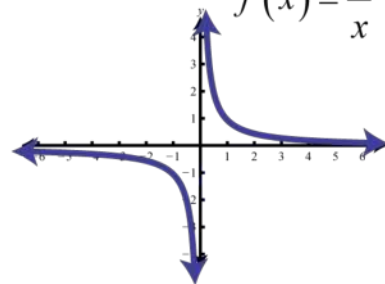
Square Root Function

$$f(x) = \sqrt{x}$$



Reciprocal Function

$$f(x) = \frac{1}{x}$$



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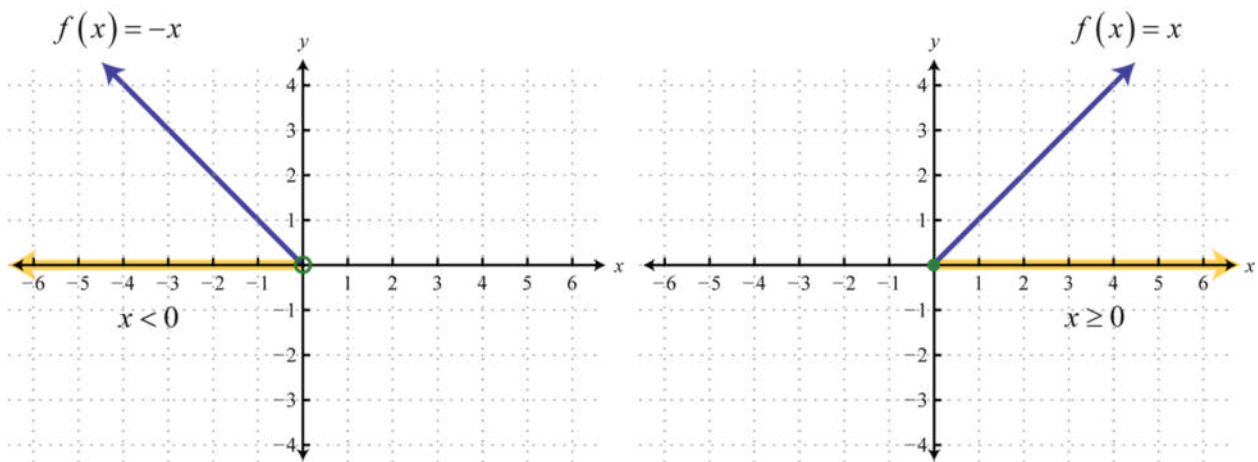
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# Piecewise Defined Functions

A **piecewise function**, or **split function**, is a function whose definition changes depending on the value in the domain. For example, we can write the absolute value function  $f(x) = |x|$  as a piecewise function:

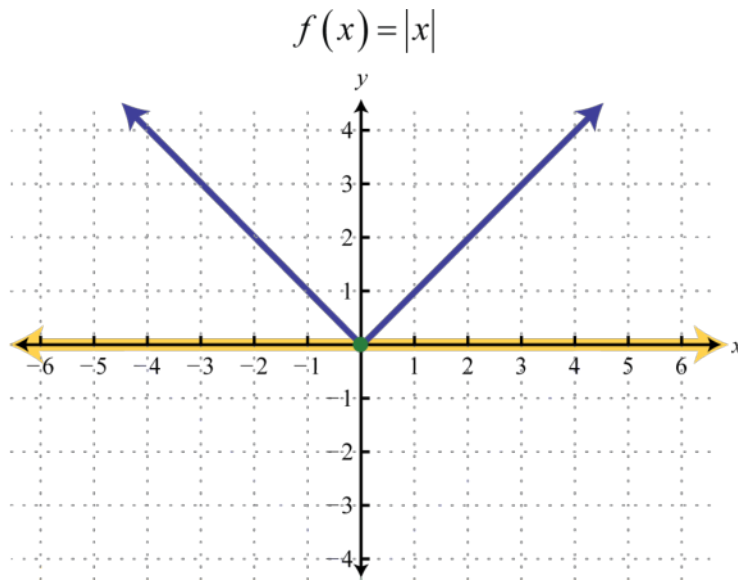
In this case, the definition used depends on the sign of the  $x$ -value. If the  $x$ -value is positive,  $x \geq 0$ , then the function is defined by  $f(x) = x$ . And if the  $x$ -value is negative,  $x < 0$ , then the function is defined by  $f(x) = -x$ .



Following is the graph of the two pieces on the same rectangular coordinate plane:

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## Example 1

Graph:

Solution:

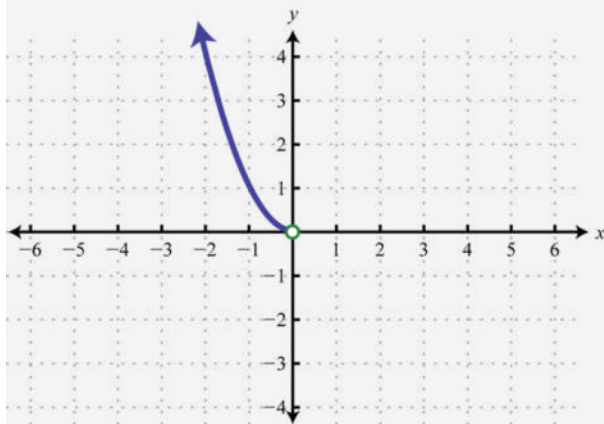
In this case, we graph the squaring function over negative  $x$ -values and the square root function over positive  $x$ -values.

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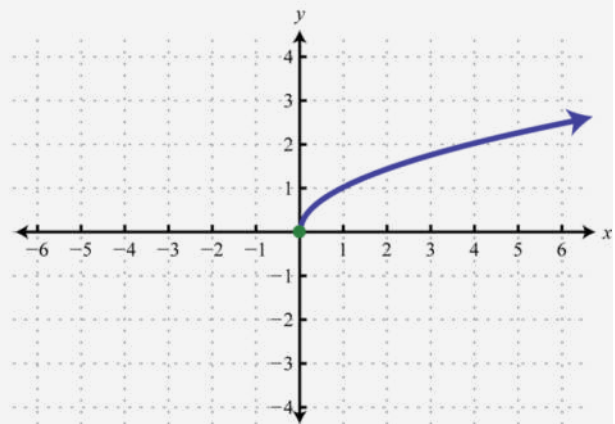
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$$g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

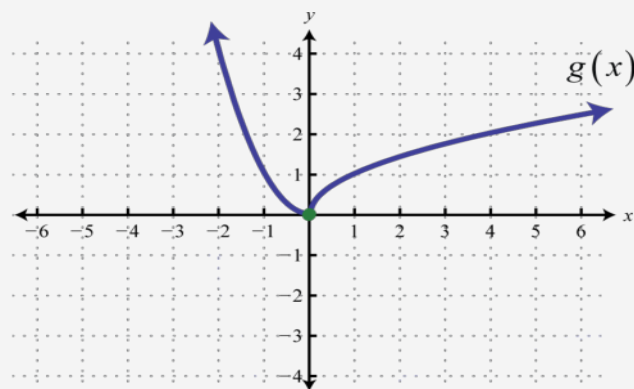


$$g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$



Notice the open dot used at the origin for the squaring function and the closed dot used for the square root function. This was determined by the inequality that defines the domain of each piece of the function. The entire function consists of each piece graphed on the same coordinate plane.

Answer:



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When evaluating, the value in the domain determines the appropriate definition to use.

## Example 2

Given the function  $h$ , find  $h(-5)$ ,  $h(0)$ , and  $h(3)$ .

Solution:

Use  $h(t) = 7t + 3$  where  $t$  is negative, as indicated by  $t < 0$ .

Where  $t$  is greater than or equal to zero, use

Answer:  $h(-5) = -32$ ,  $h(0) = 0$ , and  $h(3) = -48$

Try this! Graph:

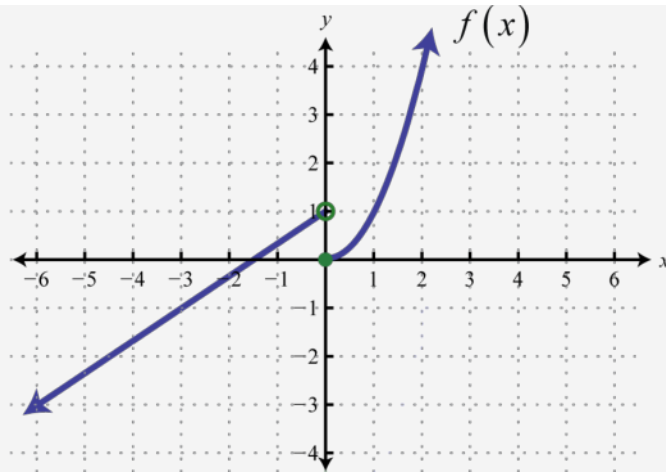
Answer:

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The definition of a function may be different over multiple intervals in the domain.

### Example 3

Graph:

Solution:

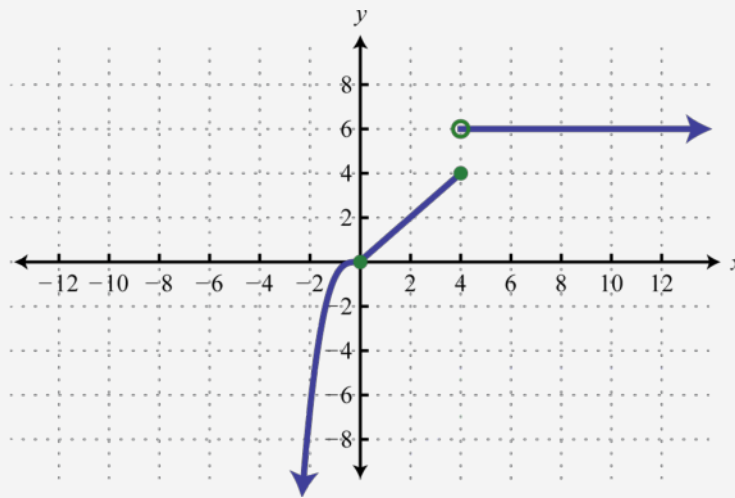
In this case, graph the cubing function over the interval  $(-\infty, 0)$ . Graph the identity function over the interval  $[0, 4]$ . Finally, graph the constant function  $f(x) = 6$  over the interval  $(4, \infty)$ . And because  $f(x) = 6$  where  $x > 4$ , we use an open dot at the point  $(4, 6)$ . Where  $x = 4$ , we use  $f(x) = x$  and thus  $(4, 4)$  is a point on the graph as indicated by a closed dot.

Answer:

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The **greatest integer function**, denoted  $f(x) = [[x]]$ , assigns the greatest integer less than or equal to any real number in its domain. For example,

This function associates any real number with the greatest integer less than or equal to it and should not be confused with rounding off.

## Example 4

Graph:  $f(x) = [[x]]$ .

Solution:

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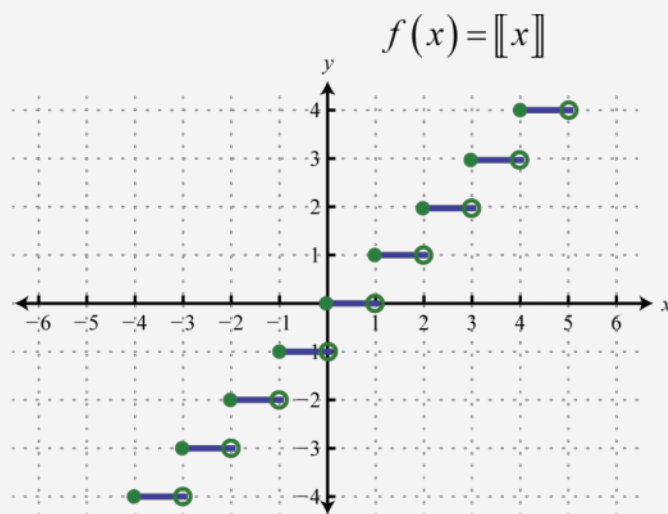
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If  $x$  is any real number, then  $y = \lfloor x \rfloor$  is the greatest integer less than or equal to  $x$ .

Using this, we obtain the following graph.

Answer:



The domain of the greatest integer function consists of all real numbers and the range consists of the set of integers. This function is often called the **floor function** and has many applications in computer science.

## KEY TAKEAWAYS

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Plot points to determine the general shape of the basic functions. The shape, as well as the domain and range, of each should be memorized.

The basic polynomial functions are:  $f(x) = c$ ,  $f(x) = x$ , , and

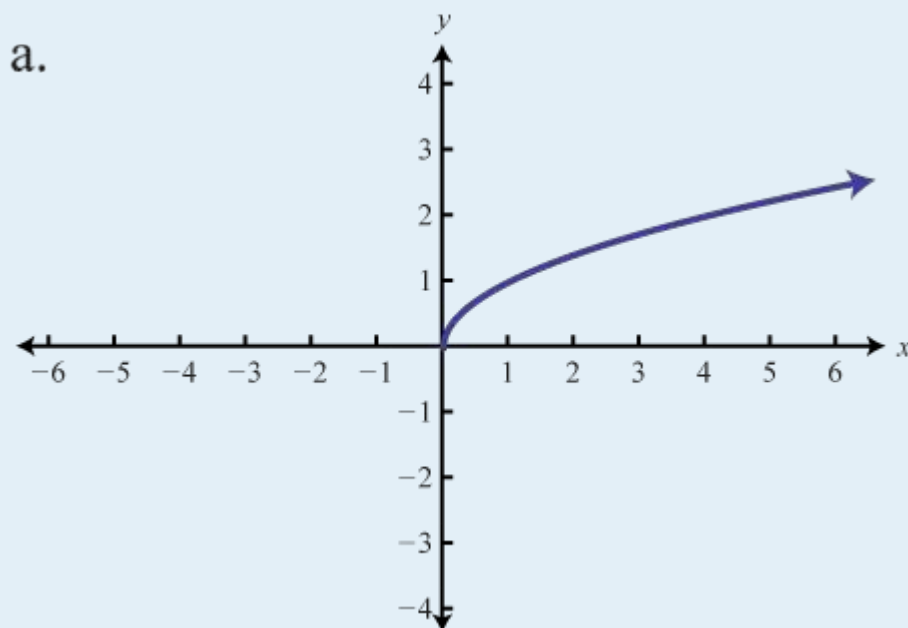
The basic nonpolynomial functions are:  $f(x) = |x|$ , , and .

A function whose definition changes depending on the value in the domain is called a piecewise function. The value in the domain determines the appropriate definition to use.

## TOPIC EXERCISES

### PART A: BASIC FUNCTIONS

Match the graph to the function definition.

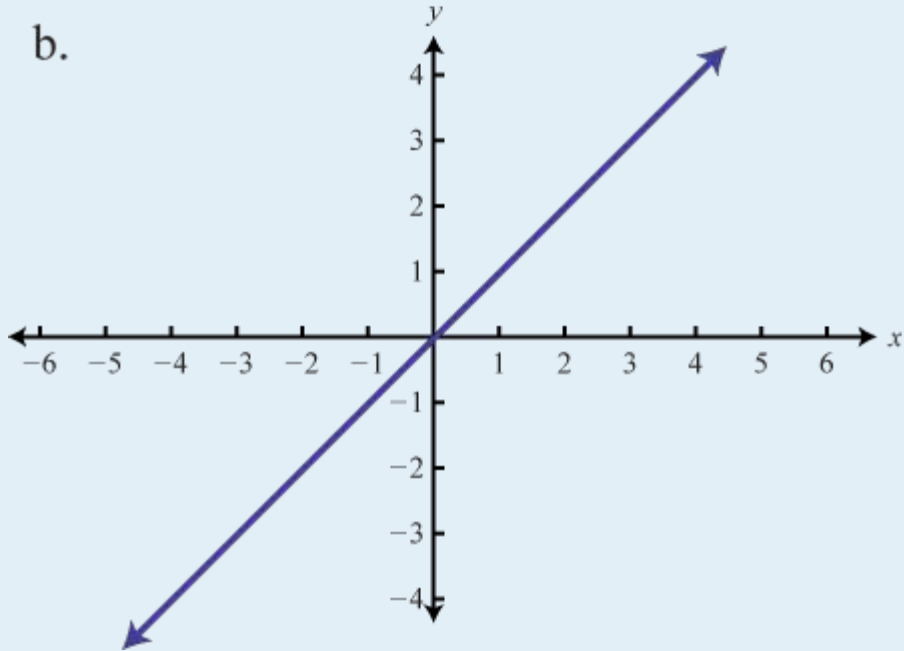


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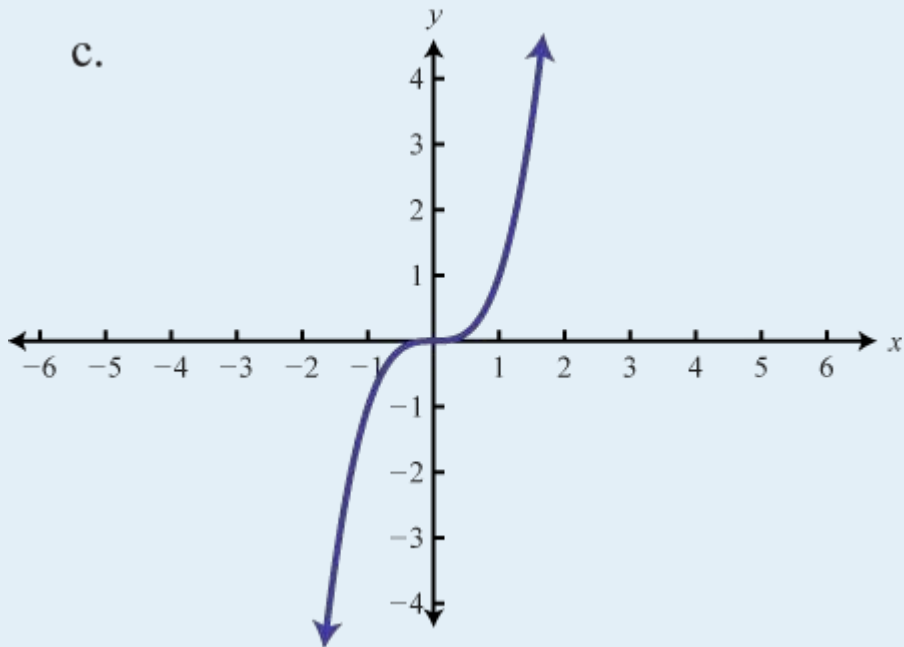
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b.



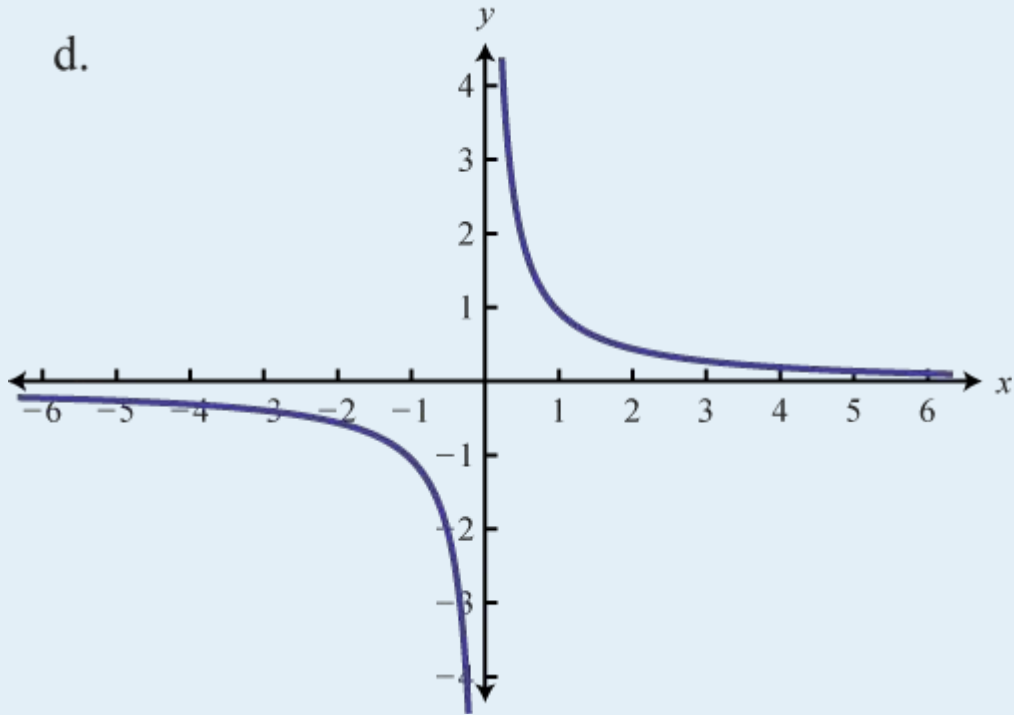
c.



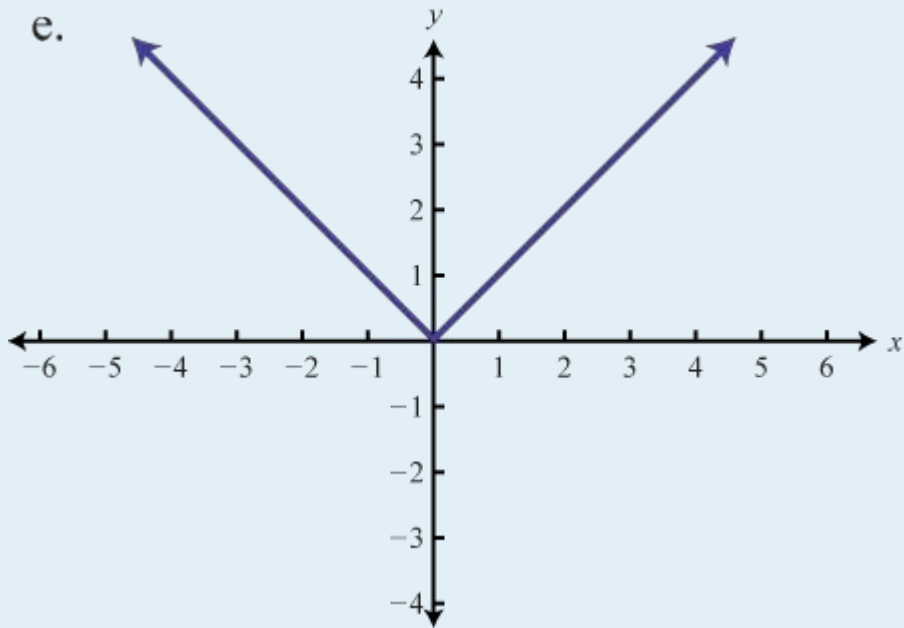
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d.



e.

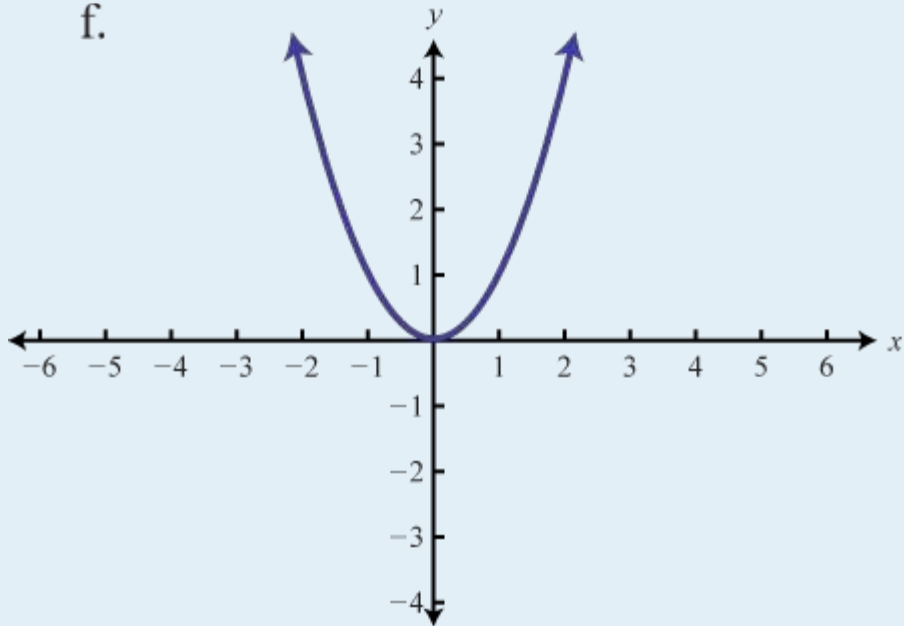


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f.



1.  $f(x) = x$

2.

3.

4.  $f(x) = |x|$

5.

6.

Evaluate.

1.  $f(x) = x$ ; find  $f(-10)$ ,  $f(0)$ , and  $f(a)$ .

2. ; find  $f(-10)$ ,  $f(0)$ , and  $f(a)$ .

3. ; find  $f(-10)$ ,  $f(0)$ , and  $f(a)$ .

4.  $f(x) = |x|$ ; find  $f(-10)$ ,  $f(0)$ , and  $f(a)$ .

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5. ; find  $f(25)$ ,  $f(0)$ , and  $f(a)$  where  $a \geq 0$ .
6. ; find  $f(-10)$ , , and  $f(a)$  where  $a \neq 0$ .
7.  $f(x) = 5$ ; find  $f(-10)$ ,  $f(0)$ , and  $f(a)$ .
8.  $f(x) = -12$ ; find  $f(-12)$ ,  $f(0)$ , and  $f(a)$ .
9. Graph  $f(x) = 5$  and state its domain and range.
10. Graph  $f(x) = -9$  and state its domain and range.

Cube root function.

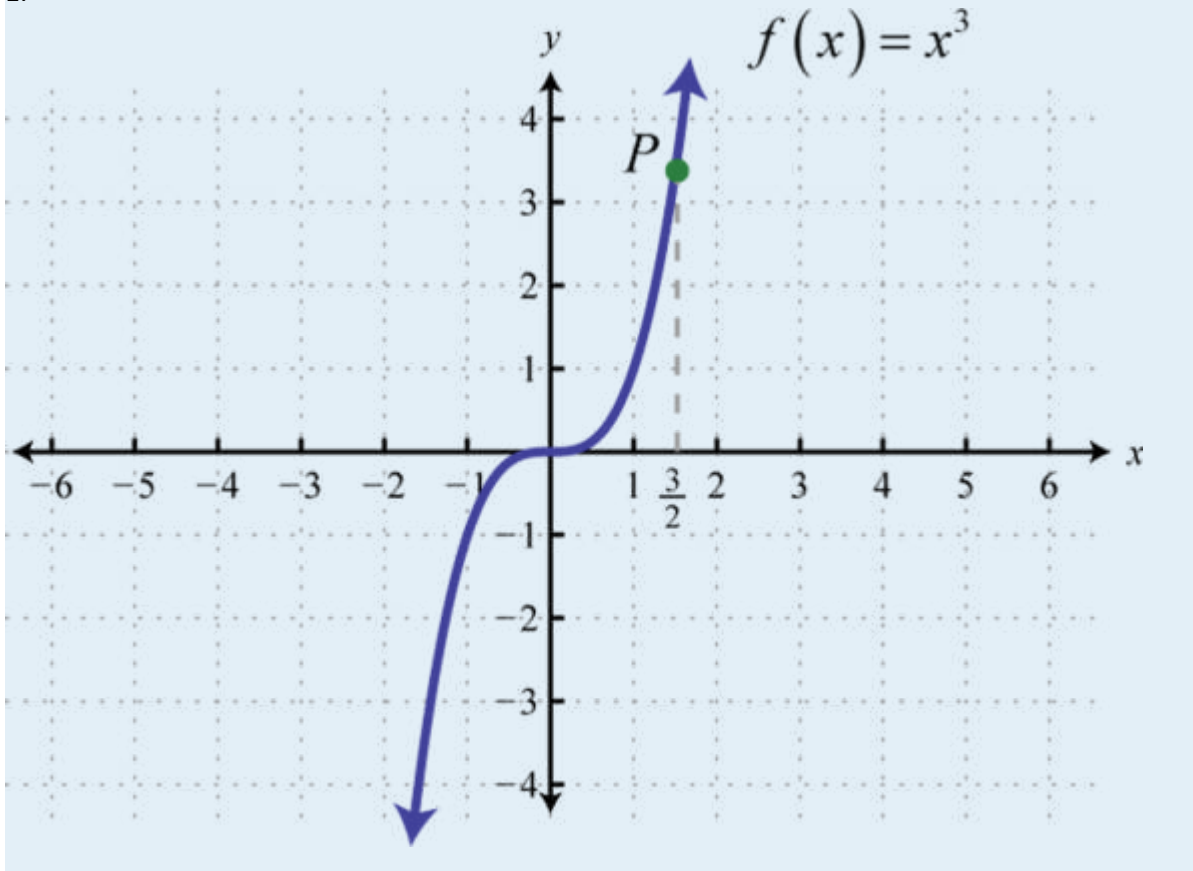
1. Find points on the graph of the function defined by with  $x$ -values in the set  $\{-8, -1, 0, 1, 8\}$ .
2. Find points on the graph of the function defined by with  $x$ -values in the set  $\{-3, -2, 1, 2, 3\}$ . Use a calculator and round off to the nearest tenth.
3. Graph the cube root function defined by by plotting the points found in the previous two exercises.
4. Determine the domain and range of the cube root function.

Find the ordered pair that specifies the point  $P$ .





1.

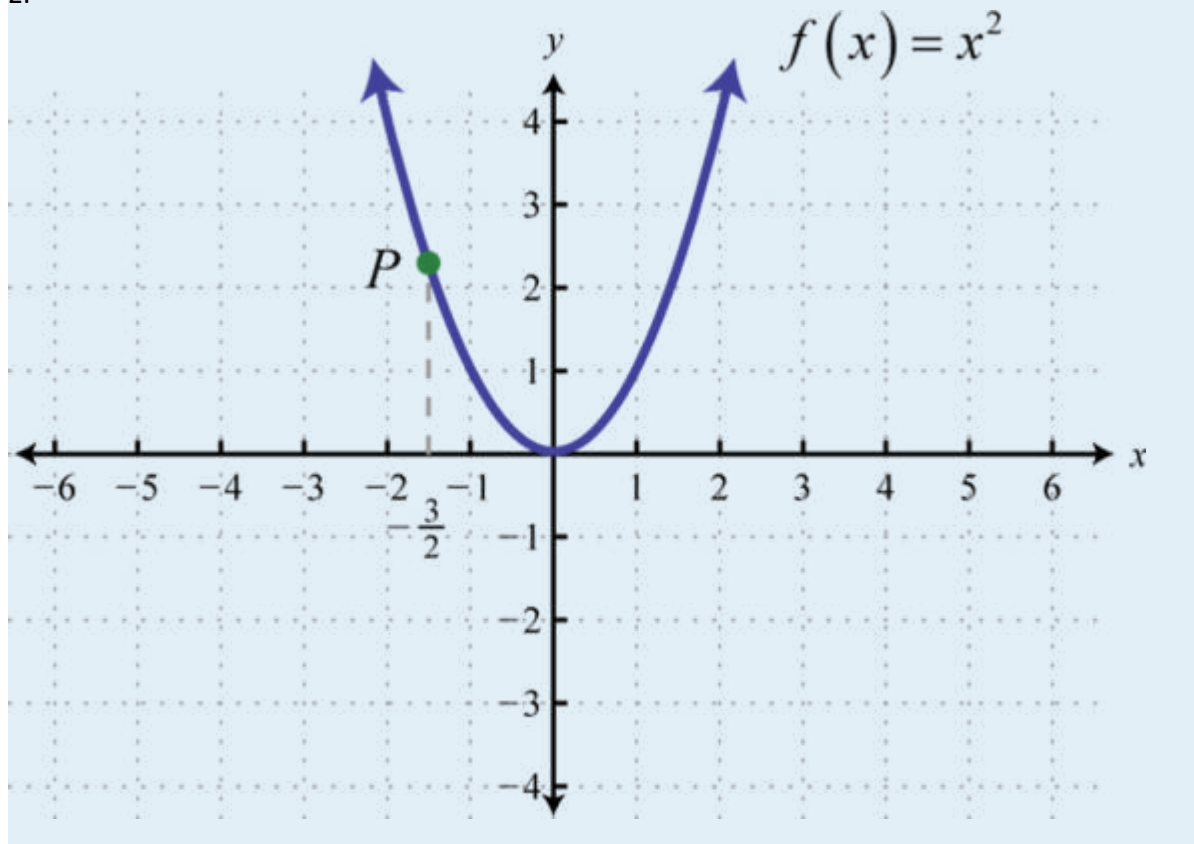


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2.

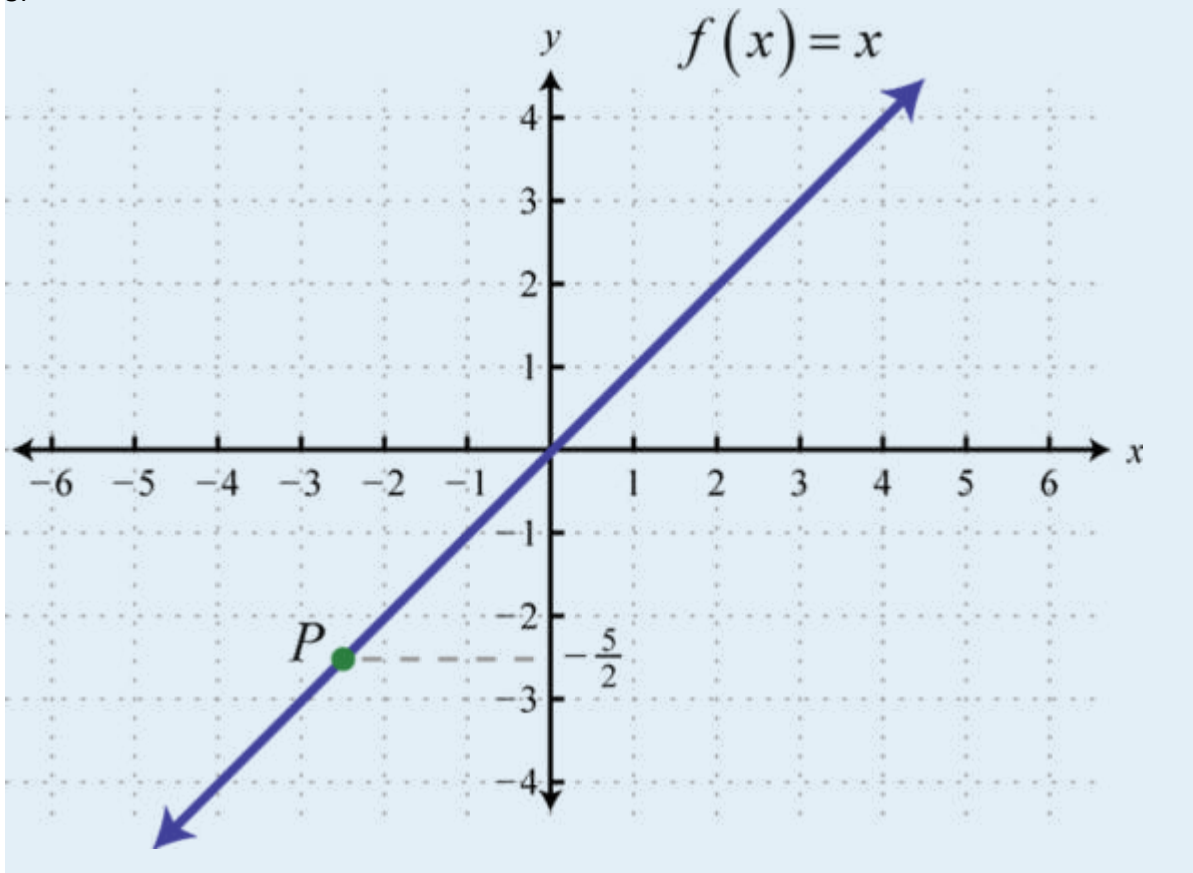


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3.

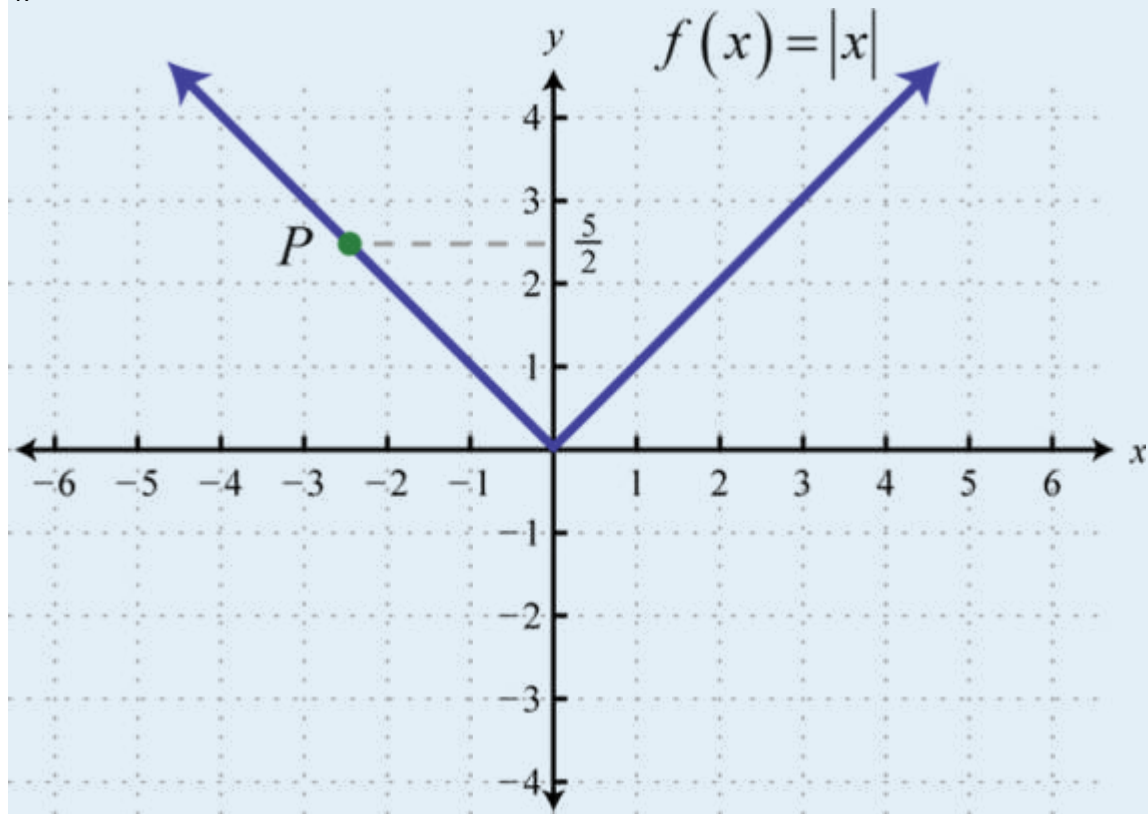


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4.



## PART B: PIECEWISE FUNCTIONS

Graph the piecewise functions.

- 25.
- 26.
- 27.
- 28.
- 29.
- 30.
- 31.
- 32.
- 33.

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34.

35.

36.

37.

38.

39.

40.

41.  $f(x) = \lceil x + 0.5 \rceil$

42.  $f(x) = \lceil x \rceil + 1$

43.  $f(x) = \lceil 0.5x \rceil$

44.  $f(x) = 2\lceil x \rceil$

Evaluate.

25.

Find  $f(-5)$ ,  $f(0)$ , and  $f(3)$ .

26. .

Find  $f(-3)$ ,  $f(0)$ , and  $f(2)$ .

27.

Find  $g(-1)$ ,  $g(1)$ , and  $g(4)$ .

28.

Find  $g(-3)$ ,  $g(-2)$ , and  $g(-1)$ .

29.

Find  $h(-2)$ ,  $h(0)$ , and  $h(4)$ .

30.

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Find  $h(-5)$ ,  $h(4)$ , and  $h(25)$ .

31.

$$f(x) = \lceil \lceil x - 0.5 \rceil \rceil$$

Find  $f(-2)$ ,  $f(0)$ , and  $f(3)$ .

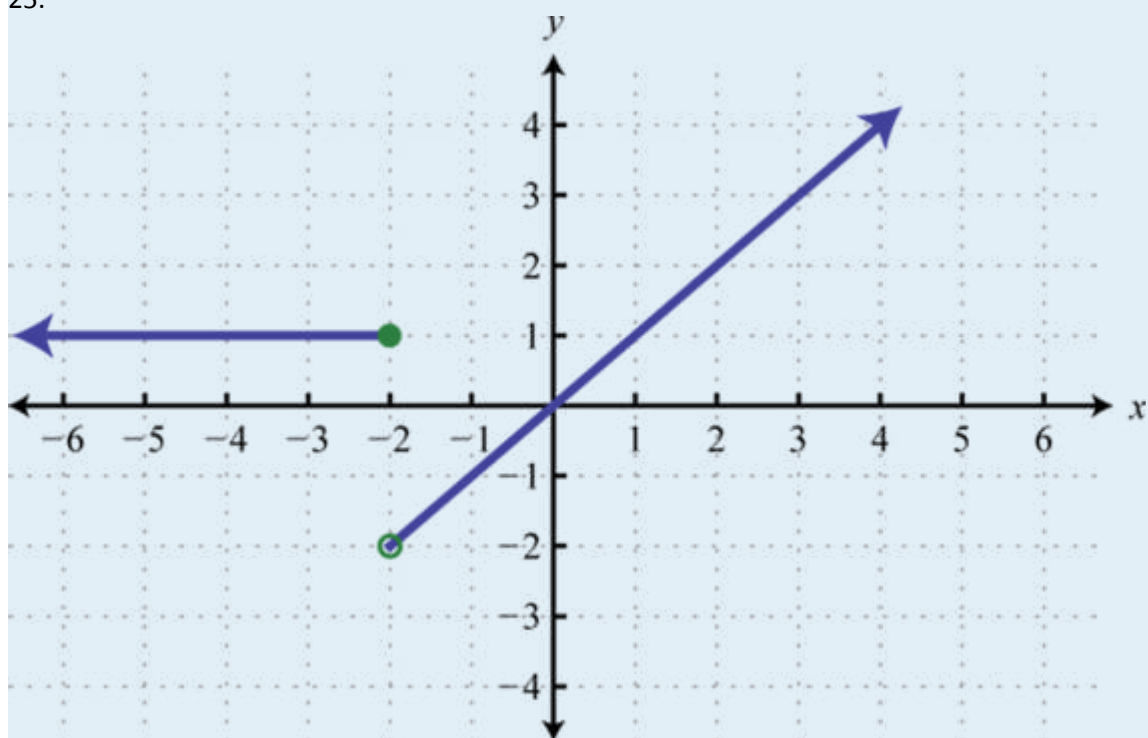
32.

$$f(x) = \lceil \lceil 2x \rceil \rceil + 1$$

Find  $f(-1.2)$ ,  $f(0.4)$ , and  $f(2.6)$ .

Evaluate given the graph of  $f$ .

25.



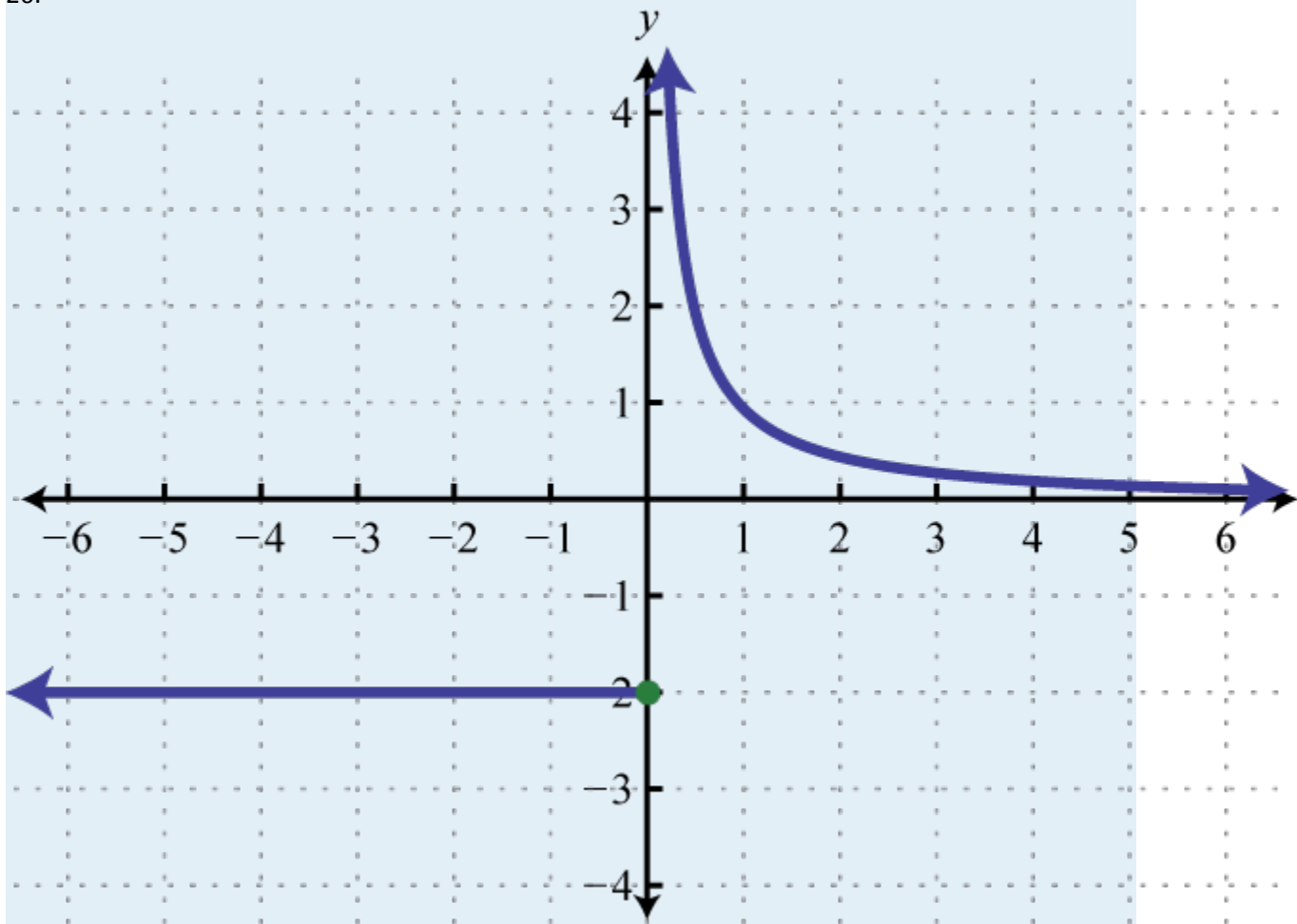
Find  $f(-4)$ ,  $f(-2)$ , and  $f(0)$ .

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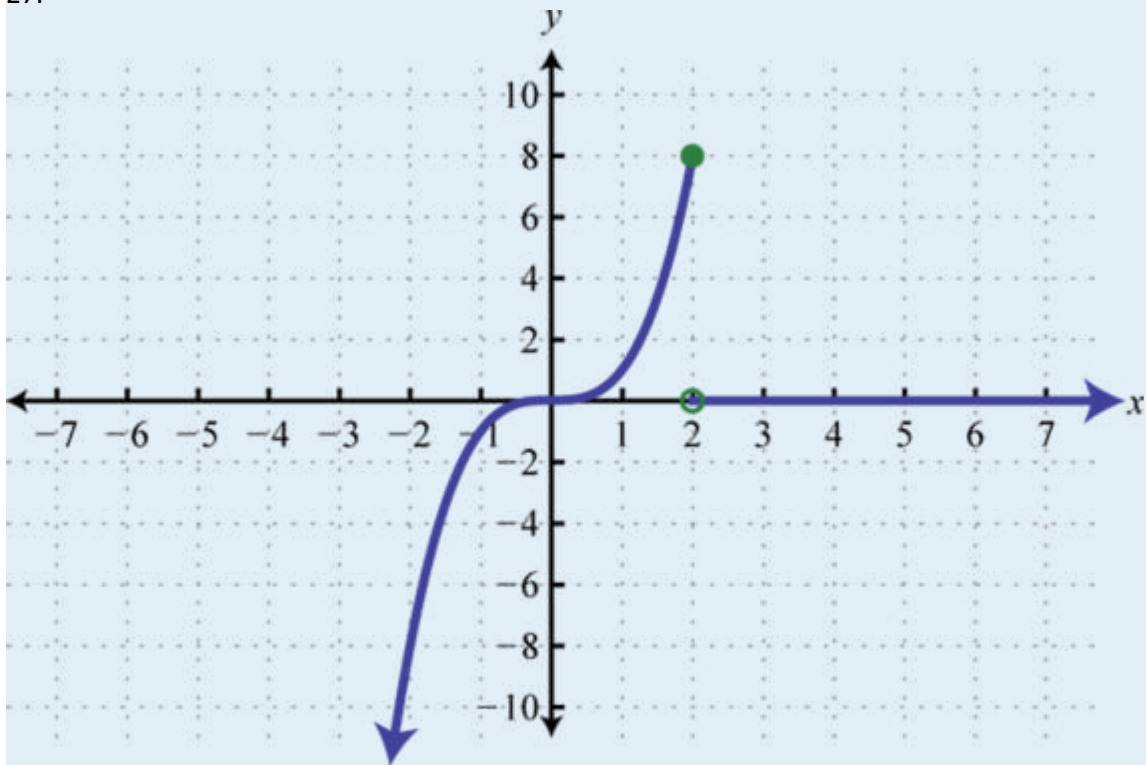
26.



Find  $f(-3)$ ,  $f(0)$ , and  $f(1)$ .



27.



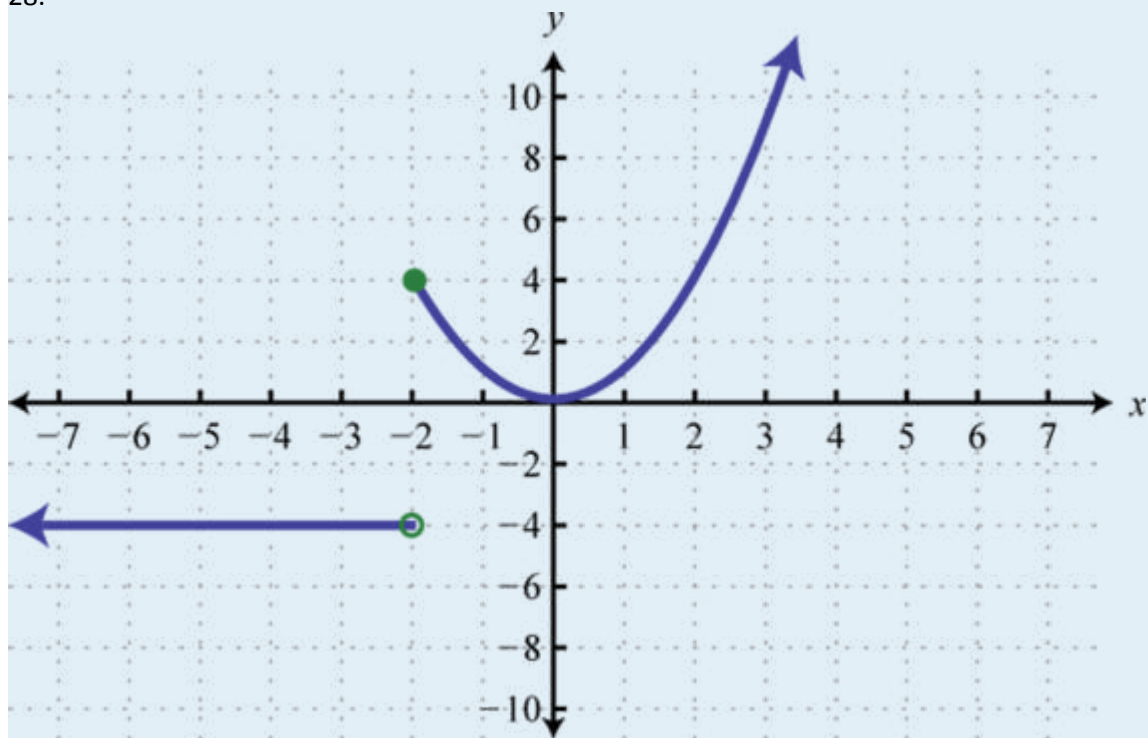
Find  $f(0)$ ,  $f(2)$ , and  $f(4)$ .

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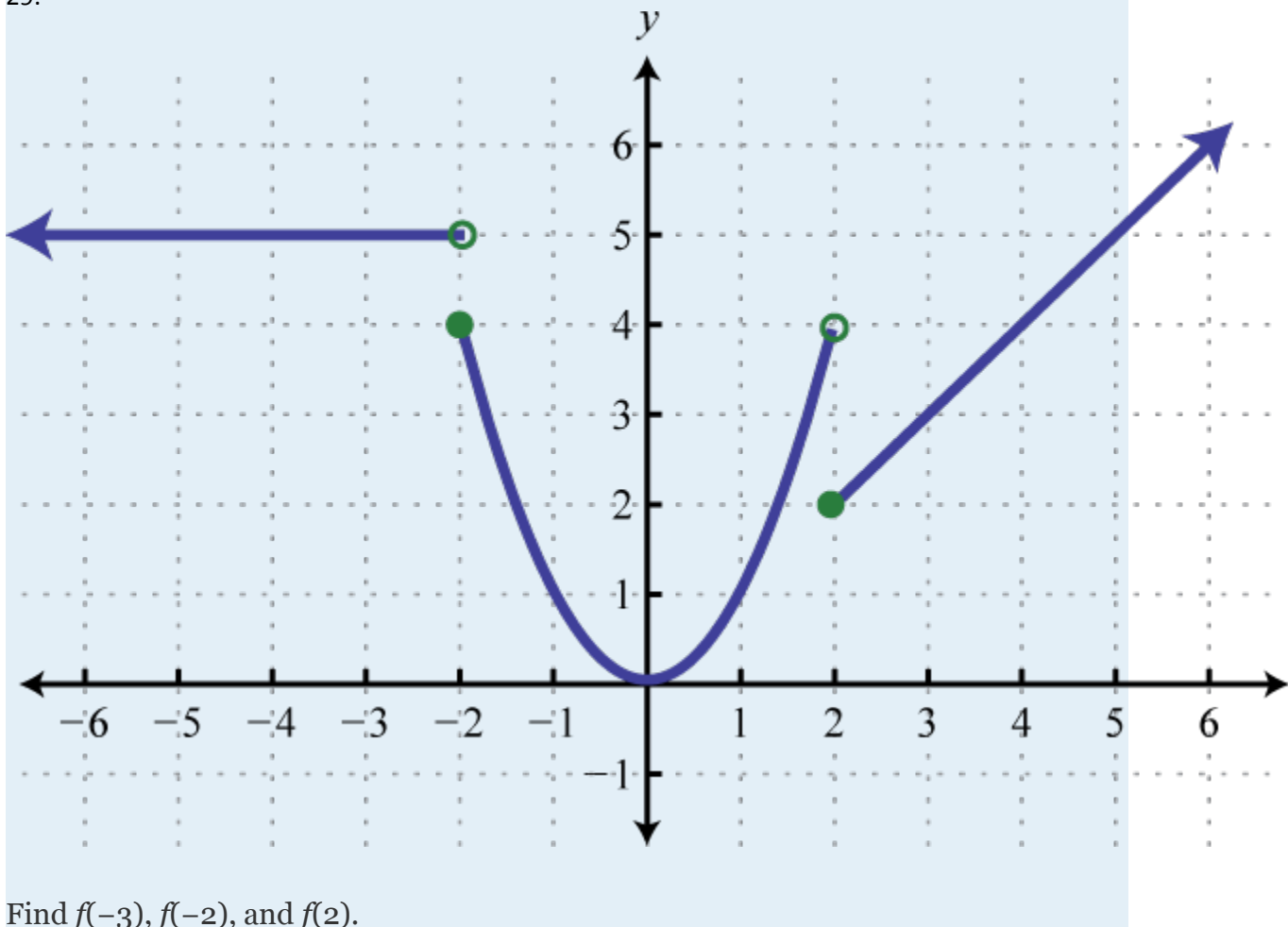
28.



Find  $f(-5)$ ,  $f(-2)$ , and  $f(2)$ .



29.

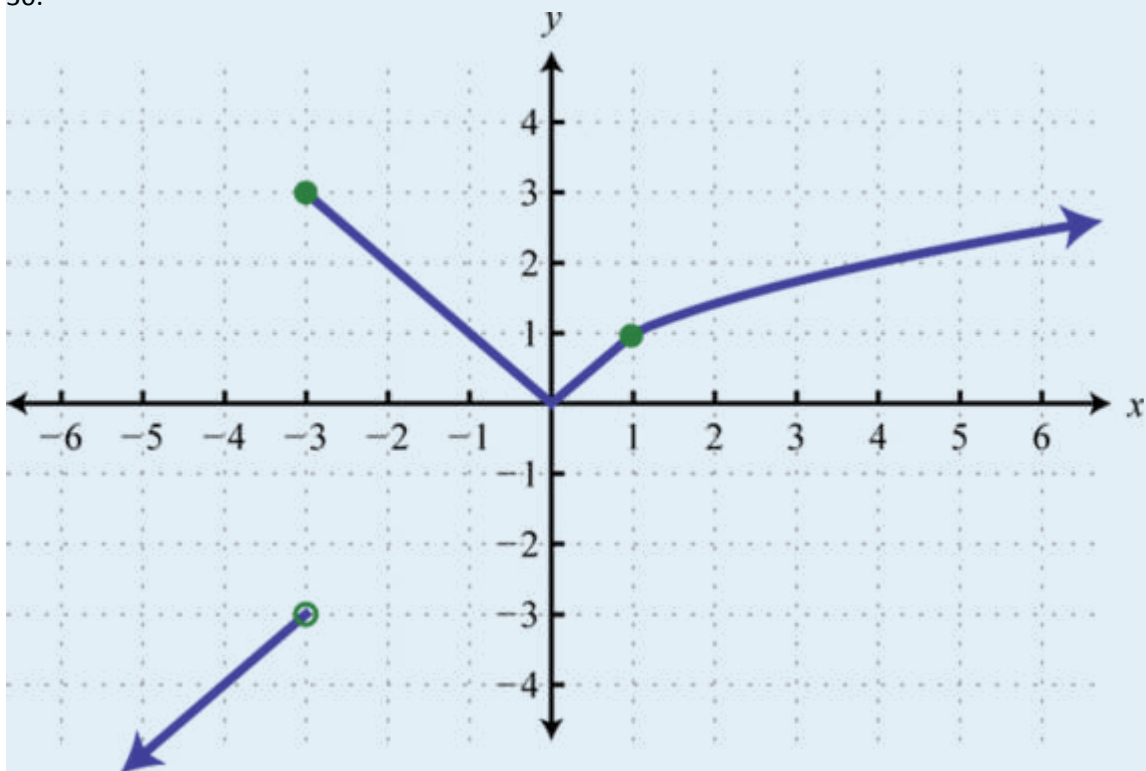


Find  $f(-3)$ ,  $f(-2)$ , and  $f(2)$ .

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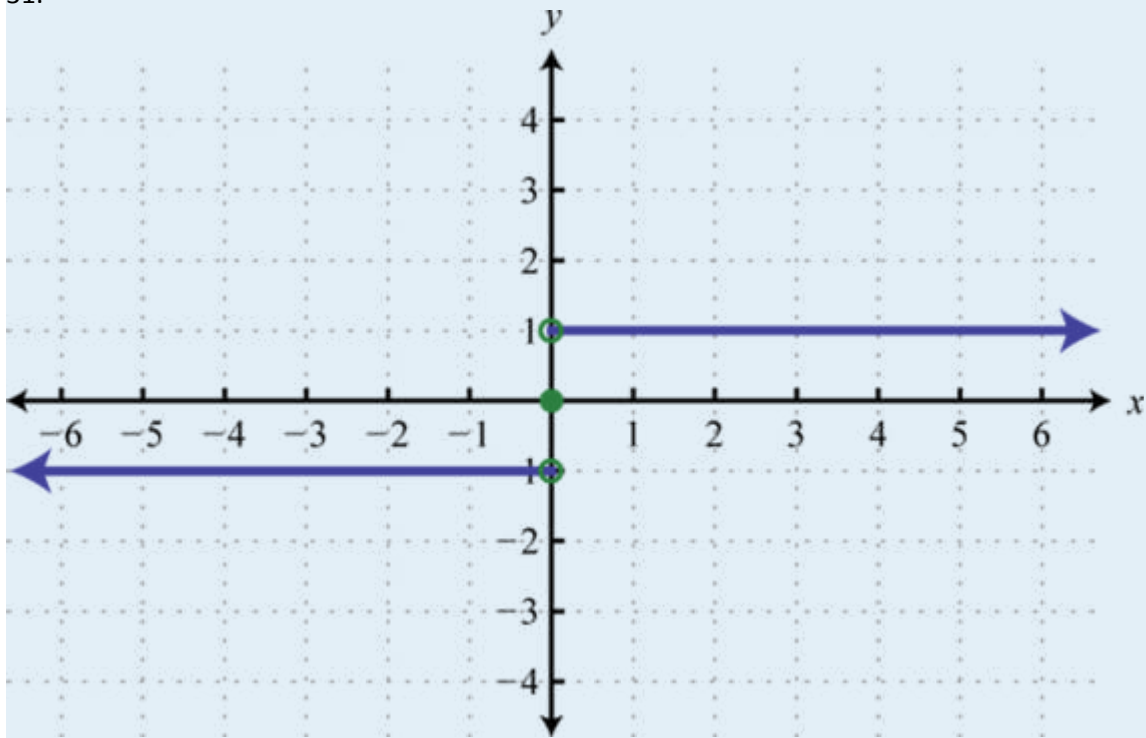
30.



Find  $f(-3)$ ,  $f(0)$ , and  $f(4)$ .



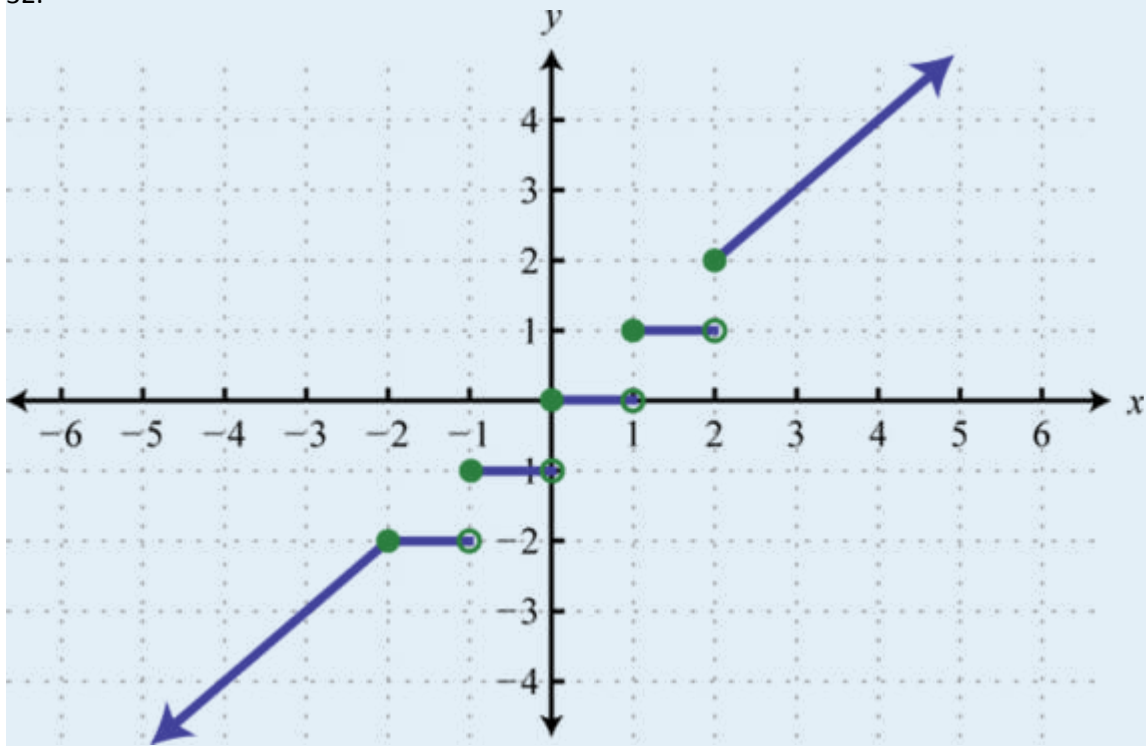
31.



Find  $f(-2)$ ,  $f(0)$ , and  $f(2)$ .



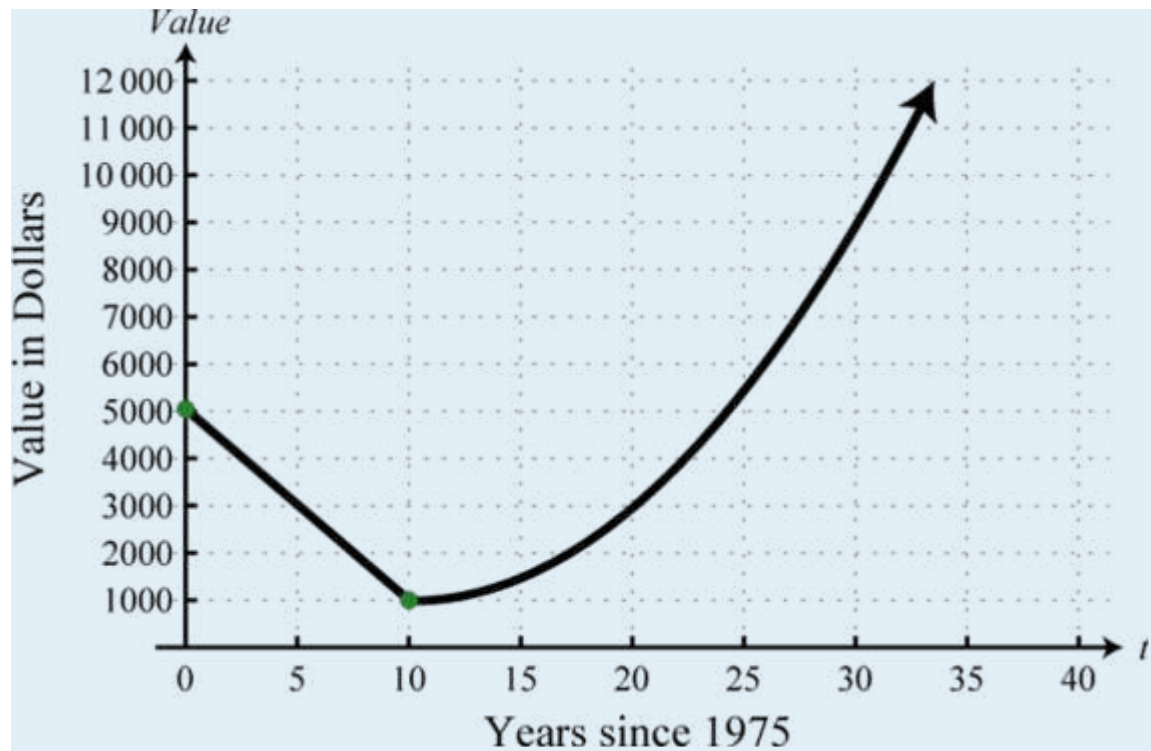
32.



Find  $f(-3)$ ,  $f(1)$ , and  $f(2)$ .

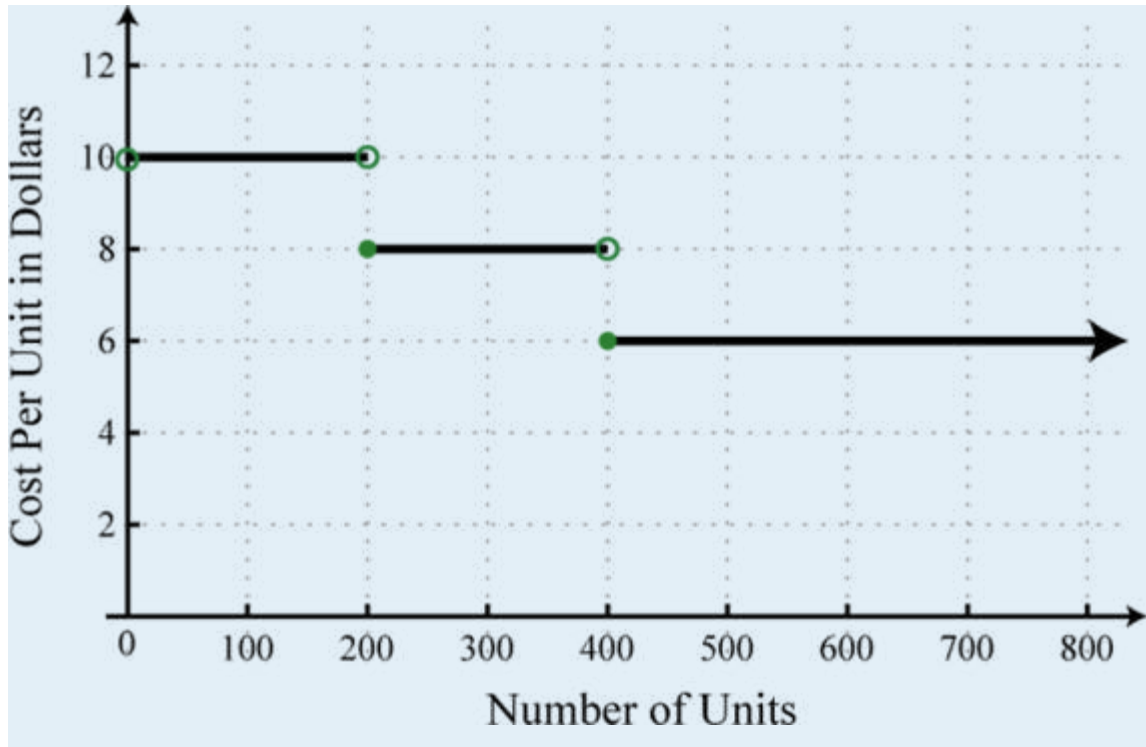
33. The value of an automobile in dollars is given in terms of the number of years since it was purchased new in 1975:





- 33.a. Determine the value of the automobile in the year 1980.
- 33.b. In what year is the automobile valued at \$9,000?
34. The cost per unit in dollars of custom lamps depends on the number of units produced according to the following graph:





34.a. What is the cost per unit if 250 custom lamps are produced?

34.b. What level of production minimizes the cost per unit?

63. An automobile salesperson earns a commission based on total sales each month according to the function:

25.a) If the salesperson's total sales for the month are \$35,500, what is her commission according to the function?

25.b) To reach the next level in the commission structure, how much more in sales will she need?

64. A rental boat costs \$32 for one hour, and each additional hour or partial hour costs \$8. Graph the cost of the rental boat and determine the cost to rent the boat for 412 hours.

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## PART C: DISCUSSION BOARD

65. Explain to a beginning algebra student what an asymptote is.
66. Research and discuss the difference between the floor and ceiling functions. What applications can you find that use these functions?

## ANSWERS

1. b
- 2.
3. c
- 4.
5. a
- 6.
7.  $f(-10) = -10, f(0) = 0, f(a) = a$
- 8.
9.  $f(-10) = -1,000, f(0) = 0,$
- 10.
11.  $f(25) = 5, f(0) = 0,$
- 12.
13.  $f(-10) = 5, f(0) = 5, f(a) = 5$
- 14.

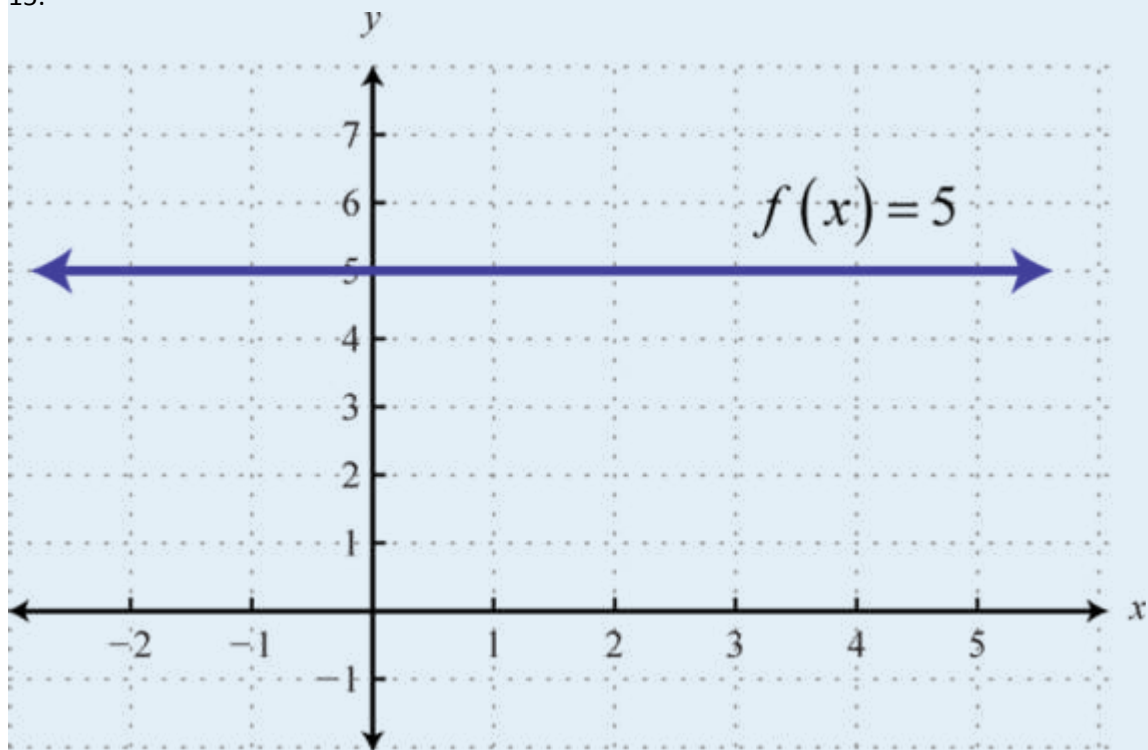
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15.



Domain: ; range: {5}

16.

17.  $\{(-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2)\}$

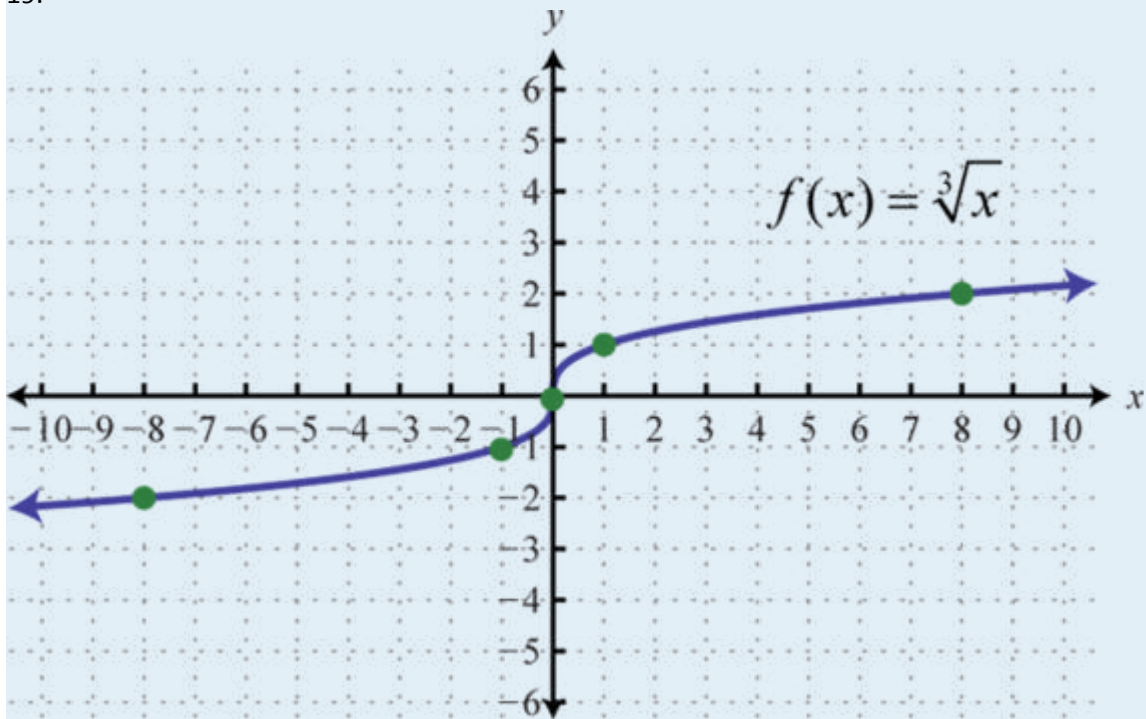
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19.



20.

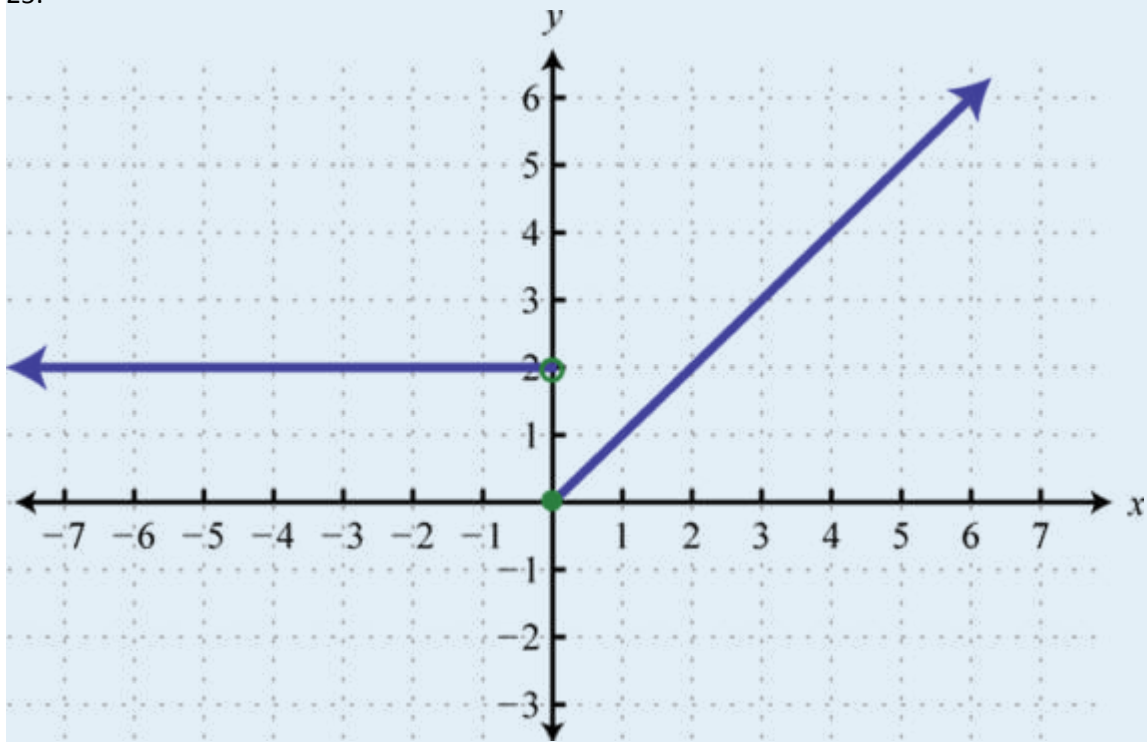
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25.

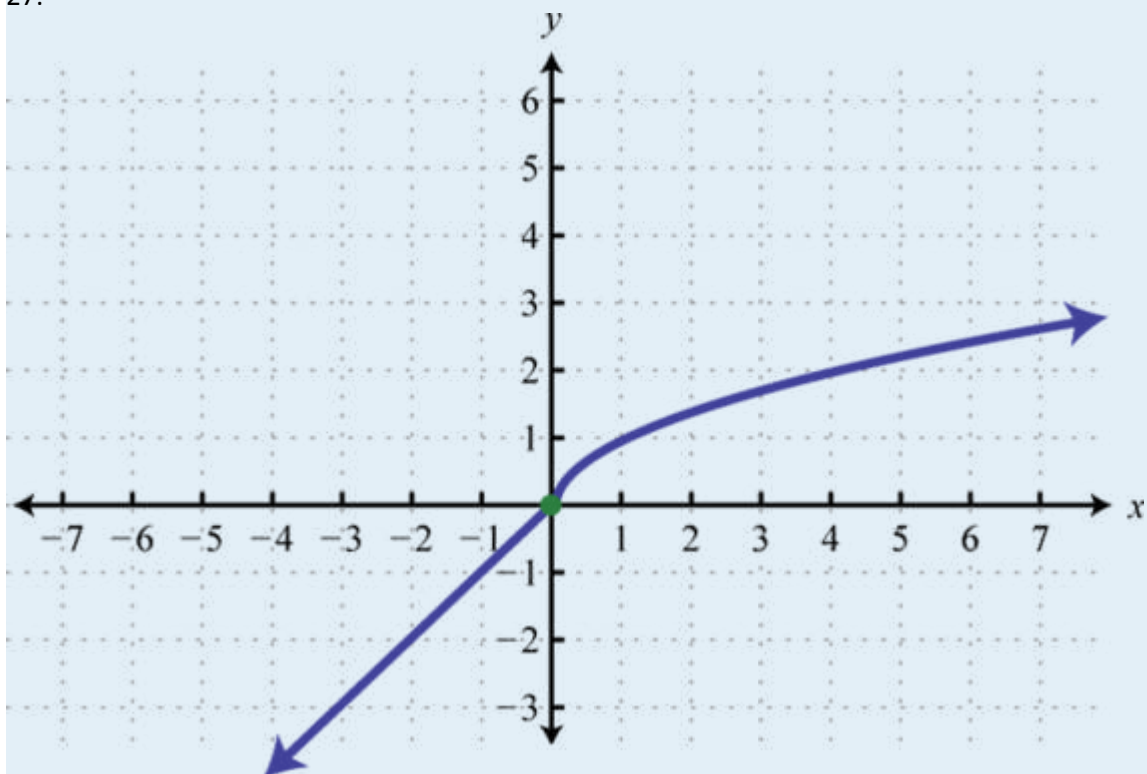


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26.

27.



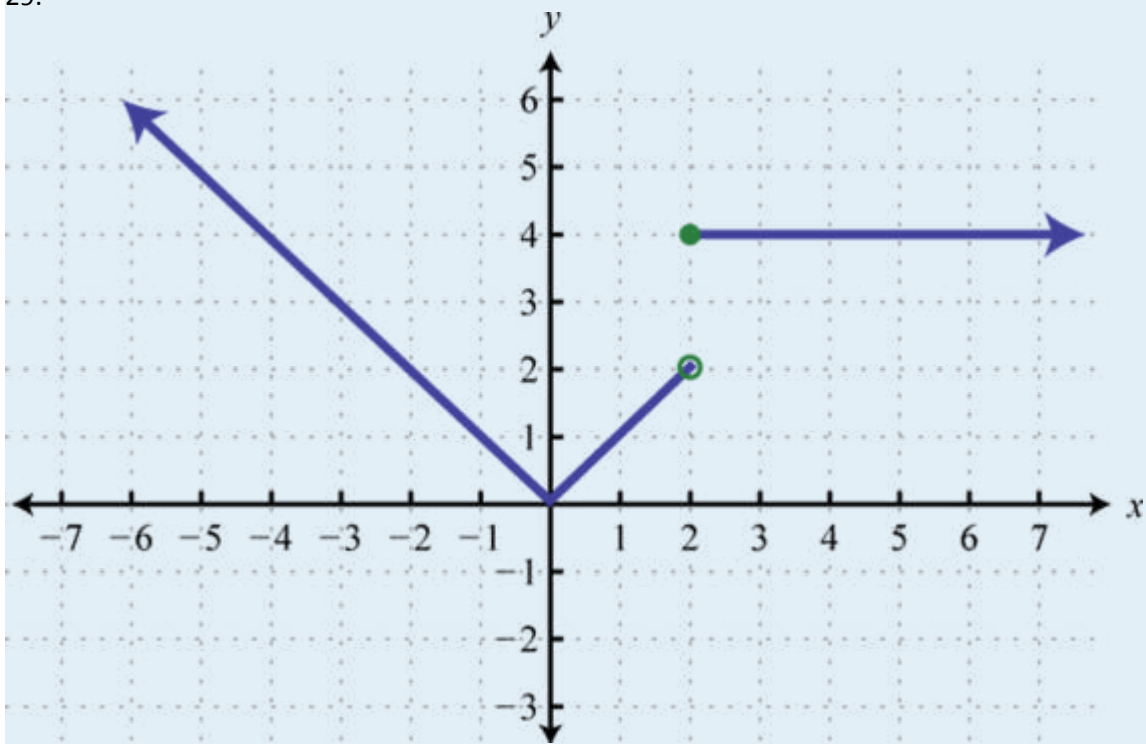
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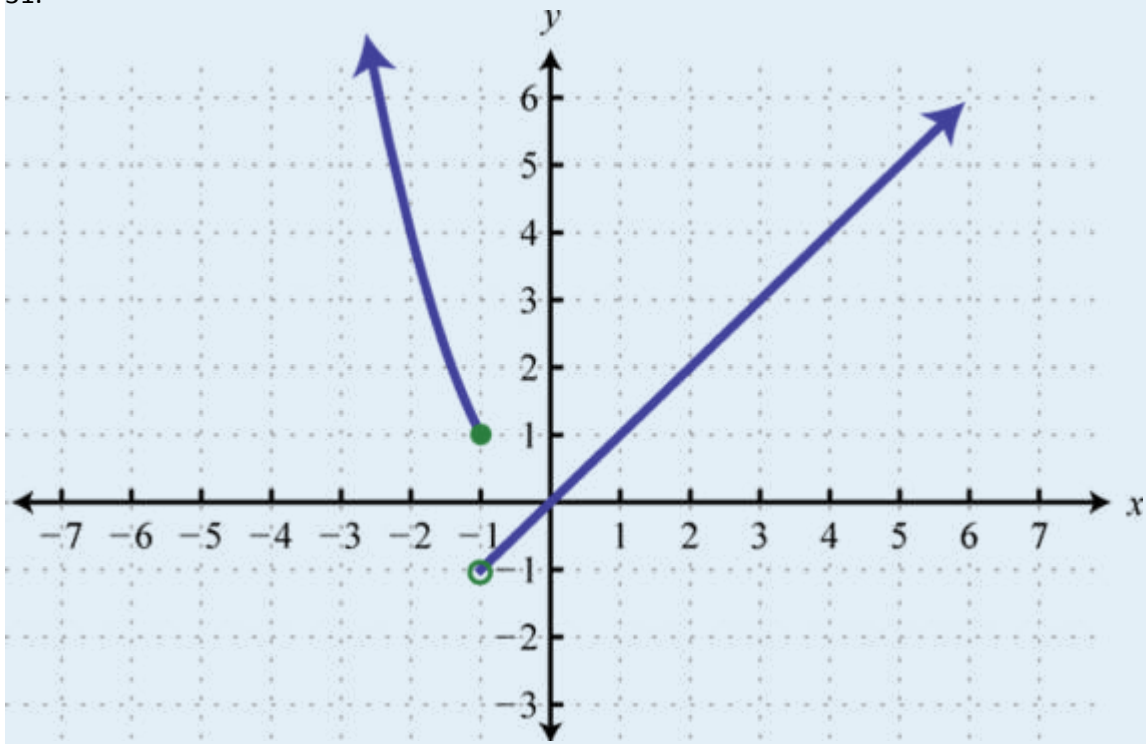


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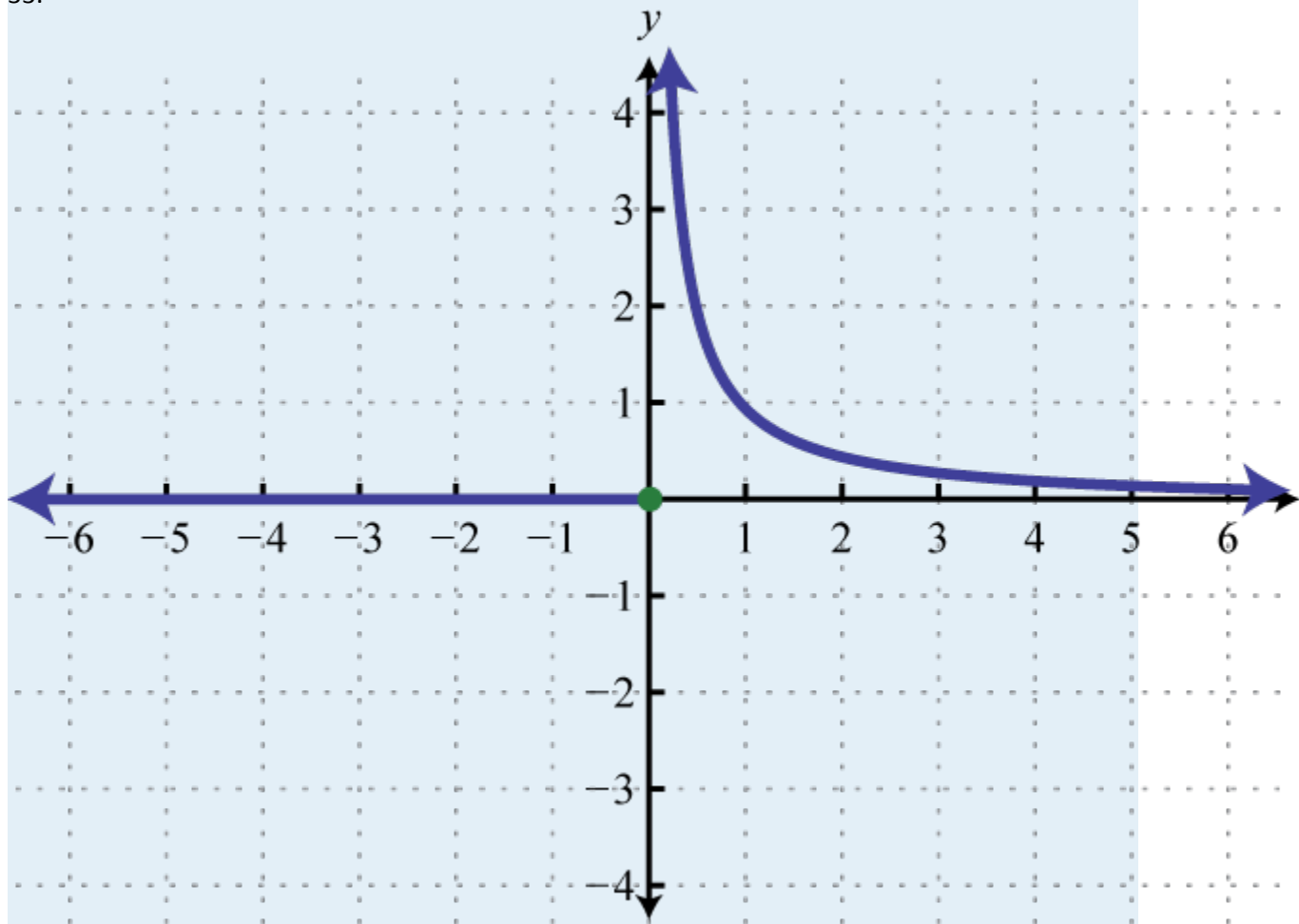


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33.

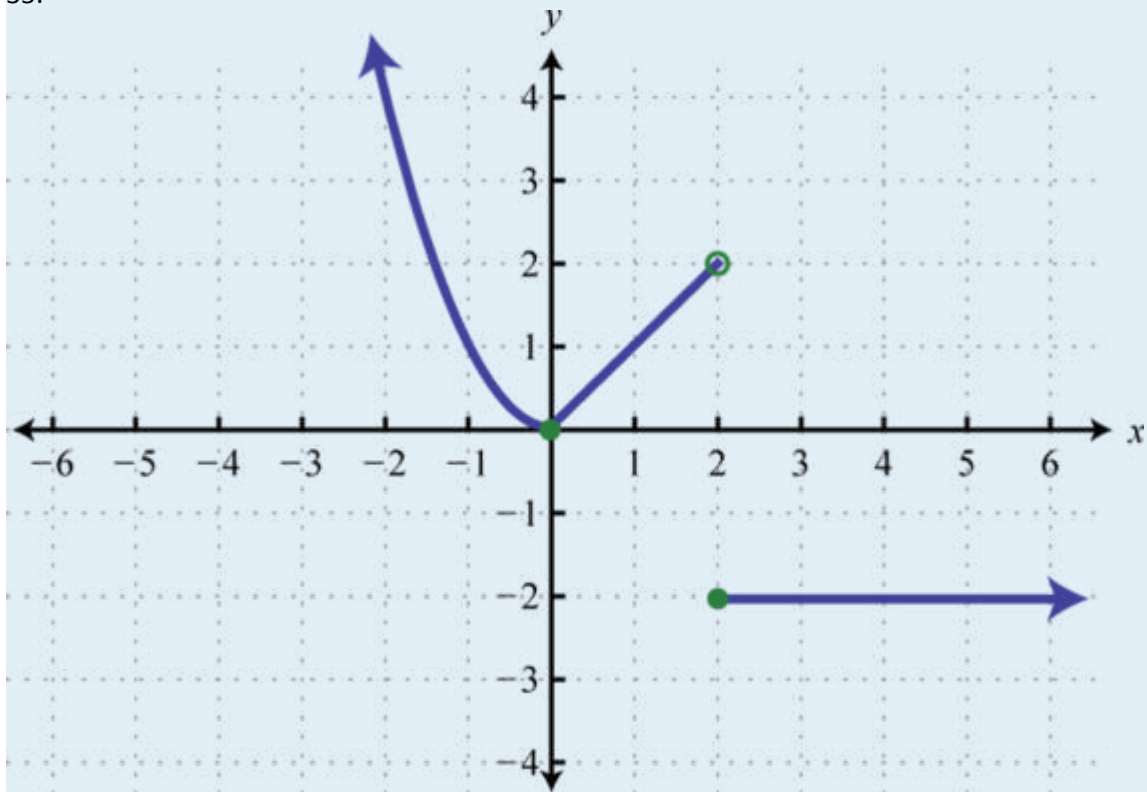


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35.



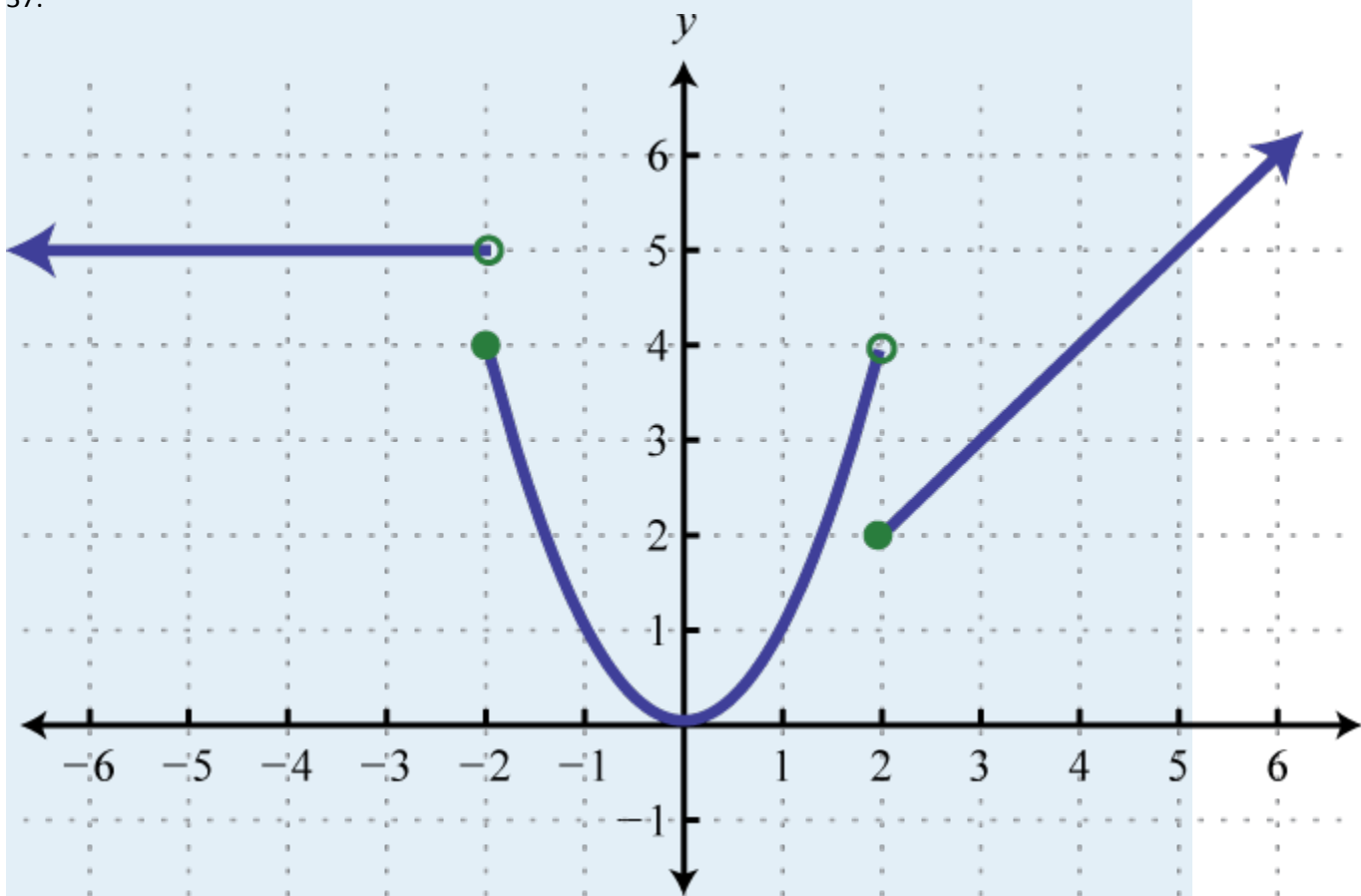
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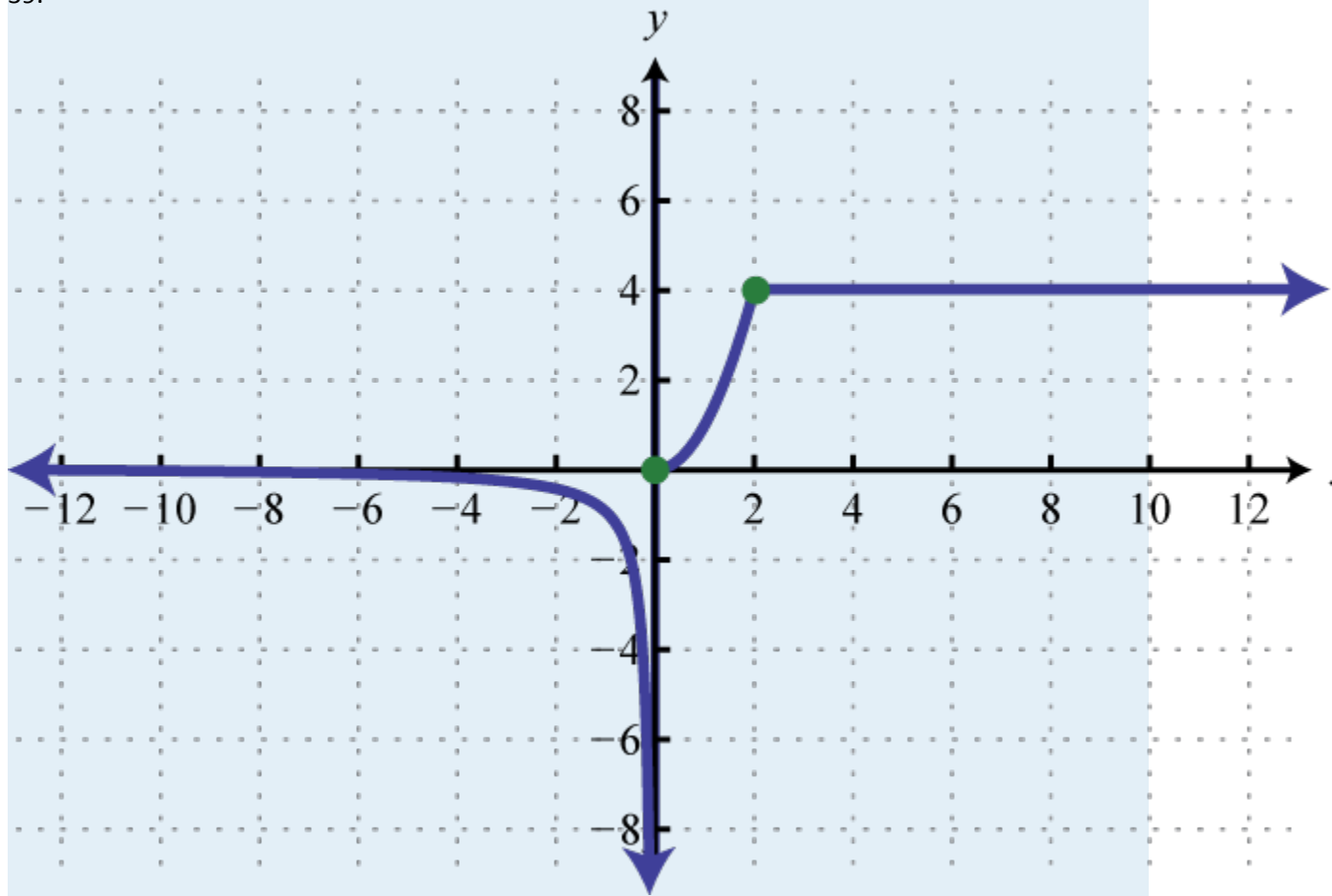


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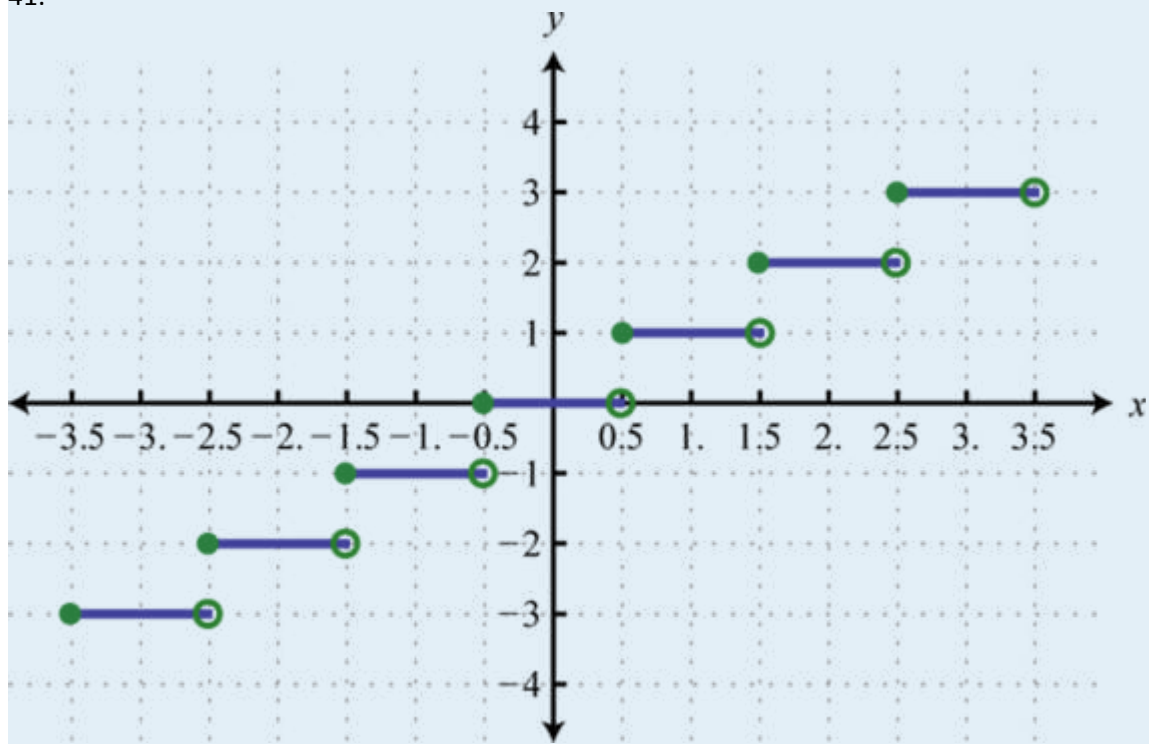


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41.

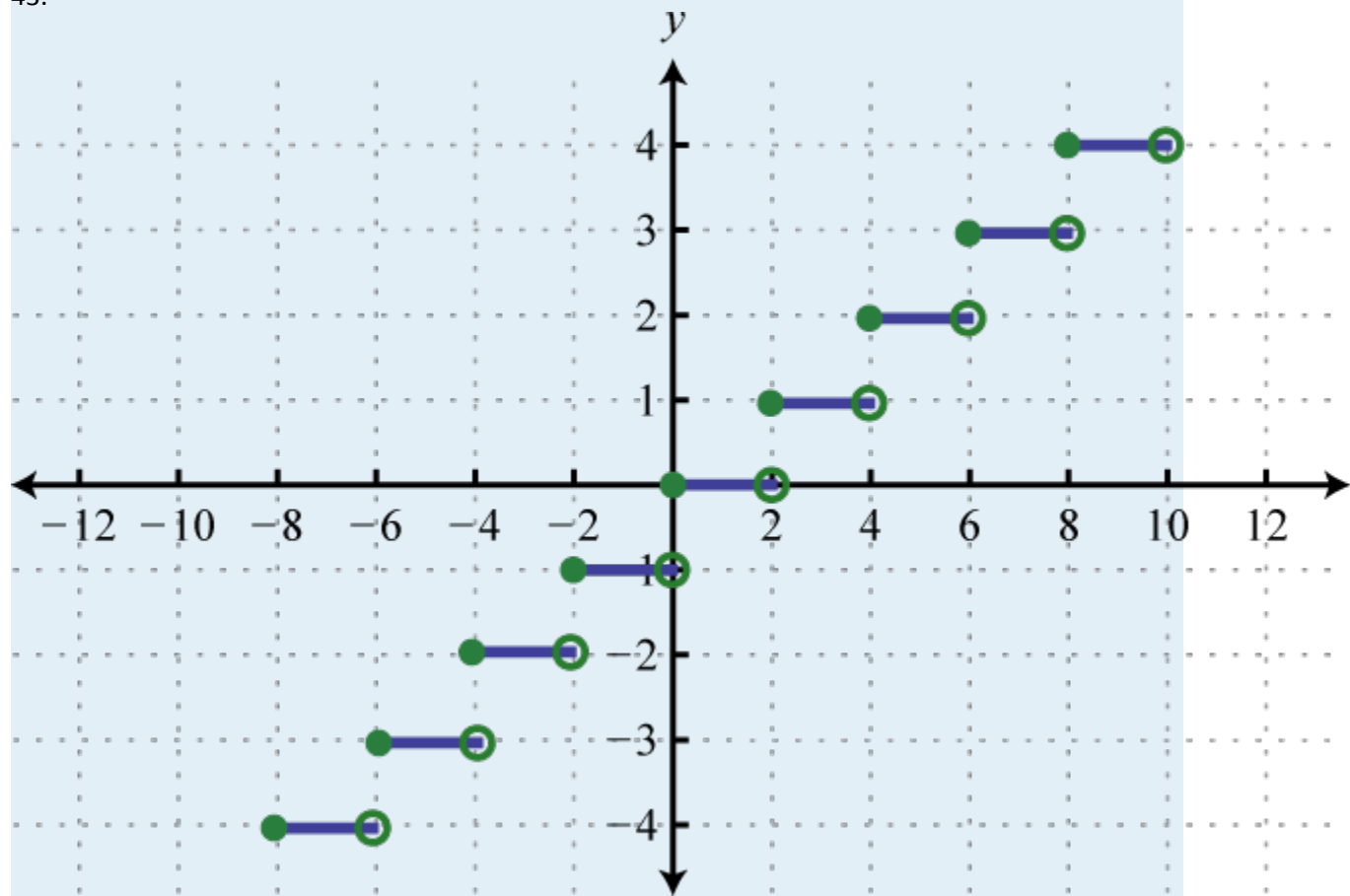


42.

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43.



44.

45.  $f(-5) = 25, f(0) = 0, \text{ and } f(3) = 5$

46.

47.  $g(-1) = -7, g(1) = 1, \text{ and } g(4) = 2$

48.

49.  $h(-2) = -5, h(0) = -3, \text{ and } h(4) = 16$

50.

51.  $f(-2) = -3, f(0) = -1, \text{ and } f(3) = 2$

52.

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53.  $f(-4) = 1, f(-2) = 1,$  and  $f(0) = 0$

54.

55.  $f(0) = 0, f(2) = 8,$  and  $f(4) = 0$

56.

57.  $f(-3) = 5, f(-2) = 4,$  and  $f(2) = 2$

58.

59.  $f(-2) = -1, f(0) = 0,$  and  $f(2) = 1$

60.

61.

a. \$3,000;

b. 2005

62.

63.

a. \$1, 775;

b. \$14, 500

64.

65. Answer may vary

## 2.5 Using Transformations to Graph Functions

### LEARNING OBJECTIVES

1. Define the rigid transformations and use them to sketch graphs.

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2. Define the non-rigid transformations and use them to sketch graphs.

## Vertical and Horizontal Translations

When the graph of a function is changed in appearance and/or location we call it a transformation. There are two types of transformations. A

**rigid transformation** changes the location of the function in a coordinate plane, but leaves the size and shape of the graph unchanged. A **non-rigid transformation** changes the size and/or shape of the graph.

A **vertical translation** is a rigid transformation that shifts a graph up or down relative to the original graph. This occurs when a constant is added to any function. If we add a positive constant to each  $y$ -coordinate, the graph will shift up. If we add a negative constant, the graph will shift down. For example, consider the functions  $g(x) = x^2 - 3$  and  $h(x) = x^2 + 3$ . Begin by evaluating for some values of the independent variable  $x$ .

$$g(x) = x^2 - 3$$

$x$	$g(x)$
-2	1
-1	-2
0	-3
1	-2
2	1

$$h(x) = x^2 + 3$$

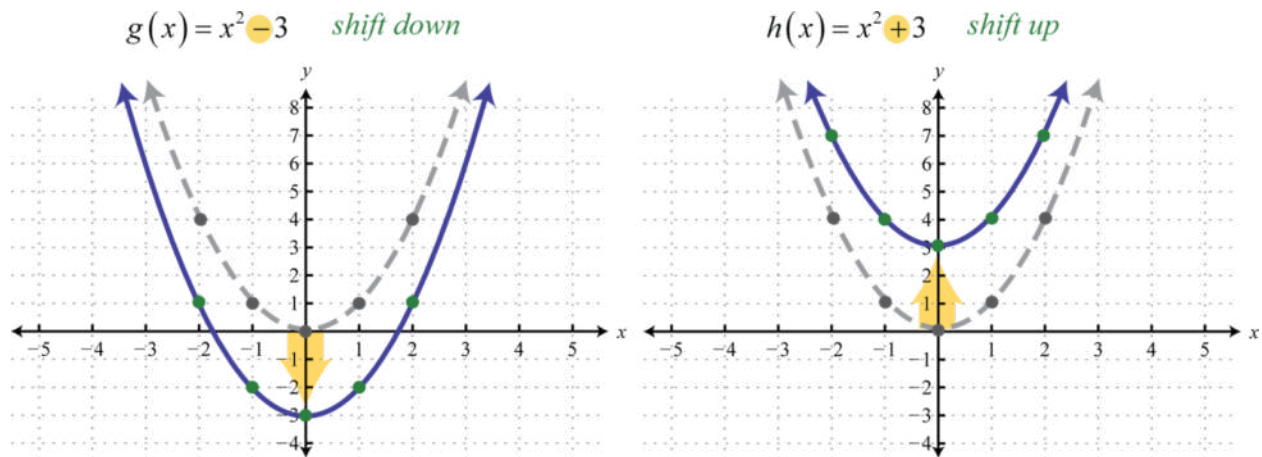
$x$	$h(x)$
-2	7
-1	4
0	3
1	4
2	7

Now plot the points and compare the graphs of the functions  $g$  and  $h$  to the basic graph of  $y = x^2$ , which is shown using a dashed grey curve below.

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The function  $g$  shifts the basic graph down 3 units and the function  $h$  shifts the basic graph up 3 units. In general, this describes the vertical translations; if  $k$  is any positive real number:

Vertical shift up $k$ units:	$F(x) = f(x) + k$

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Vertical shift down $k$ units: $F(x) = f(x) - k$	

## Example 1

Sketch the graph of

Solution:

Begin with the basic function defined by and shift the graph up 4 units.

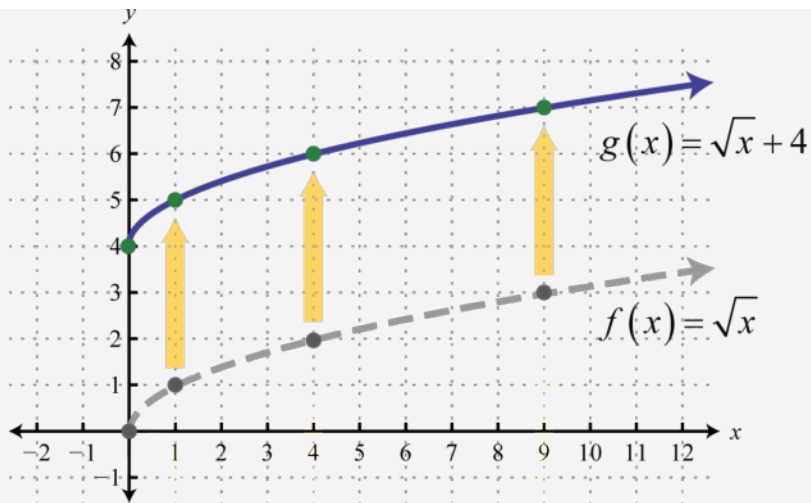
Answer:

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A **horizontal translation** is a rigid transformation that shifts a graph left or right relative to the original graph. This occurs when we add or subtract constants from the  $x$ -coordinate before the function is applied. For example, consider the functions defined by  $g(x) = (x+3)^2$  and  $h(x) = (x-3)^2$  and create the following tables:

$$g(x) = (x+3)^2$$

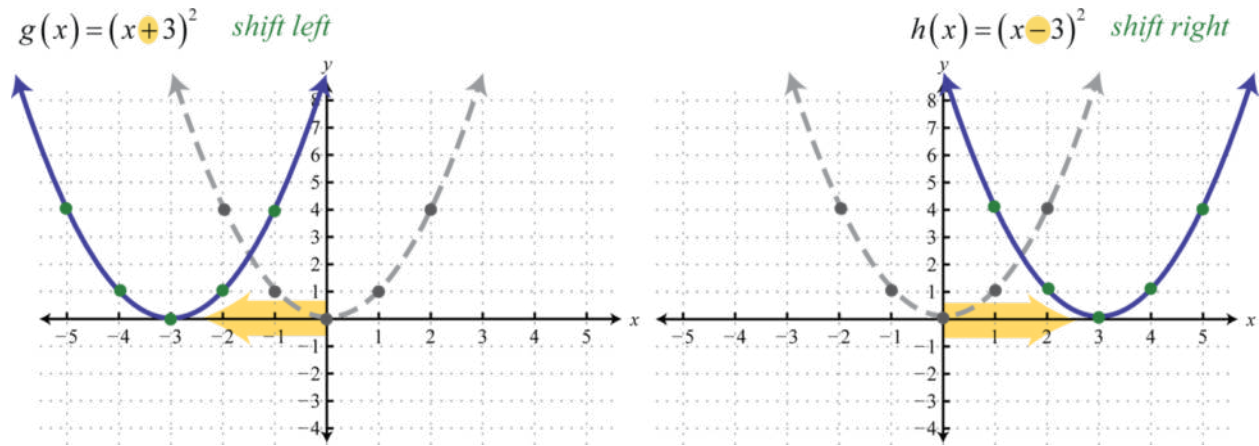
$$h(x) = (x-3)^2$$

$x$	$g(x)$
-5	4
-4	1
-3	0
-2	1
-1	4

$x$	$h(x)$
1	4
2	1
3	0
4	1
5	4



Here we add and subtract from the  $x$ -coordinates and then square the result. This produces a horizontal translation.



Note that this is the opposite of what you might expect. In general, this describes the horizontal translations; if  $h$  is any positive real number:

Horizontal shift left $h$ units: $F(x) = f(x + h)$	

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Horizontal shift right $h$ units:	$F(x) = f(x-h)$

## Example 2

Sketch the graph of

Solution:

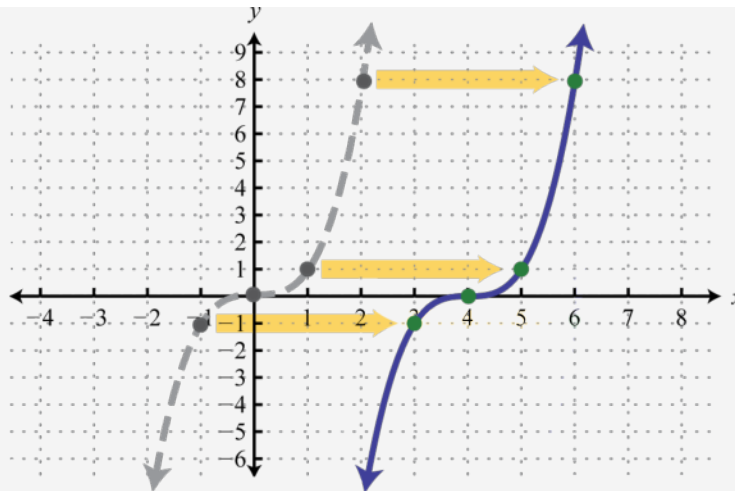
Begin with a basic cubing function defined by  $f(x) = x^3$  and shift the graph 4 units to the right.

Answer:

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It is often the case that combinations of translations occur.

### Example 3

Sketch the graph of  $g(x) = |x + 3| - 5$ .

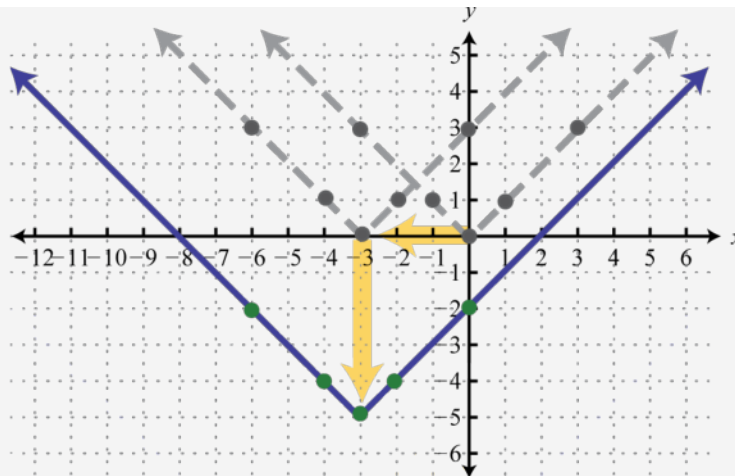
Solution:

Start with the absolute value function and apply the following transformations.

Answer:

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The order in which we apply horizontal and vertical translations does not affect the final graph.

## Example 4

Sketch the graph of

Solution:

Begin with the reciprocal function and identify the translations.

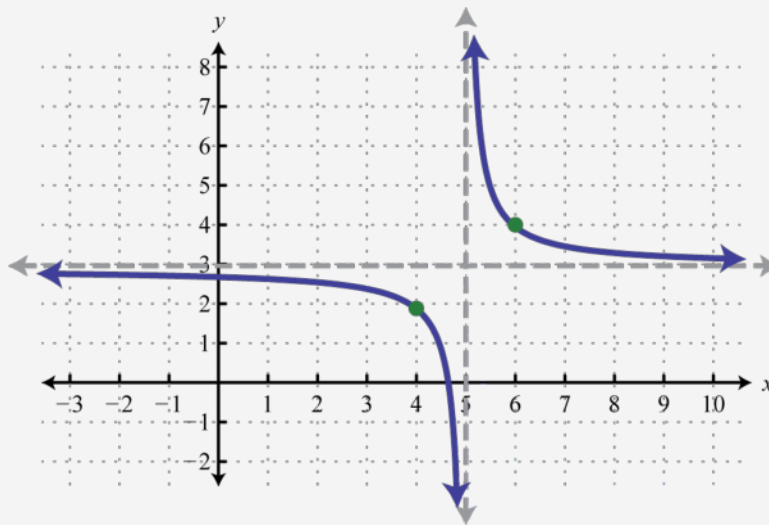
Take care to shift the vertical asymptote from the  $y$ -axis 5 units to the right and shift the horizontal asymptote from the  $x$ -axis up 3 units.

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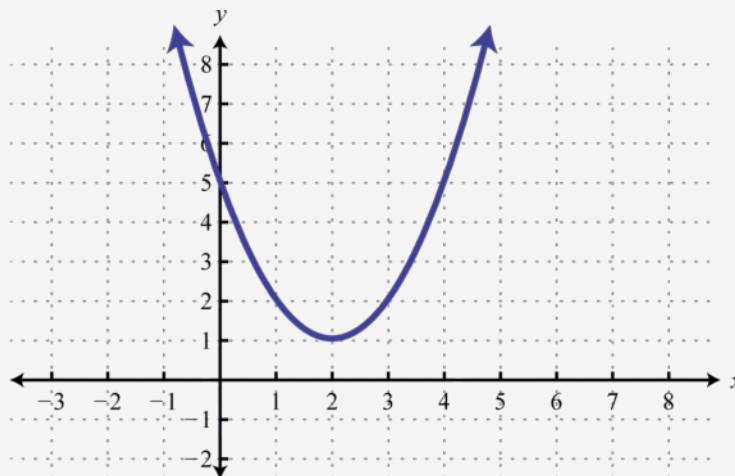


Answer:



Try this! Sketch the graph of

Answer:



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# Reflections

A **reflection** is a transformation in which a mirror image of the graph is produced about an axis. In this section, we will consider reflections about the  $x$ - and  $y$ -axis. The graph of a function is reflected about the  $x$ -axis if each  $y$ -coordinate is multiplied by  $-1$ . The graph of a function is reflected about the  $y$ -axis if each  $x$ -coordinate is multiplied by  $-1$  before the function is applied. For example, consider and

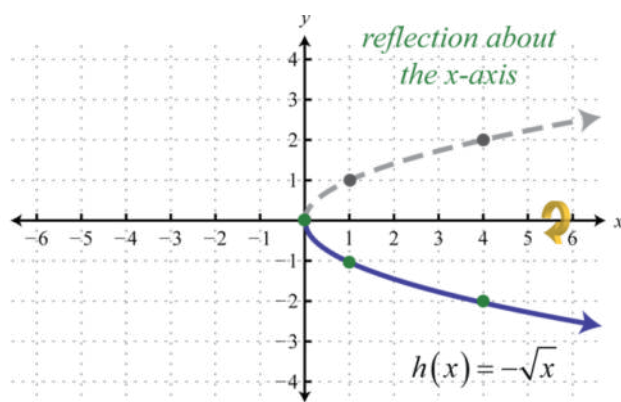
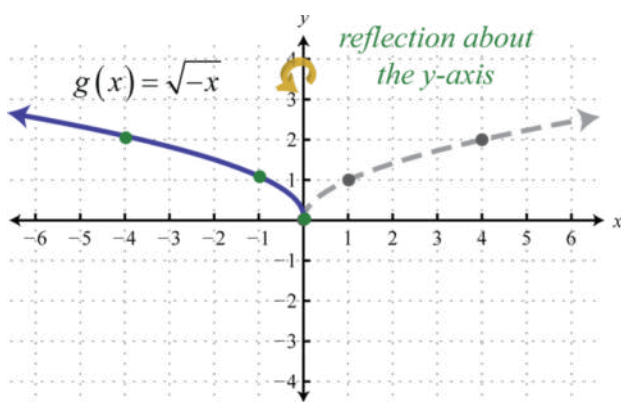
$$g(x) = \sqrt{-x}$$

$$h(x) = -\sqrt{x}$$

$x$	$g(x)$
-4	2
-1	1
0	0

$x$	$h(x)$
4	-2
1	-1
0	0

Compare the graph of  $g$  and  $h$  to the basic square root function defined by  $f(x) = \sqrt{x}$ , shown dashed in grey below:



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The first function  $g$  has a negative factor that appears “inside” the function; this produces a reflection about the  $y$ -axis. The second function  $h$  has a negative factor that appears “outside” the function; this produces a reflection about the  $x$ -axis. In general, it is true that:

Reflection about the $y$ -axis:	$F(x) = f(-x)$
Reflection about the $x$ -axis:	$F(x) = -f(x)$

When sketching graphs that involve a reflection, consider the reflection first and then apply the vertical and/or horizontal translations.

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## Example 5

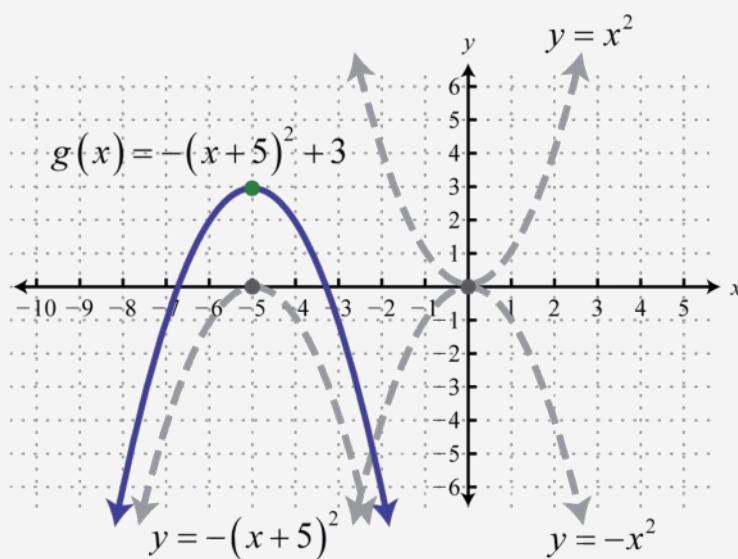
Sketch the graph of

Solution:

Begin with the squaring function and then identify the transformations starting with any reflections.

Use these translations to sketch the graph.

Answer:



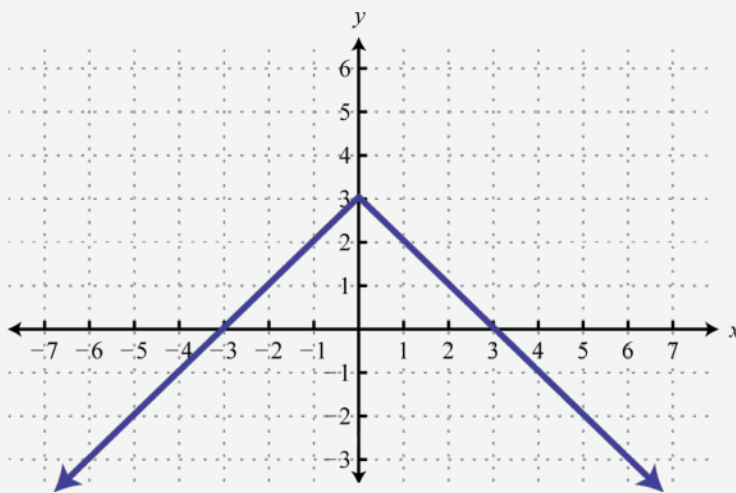
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Try this! Sketch the graph of  $g(x) = -|x| + 3$ .

Answer:



## Dilations

Horizontal and vertical translations, as well as reflections, are called rigid transformations because the shape of the basic graph is left unchanged, or rigid.

Functions that are multiplied by a real number other than 1, depending on the real number, appear to be stretched vertically or stretched horizontally. This type of non-rigid transformation is called a **dilation**. For example, we can multiply the squaring function by 4 and to see what happens to the graph.

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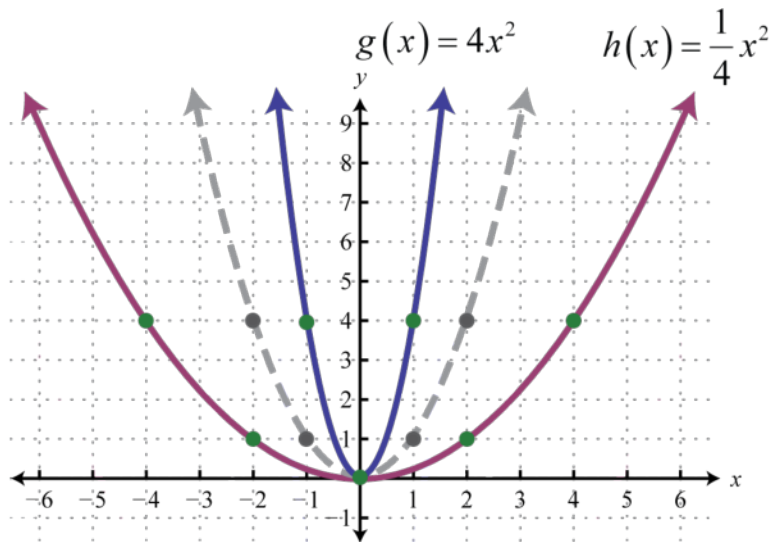
$$g(x) = 4x^2$$

$x$	$g(x)$
-1	4
0	0
1	4

$$h(x) = \frac{1}{4}x^2$$

$x$	$h(x)$
-2	1
0	0
2	1

Compare the graph of  $g$  and  $h$  to the basic squaring function defined by  $y = x^2$ , shown dashed in grey below:



The function  $g$  is steeper than the basic squaring function and its graph appears to have been stretched vertically. The function  $h$  is not as steep as the basic squaring function and appears to have been stretched horizontally.

In general, we have:

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Dilation:	$F(x) = a \cdot f(x)$

If the factor  $a$  is a nonzero fraction between  $-1$  and  $1$ , it will stretch the graph horizontally. Otherwise, the graph will be stretched vertically. If the factor  $a$  is negative, then it will produce a reflection as well.

## Example 6

Sketch the graph of  $g(x) = -2|x-5|-3$ .

Solution:

Here we begin with the product of  $-2$  and the basic absolute value function:  $y = -2|x|$ . This results in a reflection and a dilation.

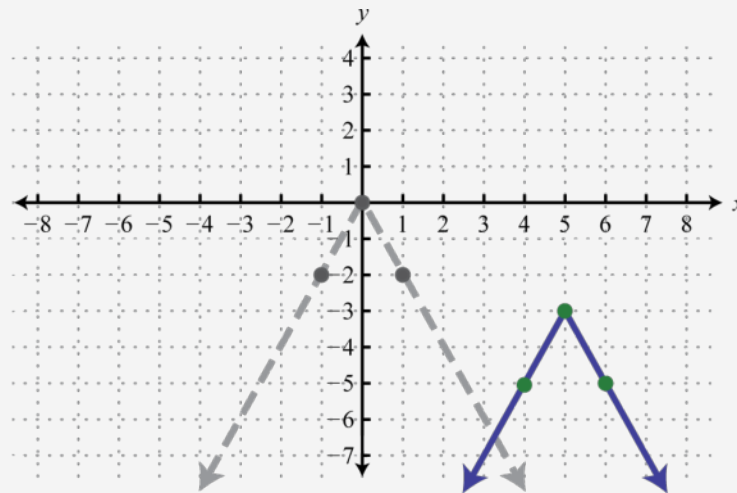
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Use the points  $\{(-1, -2), (0, 0), (1, -2)\}$  to graph the reflected and dilated function  $y = -2|x|$ . Then translate this graph 5 units to the right and 3 units down.

Answer:



In summary, given positive real numbers  $h$  and  $k$ :

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Vertical shift up  $k$  units:

$$F(x) = f(x) + k$$

Vertical shift down  $k$  units:

$$F(x) = f(x) - k$$

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Horizontal shift left $h$ units:	$F(x) = f(x + h)$
Horizontal shift right $h$ units:	$F(x) = f(x - h)$

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Reflection about the $y$ -axis:	$F(x) = f(-x)$
Reflection about the $x$ -axis:	$F(x) = -f(x)$
Dilation:	$F(x) = a \cdot f(x)$

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## KEY TAKEAWAYS

Identifying transformations allows us to quickly sketch the graph of functions. This skill will be useful as we progress in our study of mathematics. Often a geometric understanding of a problem will lead to a more elegant solution.

If a positive constant is added to a function,  $f(x) + k$ , the graph will shift up. If a positive constant is subtracted from a function,  $f(x) - k$ , the graph will shift down. The basic shape of the graph will remain the same.

If a positive constant is added to the value in the domain before the function is applied,  $f(x + h)$ , the graph will shift to the left. If a positive constant is subtracted from the value in the domain before the function is applied,  $f(x - h)$ , the graph will shift right. The basic shape will remain the same.

Multiplying a function by a negative constant,  $-f(x)$ , reflects its graph in the  $x$ -axis. Multiplying the values in the domain by  $-1$  before applying the function,  $f(-x)$ , reflects the graph about the  $y$ -axis.

When applying multiple transformations, apply reflections first.

Multiplying a function by a constant other than 1,  $a \cdot f(x)$ , produces a dilation. If the constant is a positive number greater than 1, the graph will appear to stretch vertically. If the positive constant is a fraction less than 1, the graph will appear to stretch horizontally.

## TOPIC EXERCISES

### PART A: VERTICAL AND HORIZONTAL TRANSLATIONS

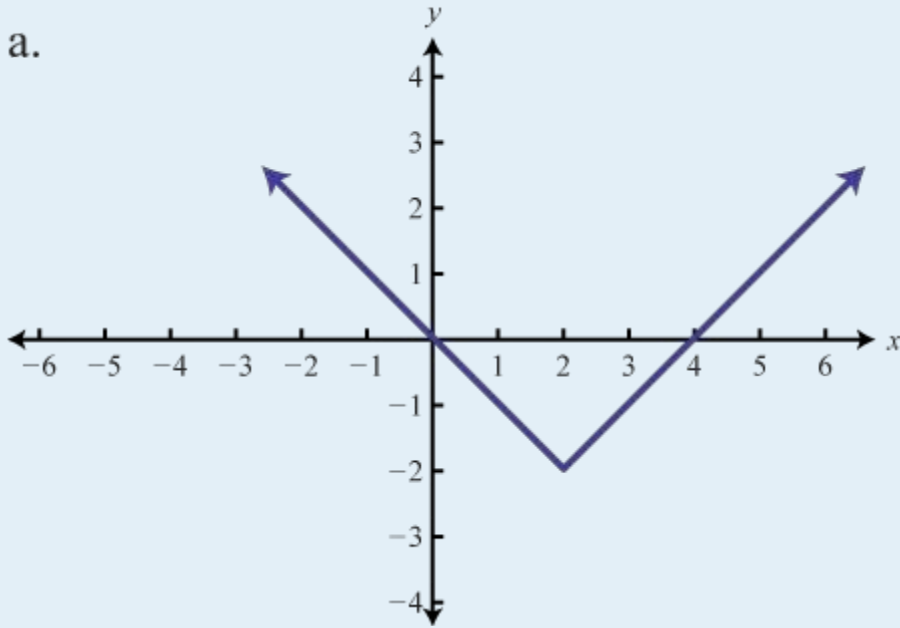
Match the graph to the function definition.

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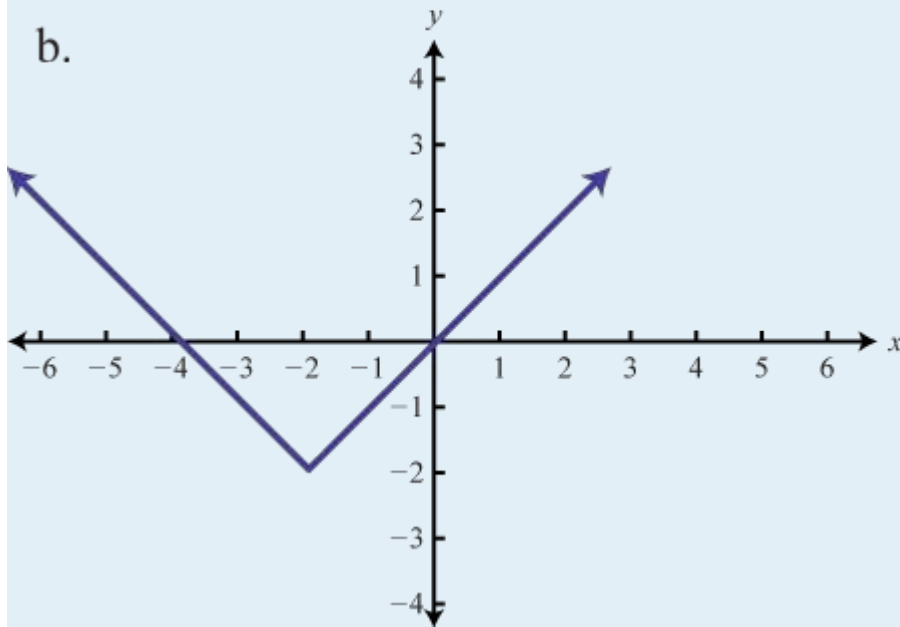
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a.

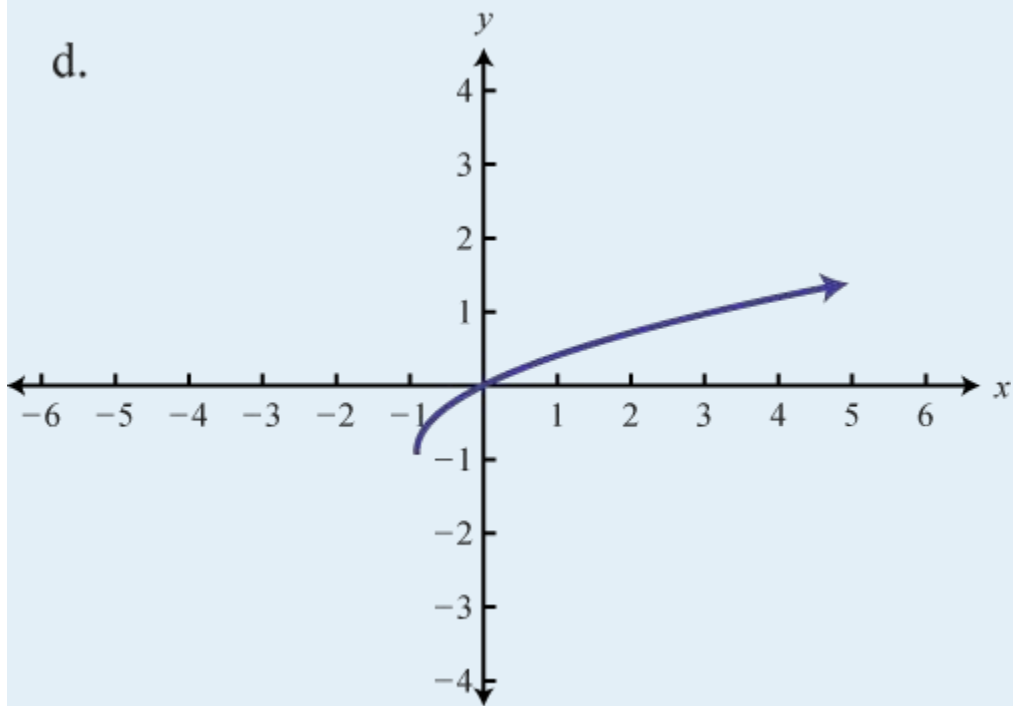
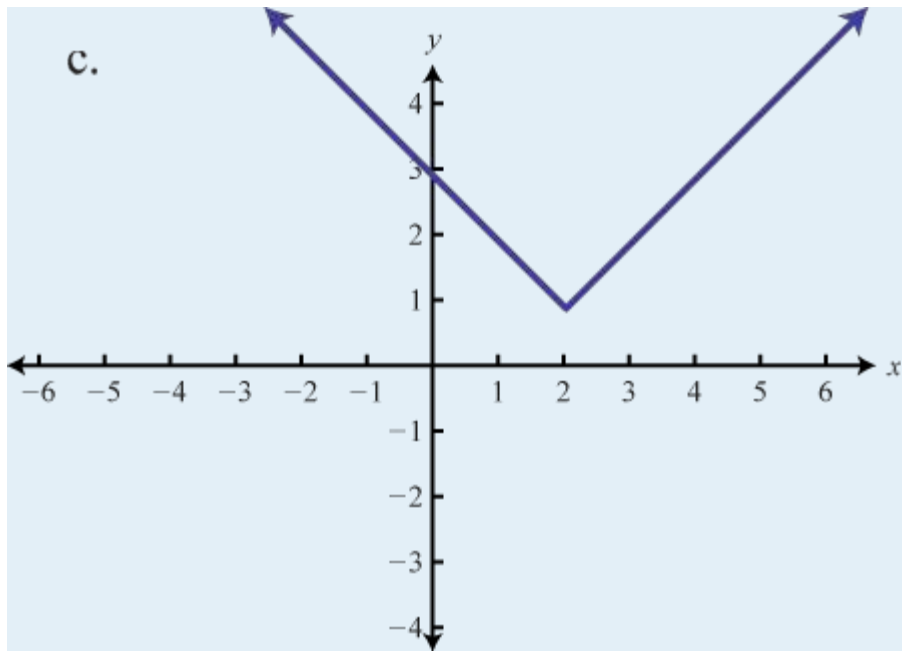


b.



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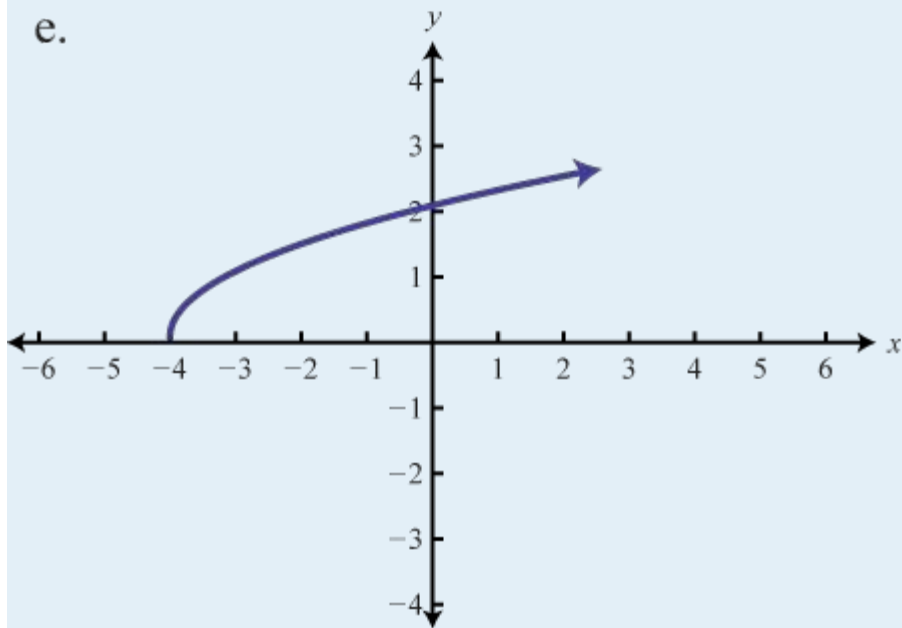




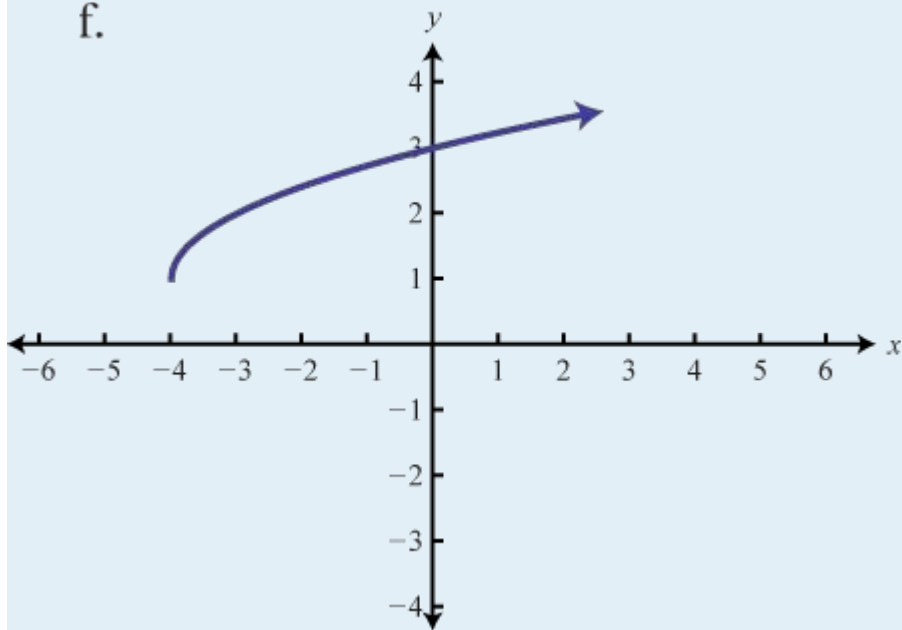
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Saylor URL: <http://www.saylor.org/books/>



e.



f.



1.



2.  $f(x) = |x-2|-2$

3.

4.  $f(x) = |x-2| + 1$

5.

6.  $f(x) = |x + 2|-2$

Graph the given function. Identify the basic function and translations used to sketch the graph. Then state the domain and range.

1.  $f(x) = x + 3$

2.  $f(x) = x-2$

3.

4.

5.

6.

7.

8.

9.  $h(x) = |x + 4|$

10.  $h(x) = |x-4|$

11.  $h(x) = |x-1|-3$

12.  $h(x) = |x + 2|-5$

13.

14.

15.

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16.

17.

18.

19.

20.

21.

22.

23.

24.

25.

26.

27.  $g(x) = -4$

28.  $g(x) = 2$

29.

30.

Graph the piecewise functions.

1.

2.

3.

4.

5.

6.

7.

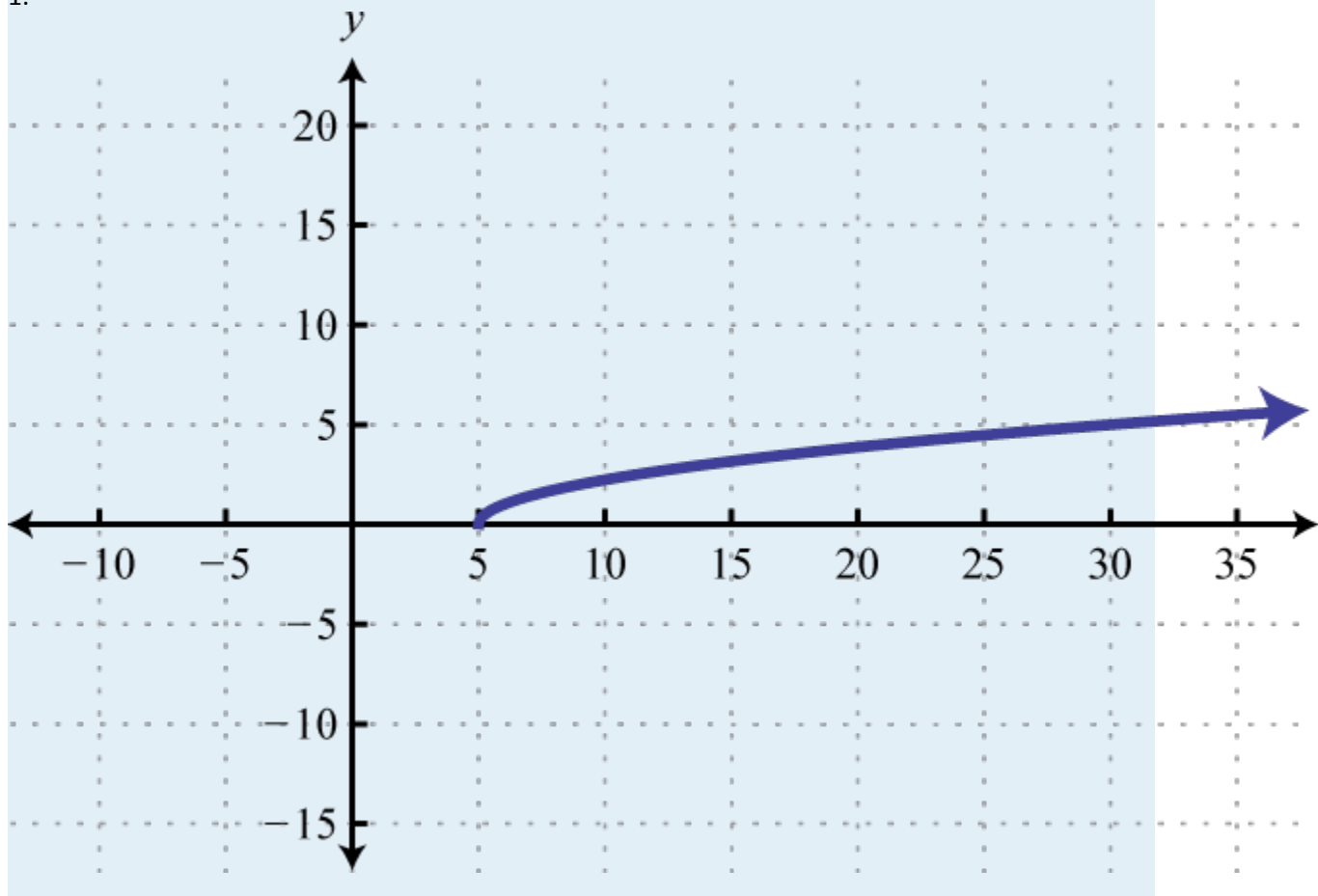
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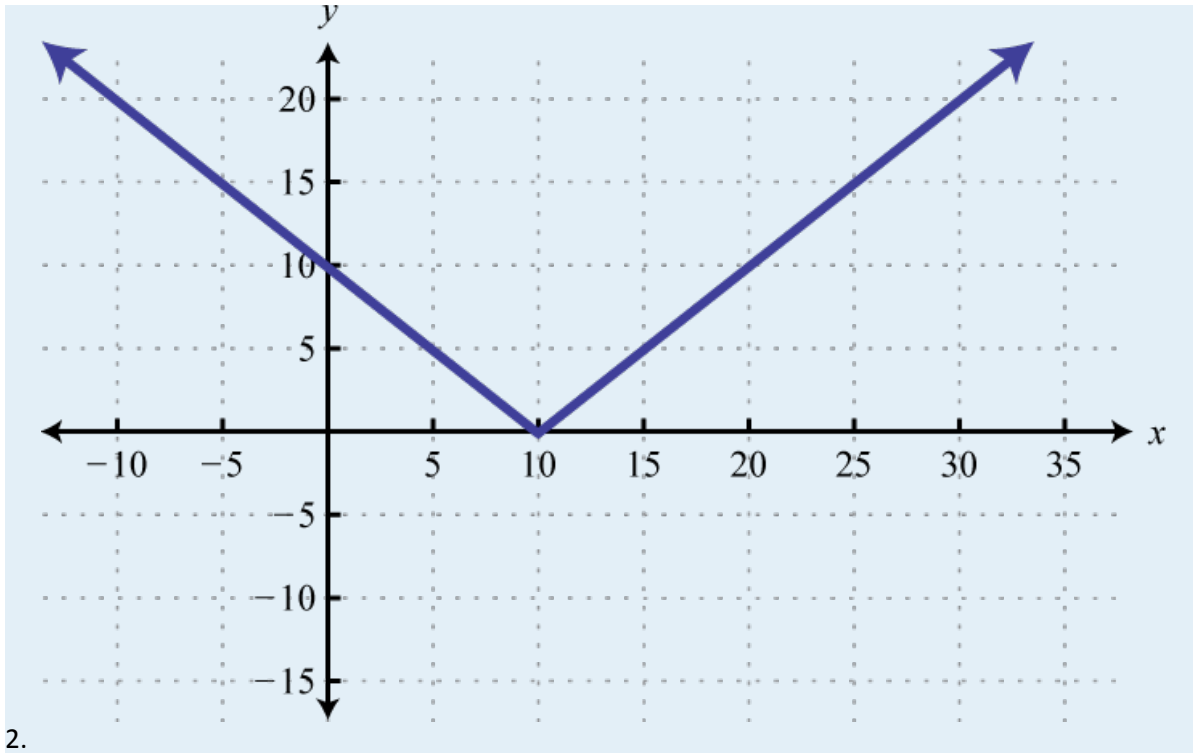
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Write an equation that represents the function whose graph is given.

1.



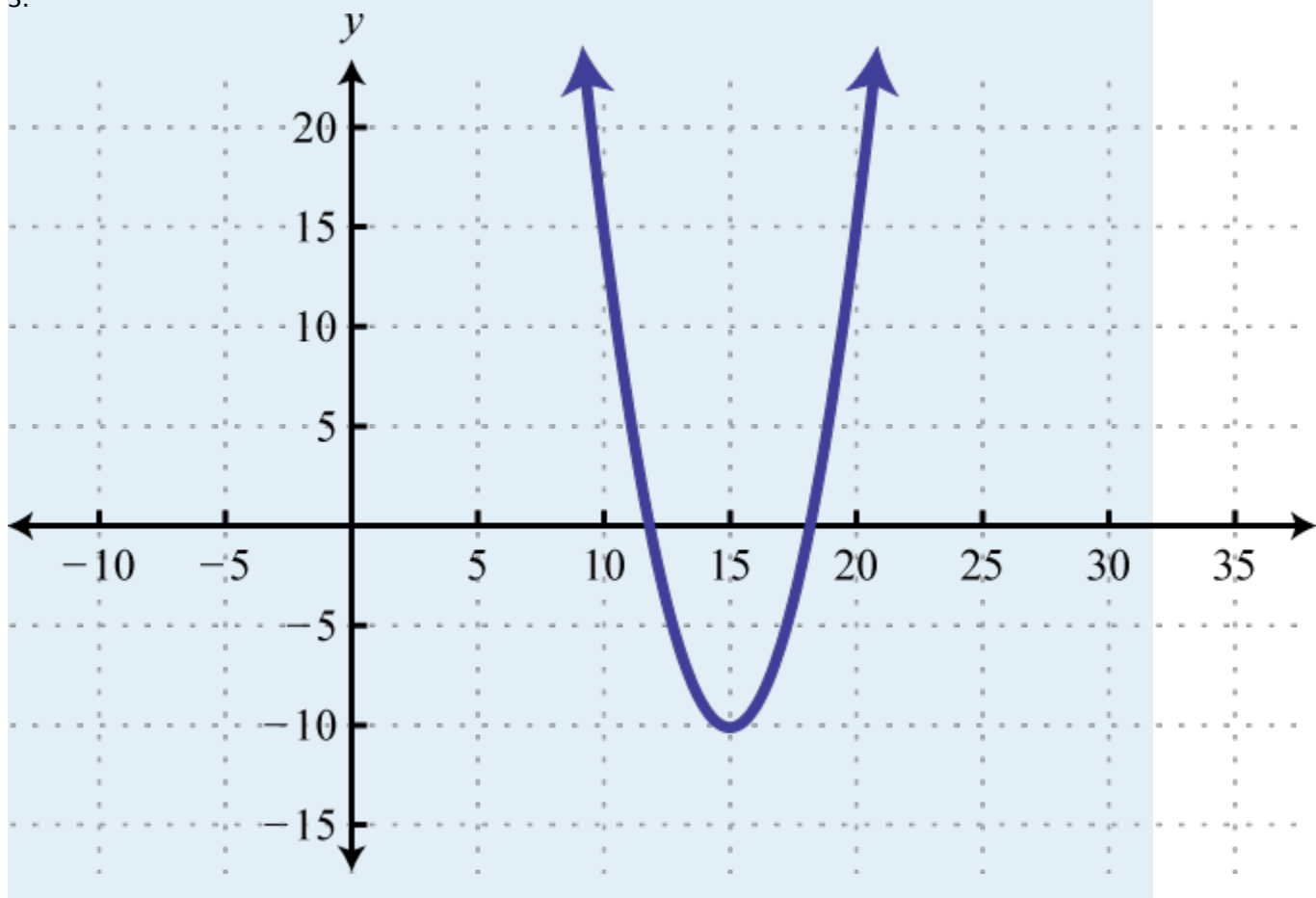


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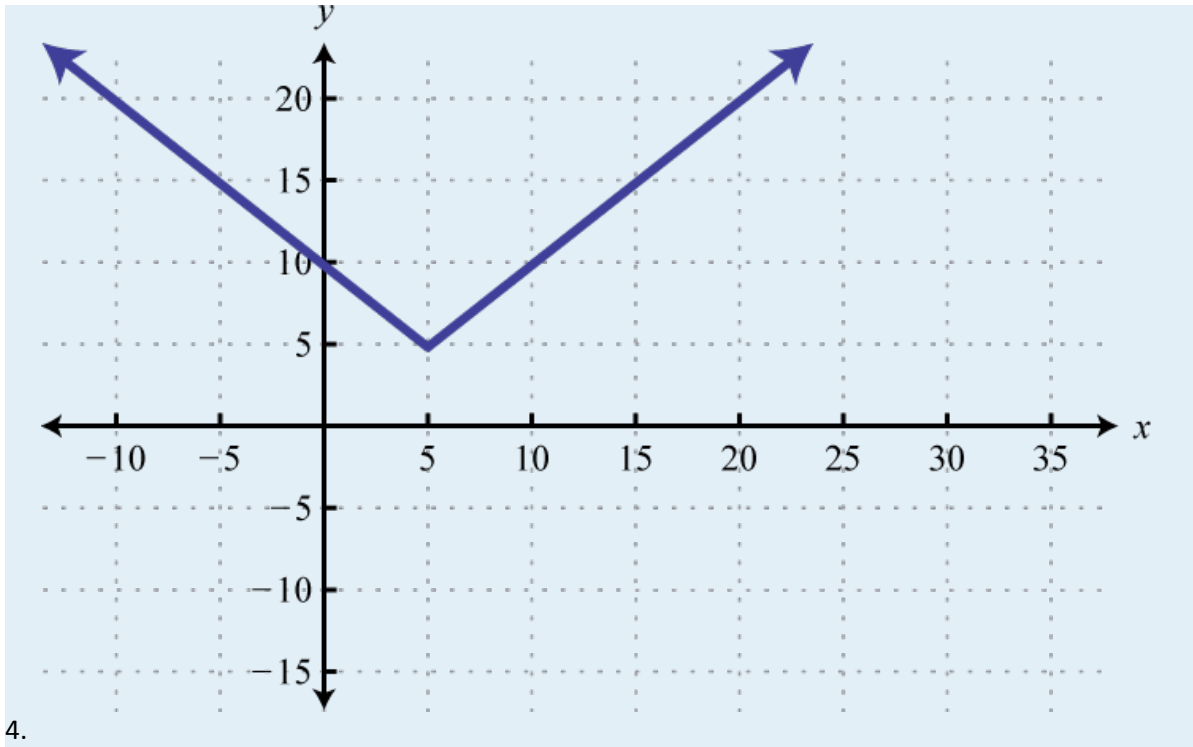
3.



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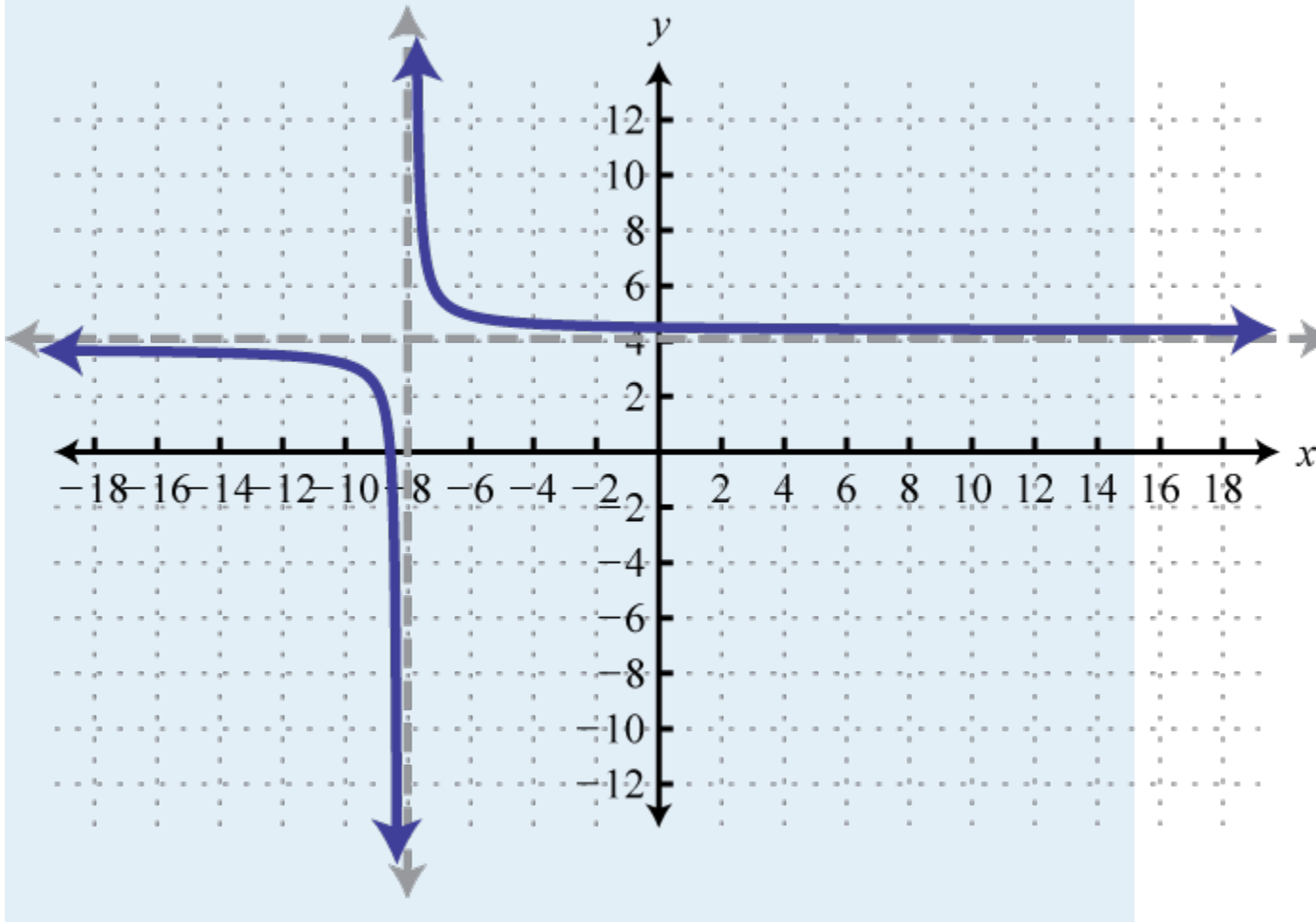


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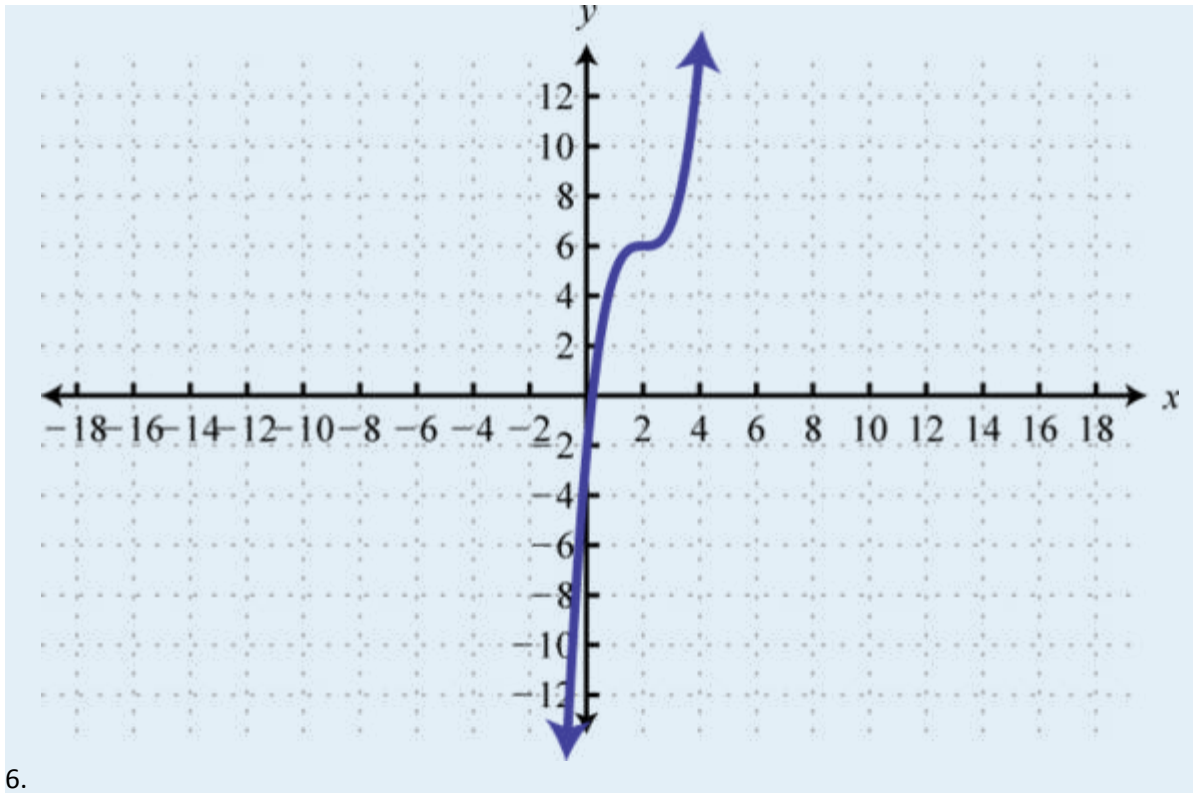
5.



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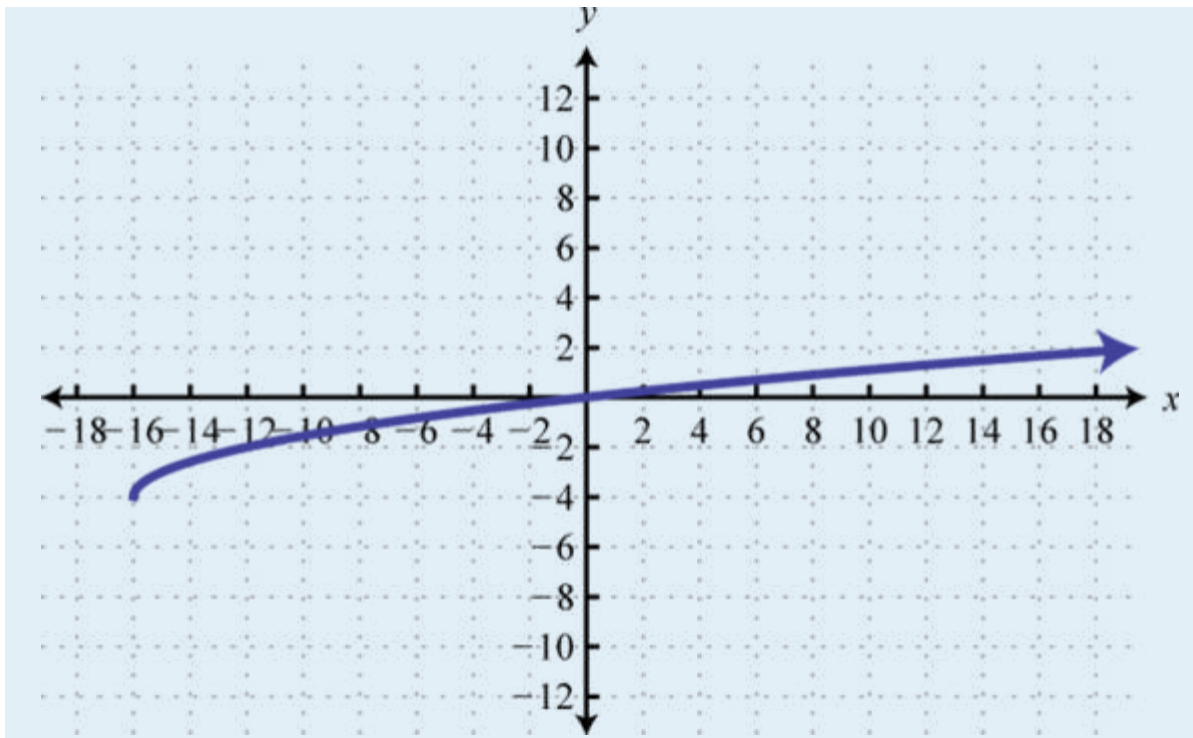
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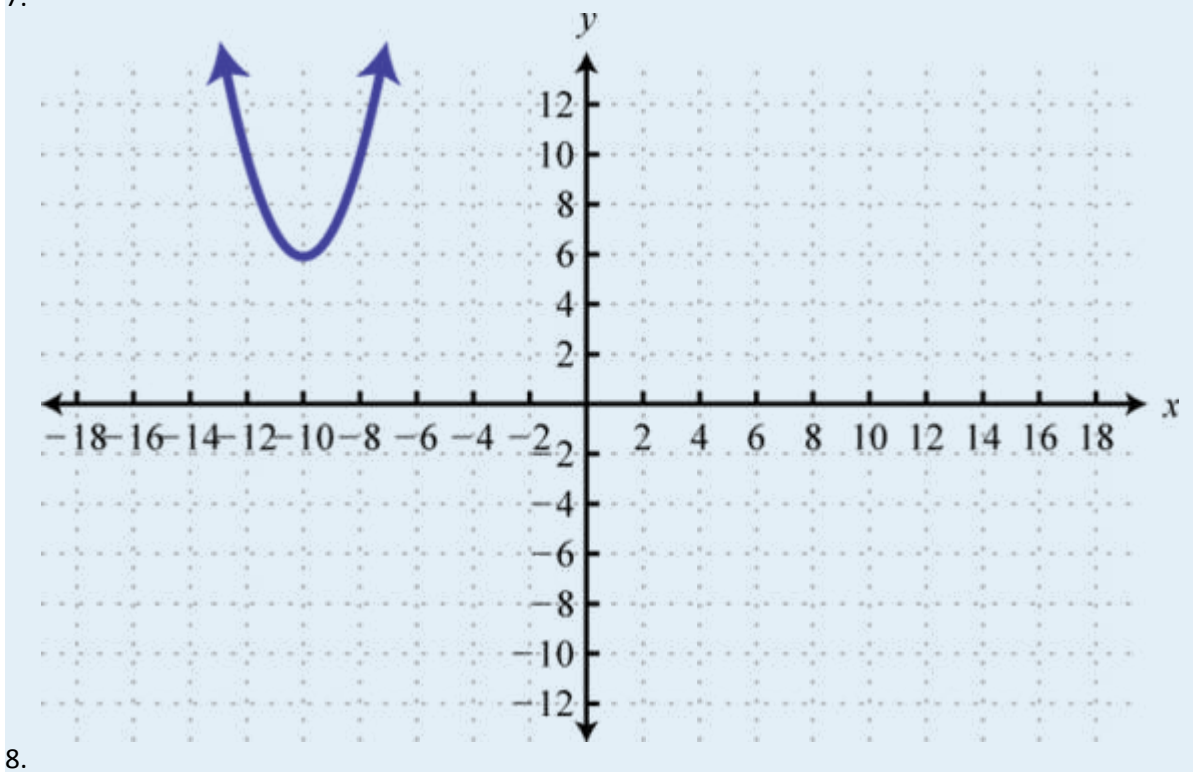


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7.



8.

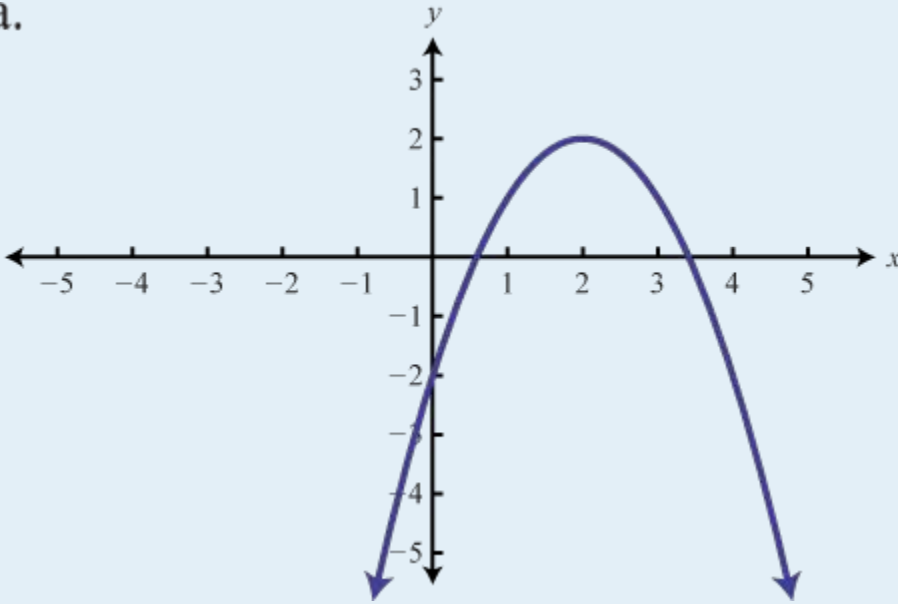
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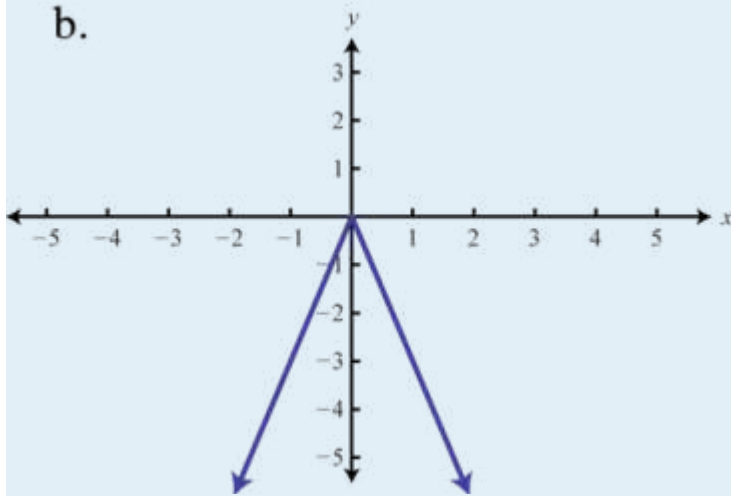
# PART B: REFLECTIONS AND DILATIONS

Match the graph the given function definition.

a.



b.

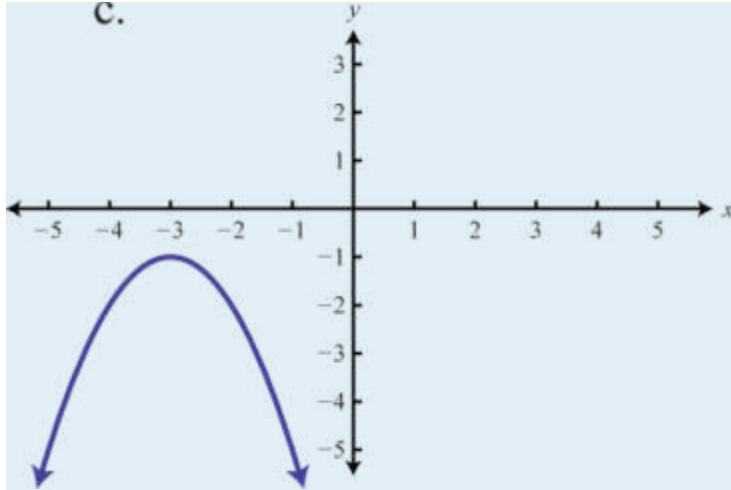


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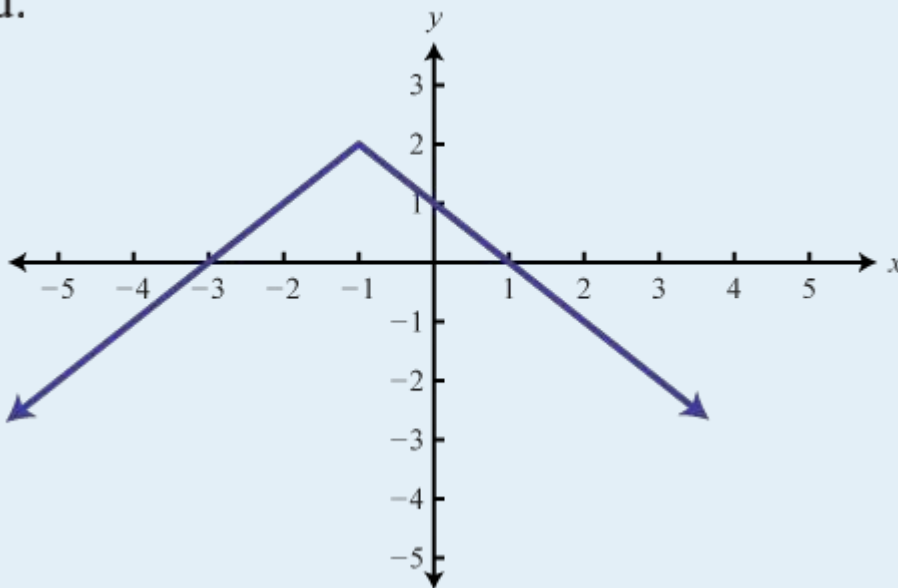
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c.



d.

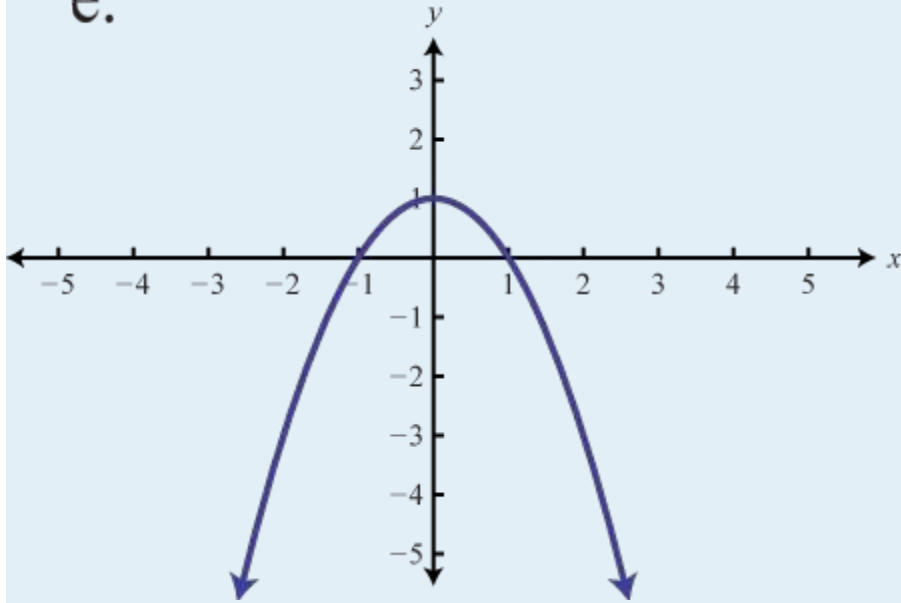


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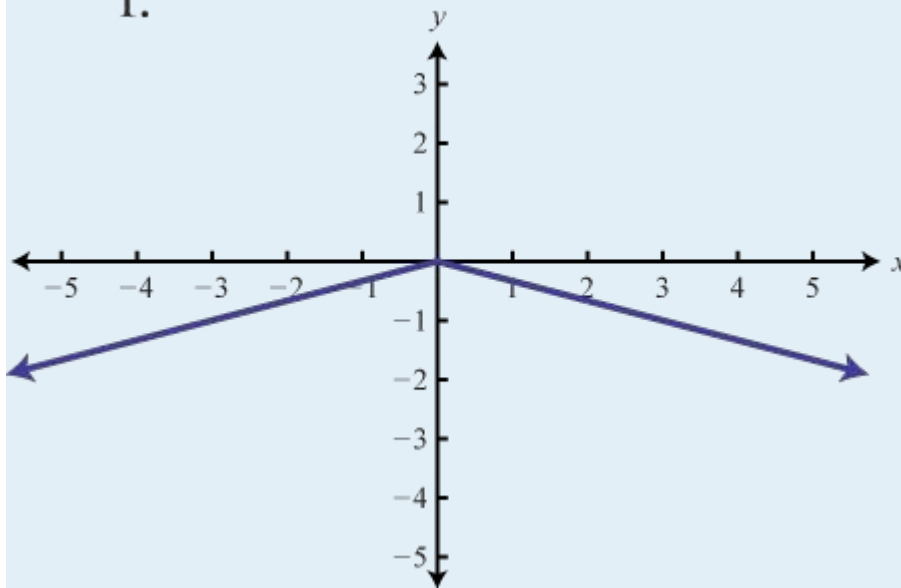
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e.



f.



53.  $f(x) = -3|x|$

54.

55.  $f(x) = -|x + 1| + 2$





56.

57.

58.

Use the transformations to graph the following functions.

53.  $f(x) = -x + 5$

54.  $f(x) = -|x| - 3$

55.  $g(x) = -|x - 1|$

56.

57.

58.

59.

60.

61.

62.

63.  $f(x) = -3|x|$

64.

65.

66.

67.

68.

69.

70.  $h(x) = -2x + 1$

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- 71.
- 72.
- 73.
- 74.
- 75.
- 76.

$$h(x) = -3|x + 4| - 2$$

## PART C: DISCUSSION BOARD

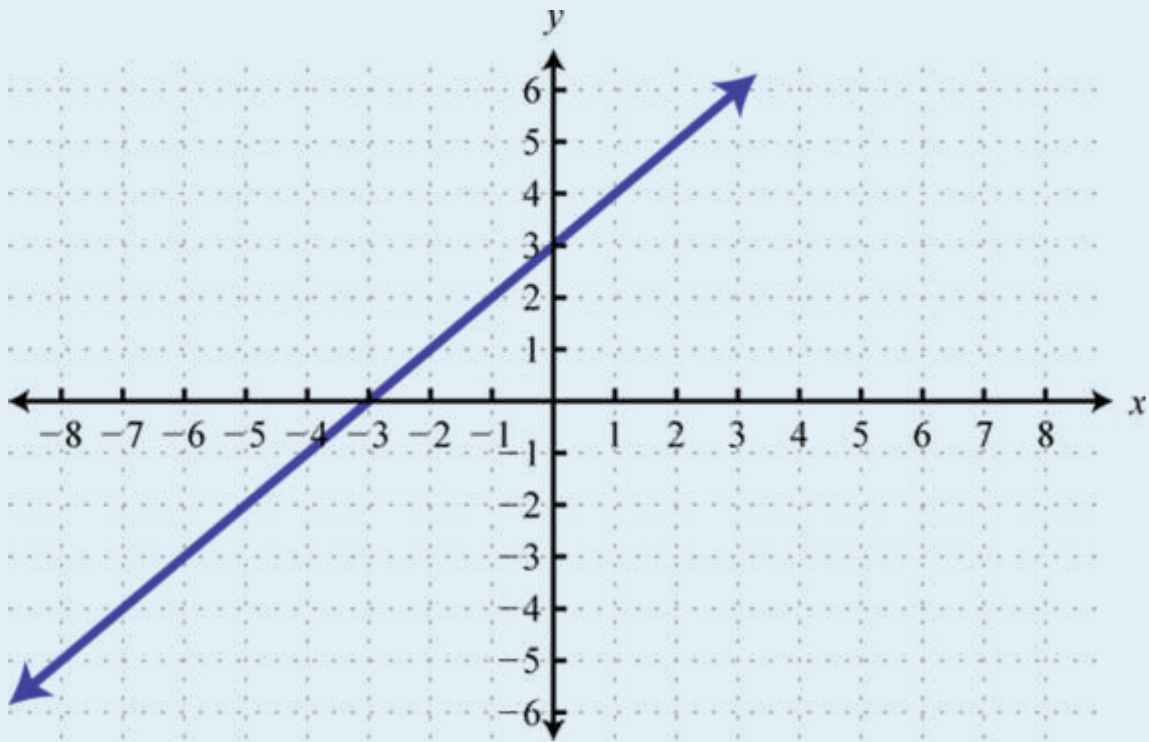
- 83. Use different colors to graph the family of graphs defined by  $y = \frac{k}{x}$ , where  $k$  is a constant. What happens to the graph when the denominator of  $k$  is very large? Share your findings on the discussion board.
- 84. Graph  $y = \frac{k}{x}$  and  $y = \frac{k}{x^2}$  on the same set of coordinate axes. What does the general shape look like? Try to find a single equation that describes the shape. Share your findings.
- 85. Explore what happens to the graph of a function when the domain values are multiplied by a factor  $a$  before the function is applied,  $f(ax)$ . Develop some rules for this situation and share them on the discussion board.

## ANSWERS

- 1. e
- 2.
- 3. d
- 4.
- 5. f



6.



7.

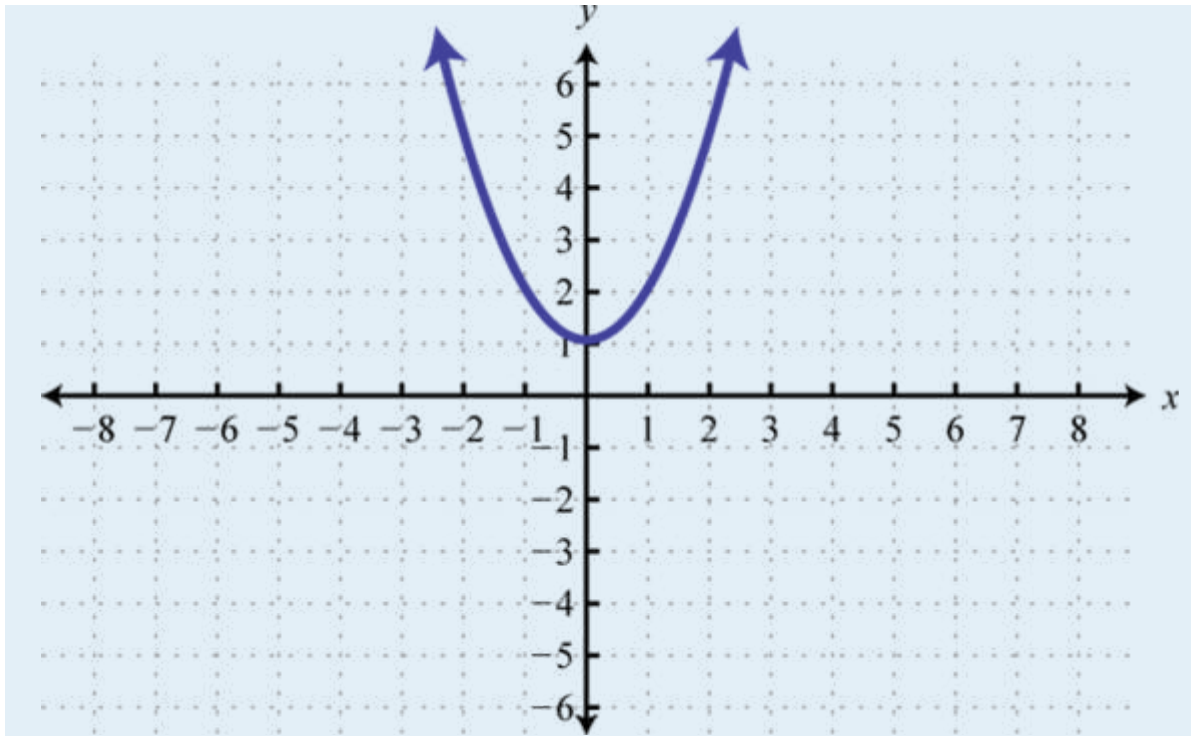
$y = x$ ; Shift up 3 units; domain: ; range:

8.

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9.

; Shift up 1 unit; domain: ; range:  $[1, \infty)$

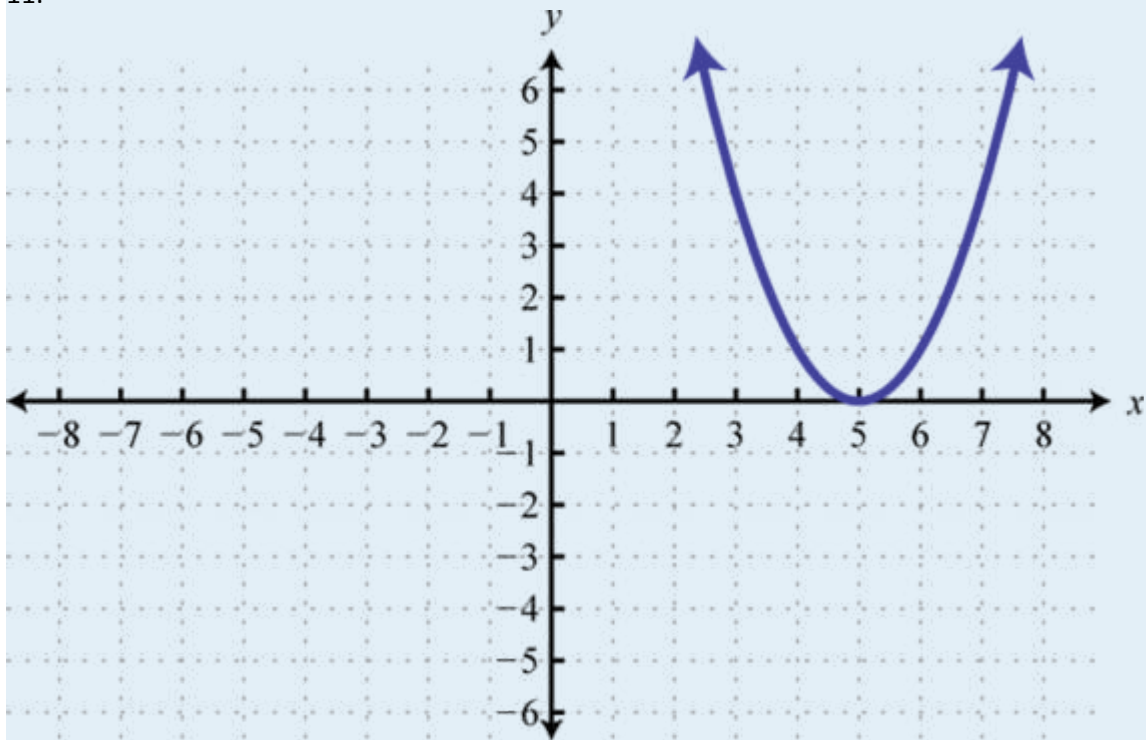
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11.



; Shift right 5 units; domain: ; range:  $[0, \infty)$

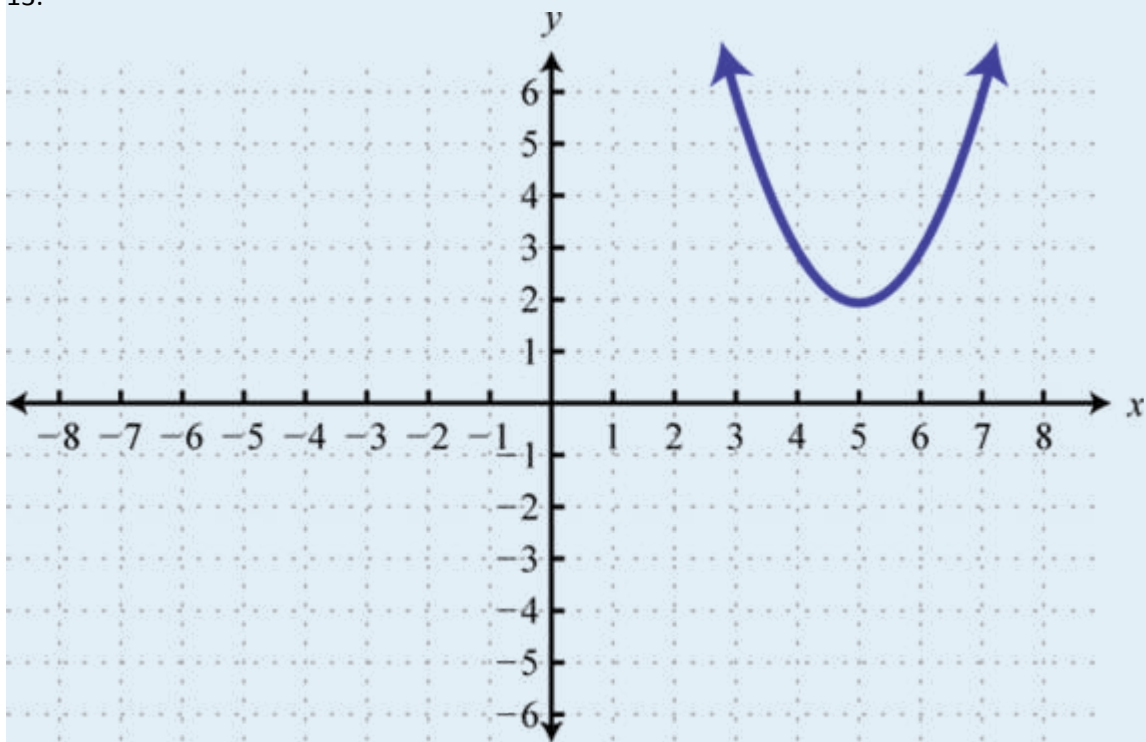
12.

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13.



; Shift right 5 units and up 2 units; domain: ; range:  $[2, \infty)$

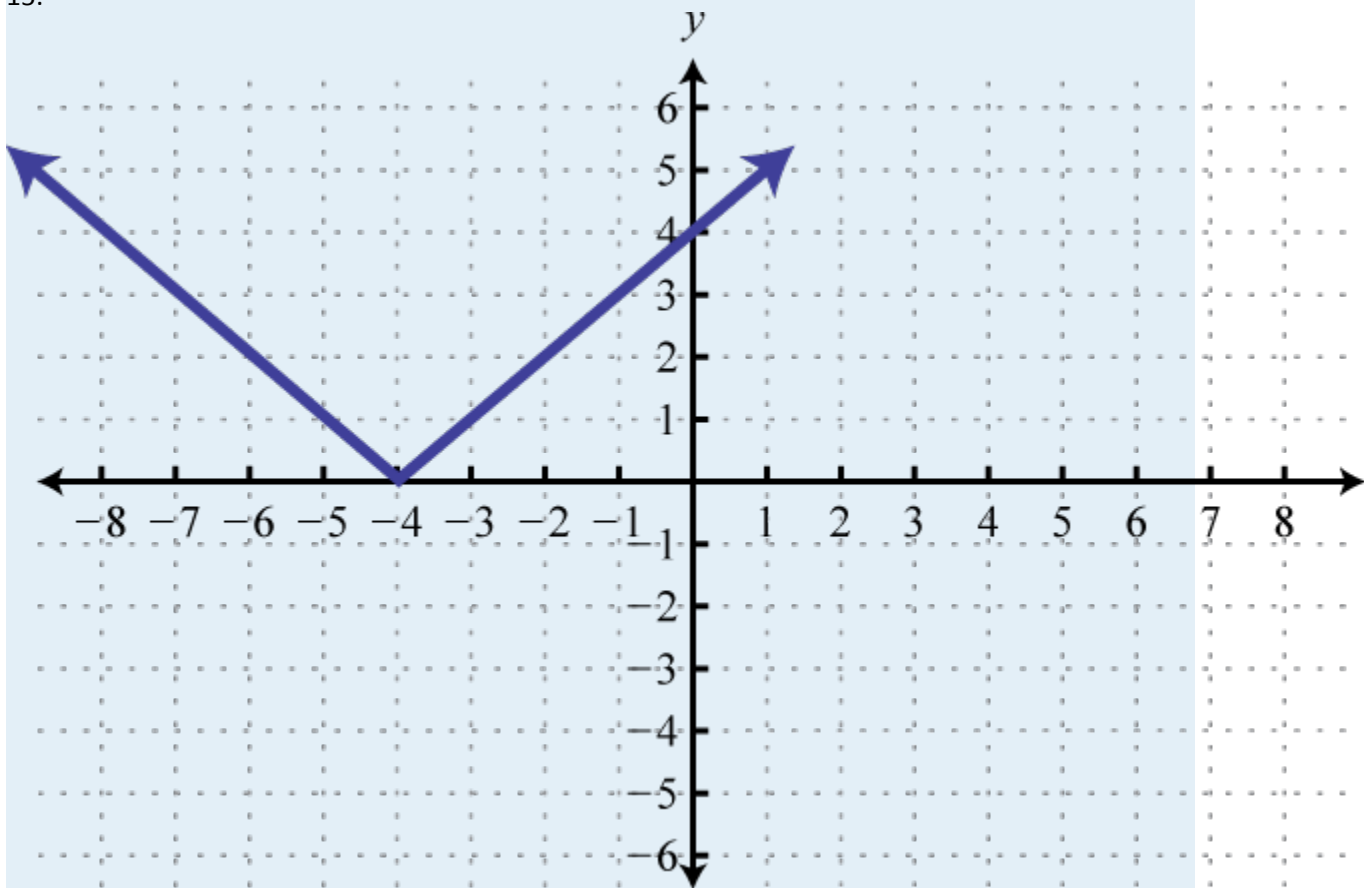
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15.

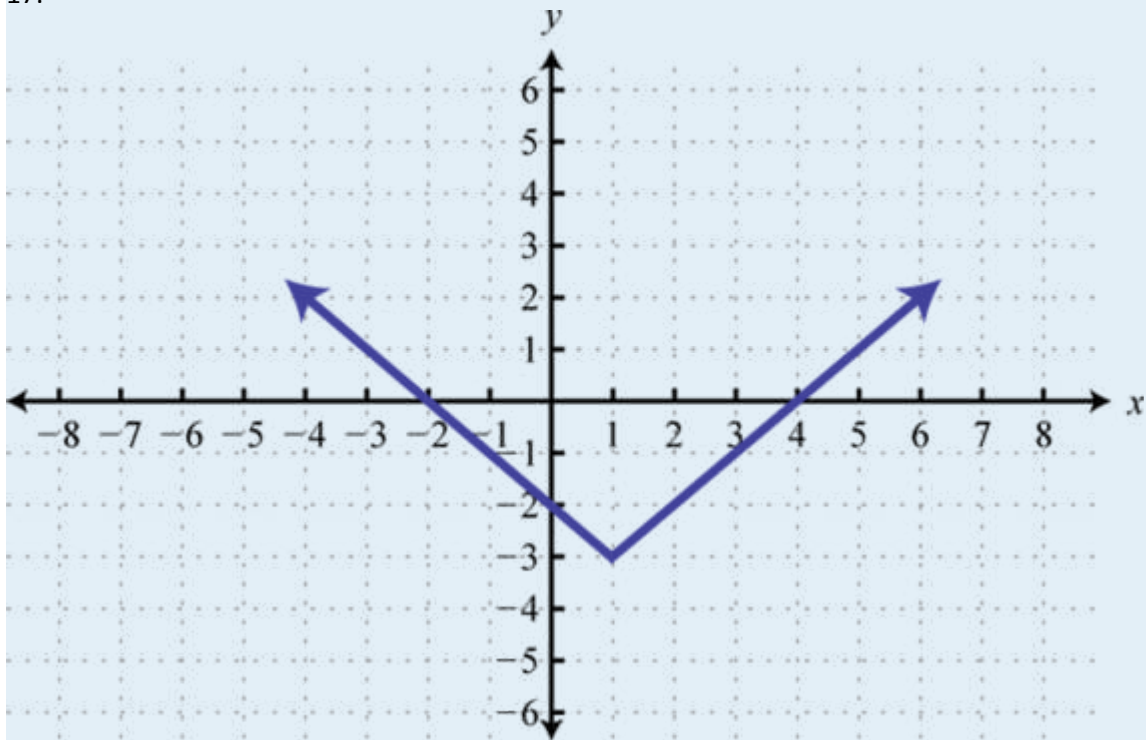


$y = |x|$ ; Shift left 4 units; domain: ; range:  $[0, \infty)$

16.



17.



$y = |x|$ ; Shift right 1 unit and down 3 units; domain: ; range:  $[-3, \infty)$

18.

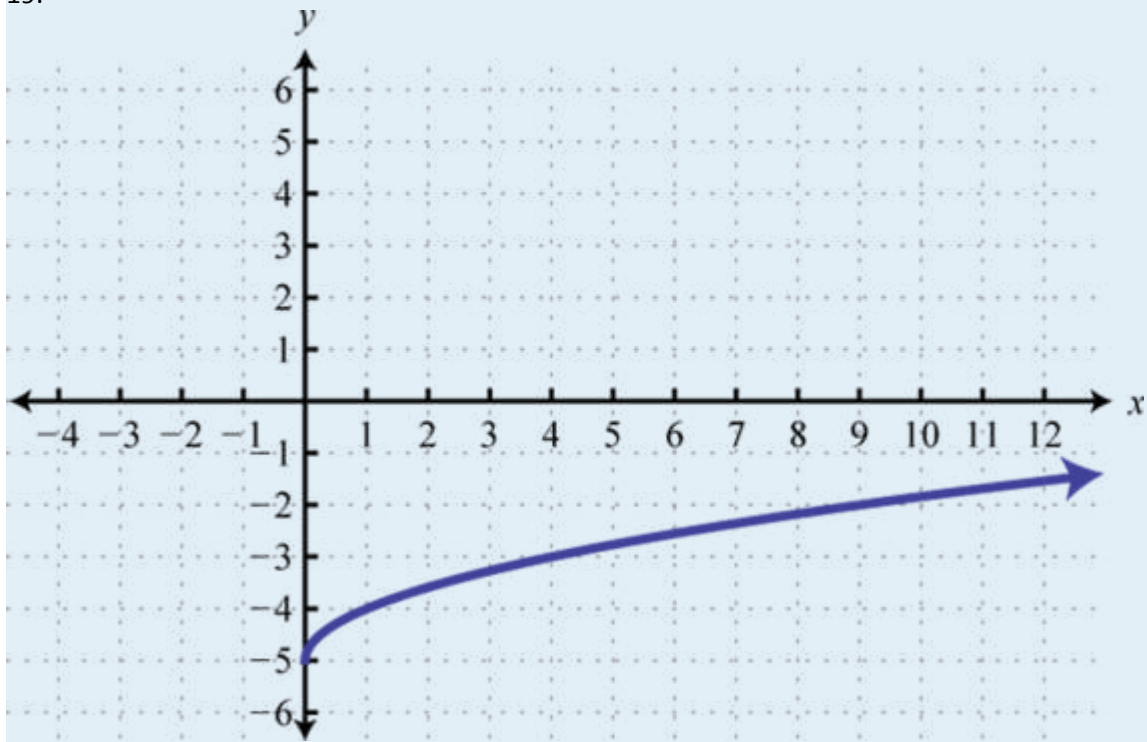
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19.



; Shift down 5 units; domain:  $[0, \infty)$ ; range:  $[-5, \infty)$

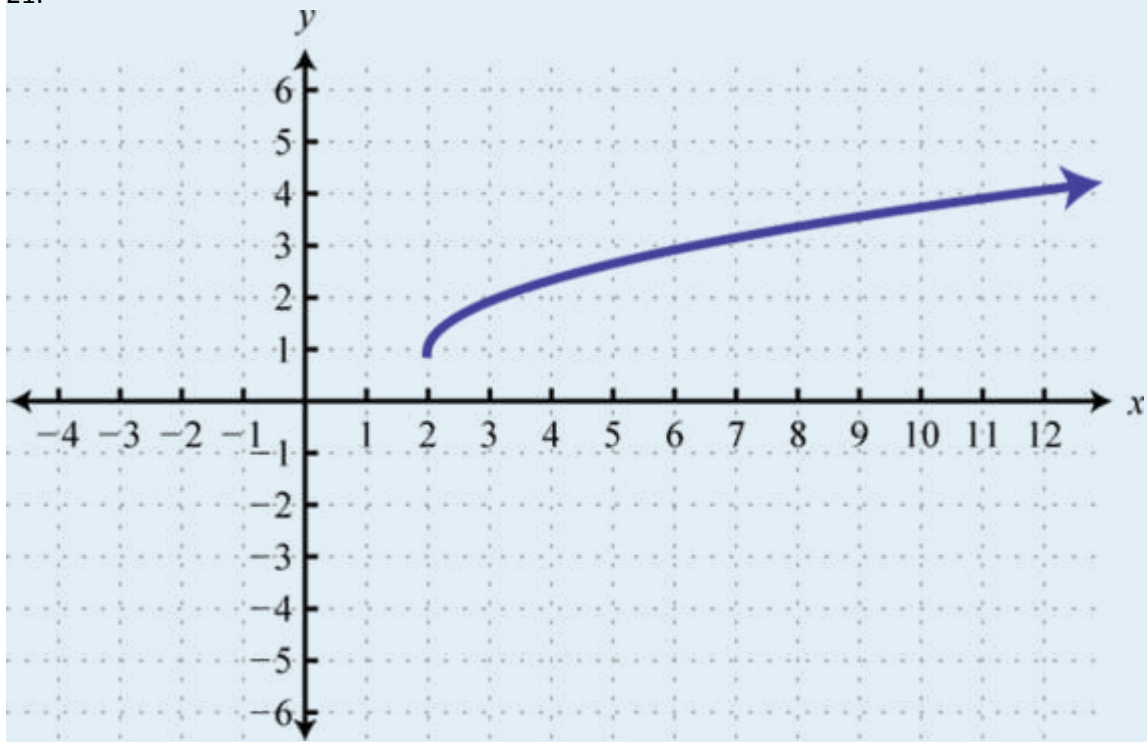
20.

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21.



; Shift right 2 units and up 1 unit; domain:  $[2, \infty)$ ; range:  $[1, \infty)$

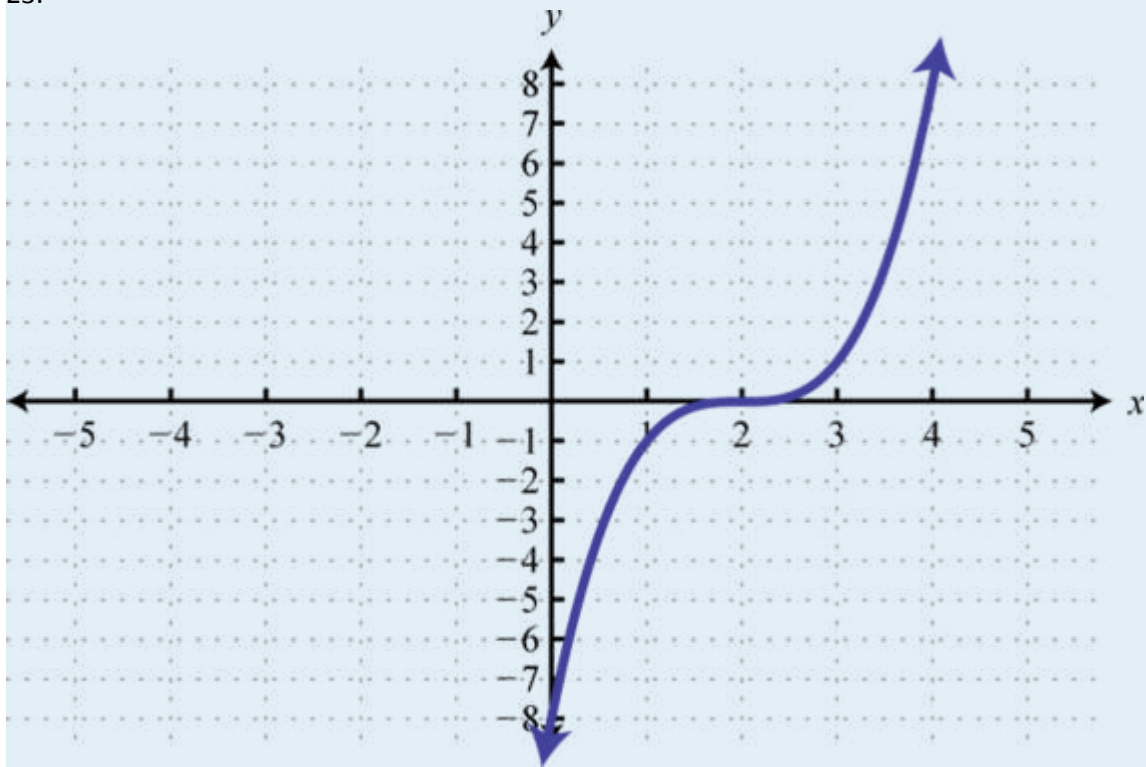
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23.



; Shift right 2 units; domain: ; range:

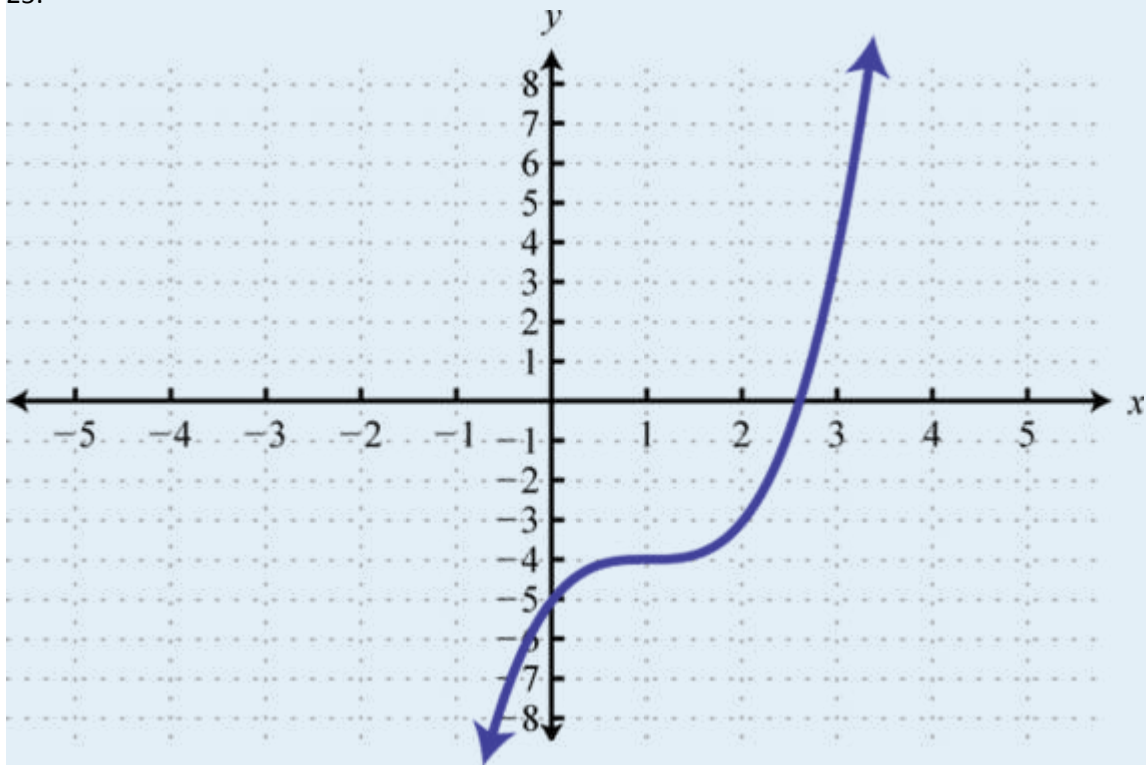
24.

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25.



; Shift right 1 unit and down 4 units; domain: ; range:

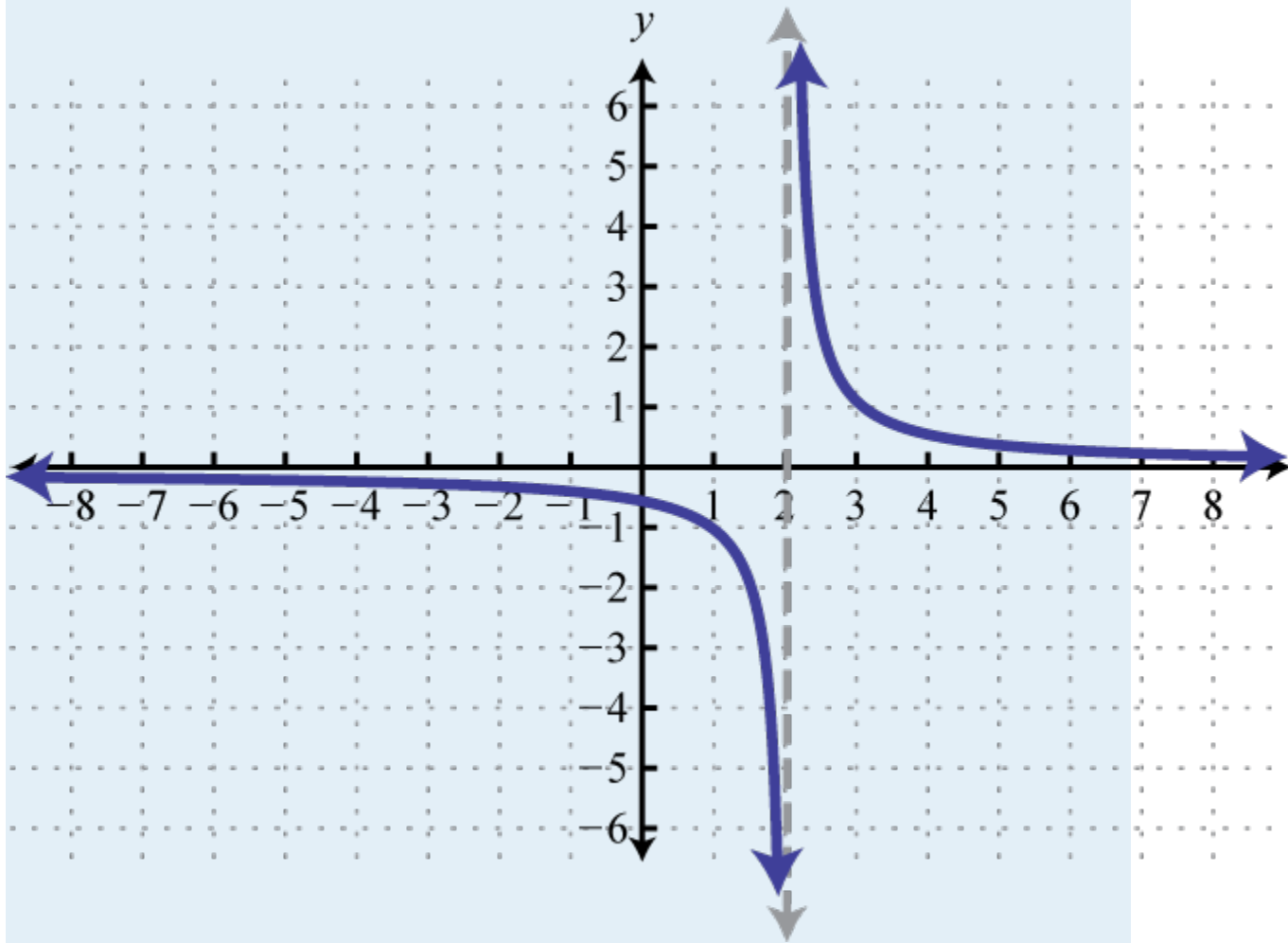
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27.



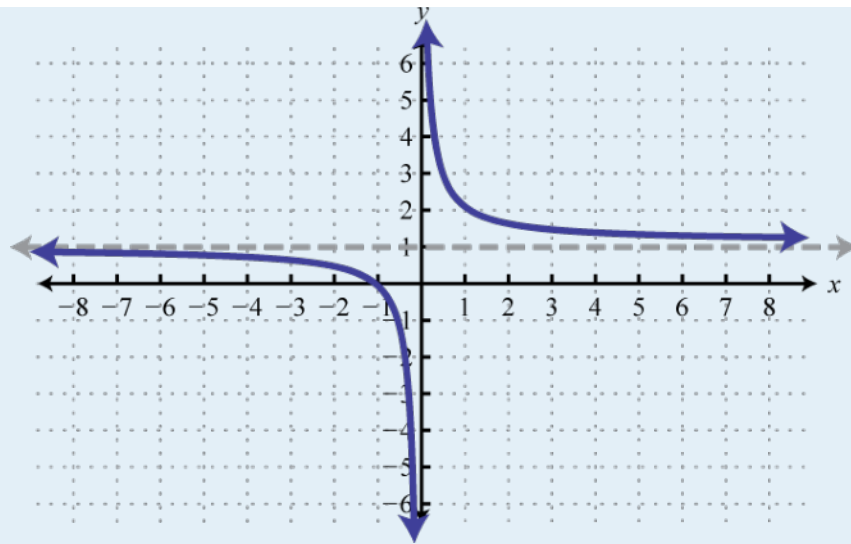
; Shift right 2 units; domain:  $(-\infty, 2) \cup (2, \infty)$ ; range:  $(-\infty, 0) \cup (0, \infty)$

28.

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29.

; Shift up 5 units; domain:  $(-\infty, 0) \cup (0, \infty)$ ; range:  $(-\infty, 1) \cup (1, \infty)$

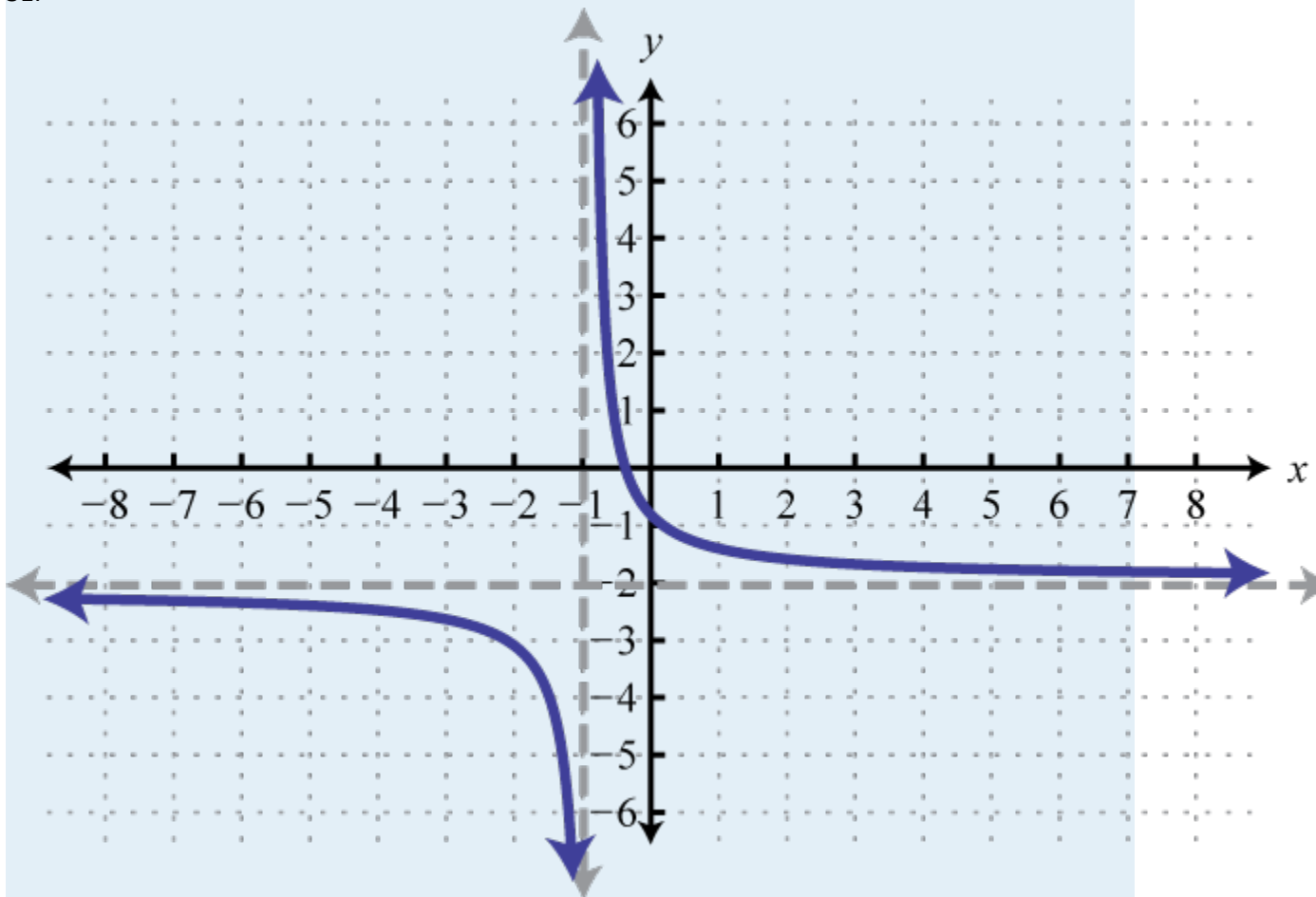
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31.

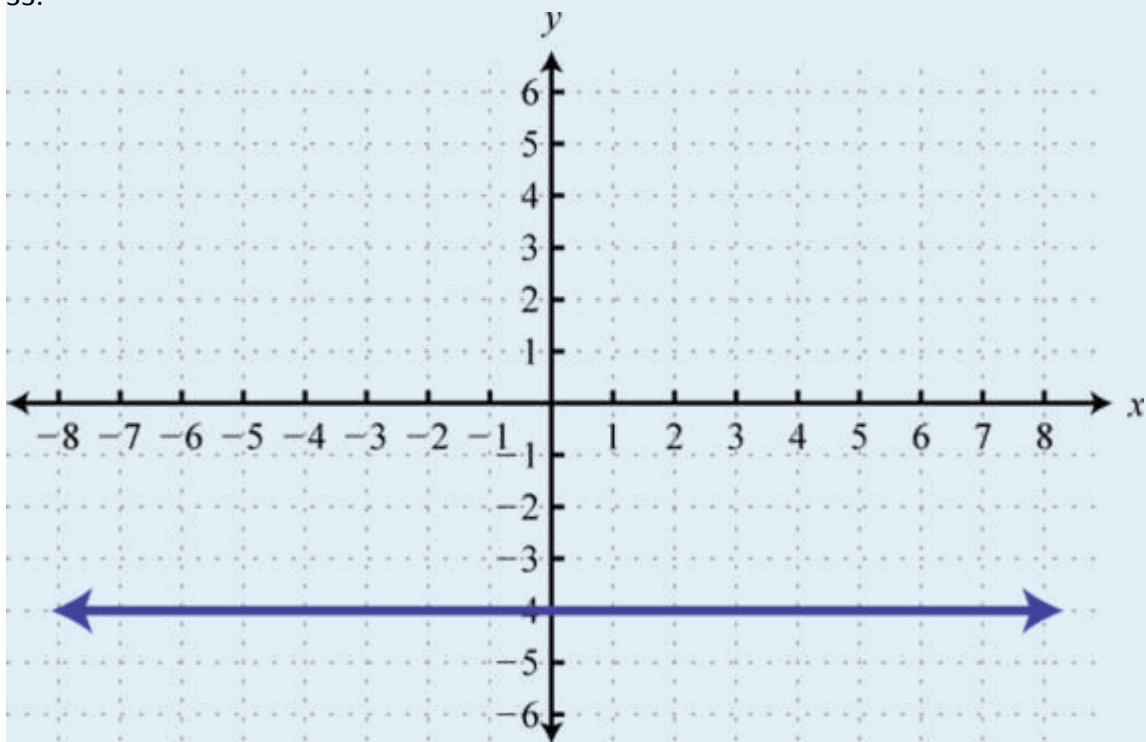


; Shift left 1 unit and down 2 units; domain:  $(-\infty, -1) \cup (-1, \infty)$ ; range:  $(-\infty, -2) \cup (-2, \infty)$

32.



33.



Basic graph  $y = -4$ ; domain: ; range:  $\{-4\}$

34.

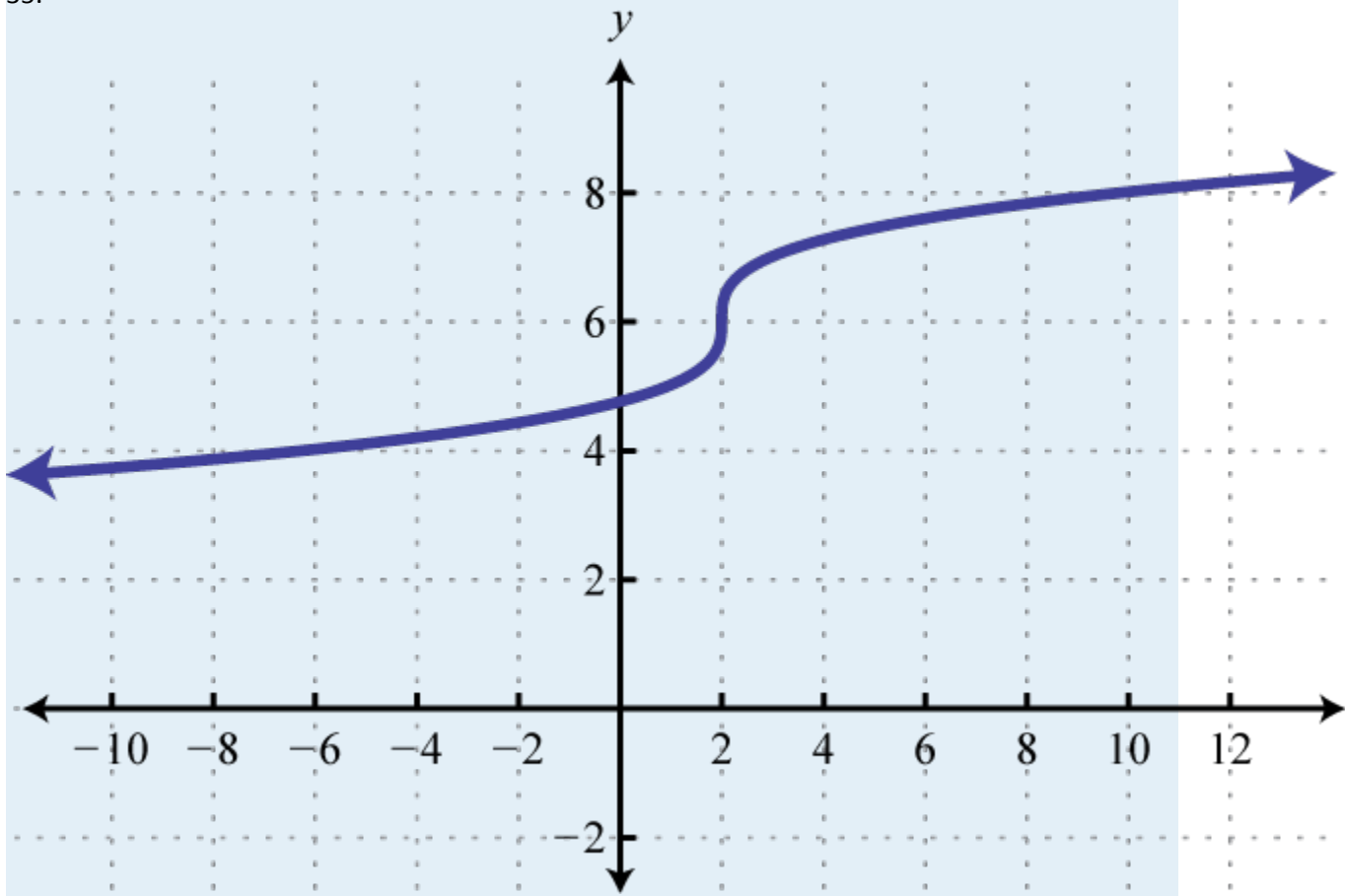
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35.

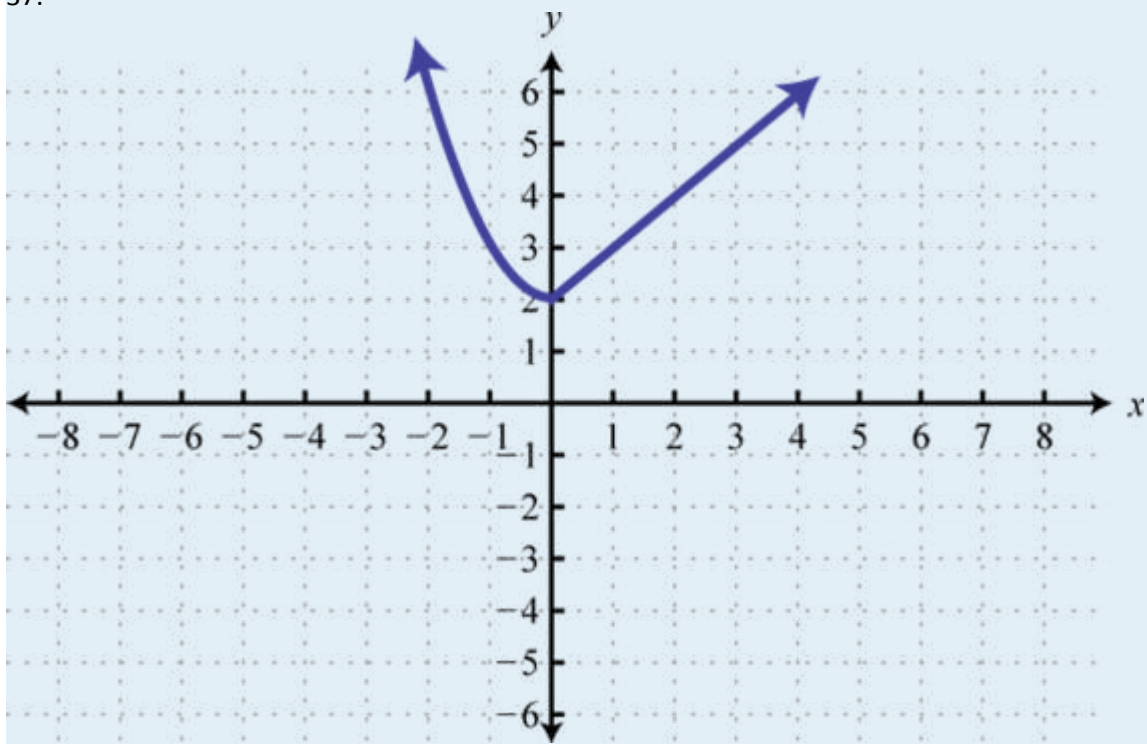


; Shift up 6 units and right 2 units; domain: ; range:

36.



37.



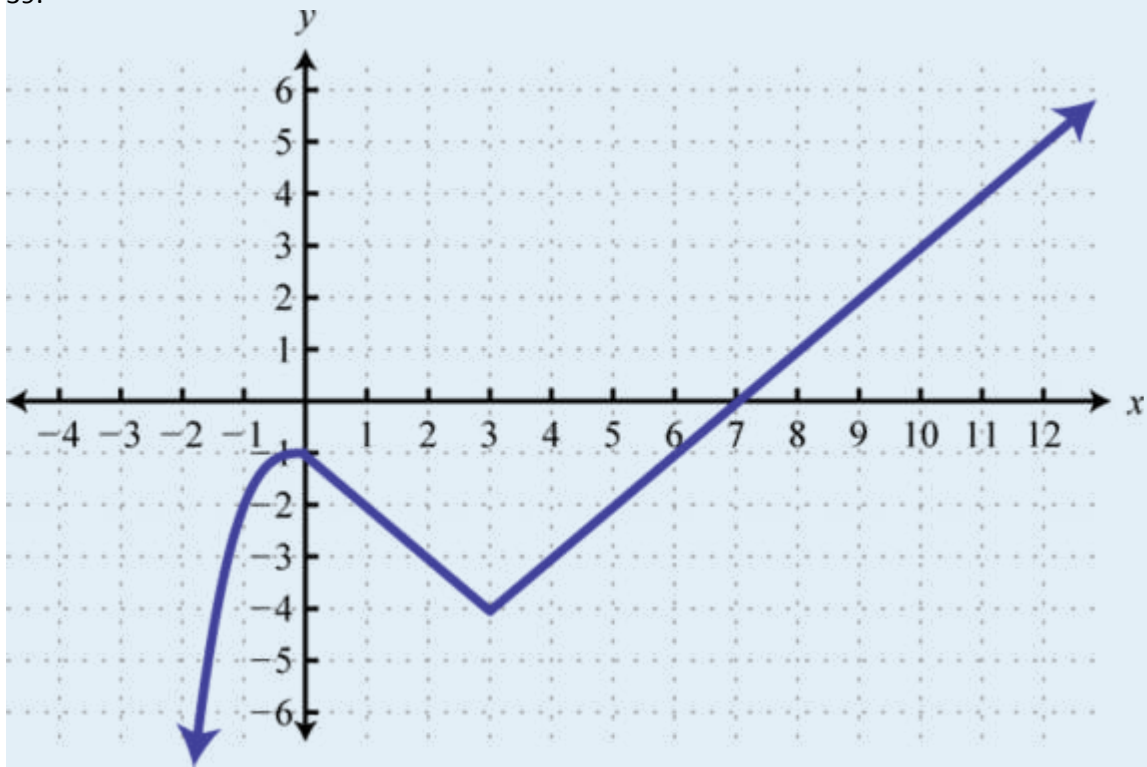
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39.

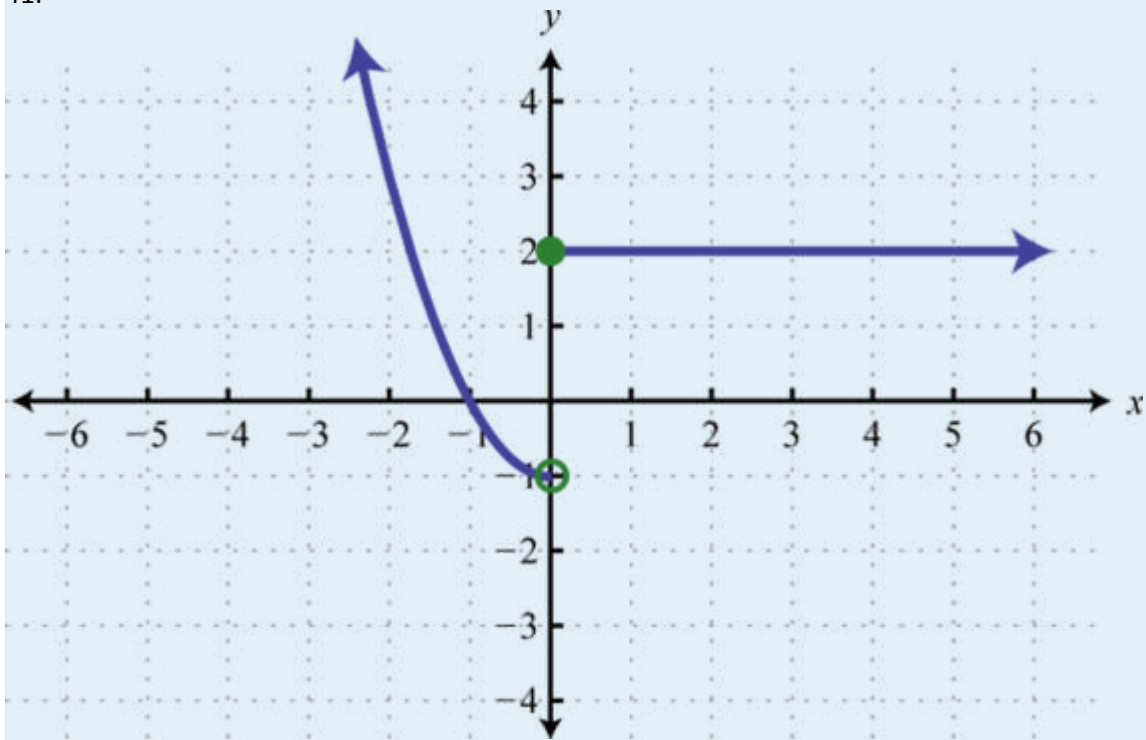


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41.

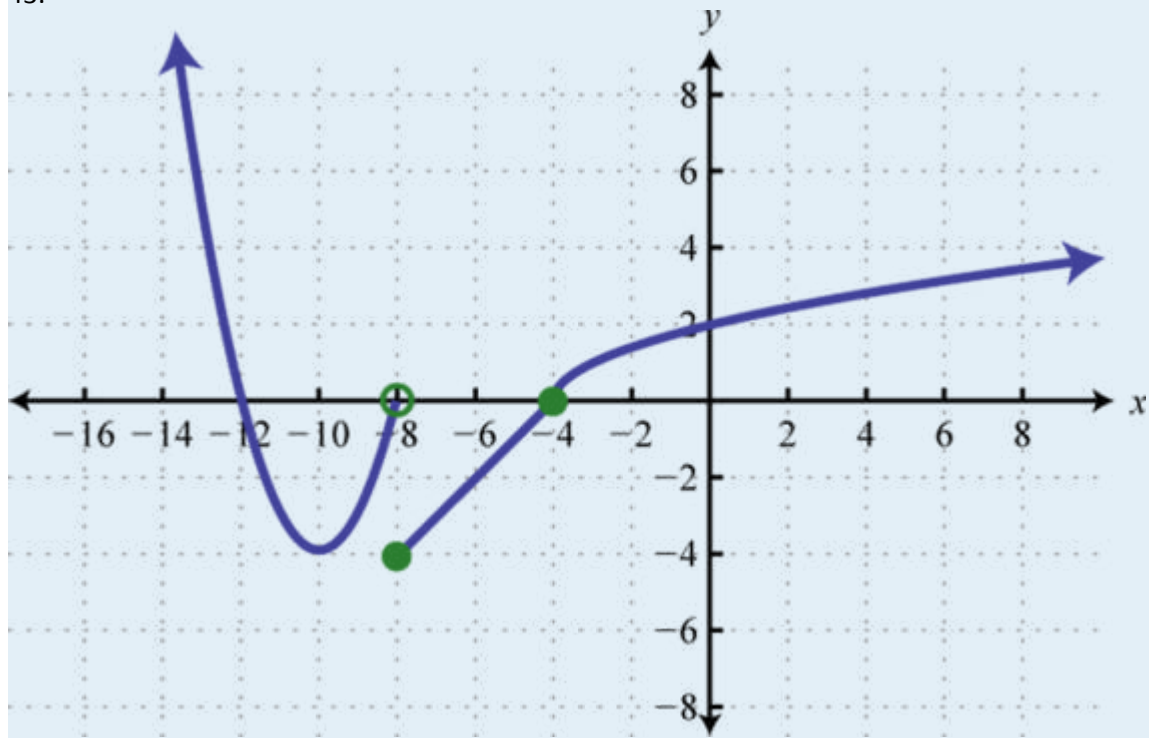


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43.



44.

45.

46.

47.

48.

49.

50.

51.

53. b

54.

55. d

56.

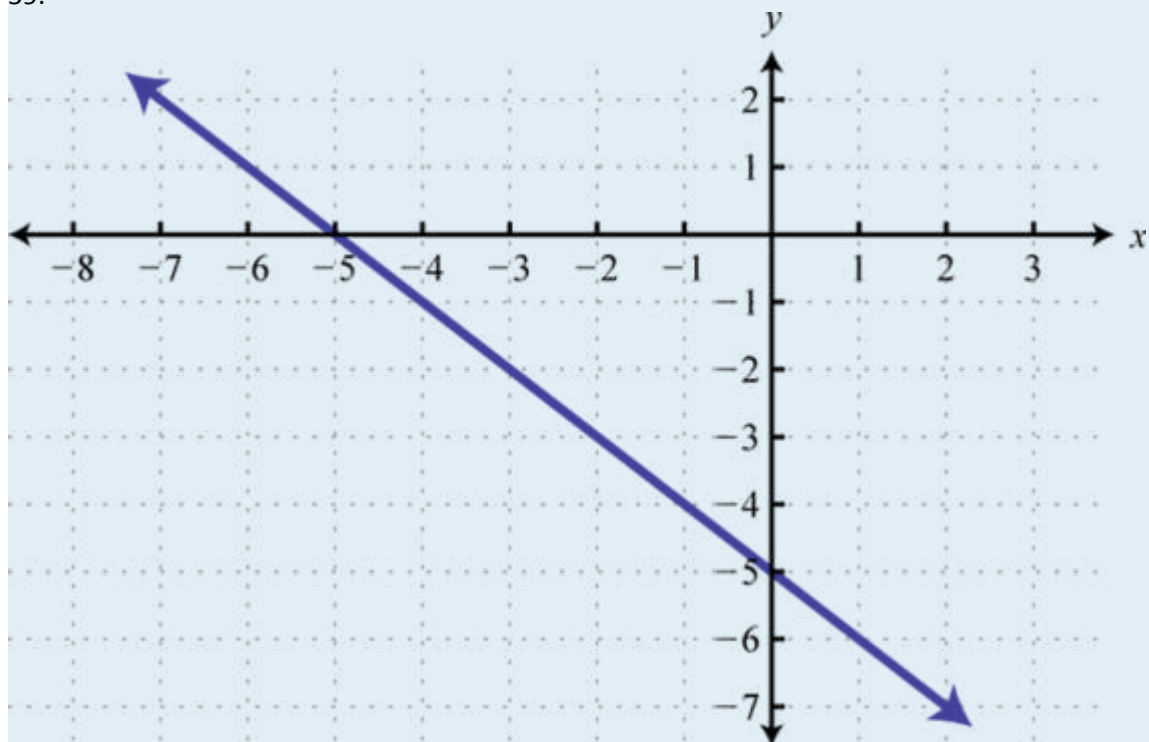
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57. f

58.

59.



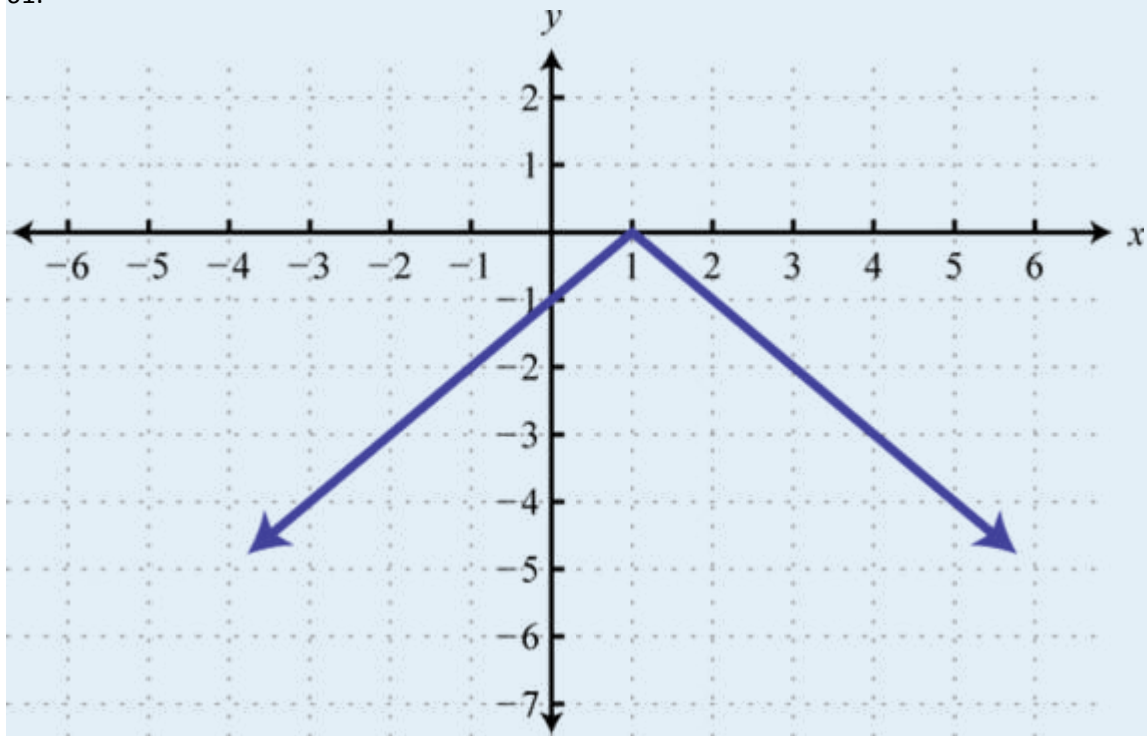
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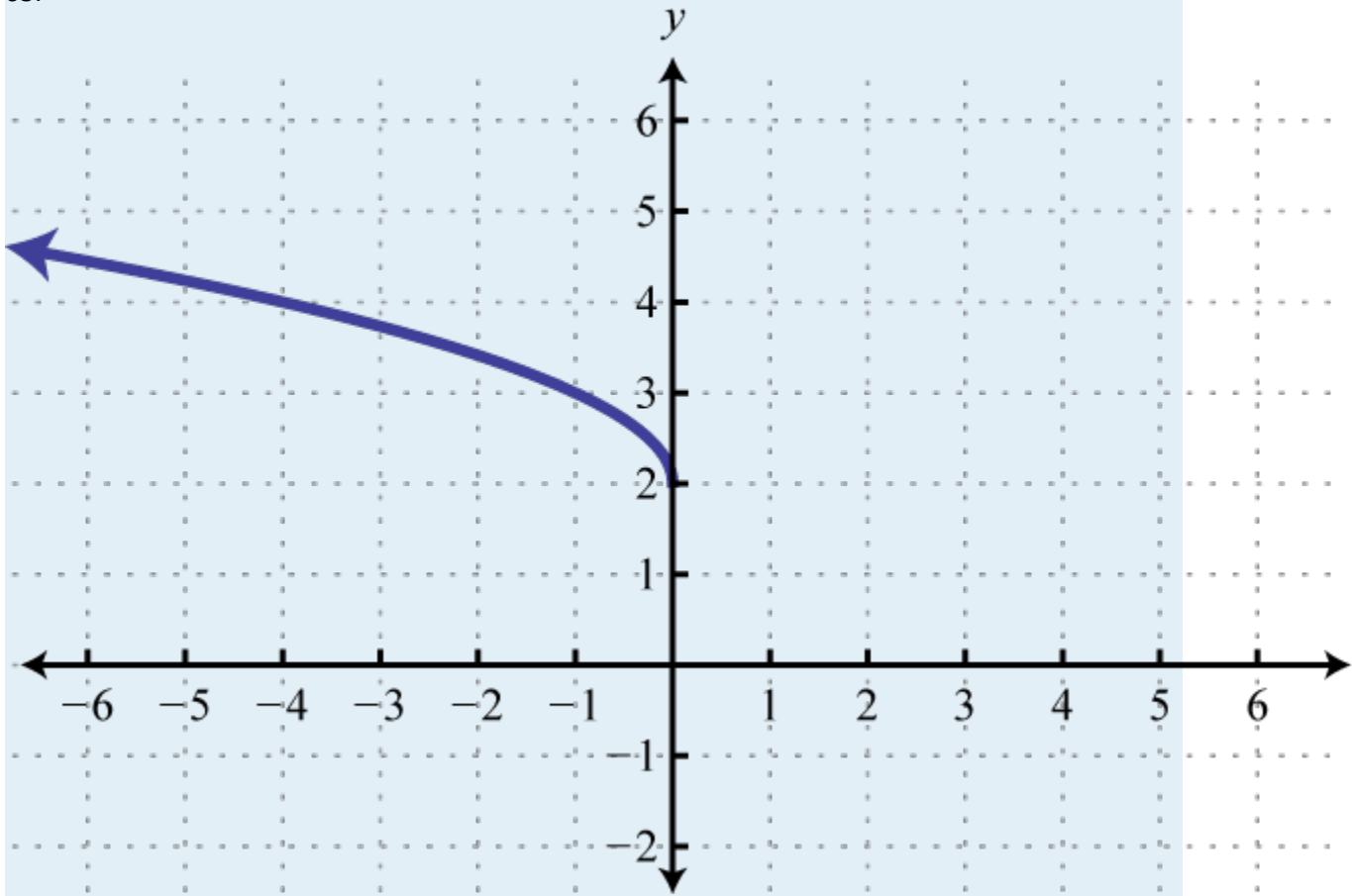


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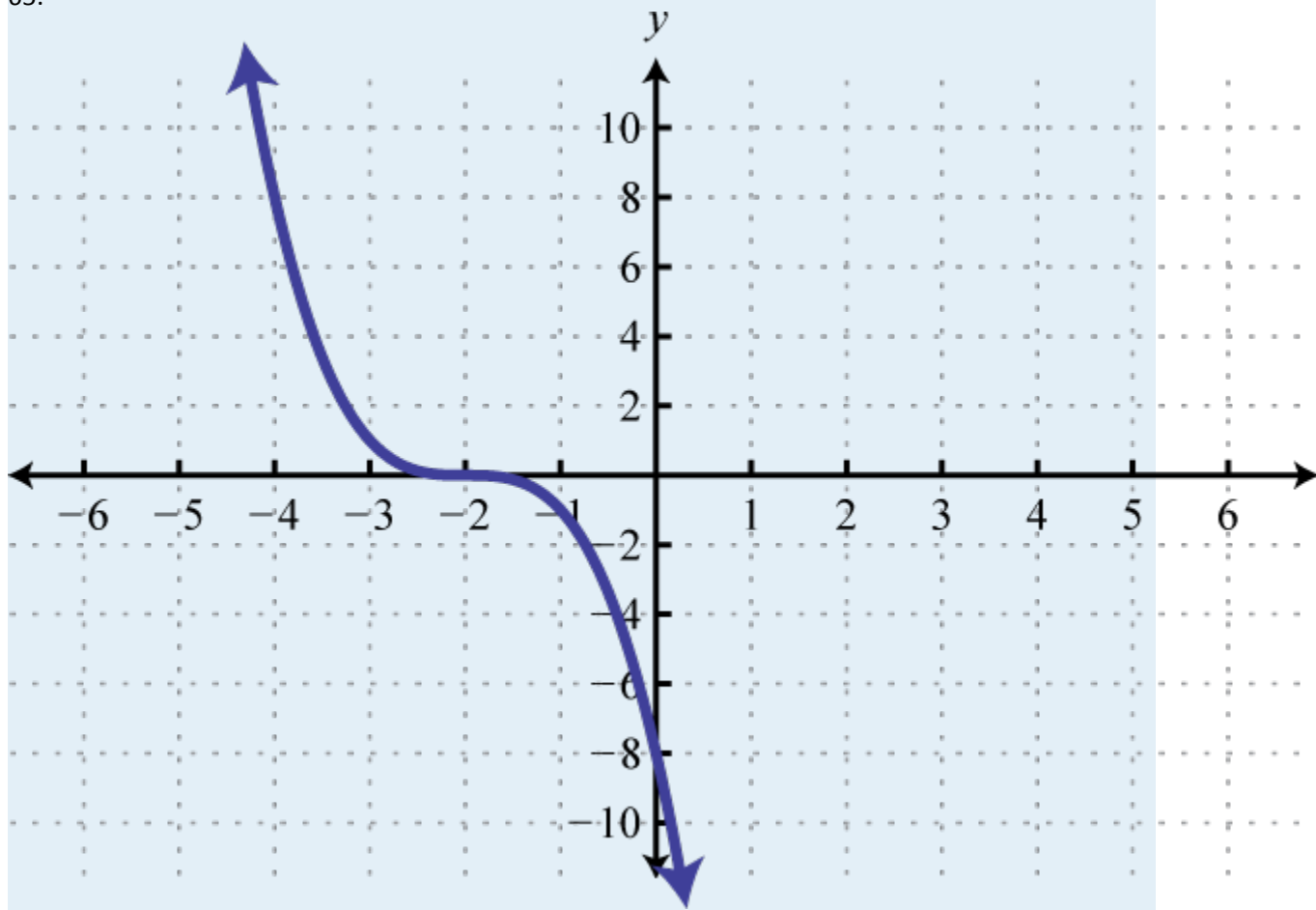
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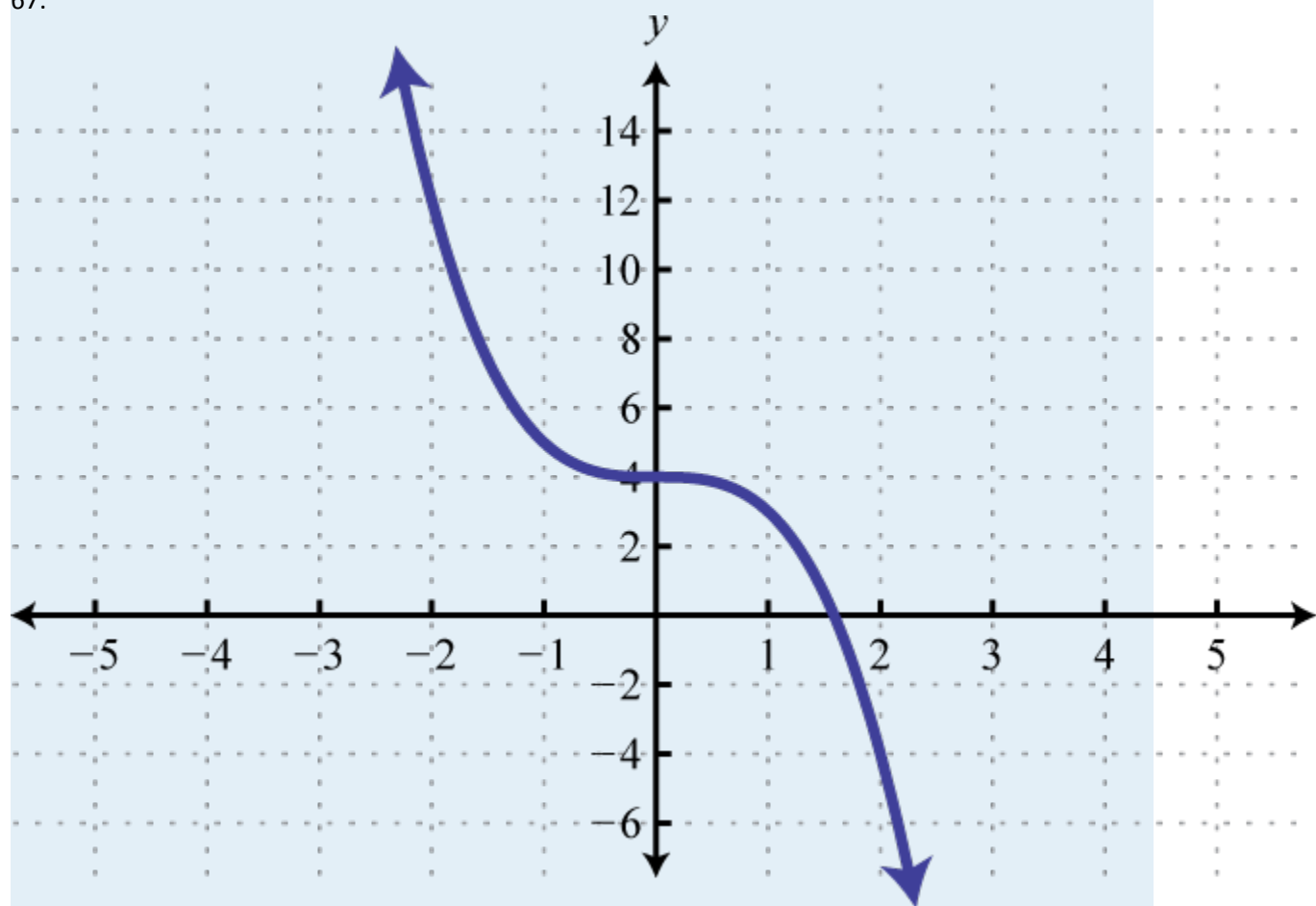


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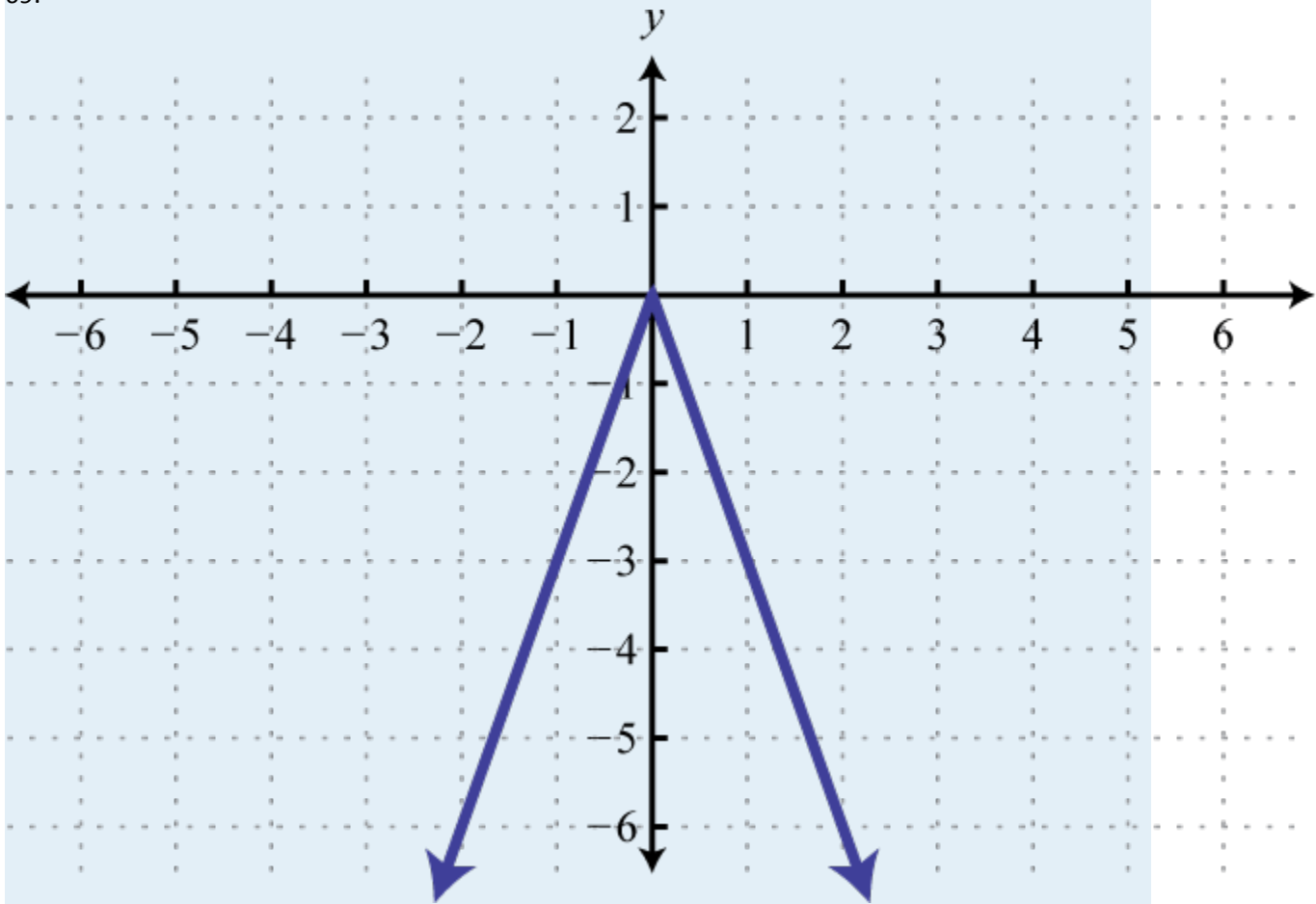


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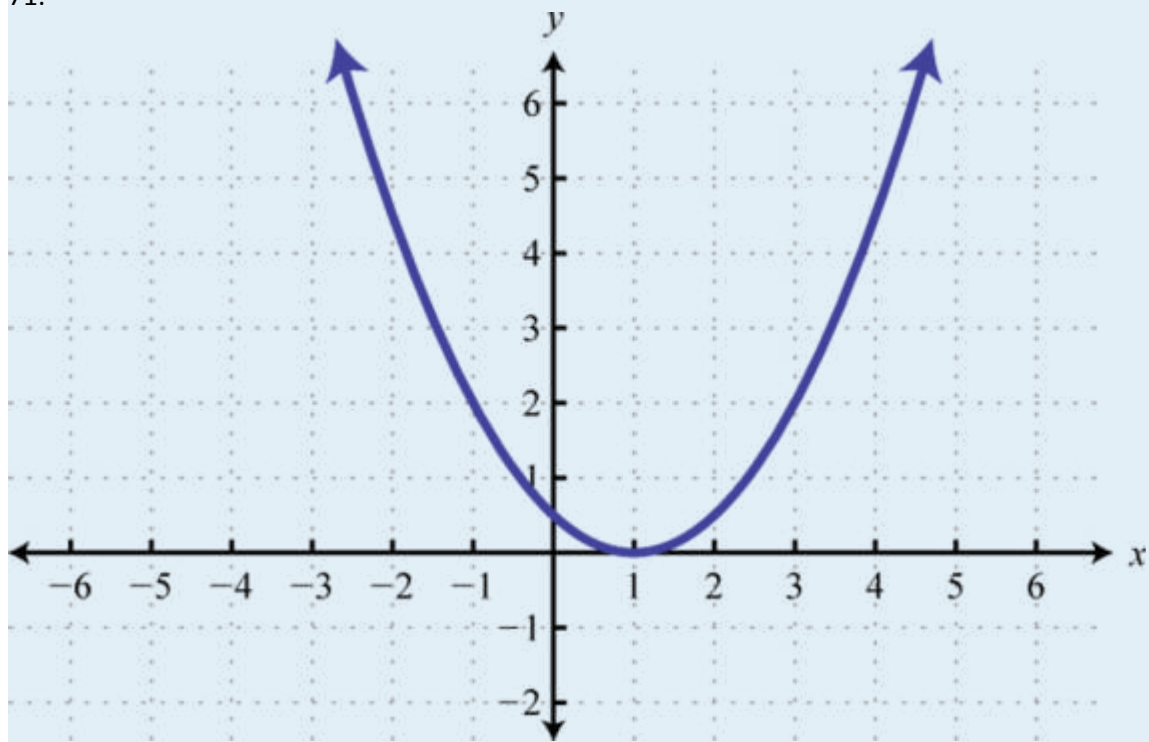


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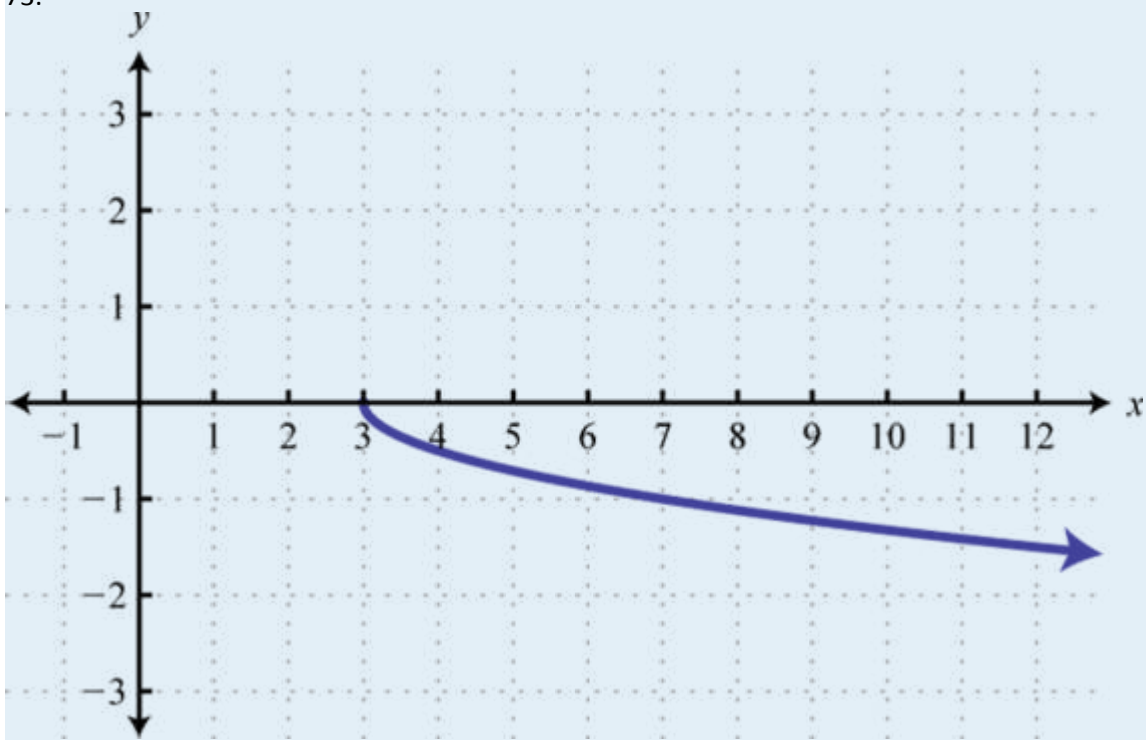


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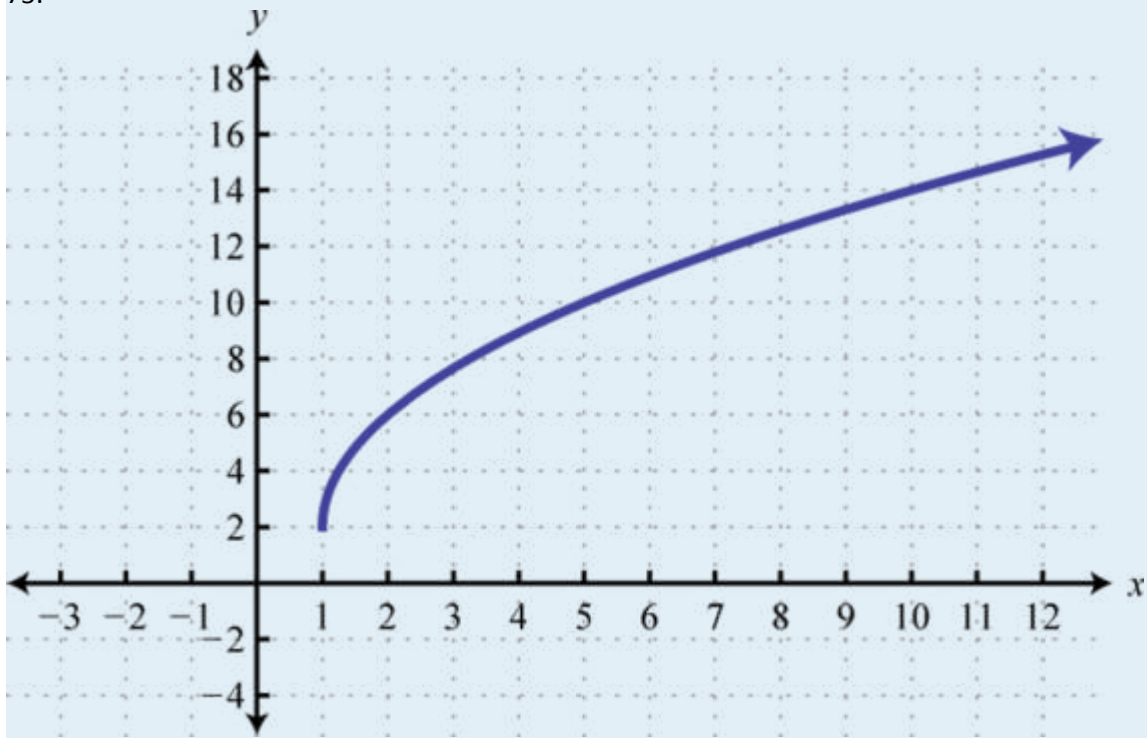


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75.

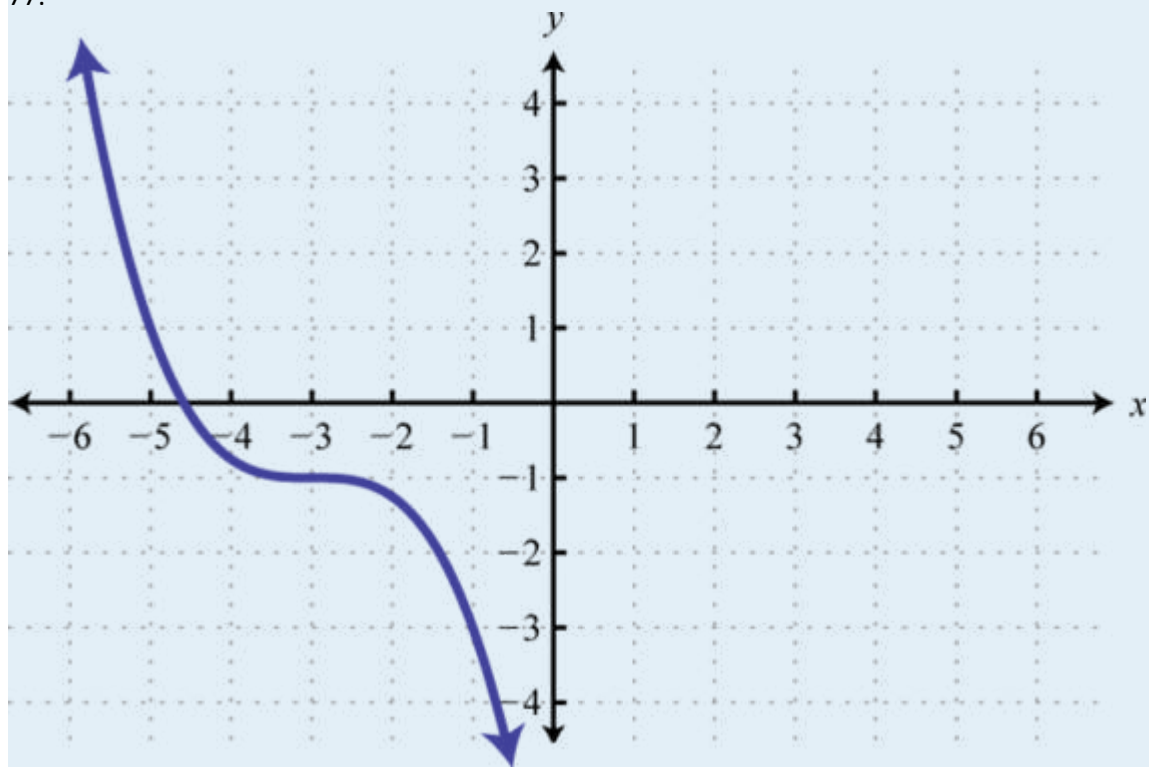


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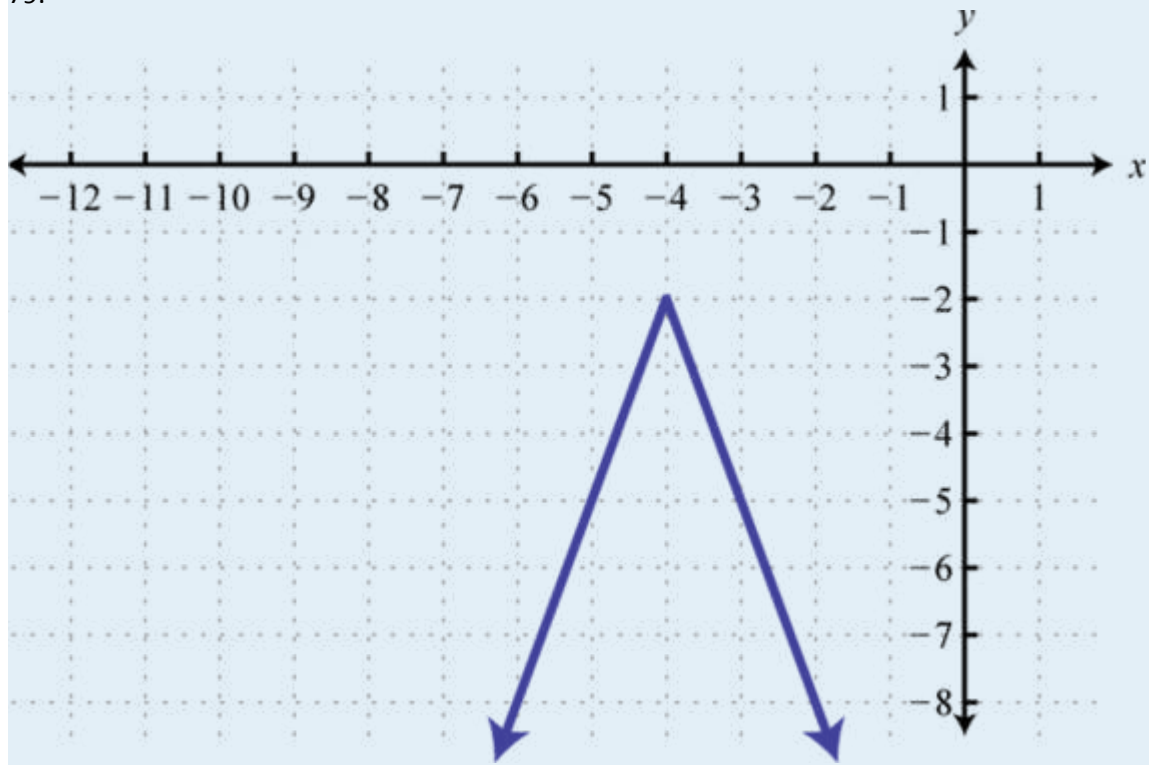


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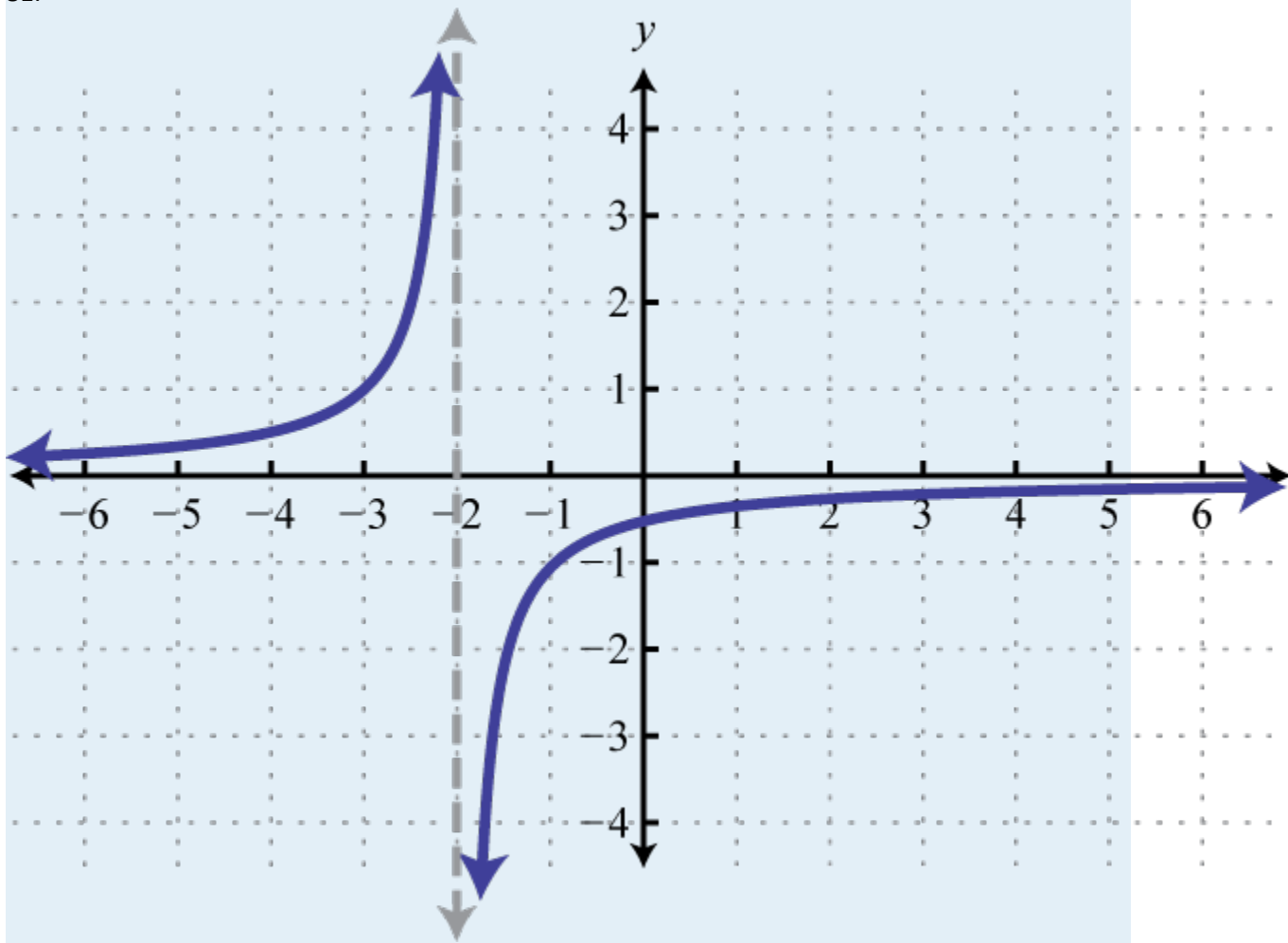
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81.



82.

83. Answer may vary

84.

85. Answer may vary



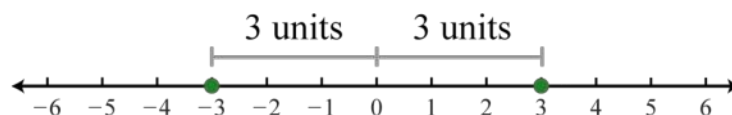
## 2.6 Solving Absolute Value Equations and Inequalities

### LEARNING OBJECTIVES

1. Review the definition of absolute value.
2. Solve absolute value equations.
3. Solve absolute value inequalities.

### Absolute Value Equations

Recall that the **absolute value** of a real number  $a$ , denoted  $|a|$ , is defined as the distance between zero (the origin) and the graph of that real number on the number line. For example,  $|-3| = 3$  and  $|3| = 3$ .



In addition, the absolute value of a real number can be defined algebraically as a piecewise function.



Given this definition,  $|3| = 3$  and  $|-3| = -(-3) = 3$ . Therefore, the equation  $|x| = 3$  has two solutions for  $x$ , namely  $\{\pm 3\}$ . In general, given any algebraic expression  $X$  and any positive number  $p$ :

$$\text{If } |X| = p \text{ then } X = -p \text{ or } X = p.$$

In other words, the **argument of the absolute value**  $X$  can be either positive or negative  $p$ . Use this theorem to solve absolute value equations algebraically.

## Example 1

Solve:  $|x + 2| = 3$ .

Solution:

In this case, the argument of the absolute value is  $x + 2$  and must be equal to 3 or  $-3$ .

$$|x + 2| = 3$$

↙  
±3

Therefore, to solve this absolute value equation, set  $x + 2$  equal to  $\pm 3$  and solve each linear equation as usual.

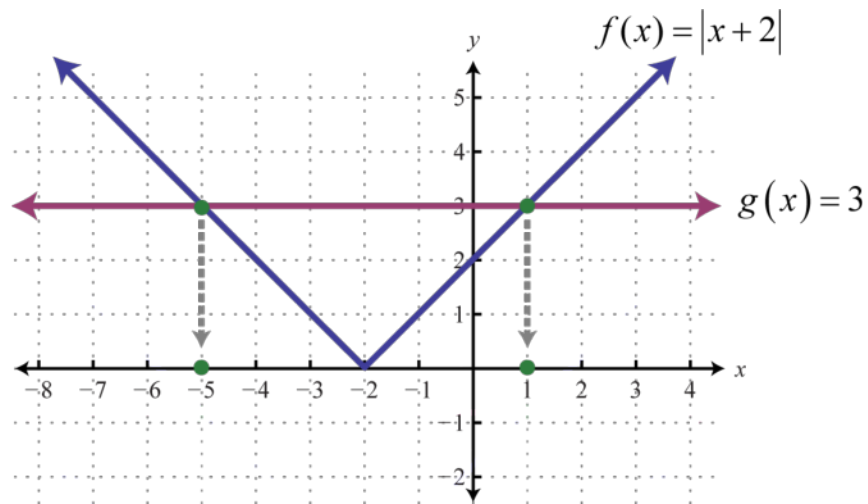
Answer: The solutions are  $-5$  and  $1$ .

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To visualize these solutions, graph the functions on either side of the equal sign on the same set of coordinate axes. In this case,  $f(x) = |x + 2|$  is an absolute value function shifted two units horizontally to the left, and  $g(x) = 3$  is a constant function whose graph is a horizontal line. Determine the  $x$ -values where  $f(x) = g(x)$ .



From the graph we can see that both functions coincide where  $x = -5$  and  $x = 1$ . The solutions correspond to the points of intersection.

## Example 2

Solve:  $|2x + 3| = 4$ .

Solution:

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Here the argument of the absolute value is  $2x + 3$  and can be equal to  $-4$  or  $4$ .

Check to see if these solutions satisfy the original equation.

<i>Check</i>	<i>Check</i>
44444 ✓	

Answer: The solutions are and

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To apply the theorem, the absolute value must be isolated. The general steps for solving absolute value equations are outlined in the following example.

### Example 3

Solve:  $2|5x-1|-3 = 9$ .

Solution:

**Step 1:** Isolate the absolute value to obtain the form  $|X| = p$ .

**Step 2:** Set the argument of the absolute value equal to  $\pm p$ . Here the argument is  $5x-1$  and  $p = 6$ .

$$5x-1 = -6 \text{ or } 5x-1 = 6$$

**Step 3:** Solve each of the resulting linear equations.

**Step 4:** Verify the solutions in the original equation.

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<b>Check</b> $x = -1$	<b>Check</b>

Answer: The solutions are  $-1$  and

**Try this!** Solve:  $2 - 7|x + 4| = -12$ .

Answer:  $-6, -2$

Not all absolute value equations will have two solutions.

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## Example 4

Solve:  $|7x-6| + 3 = 3$ .

Solution:

Begin by isolating the absolute value.

Only zero has the absolute value of zero,  $|0| = 0$ . In other words,  $|X| = 0$  has one solution, namely  $X = 0$ . Therefore, set the argument  $7x-6$  equal to zero and then solve for  $x$ .

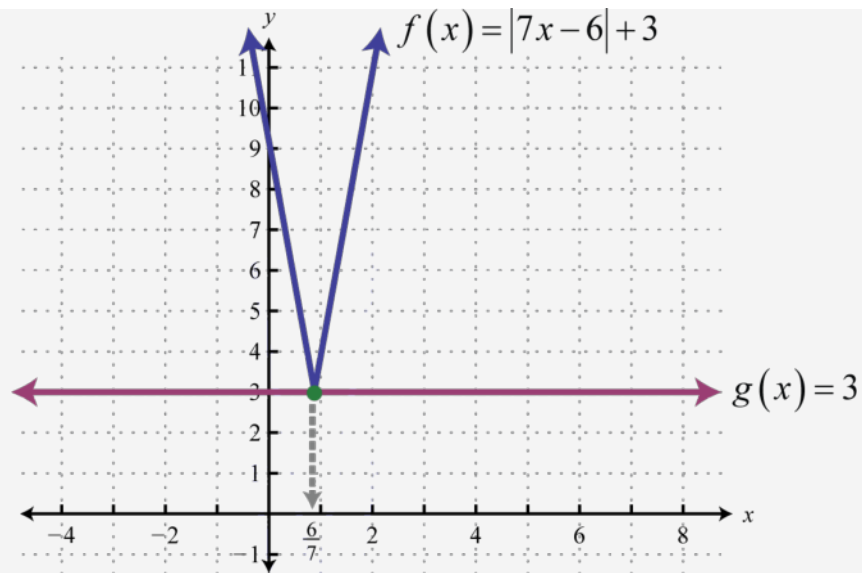
Geometrically, one solution corresponds to one point of intersection.

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Answer: The solution is

## Example 5

Solve:  $|x + 7| + 5 = 4$ .

Solution:

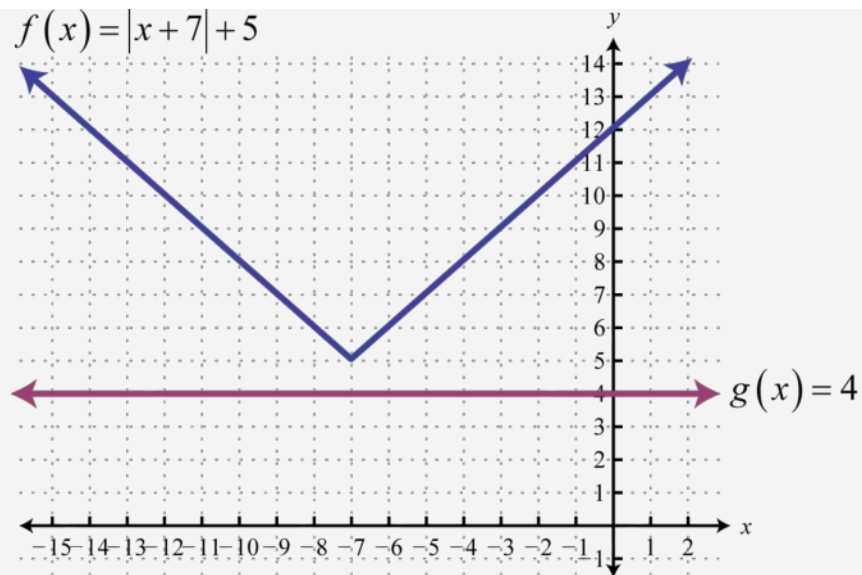
Begin by isolating the absolute value.

In this case, we can see that the isolated absolute value is equal to a negative number. Recall that the absolute value will always be positive. Therefore, we conclude that there is no solution. Geometrically, there is no point of intersection.

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Answer: There is no solution,  $\emptyset$ .

If given an equation with two absolute values of the form  $|a| = |b|$ , then  $b$  must be the same as  $a$  or opposite. For example, if  $a = 5$ , then  $b = \pm 5$  and we have:

$$|5| = |-5| \text{ or } |5| = | + 5|$$

In general, given algebraic expressions  $X$  and  $Y$ :

$$\text{If } |X| = |Y| \text{ then } X = -Y \text{ or } X = Y.$$

In other words, if two absolute value expressions are equal, then the arguments can be the same or opposite.

## Example 6

Solve:  $|2x - 5| = |x - 4|$ .

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Solution:

Set  $2x-5$  equal to  $\pm(x-4)$  and then solve each linear equation.

To check, we substitute these values into the original equation.

<b><i>Check</i></b> $x = 1$	<b><i>Check</i></b> $x = 3$

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As an exercise, use a graphing utility to graph both  $f(x) = |2x-5|$  and  $g(x) = |x-4|$  on the same set of axes. Verify that the graphs intersect where  $x$  is equal to 1 and 3.

Answer: The solutions are 1 and 3.

**Try this!** Solve:  $|x + 10| = |3x-2|$ .

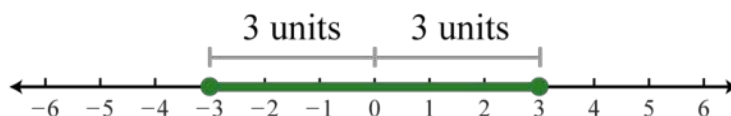
Answer:  $-2, 6$

## Absolute Value Inequalities

We begin by examining the solutions to the following inequality:

$$|x| \leq 3$$

The absolute value of a number represents the distance from the origin. Therefore, this equation describes all numbers whose distance from zero is less than or equal to 3. We can graph this solution set by shading all such numbers.



Certainly we can see that there are infinitely many solutions to  $|x| \leq 3$  bounded by  $-3$  and  $3$ . Express this solution set using set notation or interval notation as follows:

In this text, we will choose to express solutions in interval notation. In general, given any algebraic expression  $X$  and any positive number  $p$ :

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If  $|X| \leq p$  then  $-p \leq X \leq p$ .

This theorem holds true for strict inequalities as well. In other words, we can convert any absolute value inequality involving “less than” into a compound inequality which can be solved as usual.

## Example 7

Solve and graph the solution set:  $|x + 2| < 3$ .

Solution:

Bound the argument  $x + 2$  by  $-3$  and  $3$  and solve.

Here we use open dots to indicate strict inequalities on the graph as follows.



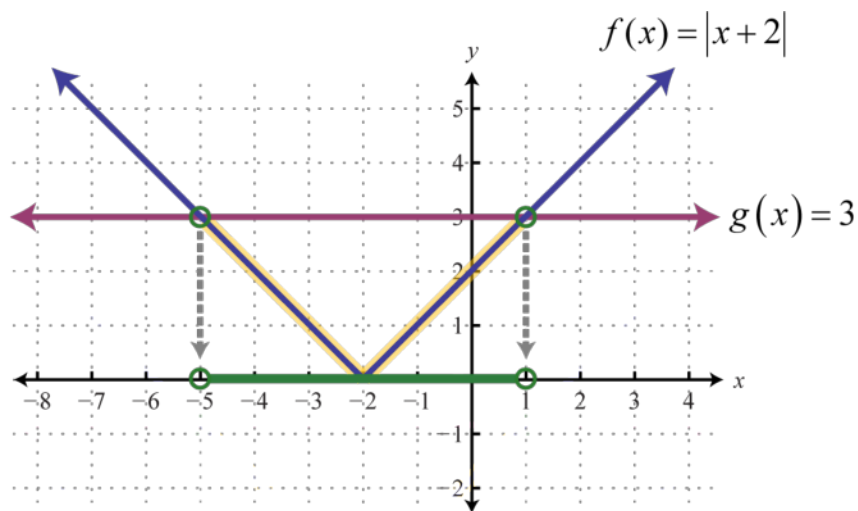
Answer: Using interval notation,  $(-5, 1)$ .

The solution to  $|x + 2| < 3$  can be interpreted graphically if we let  $f(x) = |x + 2|$  and  $g(x) = 3$  and then determine where  $f(x) < g(x)$  by graphing both  $f$  and  $g$  on the same set of axes.

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The solution consists of all  $x$ -values where the graph of  $f$  is below the graph of  $g$ . In this case, we can see that  $|x + 2| < 3$  where the  $x$ -values are between  $-5$  and  $1$ . To apply the theorem, we must first isolate the absolute value.

### Example 8

Solve:  $4|x + 3| - 7 \leq 5$ .

Solution:

Begin by isolating the absolute value.

Next, apply the theorem and rewrite the absolute value inequality as a compound inequality.

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Solve.

Shade the solutions on a number line and present the answer in interval notation. Here we use closed dots to indicate inclusive inequalities on the graph as follows:



Answer: Using interval notation,  $[-6, 0]$

**Try this!** Solve and graph the solution set:  $3 + |4x-5| < 8$ .

Answer: Interval notation:



Next, we examine the solutions to an inequality that involves “*greater than,*” as in the following example:

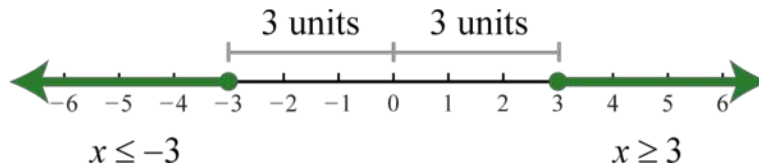
$$|x| \geq 3$$

This inequality describes all numbers whose distance from the origin is greater than or equal to 3. On a graph, we can shade all such numbers.

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There are infinitely many solutions that can be expressed using set notation and interval notation as follows:

In general, given any algebraic expression  $X$  and any positive number  $p$ :

$$\text{If } |X| \geq p \text{ then } X \leq -p \text{ or } X \geq p.$$

The theorem holds true for strict inequalities as well. In other words, we can convert any absolute value inequality involving “*greater than*” into a compound inequality that describes two intervals.

## Example 9

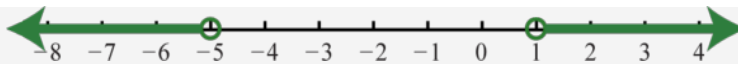
Solve and graph the solution set:  $|x + 2| > 3$ .

Solve

The argument  $x + 2$  must be less than  $-3$  or greater than  $3$ .

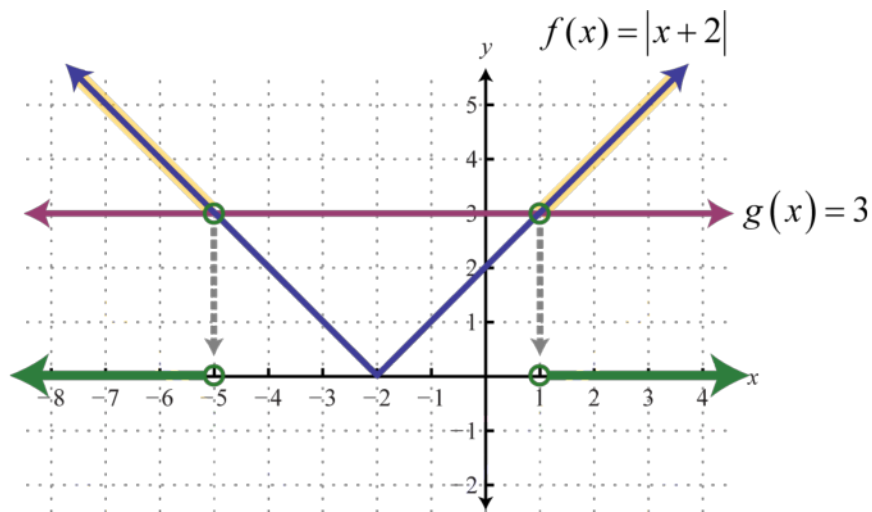






Answer: Using interval notation,  $(-\infty, -5) \cup (1, \infty)$ .

The solution to  $|x + 2| > 3$  can be interpreted graphically if we let  $f(x) = |x + 2|$  and  $g(x) = 3$  and then determine where  $f(x) > g(x)$  by graphing both  $f$  and  $g$  on the same set of axes.



The solution consists of all  $x$ -values where the graph of  $f$  is above the graph of  $g$ . In this case, we can see that  $|x + 2| > 3$  where the  $x$ -values are less than  $-5$  or are greater than  $1$ . To apply the theorem we must first isolate the absolute value.

## Example 10

Solve:  $3 + 2|4x - 7| \geq 13$ .

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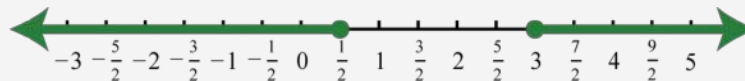
Solution:

Begin by isolating the absolute value.

Next, apply the theorem and rewrite the absolute value inequality as a compound inequality.

Solve.

Shade the solutions on a number line and present the answer using interval notation.



Answer: Using interval notation,

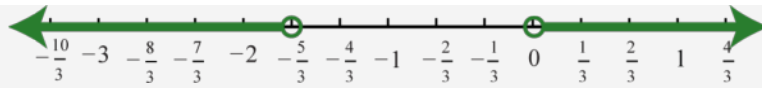
**Try this!** Solve and graph:  $3|6x + 5| - 2 > 13$ .

Answer: Using interval notation,

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Up to this point, the solution sets of linear absolute value inequalities have consisted of a single bounded interval or two unbounded intervals. This is not always the case.

### Example 11

Solve and graph:  $|2x-1| + 5 > 2$ .

Solution:

Begin by isolating the absolute value.

Notice that we have an absolute value greater than a negative number. For any real number  $x$  the absolute value of the argument will always be positive. Hence, any real number will solve this inequality.

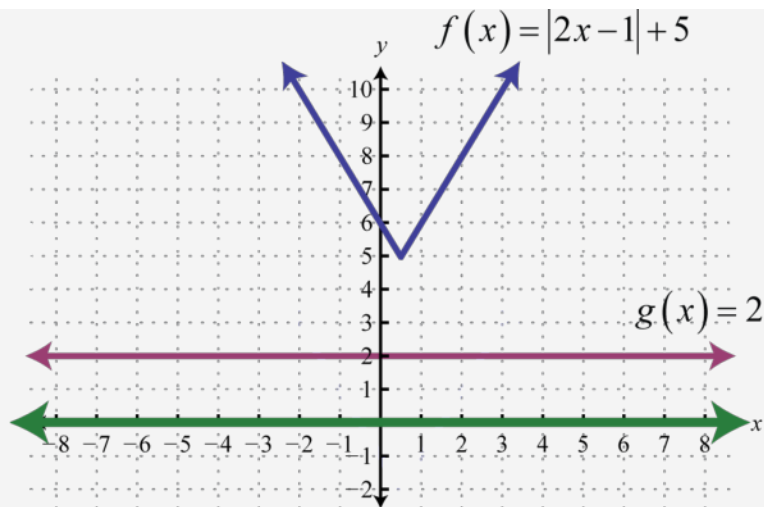


Geometrically, we can see that  $f(x) = |2x-1| + 5$  is always greater than  $g(x) = 2$ .

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Answer: All real numbers, .

### Example 12

Solve and graph:  $|x + 1| + 4 \leq 3$ .

Solution:

Begin by isolating the absolute value.

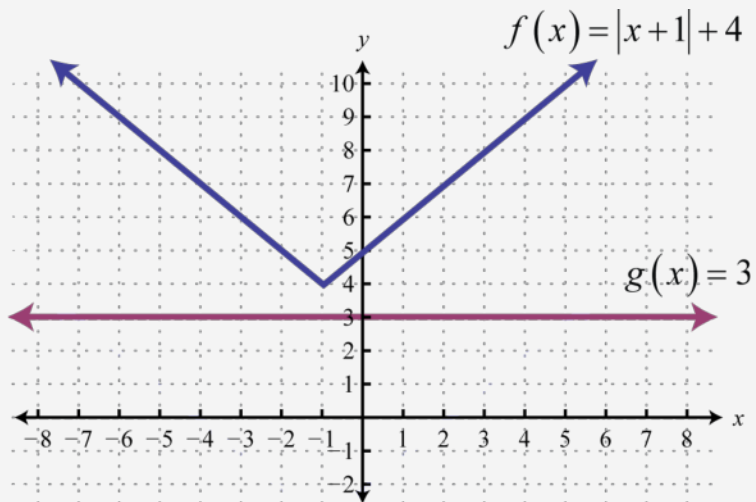
In this case, we can see that the isolated absolute value is to be less than or equal to a negative number. Again, the absolute value will always be positive; hence, we can conclude that there is no solution.

Geometrically, we can see that  $f(x) = |x + 1| + 4$  is never less than  $g(x) = 3$ .

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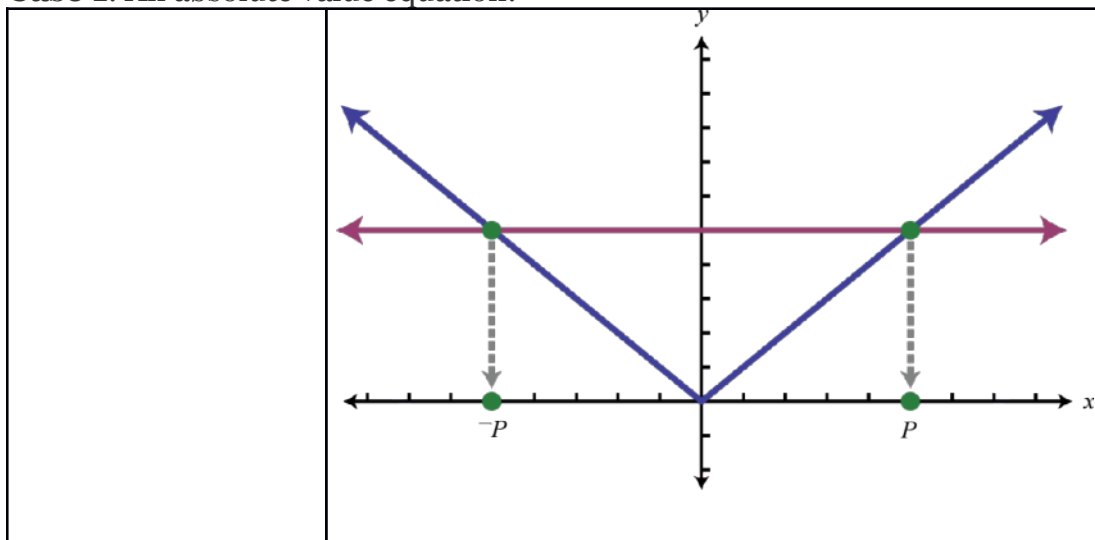




Answer:  $\emptyset$

In summary, there are three cases for absolute value equations and inequalities. The relations  $=$ ,  $<$ ,  $\leq$ ,  $>$ , and  $\geq$  determine which theorem to apply.

**Case 1: An absolute value equation:**

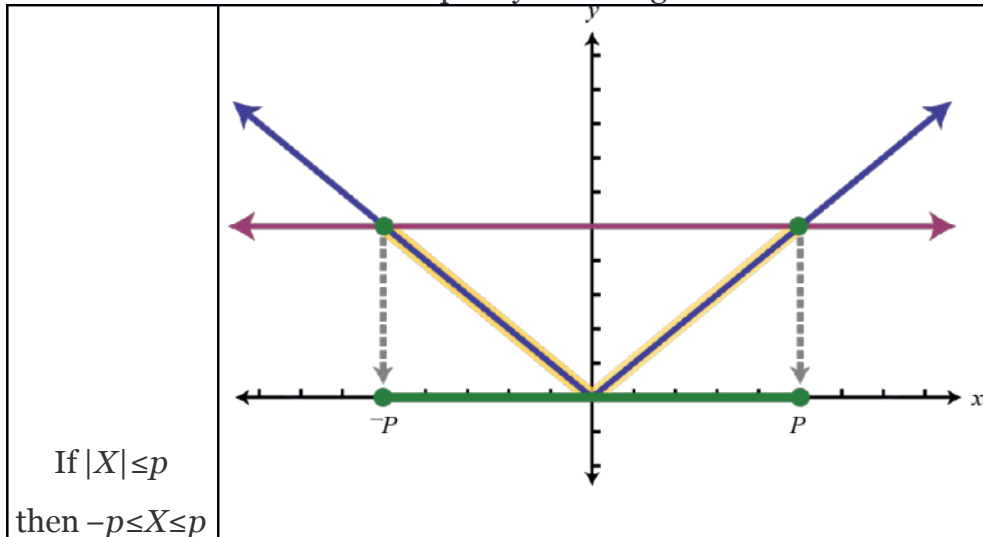


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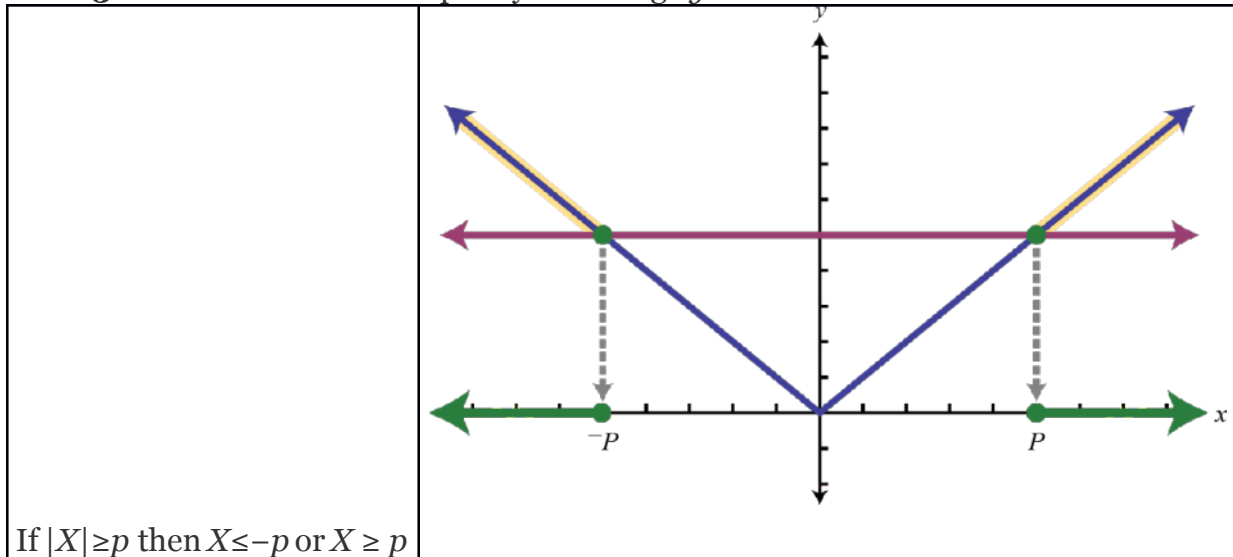
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**Case 2:** An absolute value inequality involving “less than.”



**Case 3:** An absolute value inequality involving “greater than.”



## KEY TAKEAWAYS

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To solve an absolute value equation, such as  $|X| = p$ , replace it with the two equations  $X = -p$  and  $X = p$  and then solve each as usual. Absolute value equations can have up to two solutions.

To solve an absolute value inequality involving “less than,” such as  $|X| \leq p$ , replace it with the compound inequality  $-p \leq X \leq p$  and then solve as usual.

To solve an absolute value inequality involving “greater than,” such as  $|X| \geq p$ , replace it with the compound inequality  $X \leq -p$  or  $X \geq p$  and then solve as usual.

Remember to isolate the absolute value before applying these theorems.

## TOPIC EXERCISES

### PART A: ABSOLUTE VALUE EQUATIONS SOLVE.

1.  $|x| = 9$
2.  $|x| = 1$
3.  $|x-7| = 3$
4.  $|x-2| = 5$
5.  $|x + 12| = 0$
6.  $|x + 8| = 0$
7.  $|x + 6| = -1$
8.  $|x-2| = -5$
9.  $|2y-1| = 13$
10.  $|3y-5| = 16$
11.  $|-5t + 1| = 6$

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12.  $|-6t + 2| = 8$
- 13.
- 14.
15.  $|0.2x + 1.6| = 3.6$
16.  $|0.3x - 1.2| = 2.7$
17.  $|5(y - 4) + 5| = 15$
18.  $|2(y - 1) - 3y| = 4$
19.  $|5x - 7| + 3 = 10$
20.  $|3x - 8| - 2 = 6$
21.  $9 + |7x + 1| = 9$
22.  $4 - |2x - 3| = 4$
23.  $3|x - 8| + 4 = 25$
24.  $2|x + 6| - 3 = 17$
25.  $9 + 5|x - 1| = 4$
26.  $11 + 6|x - 4| = 5$
27.  $8 - 2|x + 1| = 4$
28.  $12 - 5|x - 2| = 2$
- 29.
- 30.
31.  $-2|7x + 1| - 4 = 2$
32.  $-3|5x - 3| + 2 = 5$

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33.  $1.2|t-2.8|-4.8 = 1.2$

34.  $3.6|t + 1.8|-2.6 = 8.2$

35.

36.

37.  $|5x-7| = |4x-2|$

38.  $|8x-3| = |7x-12|$

39.  $|5y + 8| = |2y + 3|$

40.  $|7y + 2| = |5y-2|$

41.  $|5(x-2)| = |3x|$

42.  $|3(x + 1)| = |7x|$

43.

44.

45.  $|1.5t-3.5| = |2.5t + 0.5|$

46.  $|3.2t-1.4| = |1.8t + 2.8|$

47.  $|5-3(2x + 1)| = |5x + 2|$

48.  $|3-2(3x-2)| = |4x-1|$

**Assume all variables in the denominator are nonzero.**

1. Solve for  $x$ :  $p|ax + b|-q = 0$

2. Solve for  $x$ :  $|ax + b| = |p + q|$

## **PART B: ABSOLUTE VALUE INEQUALITIES**

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**Solve and graph the solution set. In addition, give the solution set in interval notation.**

51.  $|x| < 5$

52.  $|x| \leq 2$

53.  $|x + 3| \leq 1$

54.  $|x - 7| < 8$

55.  $|x - 5| < 0$

56.  $|x + 8| < -7$

57.  $|2x - 3| \leq 5$

58.  $|3x - 9| < 27$

59.  $|5x - 3| \leq 0$

60.  $|10x + 5| < 25$

61.

62.

63.  $|x| \geq 5$

64.  $|x| > 1$

65.  $|x + 2| > 8$

66.  $|x - 7| \geq 11$

67.  $|x + 5| \geq 0$

68.  $|x - 12| > -4$

69.  $|2x - 5| \geq 9$

70.  $|2x + 3| \geq 15$

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71.  $|4x-3|>9$

72.  $|3x-7|\geq 2$

73.

74.

75.

**Solve and graph the solution set.**

51.  $|3(2x-1)|>15$

52.  $|3(x-3)|\leq 21$

53.  $-5|x-4|>-15$

54.  $-3|x+8|\leq -18$

55.  $6-3|x-4|<3$

56.  $5-2|x+4|\leq -7$

57.  $6-|2x+5|<-5$

58.  $25-|3x-7|\geq 18$

59.  $|2x+25|-4\geq 9$

60.  $|3(x-3)|-8<-2$

61.  $2|9x+5|+8>6$

62.  $3|4x-9|+4<-1$

63.  $5|4-3x|-10\leq 0$

64.  $6|1-4x|-24\geq 0$

65.  $3-2|x+7|>-7$

66.  $9-7|x-4|<-12$

67.  $|5(x-4)+5|>15$

68.  $|3(x-9)+6|\leq 3$

69.

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70.

71.  $12 + 4|2x-1| \leq 12$

72.  $3-6|3x-2| \geq 3$

73.  $12|2x-1| + 3 < 4$

74.  $2||12x + 23|| - 3 \leq -1$

75.  $7 - |-4 + 2(3-4x)| > 5$

76.  $9 - |6 + 3(2x-1)| \geq 8$

77.

**Assume all variables in the denominator are nonzero.**

51. Solve for  $x$  where  $a, p > 0$ :  $p|ax + b| - q \leq 0$

52. Solve for  $x$  where  $a, p > 0$ :  $p|ax + b| - q \geq 0$

53.

**Given the graph of  $f$  and  $g$ , determine the  $x$ -values where:**

51.1.  $f(x) = g(x)$

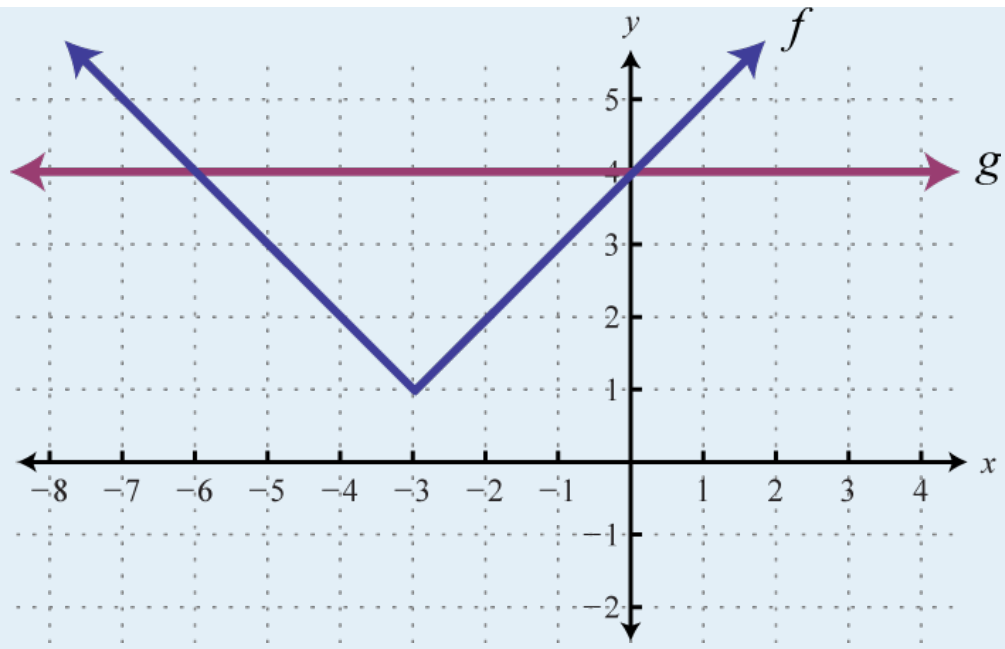
51.2.  $f(x) > g(x)$

51.3.  $f(x) < g(x)$

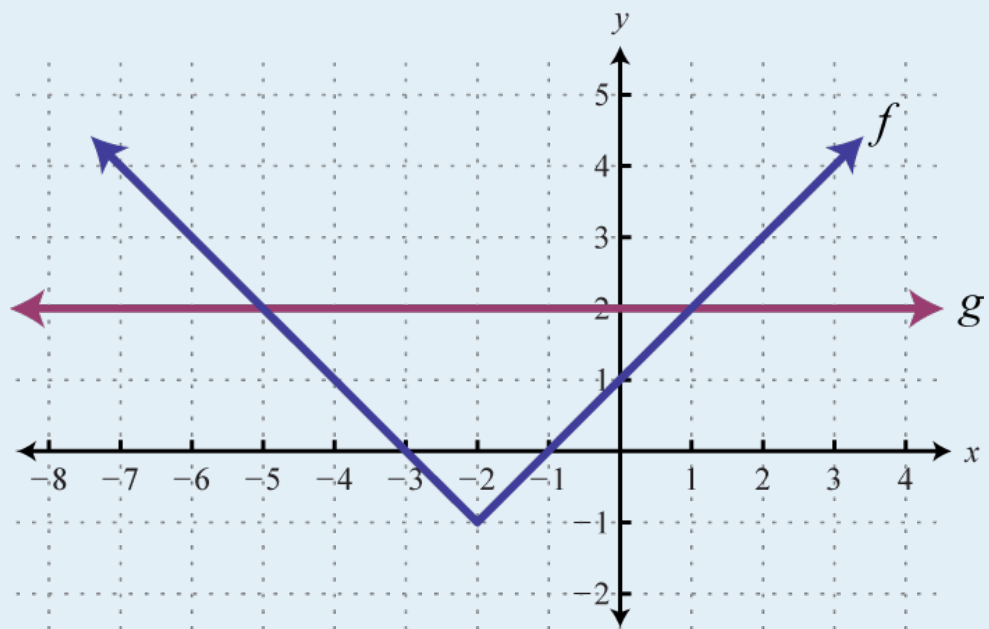
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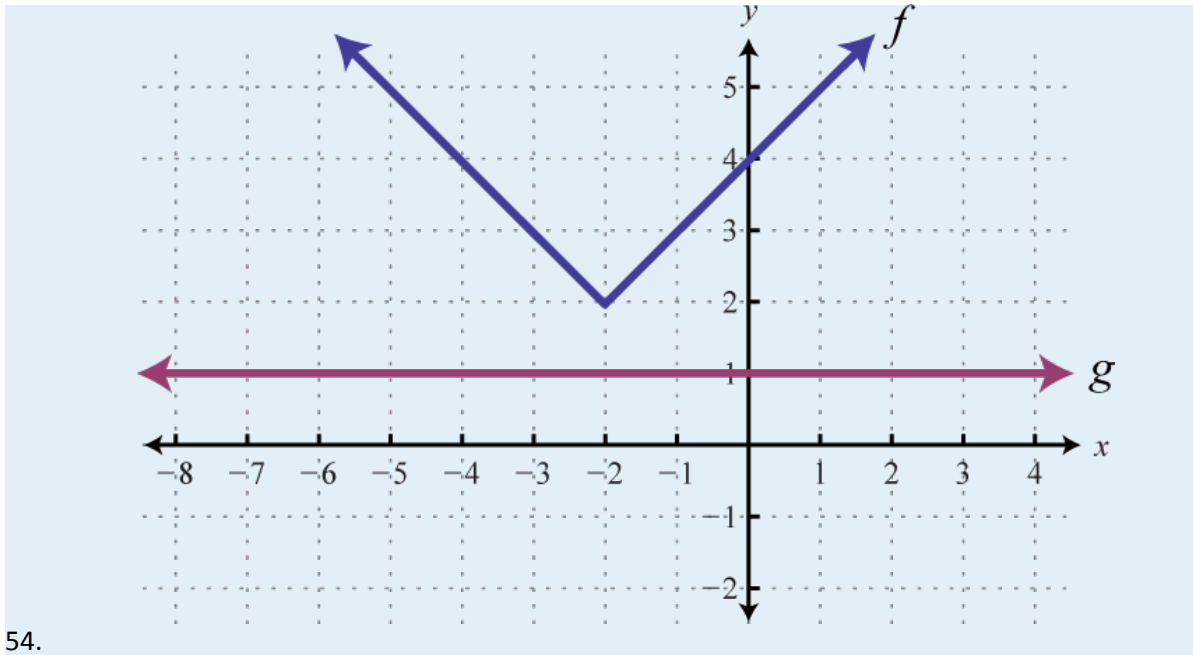


52.



53.



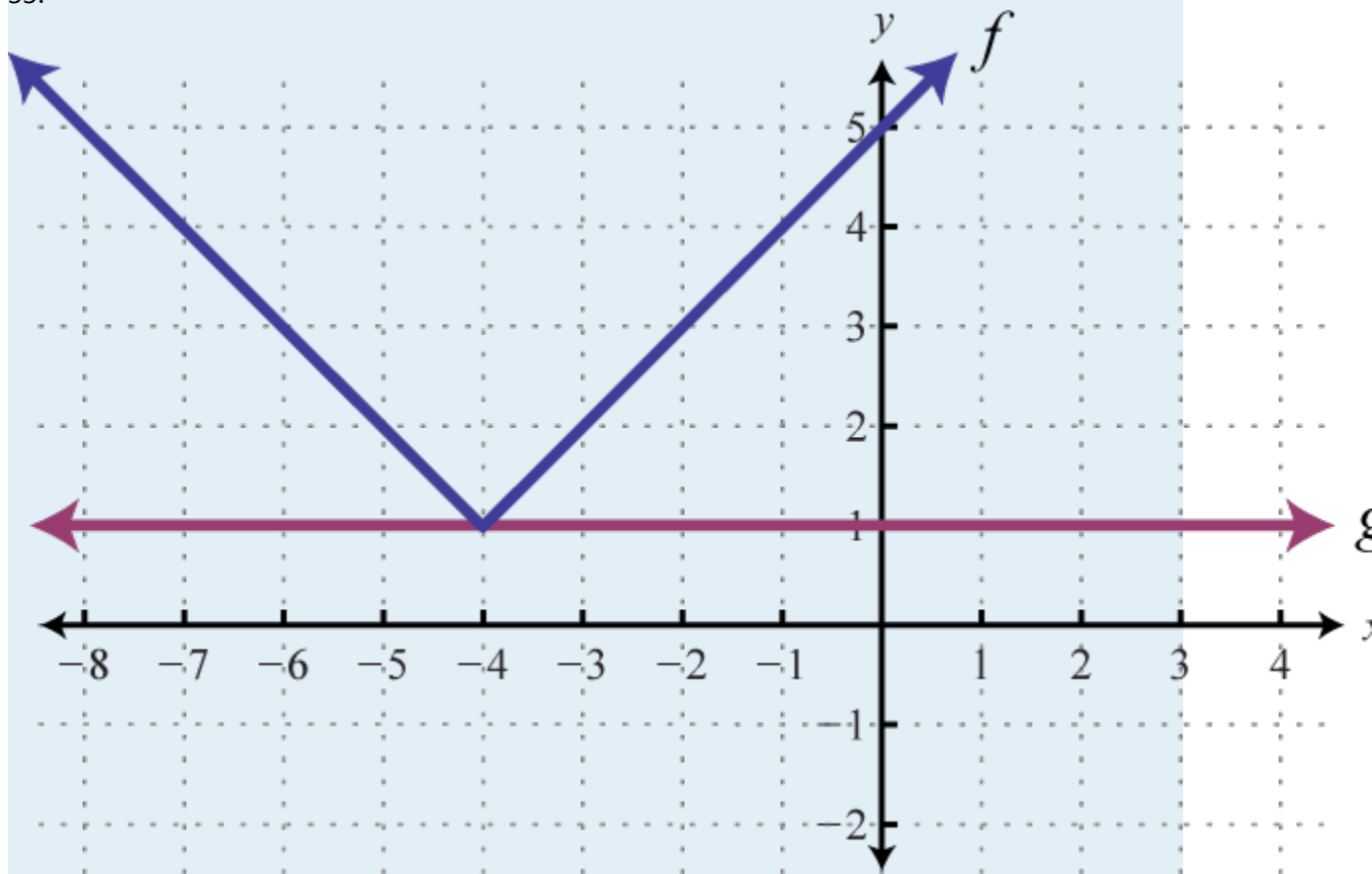


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55.



## PART C: DISCUSSION BOARD

109. Make three note cards, one for each of the three cases described in this section. On one side write the theorem, and on the other write a complete solution to a representative example. Share your strategy for identifying and solving absolute value equations and inequalities on the discussion board.

110. Make your own examples of absolute value equations and inequalities that have no solution, at least one for each case described in this section. Illustrate your examples with a graph.

## ANSWERS

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1.  $-9, 9$
- 2.
3.  $4, 10$
- 4.
5.  $-12$
- 6.
7.  $\emptyset$
- 8.
9.  $-6, 7$
- 10.
11.  $-1,$
- 12.
13.  $1,$
- 14.
15.  $-26, 10$
- 16.
17.  $0, 6$
- 18.
19.  $0,$
- 20.
- 21.
- 22.
23.  $1, 15$
- 24.
25.  $\emptyset$
- 26.

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27.  $-3, 1$

28.

29.  $4, 6$

30.

31.  $\emptyset$

32.

33.  $-2.2, 7.8$

34.

35.

36.

37.  $1, 5$

38.

39.

40.  $, 5$

41.

42.  $, 1$

43.

44.  $-4, 0.75$

45.

46.  $0, 4$

47.

48.

51.  $(-5, 5);$

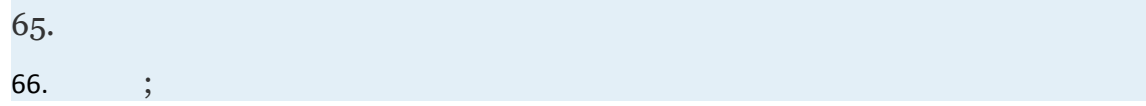
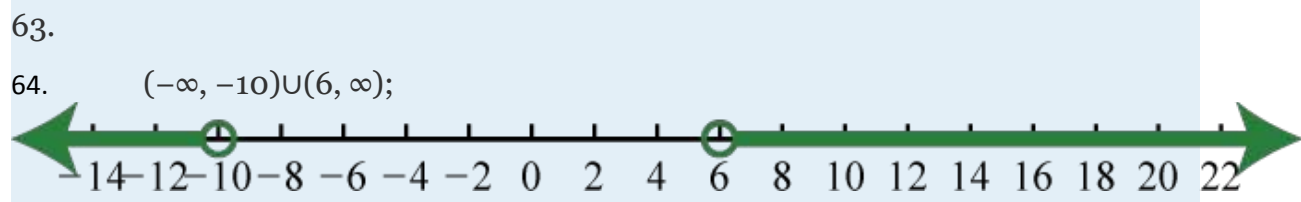
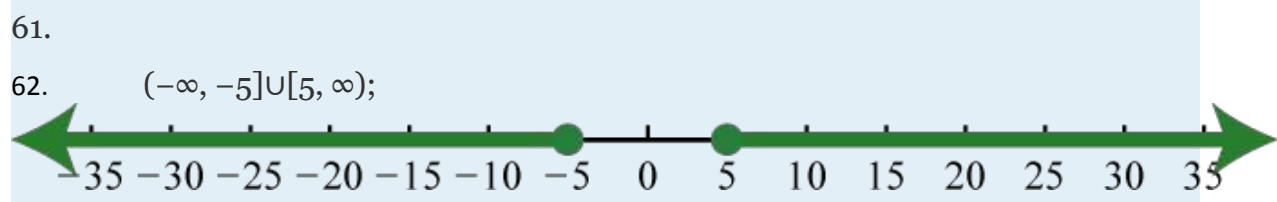
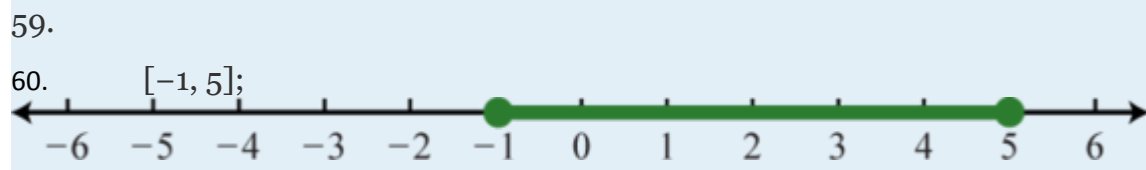
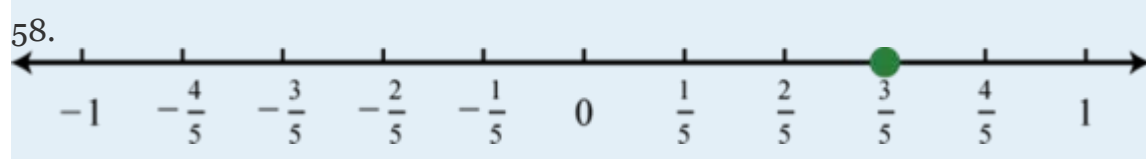
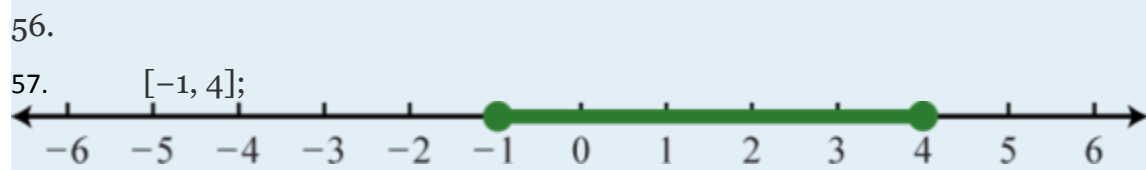
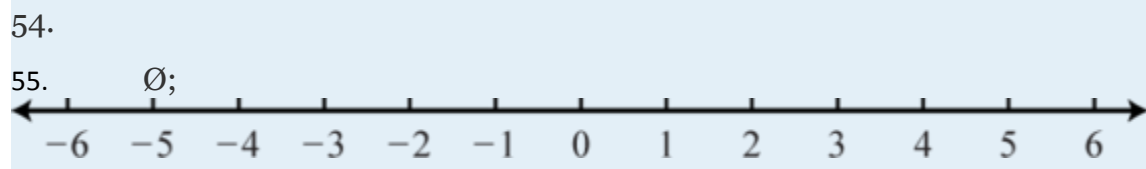
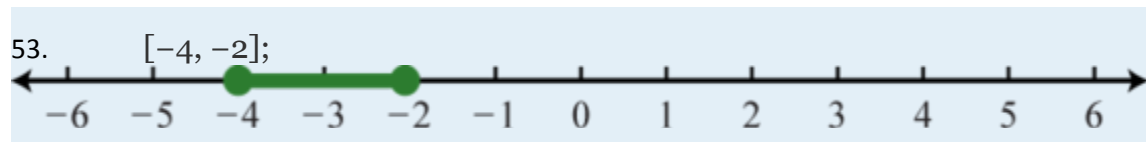


52.

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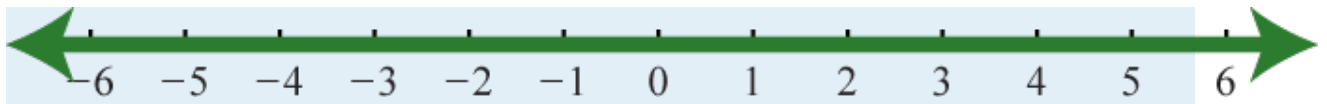
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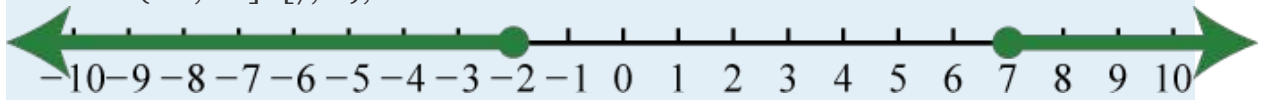
Attributed to John Redden  
 Saylor URL: <http://www.saylor.org/books/>





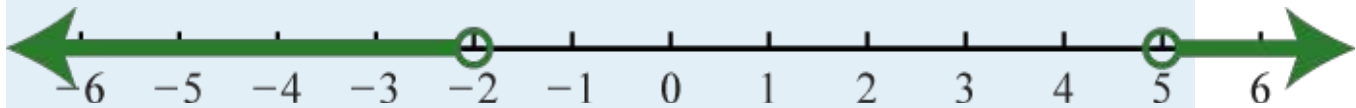
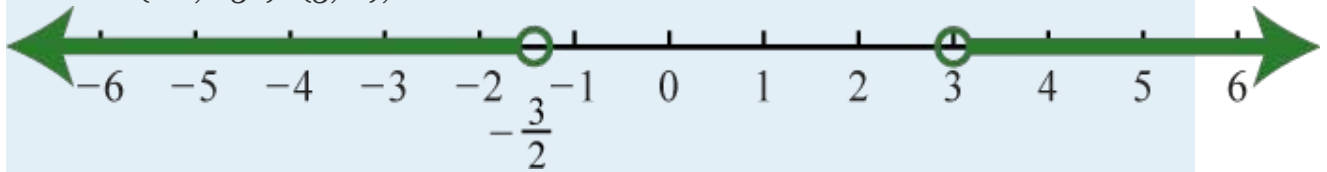
67.

68.  $(-\infty, -2] \cup [7, \infty)$ ;



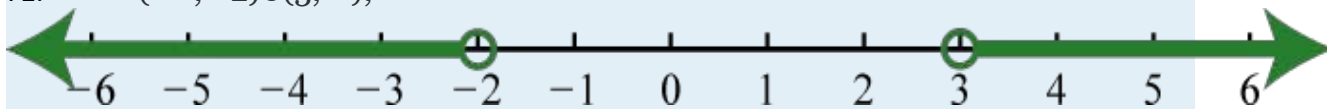
69.

70.  $(-\infty, -\frac{3}{2}) \cup (3, \infty)$ ;



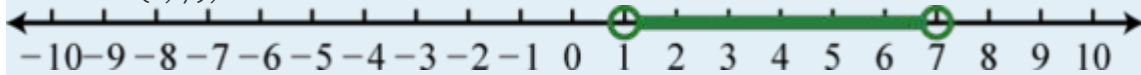
71.

72.  $(-\infty, -2) \cup (3, \infty)$ ;



73.

74.  $(1, 7)$ ;



75.

76.  $(-\infty, 3) \cup (5, \infty)$ ;



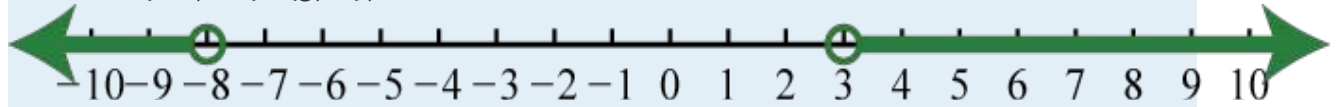
Attributed to John Redden

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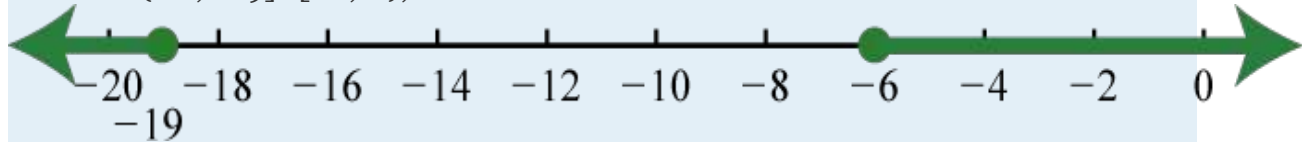
77.

78.  $(-\infty, -8) \cup (3, \infty)$ ;



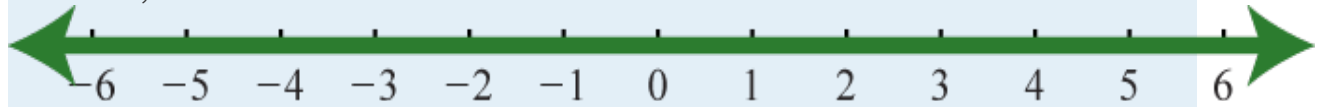
79.

80.  $(-\infty, -19] \cup [-6, \infty)$ ;



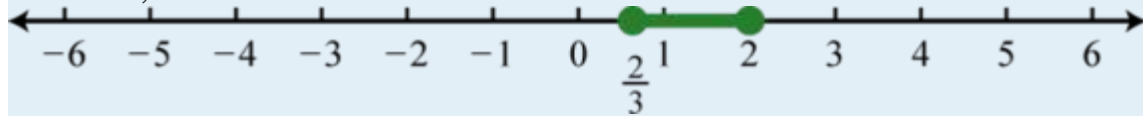
81.

82. ;



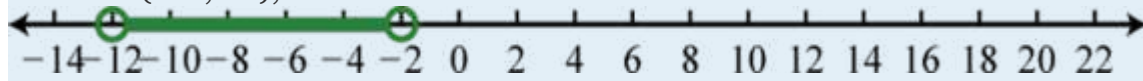
83.

84. ;



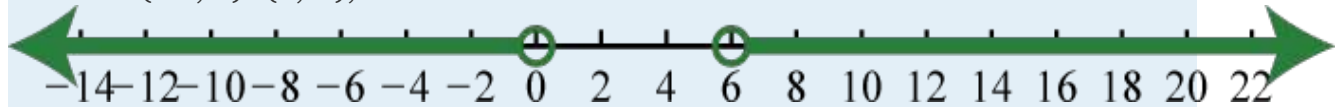
85.

86.  $(-12, -2)$ ;



87.

88.  $(-\infty, 0) \cup (6, \infty)$ ;

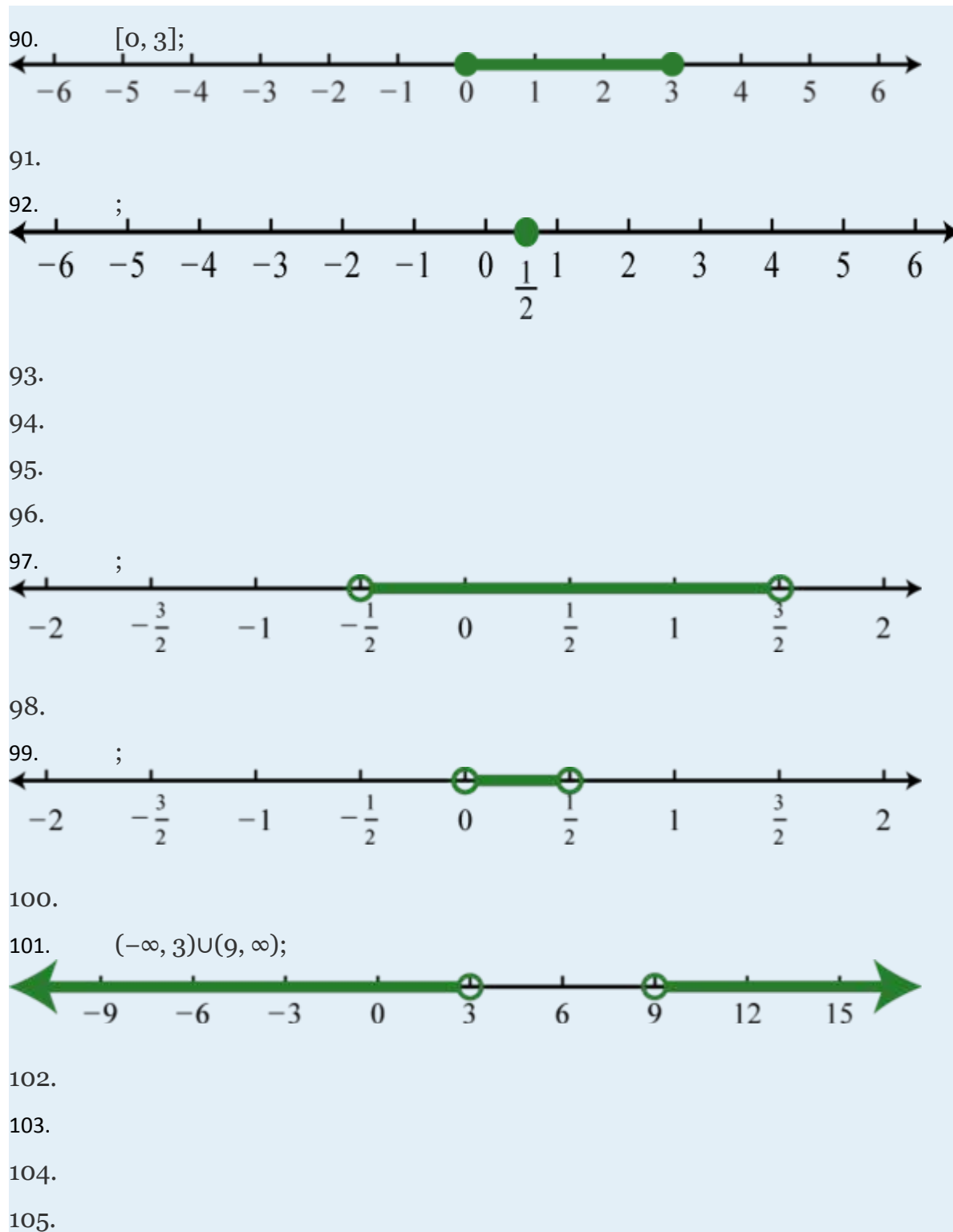


89.

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- a.  $-6, 0$ ;
  - b.  $(-\infty, -6) \cup (0, \infty)$ ;
  - c.  $(-6, 0)$
- 106.
- 107.
- a.  $\emptyset$ ;
  - b.  $;$
  - c.  $\emptyset$
- 108.
109. Answer may vary

## 2.7 Solving Inequalities with Two Variables

### LEARNING OBJECTIVES

1. Identify and check solutions to inequalities with two variables.
2. Graph solution sets of linear inequalities with two variables.

## Solutions to Inequalities with Two Variables

We know that a linear equation with two variables has infinitely many ordered pair solutions that form a line when graphed. A **linear inequality with two variables**, on the other hand, has a solution set consisting of a region that defines half of the plane.

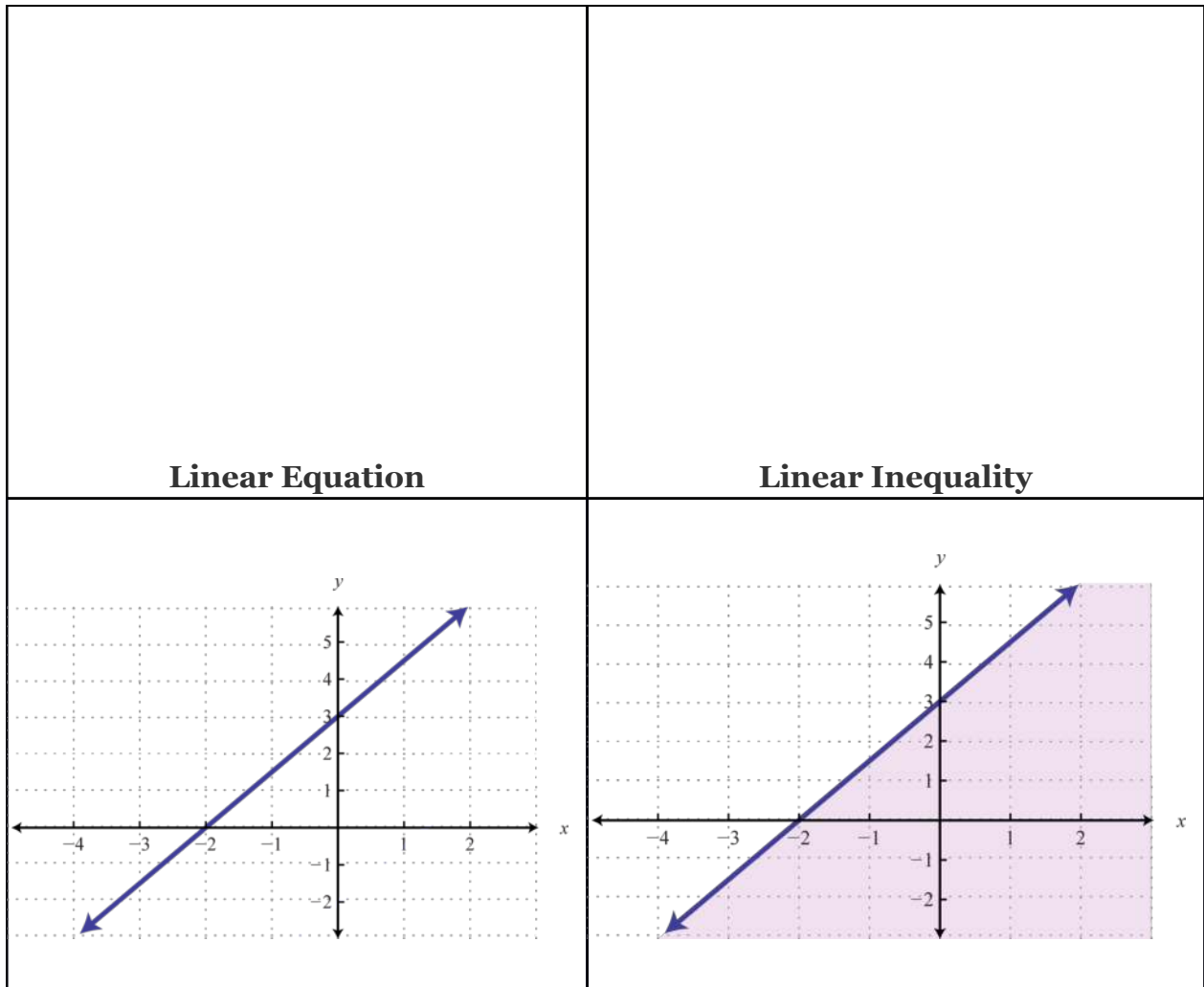
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<p style="text-align: center;"><b>Linear Equation</b></p>	<p style="text-align: center;"><b>Linear Inequality</b></p>





For the inequality, the line defines the boundary of the region that is shaded. This indicates that any ordered pair in the shaded region, including the boundary line, will satisfy the inequality. To see that this is the case, choose a few **test points** and substitute them into the inequality.





Test point	
$(0, 0)$	✓
$(2, 1)$	✓

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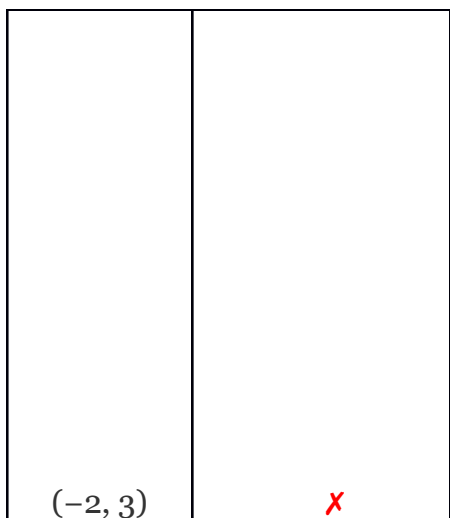


$(-2, -1)$	✓
------------	---

Also, we can see that ordered pairs outside the shaded region do not solve the linear inequality.

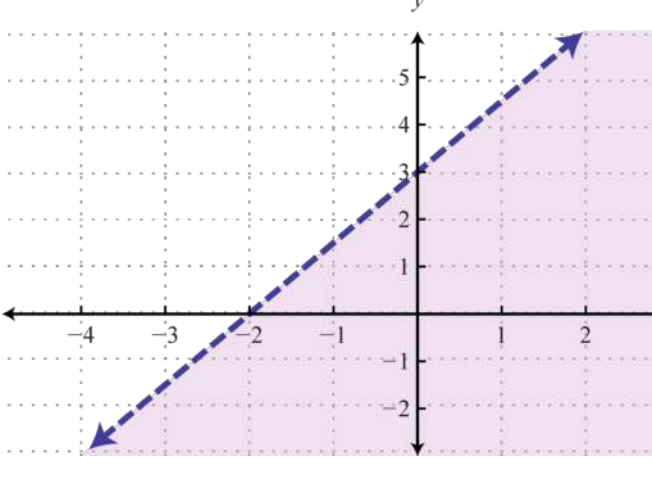
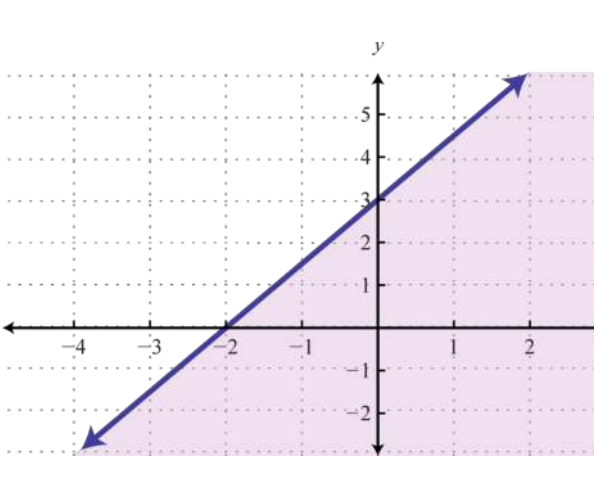
Test point	
------------	--





The graph of the solution set to a linear inequality is always a region. However, the boundary may not always be included in that set. In the previous example, the line was part of the solution set because of the “or equal to” part of the inclusive inequality  $\leq$ . If given a strict inequality  $<$ , we would then use a dashed line to indicate that those points are not included in the solution set.

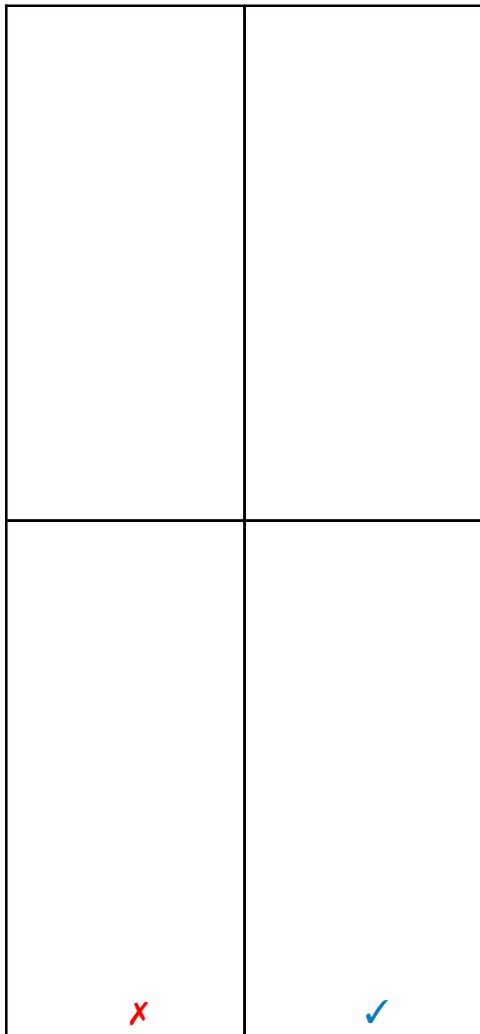


Non-Inclusive Boundary	Inclusive Boundary
 <p>A coordinate plane with x and y axes ranging from -4 to 2. A dashed blue line passes through the points (-2, 0) and (0, 2). The region below and to the right of this line is shaded light purple. The line itself is not shaded, indicating a non-inclusive boundary.</p>	 <p>A coordinate plane with x and y axes ranging from -4 to 2. A solid blue line passes through the points (-2, 0) and (0, 2). The region below and to the right of this line is shaded light purple. The line itself is shaded, indicating an inclusive boundary.</p>

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Consider the point  $(0, 3)$  on the boundary; this ordered pair satisfies the linear equation. It is the “or equal to” part of the inclusive inequality that makes the ordered pair part of the solution set.



So far we have seen examples of inequalities that were “less than.” Now consider the following graphs with the same boundary:

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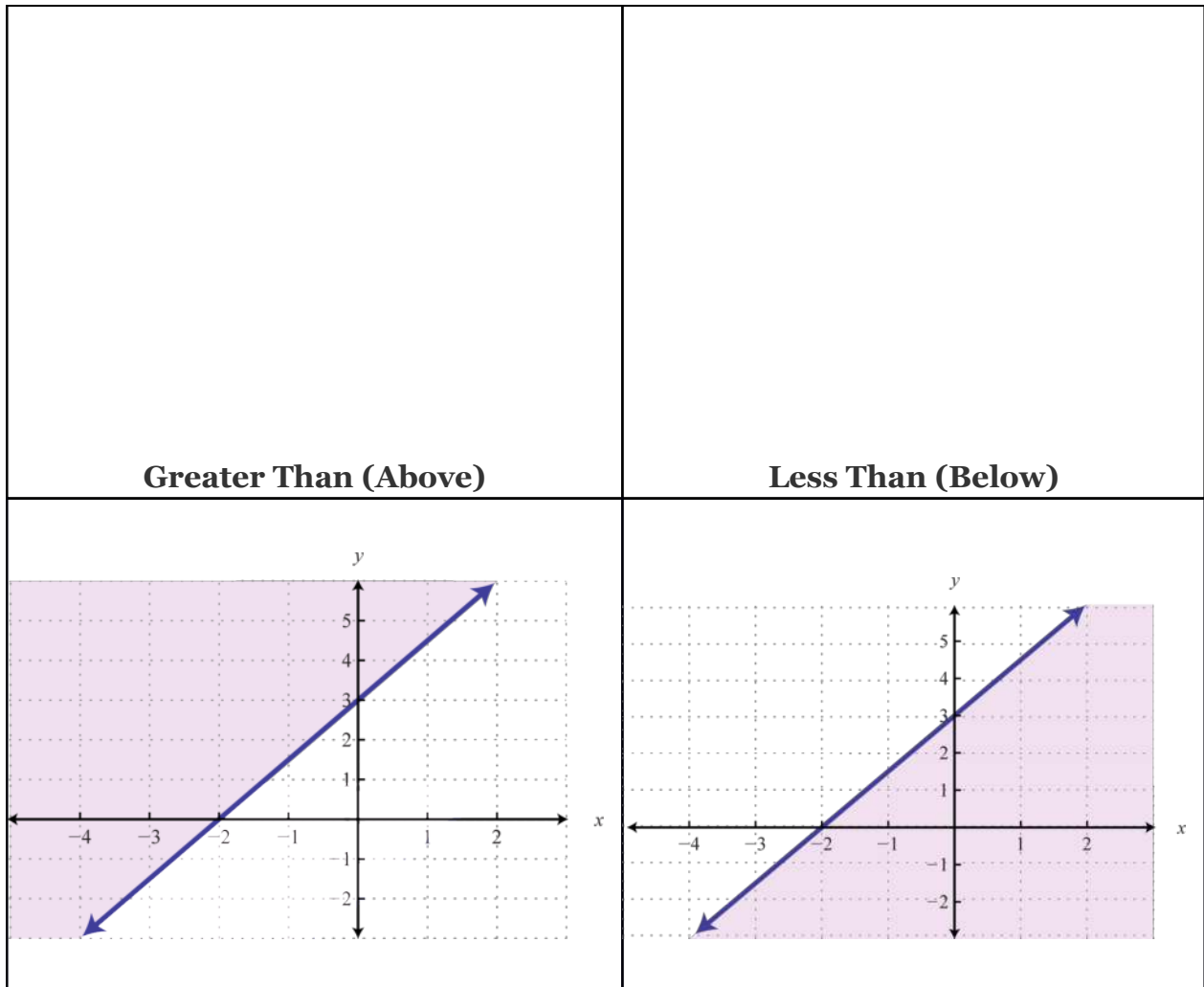


<b>Greater Than (Above)</b>	<b>Less Than (Below)</b>

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Given the graphs above, what might we expect if we use the origin (0, 0) as a test point?

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x	✓

## Example 1

Determine whether or not  $(2, 12)$  is a solution to  $5x - 2y < 10$ .

Solution:

Substitute the  $x$ - and  $y$ -values into the equation and see if a true statement is obtained.



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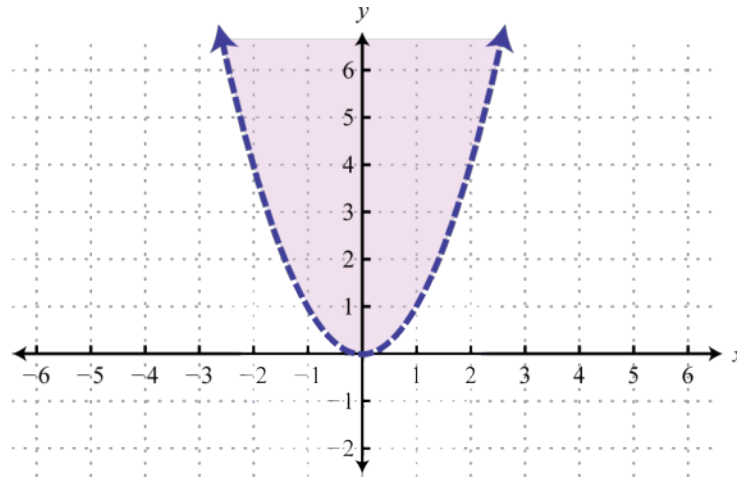
Saylor URL: <http://www.saylor.org/books/>





Answer: is a solution.

These ideas and techniques extend to nonlinear inequalities with two variables. For example, all of the solutions to are shaded in the graph below.



The boundary of the region is a parabola, shown as a dashed curve on the graph, and is not part of the solution set. However, from the graph we expect the ordered pair  $(-1, 4)$  to be a solution. Furthermore, we expect that ordered pairs that are not in the shaded region, such as  $(-3, 2)$ , will not satisfy the inequality.

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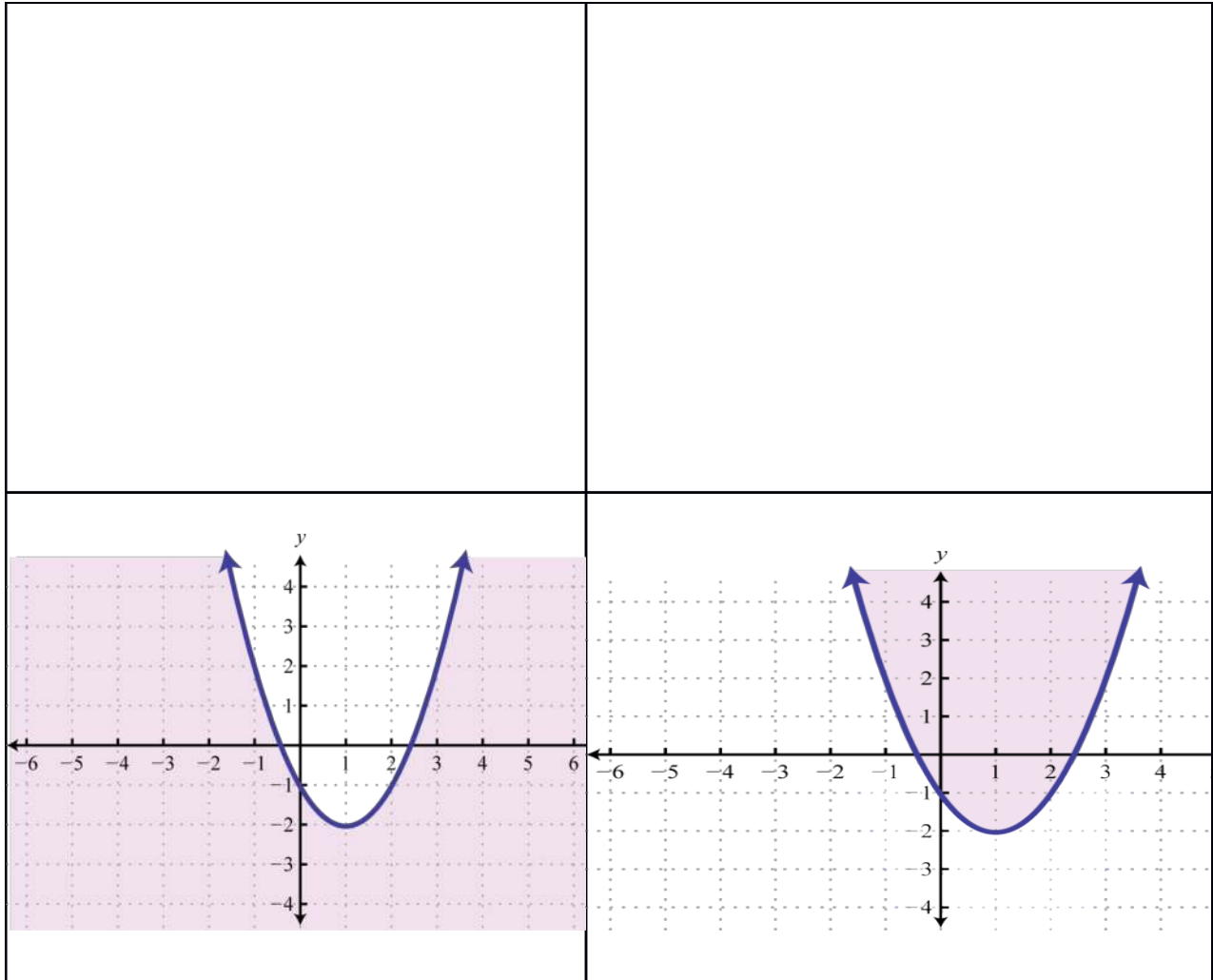


<i>Check (-1, 4)</i>	<i>Check (-3, 2)</i>
✓	✗

Following are graphs of solutions sets of inequalities with inclusive parabolic boundaries.

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You are encouraged to test points in and out of each solution set that is graphed above.

Try this! Is  $(-3, -2)$  a solution to

Answer: No

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# Graphing Solutions to Inequalities with Two Variables

Solutions to linear inequalities are a shaded half-plane, bounded by a solid line or a dashed line. This boundary is either included in the solution or not, depending on the given inequality. If we are given a strict inequality, we use a dashed line to indicate that the boundary is not included. If we are given an inclusive inequality, we use a solid line to indicate that it is included. The steps for graphing the solution set for an inequality with two variables are shown in the following example.

## Example 2

Graph the solution set  $y > -3x + 1$ .

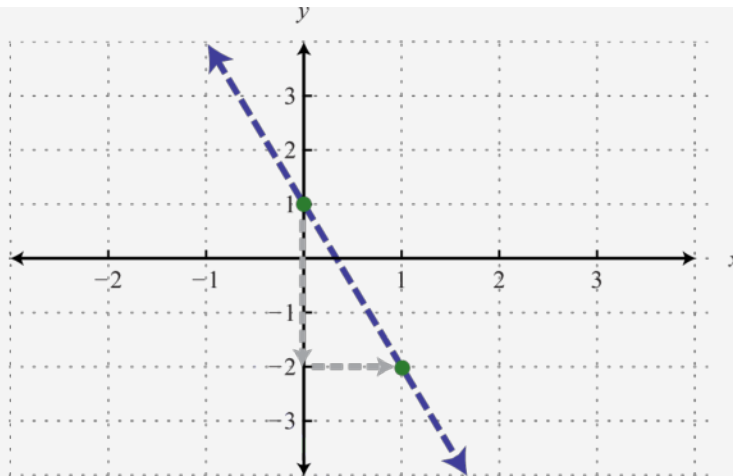
Solution:

Step 1: Graph the boundary. Because of the strict inequality, we will graph the boundary  $y = -3x + 1$  using a dashed line. We can see that the slope is  $-3$  and the  $y$ -intercept is  $(0, 1)$ .

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Step 2: Test a point that is *not* on the boundary. A common test point is the origin, (0, 0). The test point helps us determine which half of the plane to shade.

Test point	$y > -3x + 1$
------------	---------------



$(0, 0)$	$0 > -3(0) + 1$ $0 > 1$ ✗

Step 3: Shade the region containing the solutions. Since the test point  $(0, 0)$  was not a solution, it does not lie in the region containing all the ordered pair solutions. Therefore, shade the half of the plane that does not contain this test point. In this case, shade above the boundary line.

Answer:



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Consider the problem of shading above or below the boundary line when the inequality is in slope-intercept form. If  $y > mx + b$ , then shade above the line. If  $y < mx + b$ , then shade below the line. Shade with caution; sometimes the boundary is given in standard form, in which case these rules do not apply.

### Example 3

Graph the solution set  $2x - 5y \geq -10$ .

Solution:

Here the boundary is defined by the line  $2x - 5y = -10$ . Since the inequality is inclusive, we graph the boundary using a solid line. In this case, graph the boundary line using intercepts.

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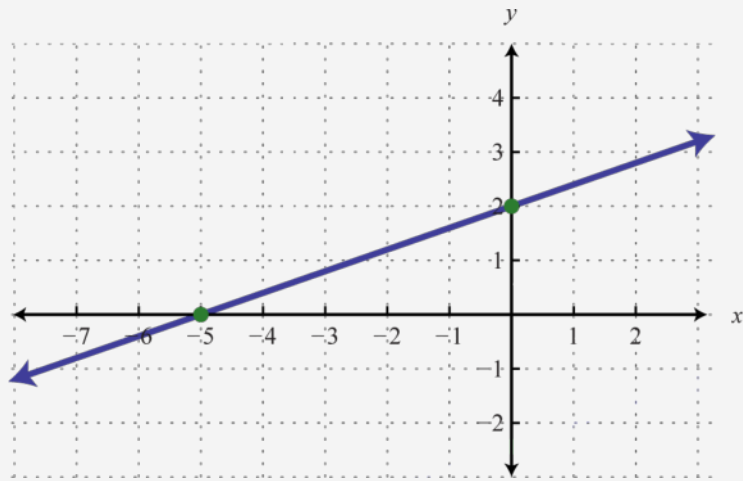
<b>To find the <math>x</math>-intercept, set <math>y = 0</math>. To find the <math>y</math>-intercept, set <math>x = 0</math>.</b>	
$x$ -intercept: $(-5, 0)$	$y$ -intercept: $(0, 2)$

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Next, test a point; this helps decide which region to shade.

Test point	$2x - 5y \geq -10$

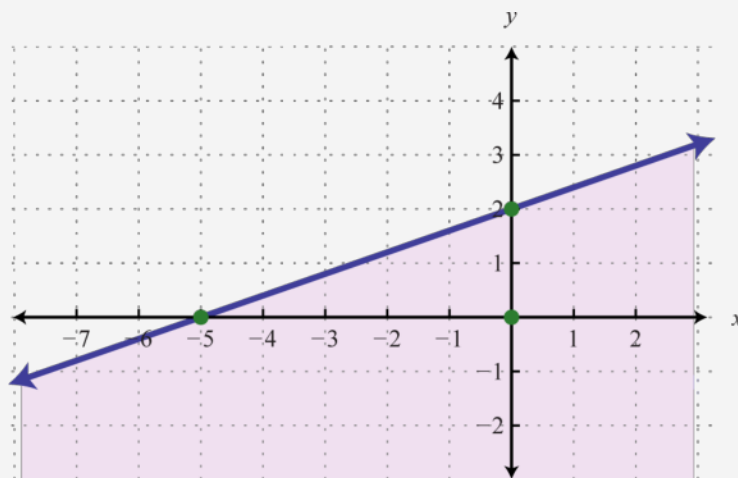
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$(0, 0)$	$2(0) - 5(0) \geq$ $\geq -10 - 10 \quad \checkmark$

Since the test point is in the solution set, shade the half of the plane that contains it.

Answer:



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In this example, notice that the solution set consists of all the ordered pairs below the boundary line. This may seem counterintuitive because the original inequality involved “greater than”  $\geq$ . This illustrates that it is a best practice to actually test a point. Solve for  $y$  and you see that the shading is correct.

In slope-intercept form, you can see that the region below the boundary line should be shaded. An alternate approach is to first express the boundary in slope-intercept form, graph it, and then shade the appropriate region.

## Example 4

Graph the solution set  $y < 2$ .

Solution:

First, graph the boundary line  $y = 2$  with a dashed line because of the strict inequality. Next, test a point.

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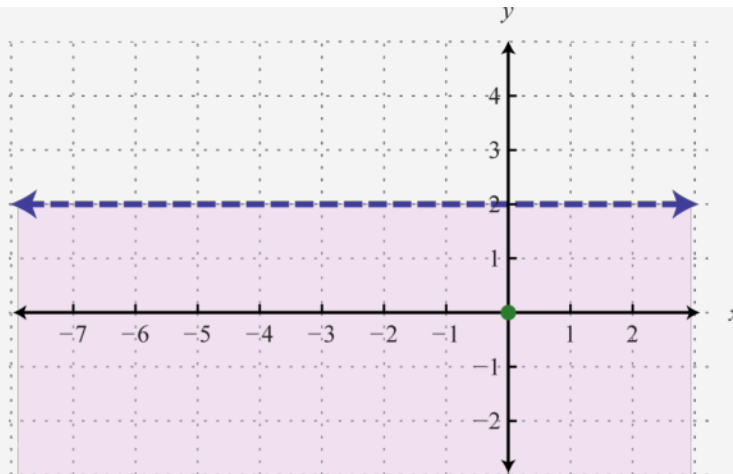
Test point	$y < 2$
$(0, 0)$	$0 < 2$ ✓

In this case, shade the region that contains the test point.

Answer:

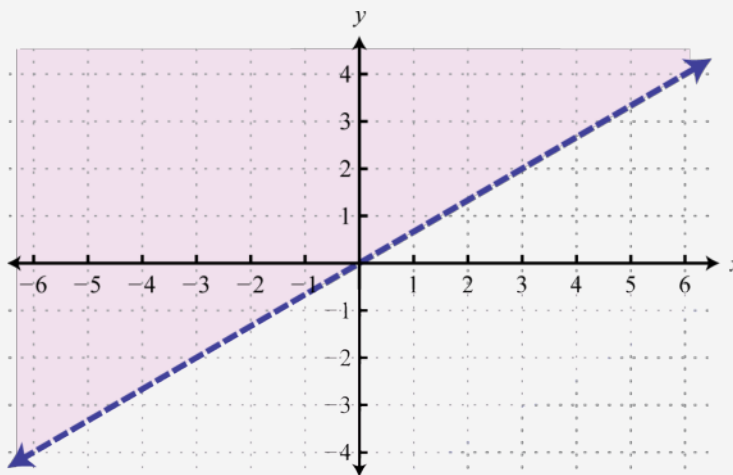
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Try this! Graph the solution set  $2x - 3y < 0$ .

Answer:



The steps are the same for nonlinear inequalities with two variables. Graph the boundary first and then test a point to determine which region contains the solutions.

## Example 5

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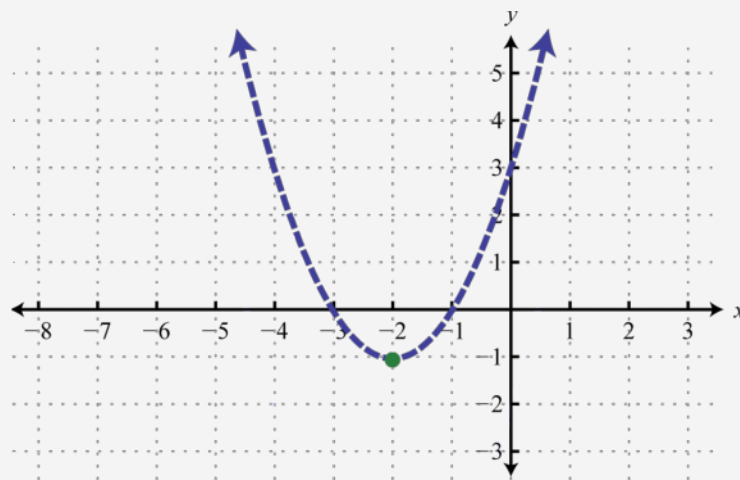
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Graph the solution set

Solution:

The boundary is a basic parabola shifted 2 units to the left and 1 unit down. Begin by drawing a dashed parabolic boundary because of the strict inequality.



Next, test a point.

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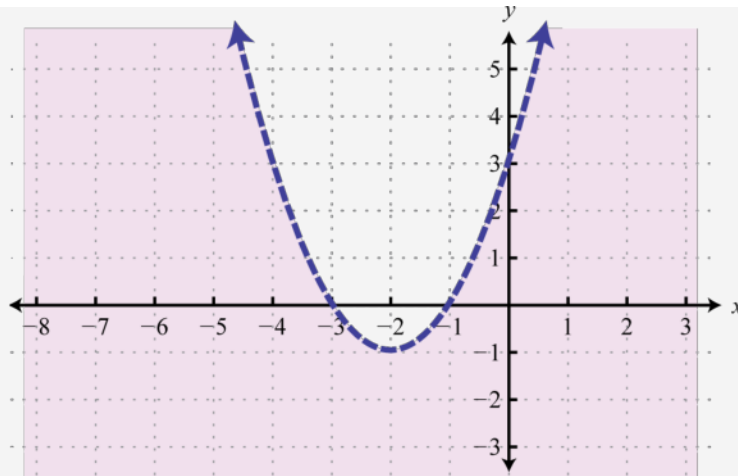
Test point	
(0, 0)	✓

In this case, shade the region that contains the test point (0, 0).

Answer:

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## Example 6

Graph the solution set

Solution:

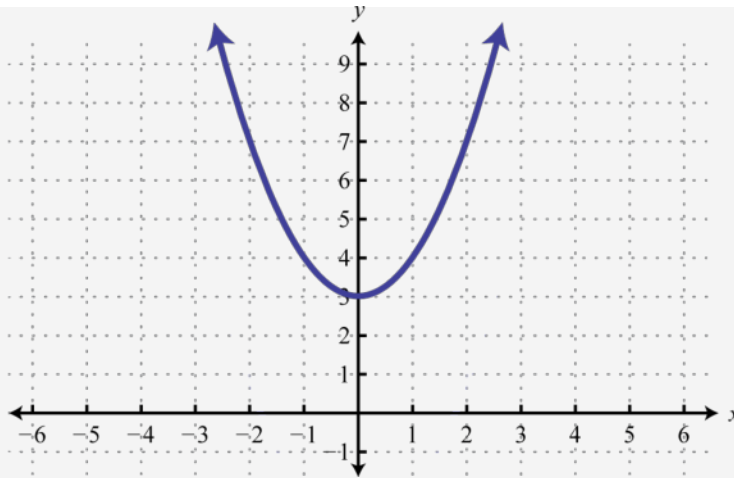
The boundary is a basic parabola shifted 3 units up. It is graphed using a solid curve because of the inclusive inequality.

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Next, test a point.

Test point	
------------	--

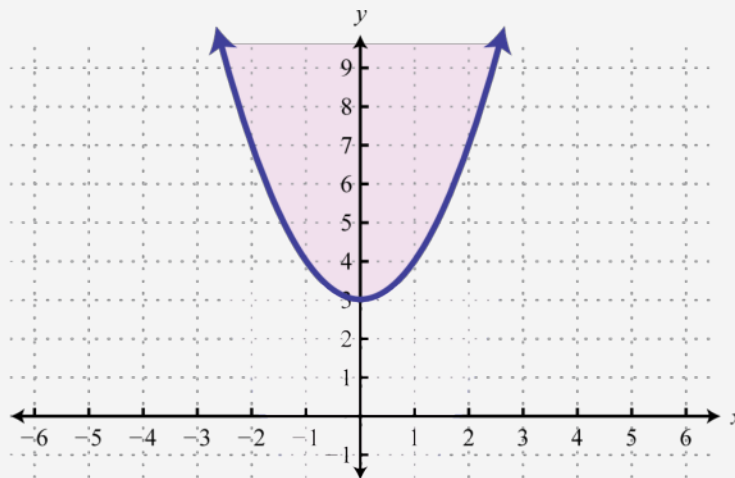
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(0, 0)	x
--------	---

In this case, shade the region that does not contain the test point (0, 0).

Answer:

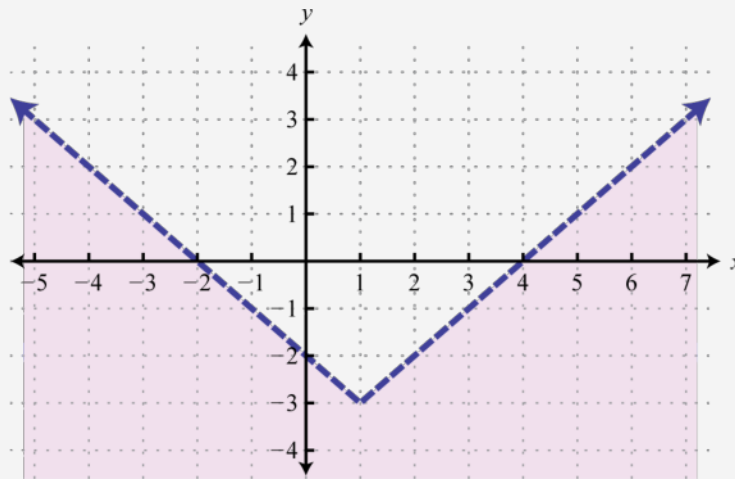


Try this! Graph the solution set  $y < |x-1| - 3$ .

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Answer:



## KEY TAKEAWAYS

Linear inequalities with two variables have infinitely many ordered pair solutions, which can be graphed by shading in the appropriate half of a rectangular coordinate plane.

To graph the solution set of an inequality with two variables, first graph the boundary with a dashed or solid line depending on the inequality. If given a strict inequality, use a dashed line for the boundary. If given an inclusive inequality, use a solid line. Next, choose a test point not on the boundary. If the test point solves the inequality, then shade the region that contains it; otherwise, shade the opposite side.

Check your answer by testing points in and out of the shading region to verify that they solve the inequality or not.

## TOPIC EXERCISES

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## PART A: SOLUTIONS TO INEQUALITIES WITH TWO VARIABLES

Is the ordered pair a solution to the given inequality?

1.  $5x - y > -2$ ;  $(-3, -4)$

2.  $4x - y < -8$ ;  $(-3, -10)$

3.

4.

5.

6.

7. ;  $(-1, 1)$

8. ;  $(-2, 0)$

9. ;  $(3, 4)$

10. ;  $(-1, -2)$

11.  $y > 3 - |x|$ ;  $(-4, -3)$

12.  $y < |x| - 8$ ;  $(5, -7)$

13.  $y > |2x - 1| - 3$ ;  $(-1, 3)$

14.  $y < |3x - 2| + 2$ ;  $(-2, 10)$

## PART B: GRAPHING SOLUTIONS TO INEQUALITIES WITH TWO VARIABLES.

Graph the solution set.

15.  $y < 2x - 1$

16.  $y > -4x + 1$

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17.

18.

19.  $2x + 3y \leq 18$

20.  $5x + 2y \leq 8$

21.  $6x - 5y > 30$

22.  $8x - 6y < 24$

23.  $3x - 4y < 0$

24.  $x - 3y > 0$

25.  $x + y \geq 0$

26.  $x - y \geq 0$

27.  $y \leq -2$

28.  $y > -3$

29.  $x < -2$

30.  $x \geq -3$

31.

32.

33.

34.

35.  $5x \leq -4y - 12$

36.  $-4x \leq 12 - 3y$

37.  $4y + 2 < 3x$

38.  $8x < 9 - 6y$

39.  $5 \geq 3x - 15y$

40.  $2x \geq 6 - 9y$

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41. Write an inequality that describes all points in the upper half-plane above the  $x$ -axis.
42. Write an inequality that describes all points in the lower half-plane below the  $x$ -axis.
43. Write an inequality that describes all points in the half-plane left of the  $y$ -axis.
44. Write an inequality that describes all points in the half-plane right of the  $y$ -axis.
45. Write an inequality that describes all ordered pairs whose  $y$ -coordinate is at least  $k$  units.
46. Write an inequality that describes all ordered pairs whose  $x$ -coordinate is at most  $k$  units.

Graph the solution set.

- 15.
- 16.
- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.

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25.  $y \geq |x| - 2$   
26.  $y < |x| + 1$   
27.  $y < |x - 3|$   
28.  $y \leq |x + 2|$   
29.  $y > -|x + 1|$   
30.  $y \leq -|x - 2|$   
31.  $y \geq |x + 3| - 2$   
32.  $y \geq |x - 2| - 1$   
33.  $y < -|x + 4| + 2$   
34.  $y > -|x - 4| - 1$

35.

36.

37.

38.

39. A rectangular pen is to be constructed with at most 200 feet of fencing. Write a linear inequality in terms of the length  $l$  and the width  $w$ . Sketch the graph of all possible solutions to this problem.

40. A company sells one product for \$8 and another for \$12. How many of each product must be sold so that revenues are at least \$2,400? Let  $x$  represent the number of products sold at \$8 and let  $y$  represent the number of products sold at \$12. Write a linear inequality in terms of  $x$  and  $y$  and sketch the graph of all possible solutions.

## ANSWERS

1. No

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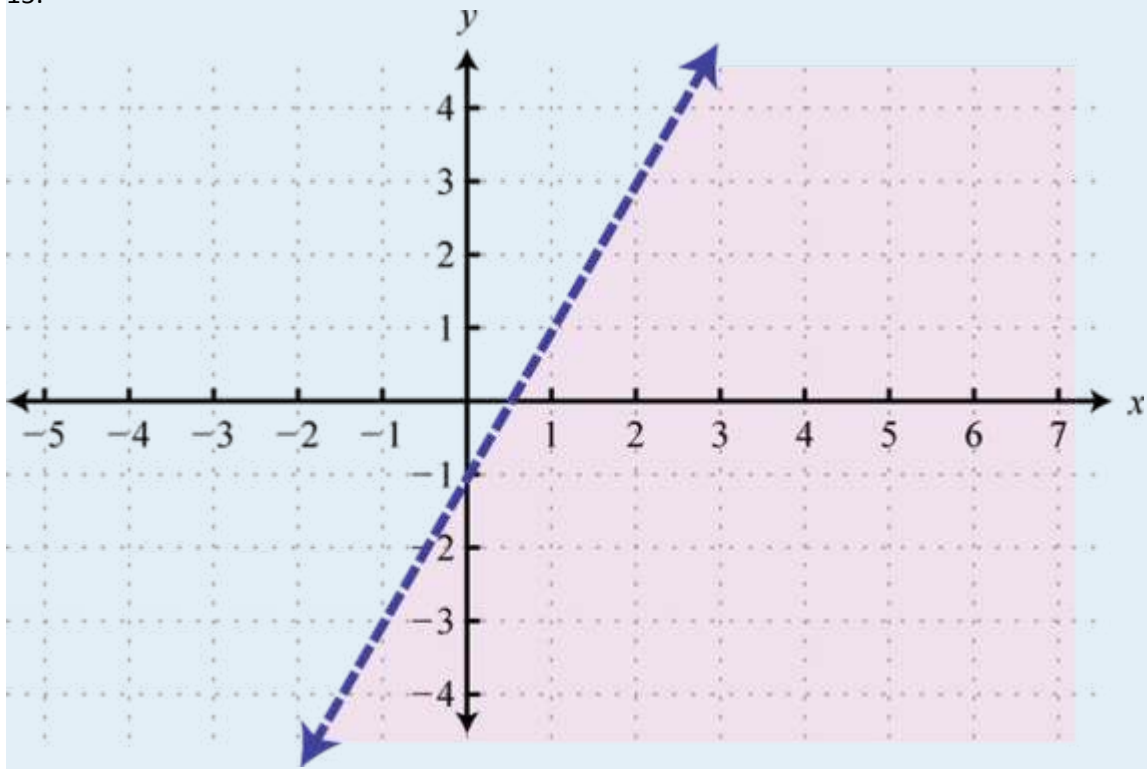


- 2.
3. Yes
- 4.
5. Yes
- 6.
7. No
- 8.
9. No
- 10.
11. No
- 12.
13. Yes
- 14.





15.

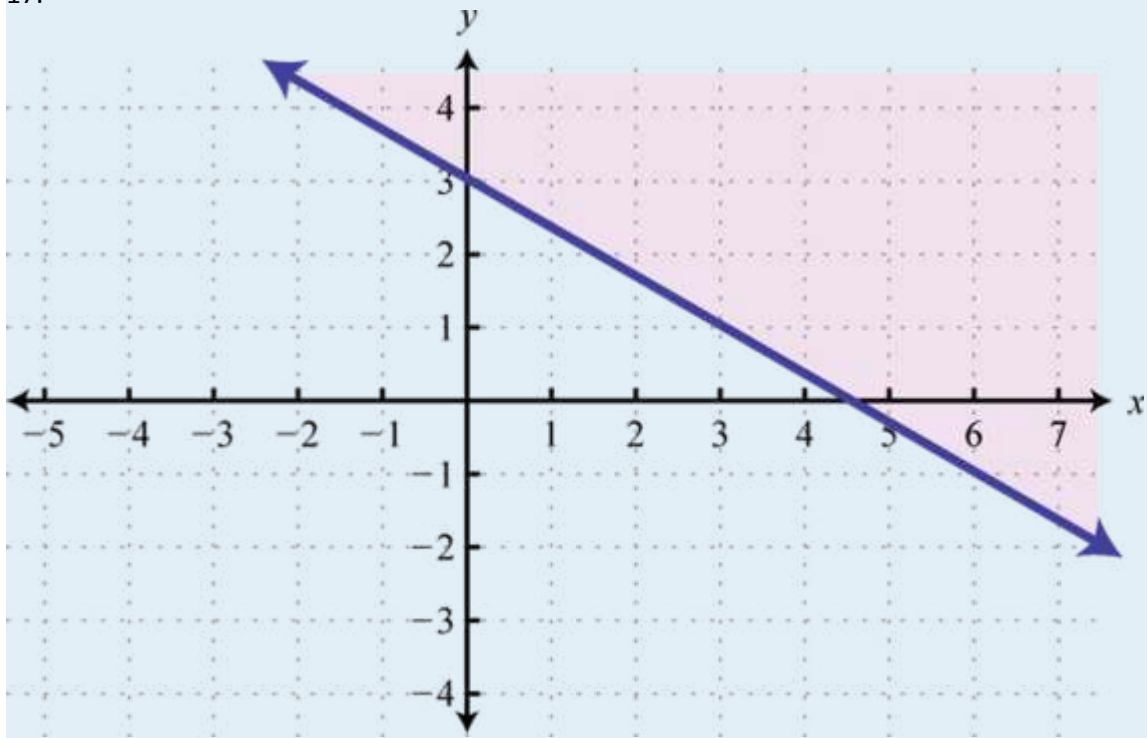


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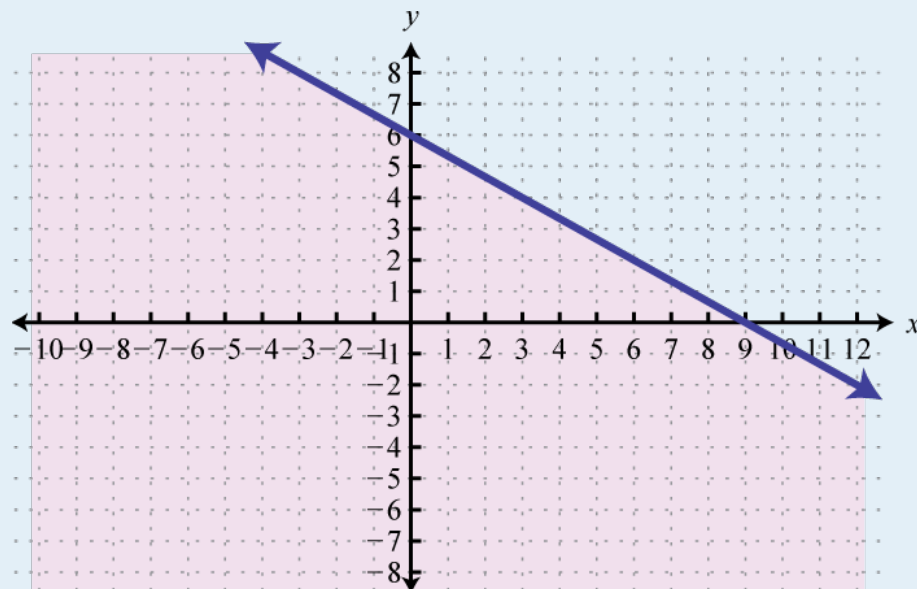
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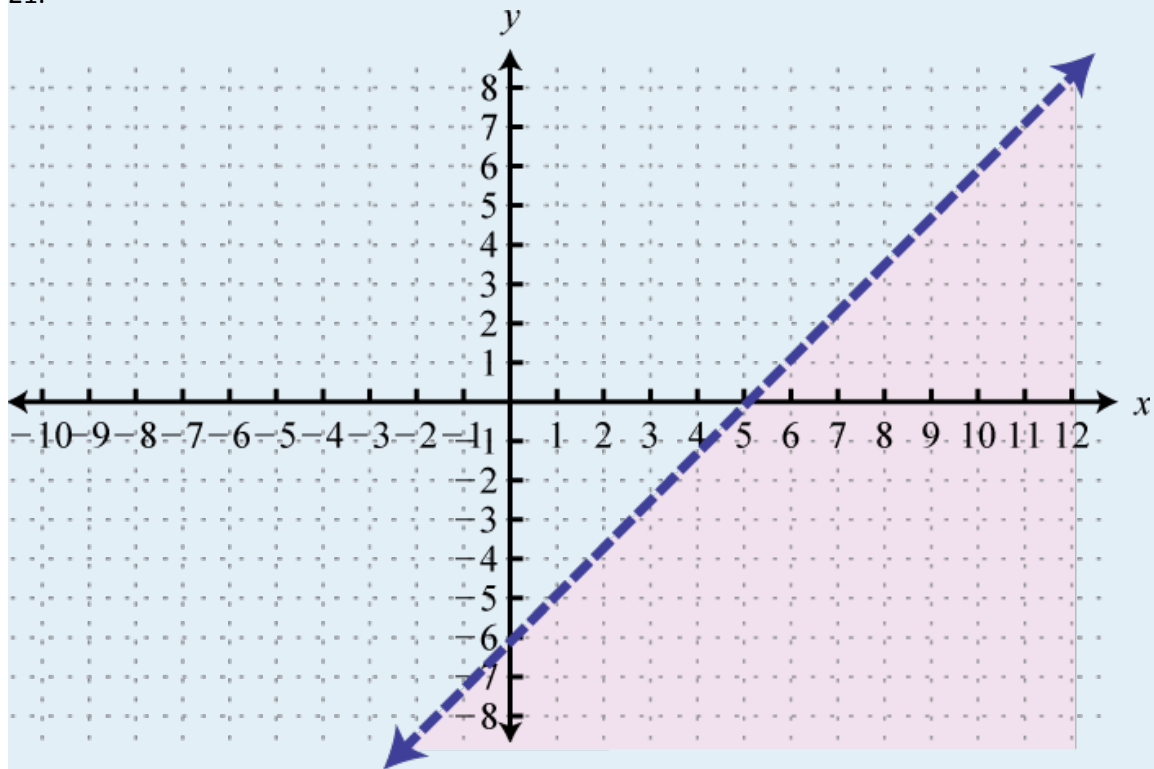
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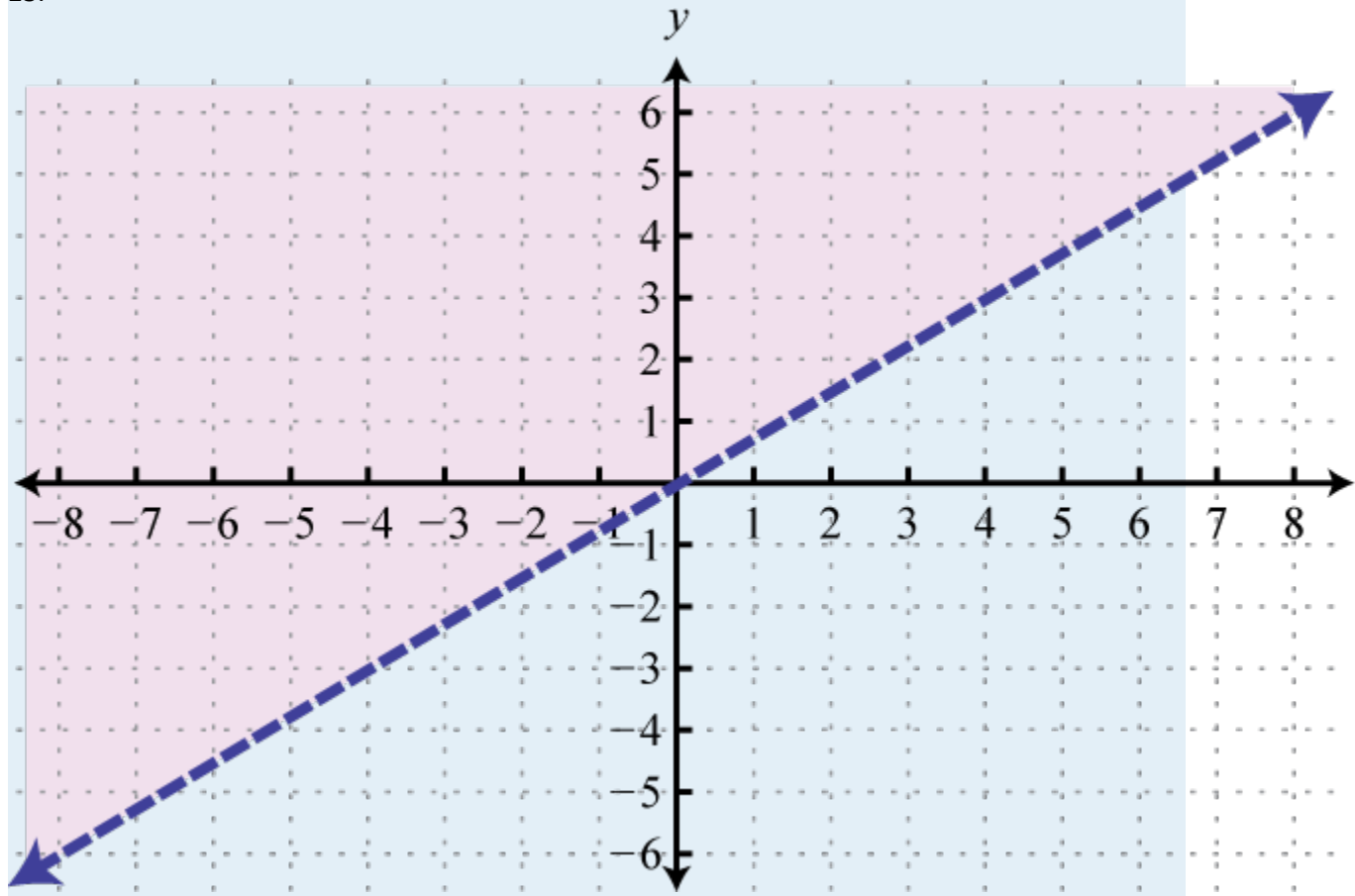


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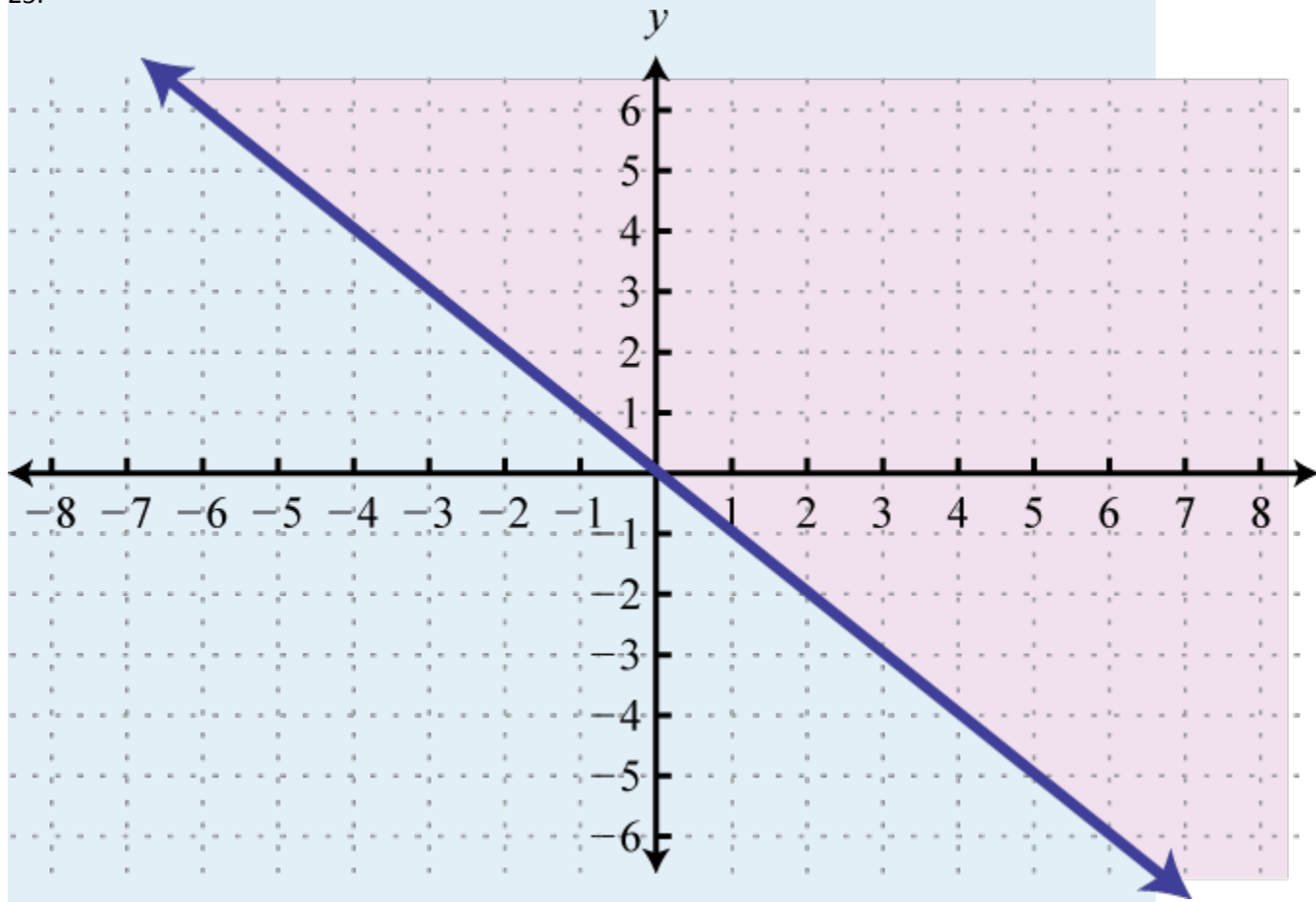


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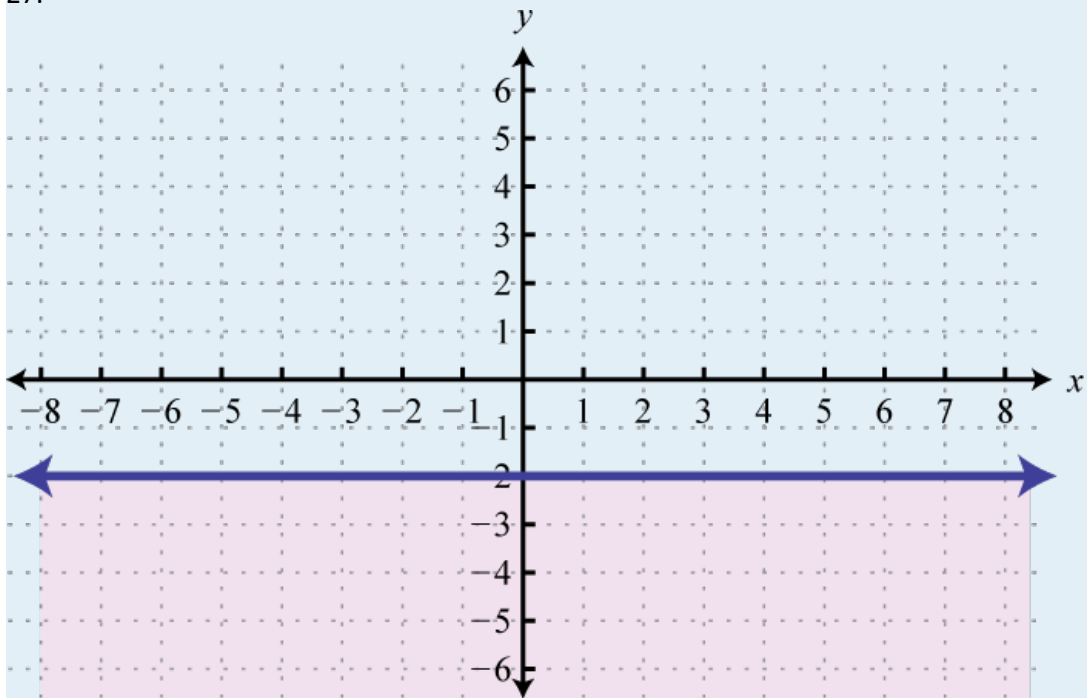


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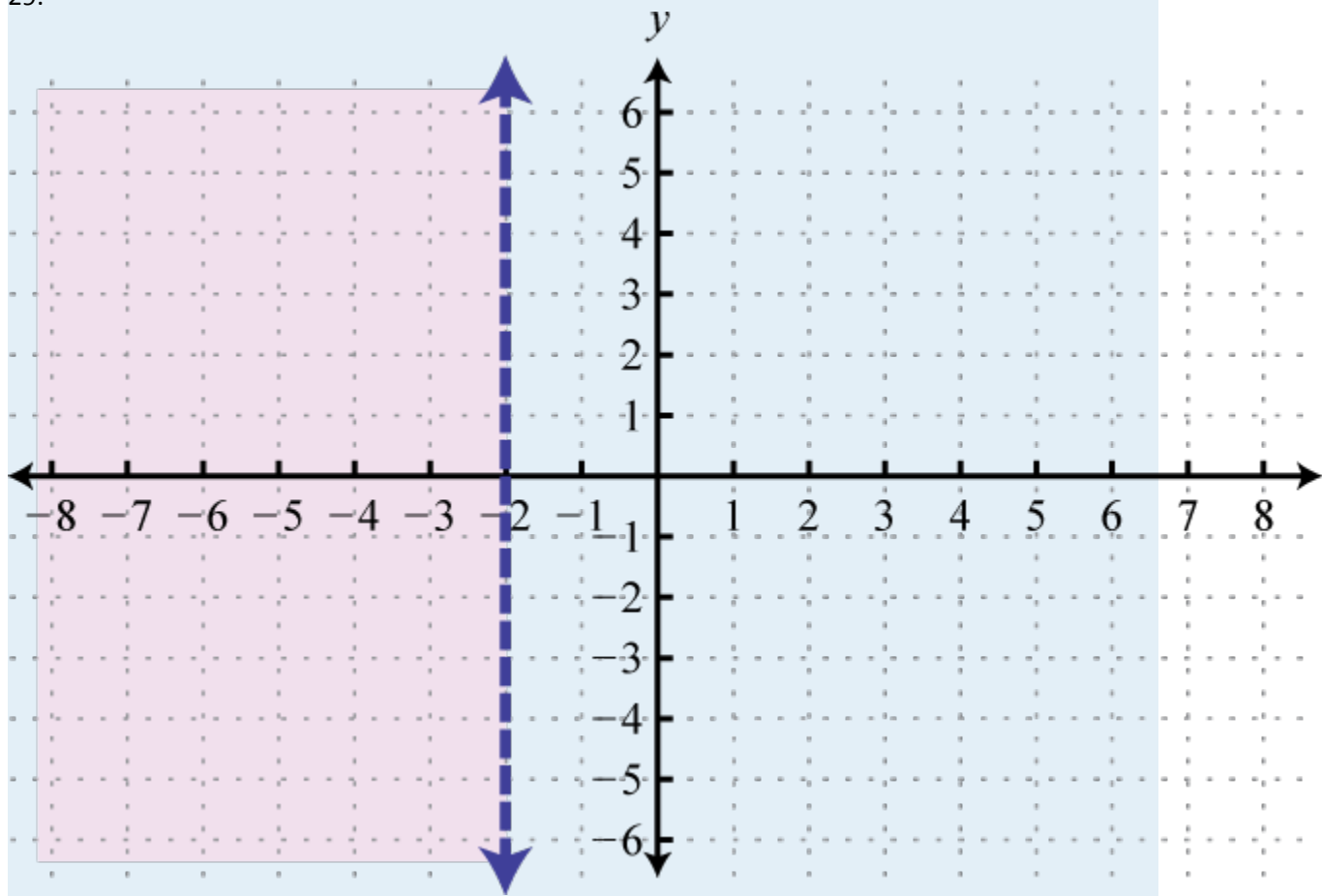


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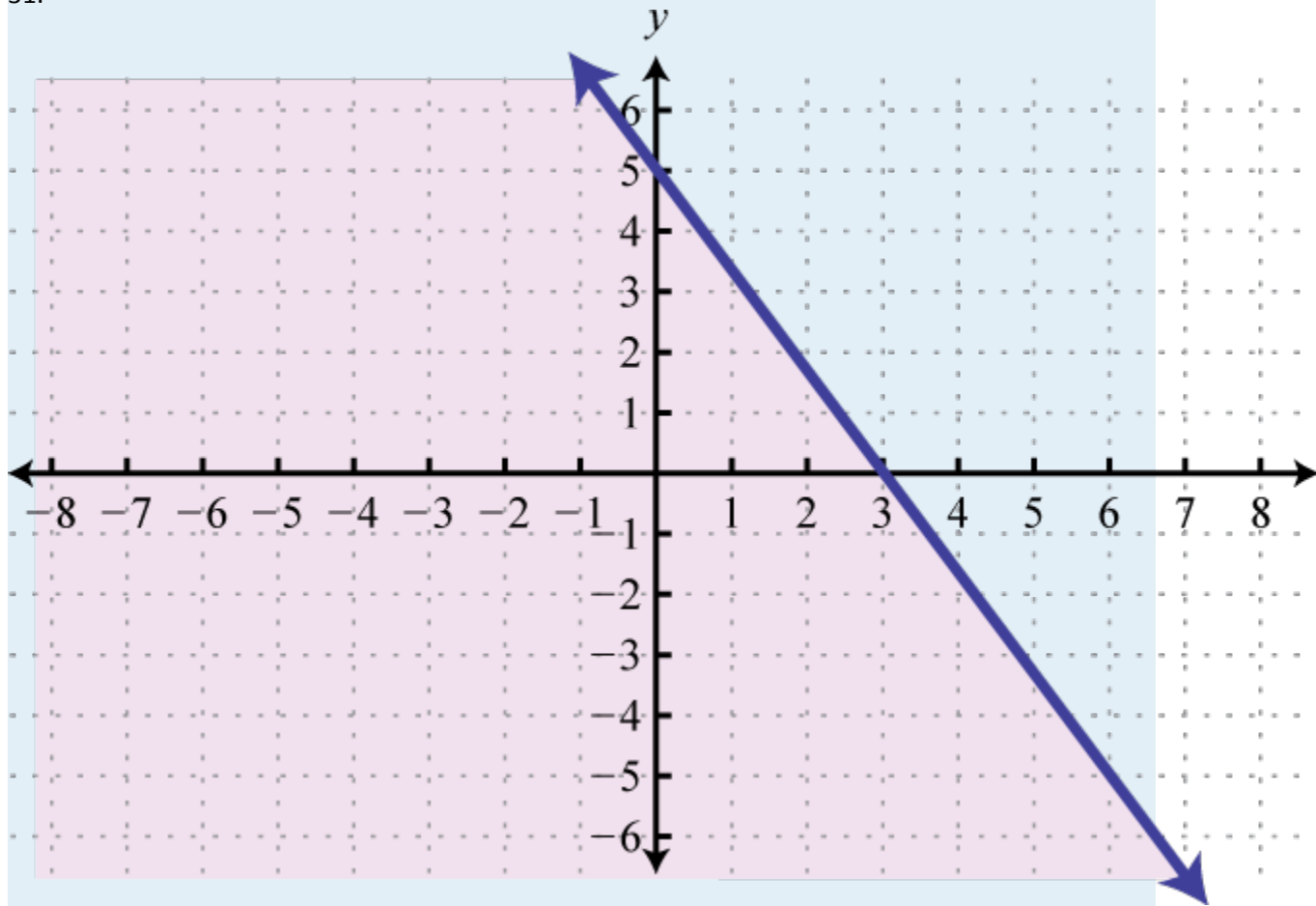


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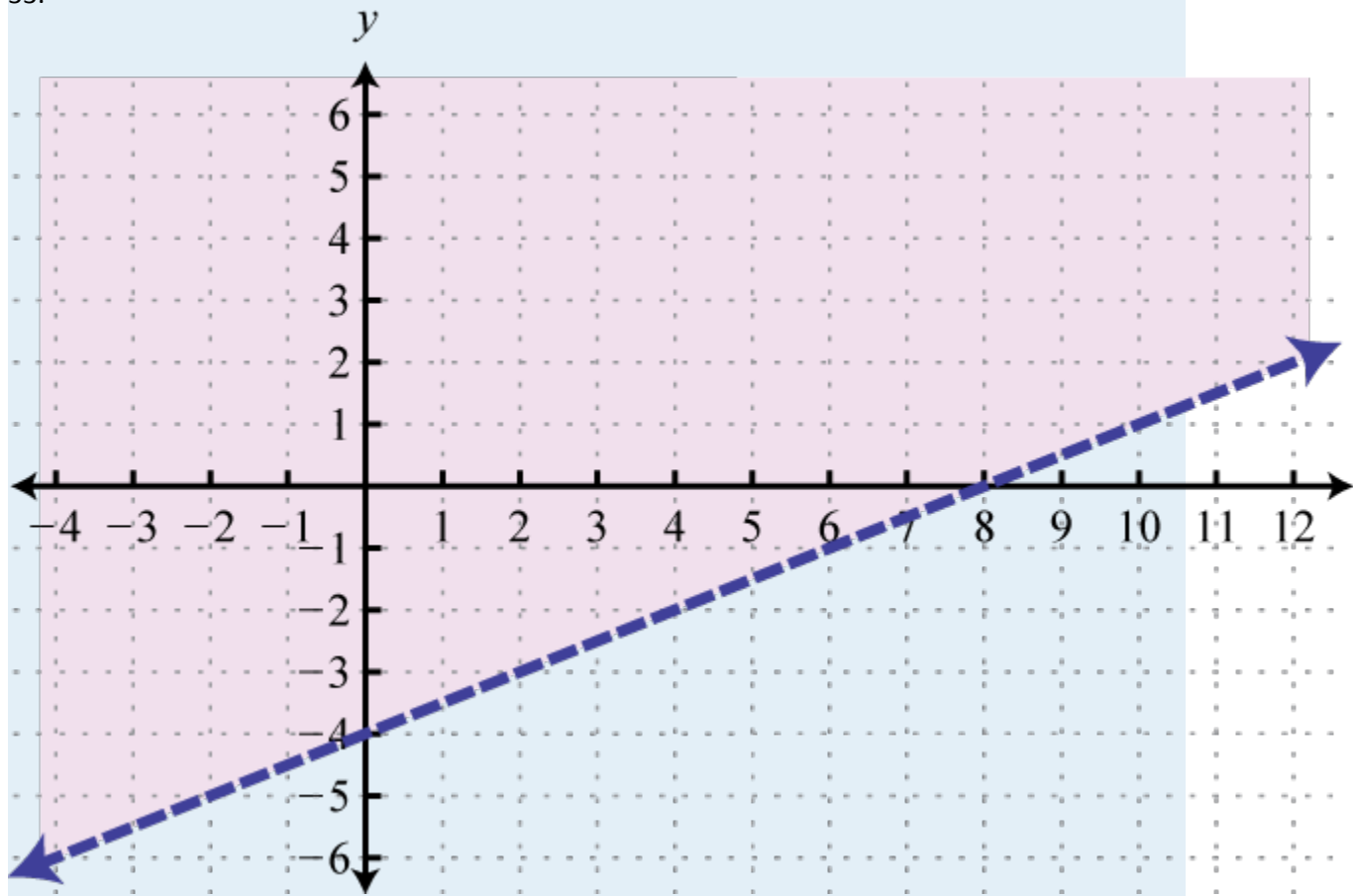
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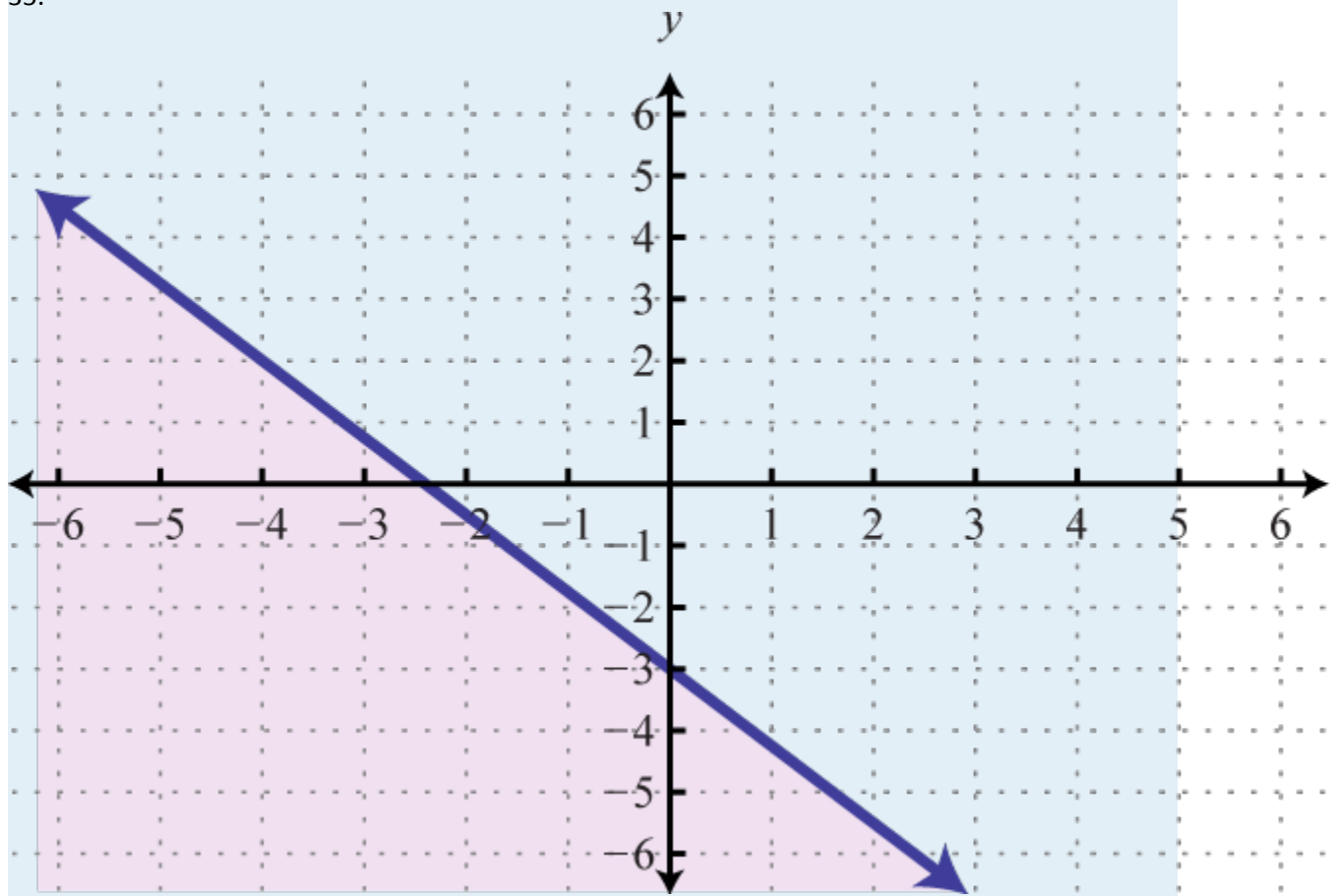


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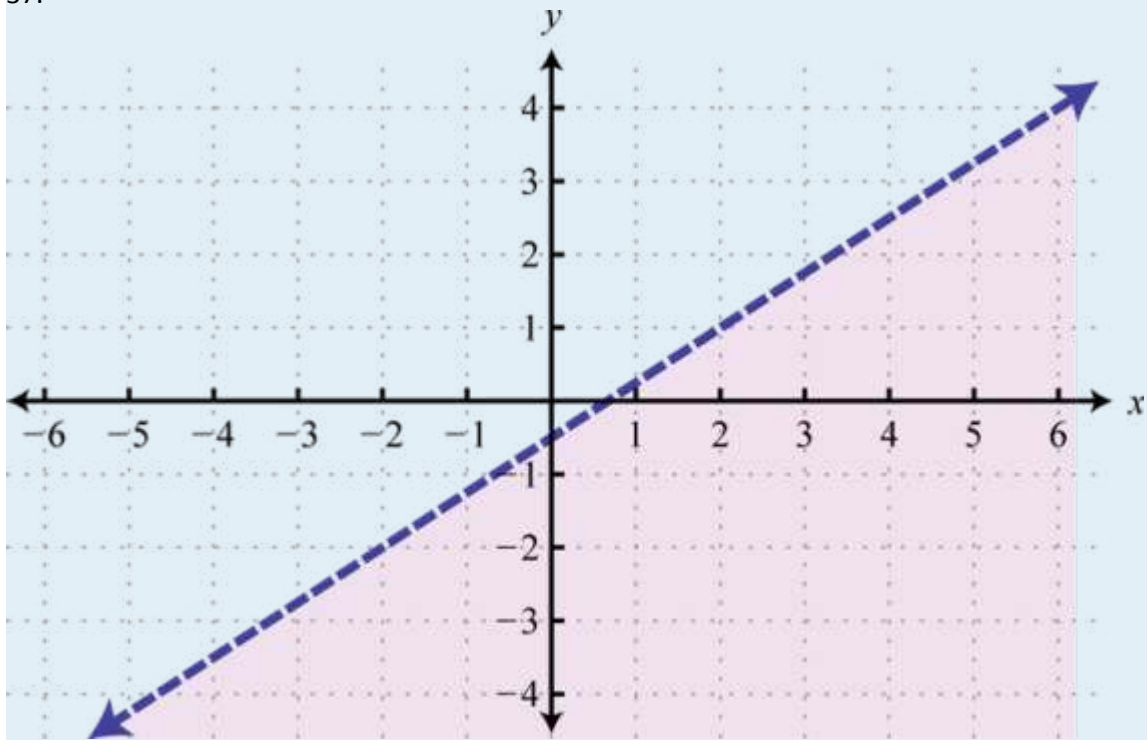
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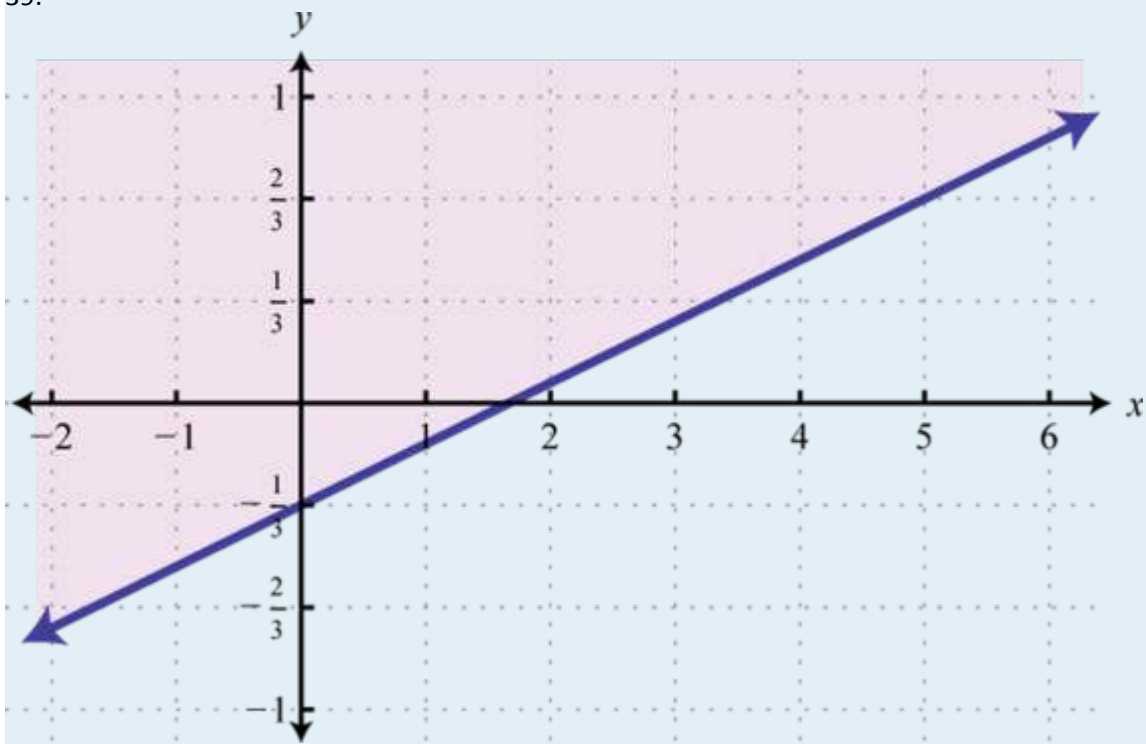


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39.



40.

41.  $y > 0$

42.

43.  $x < 0$

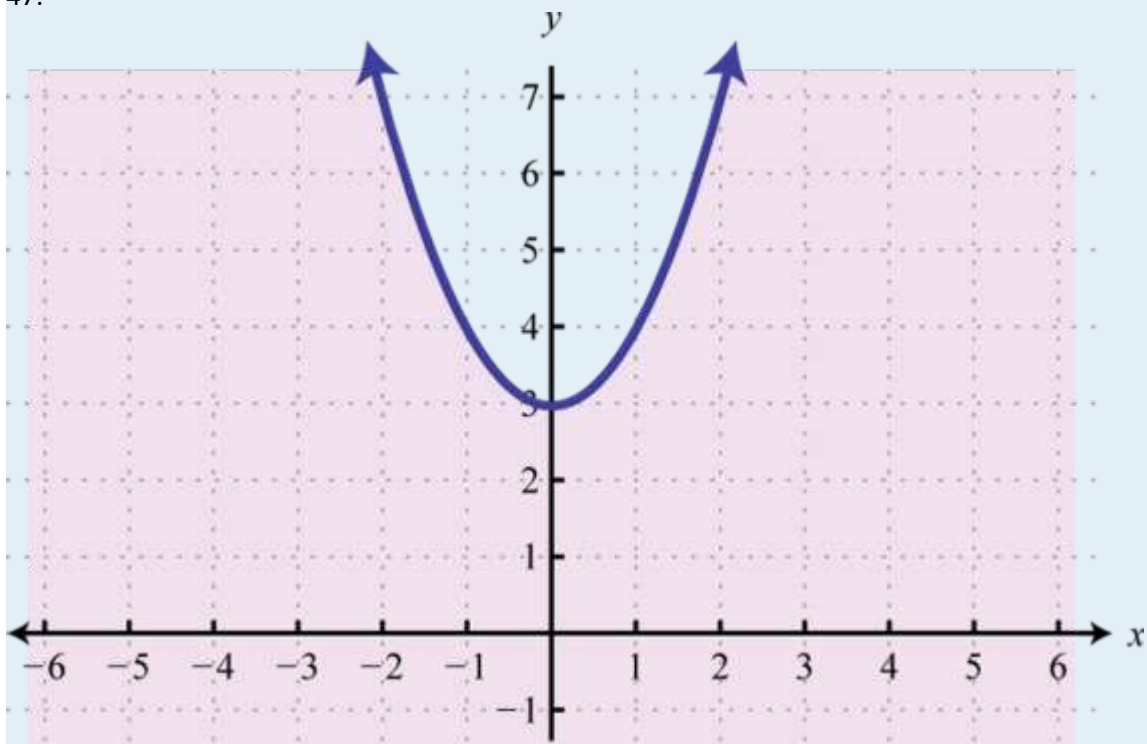
44.

45.  $y \geq k$

46.



47.

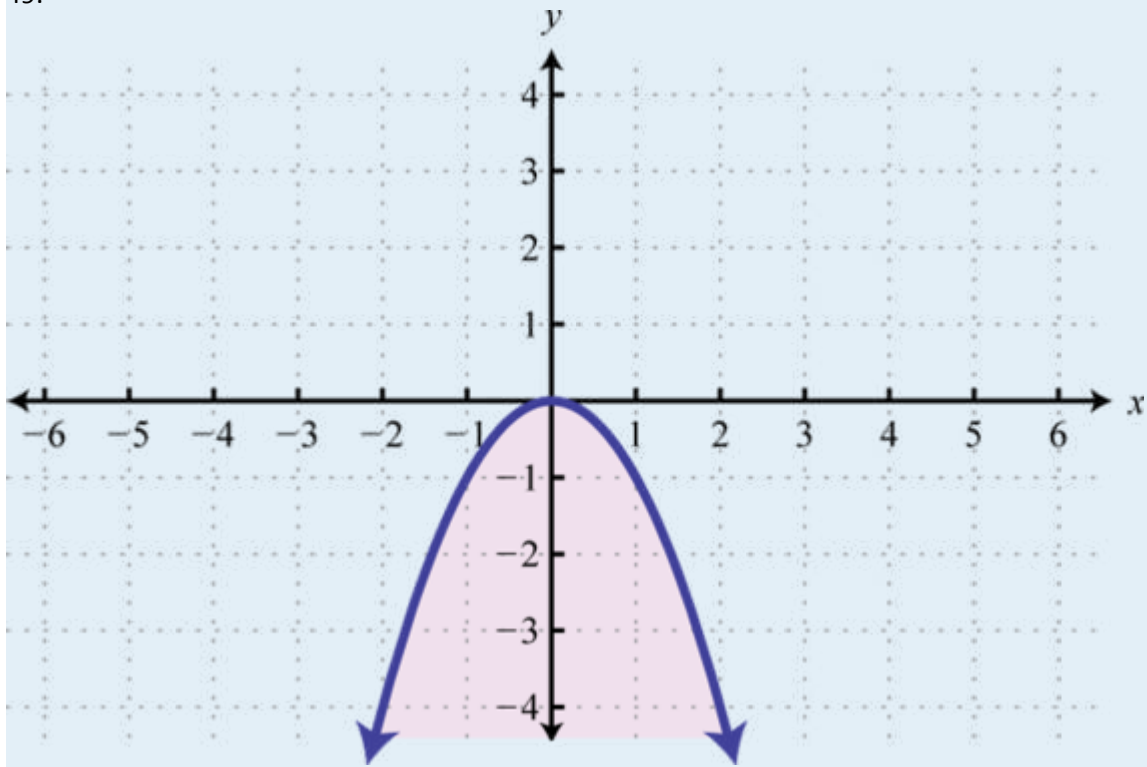


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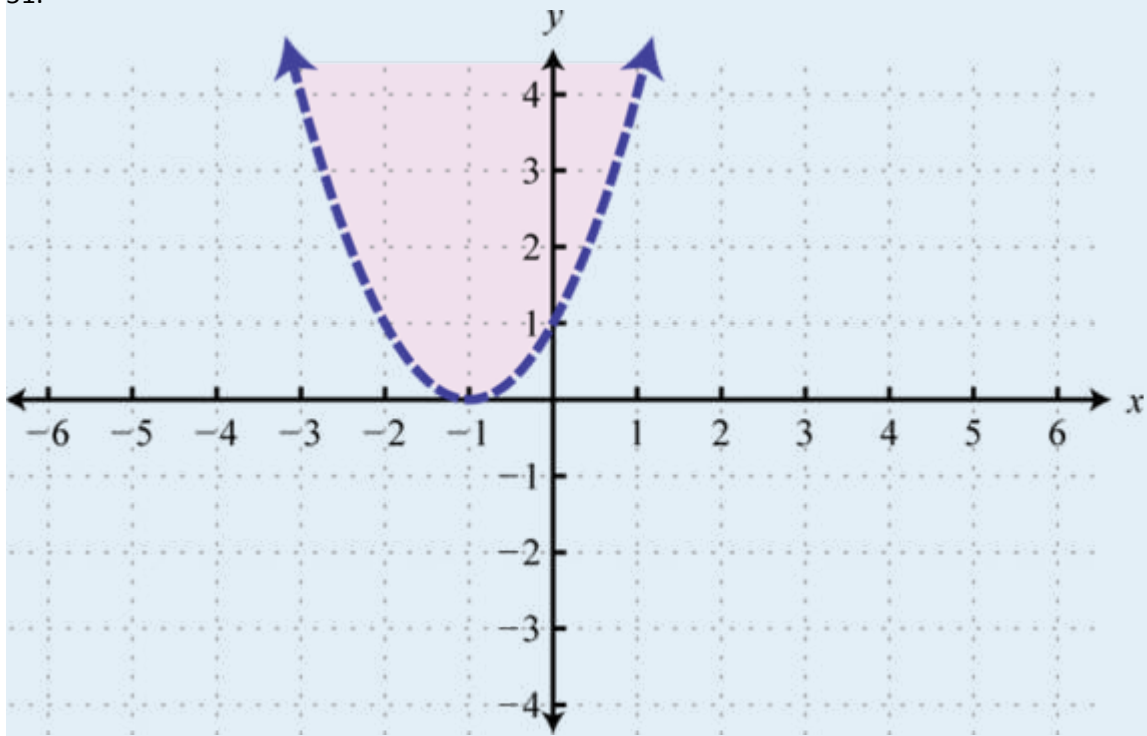


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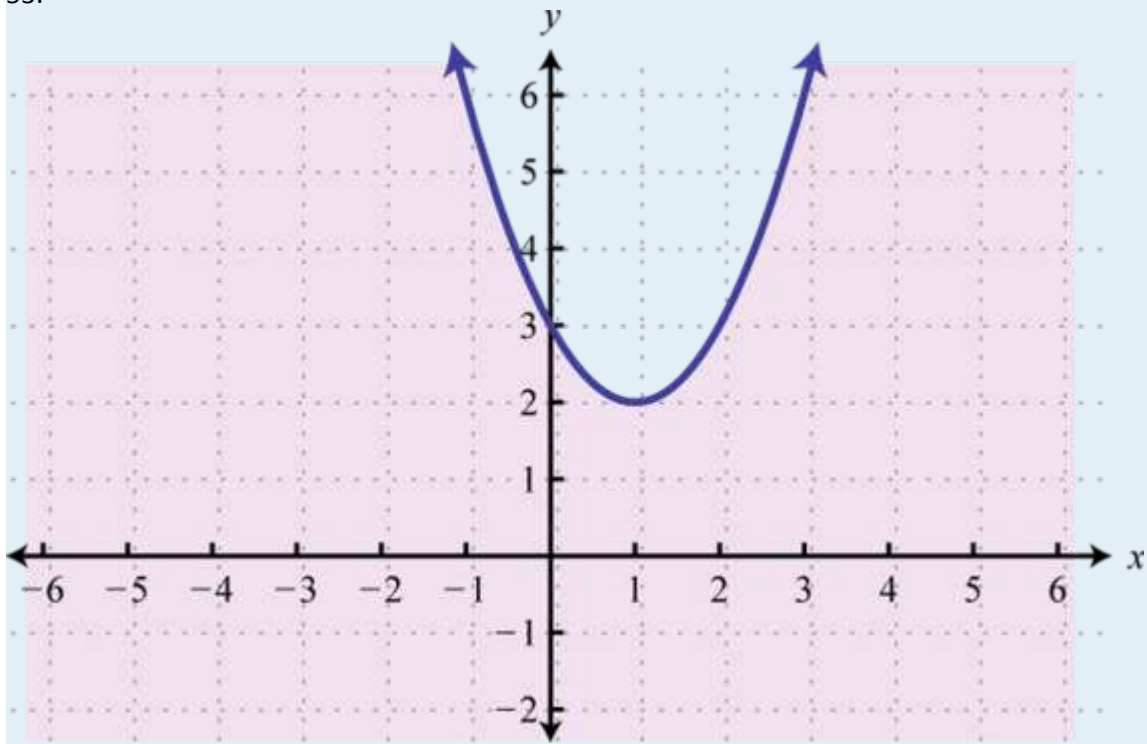


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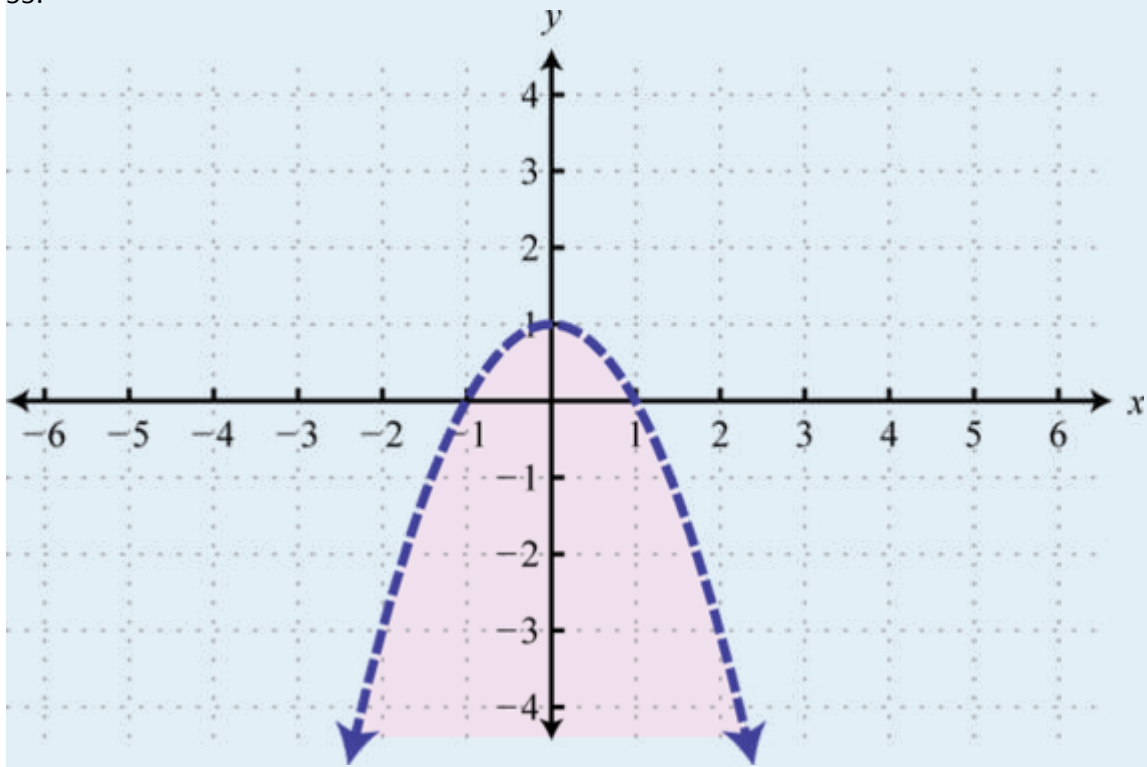
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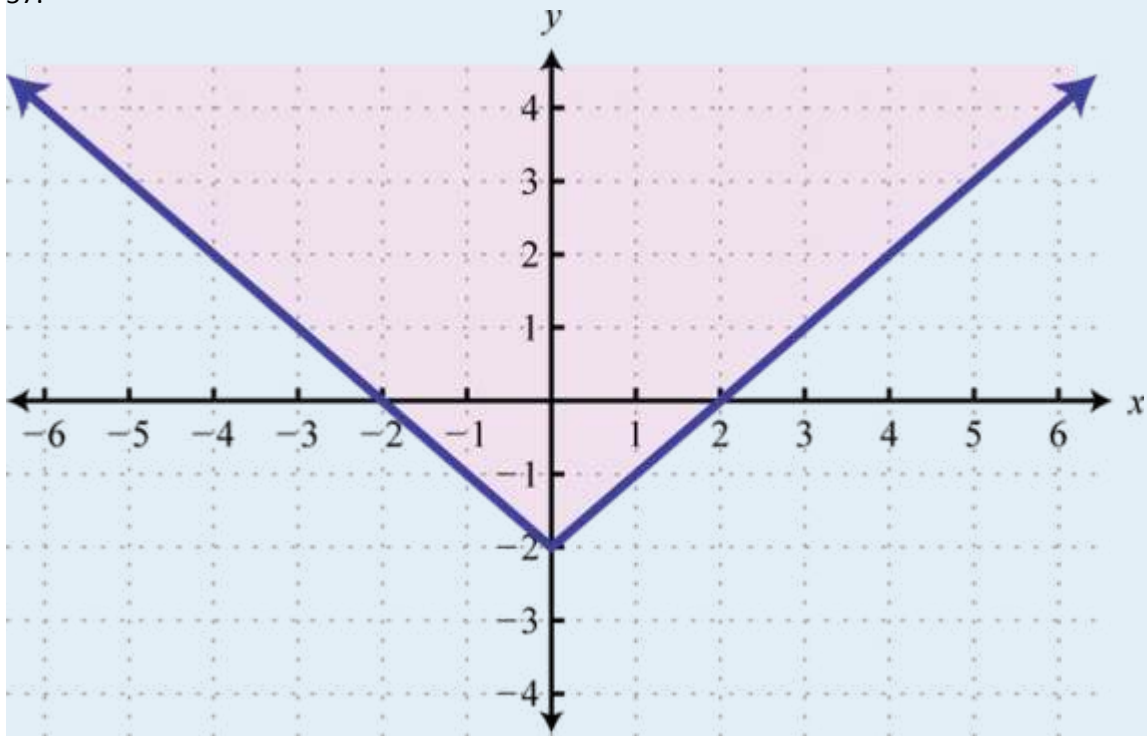


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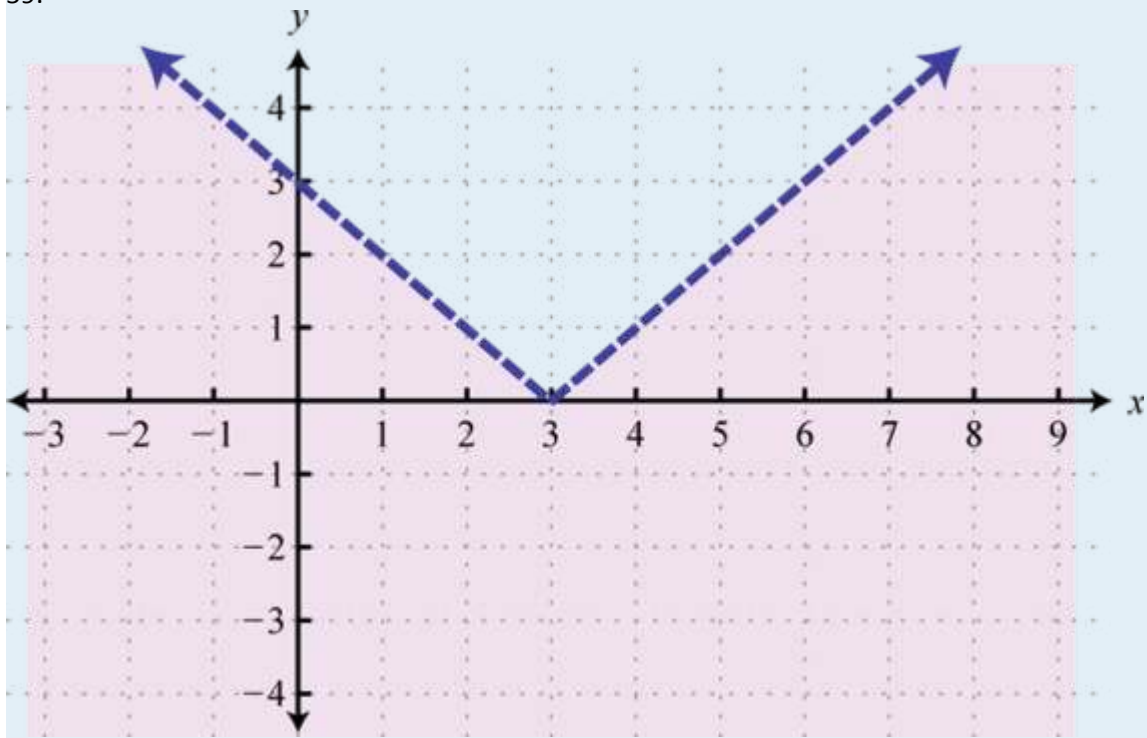


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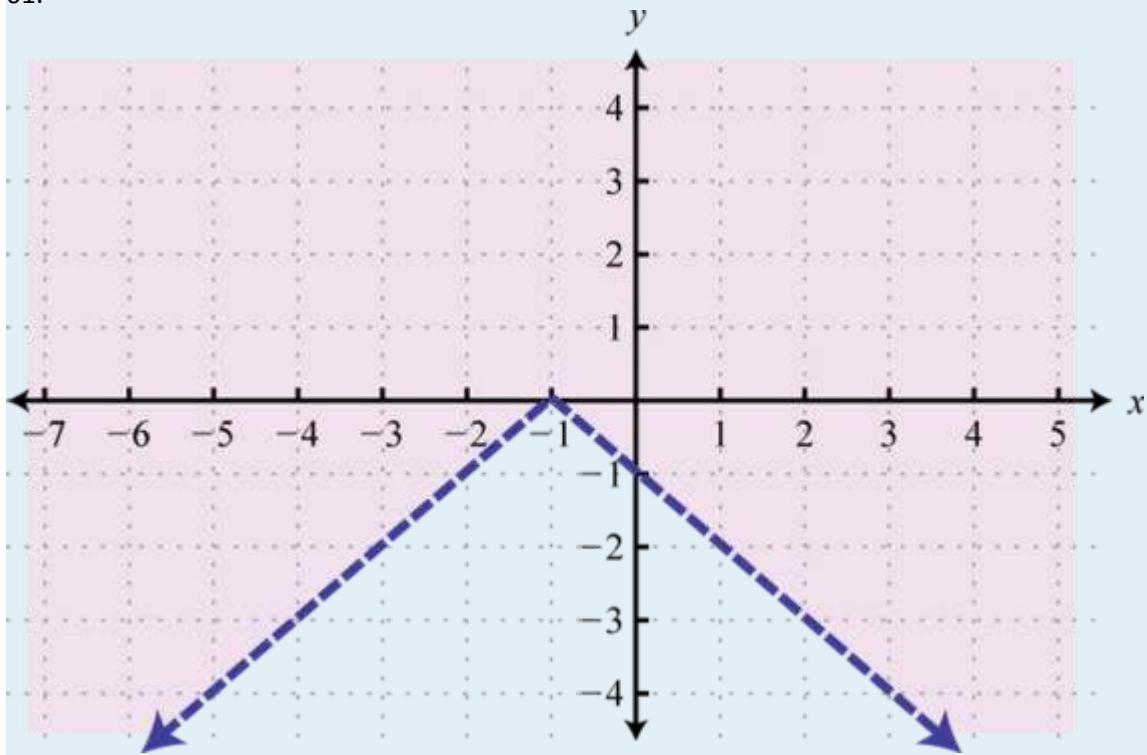


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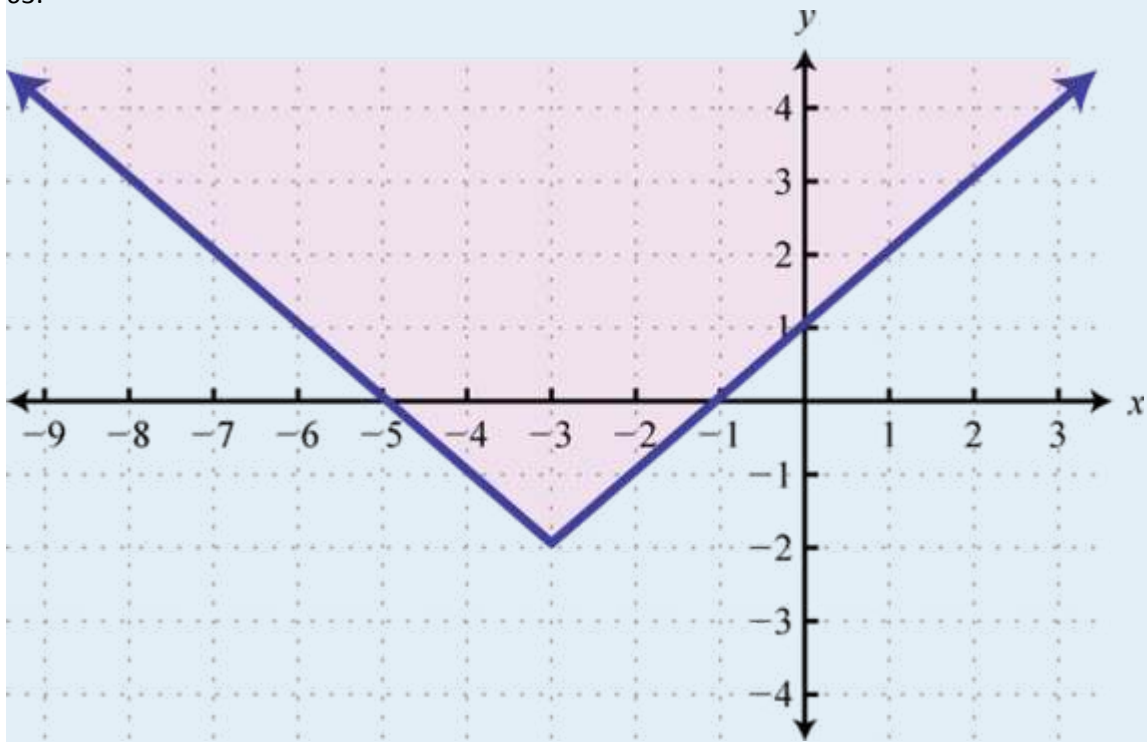


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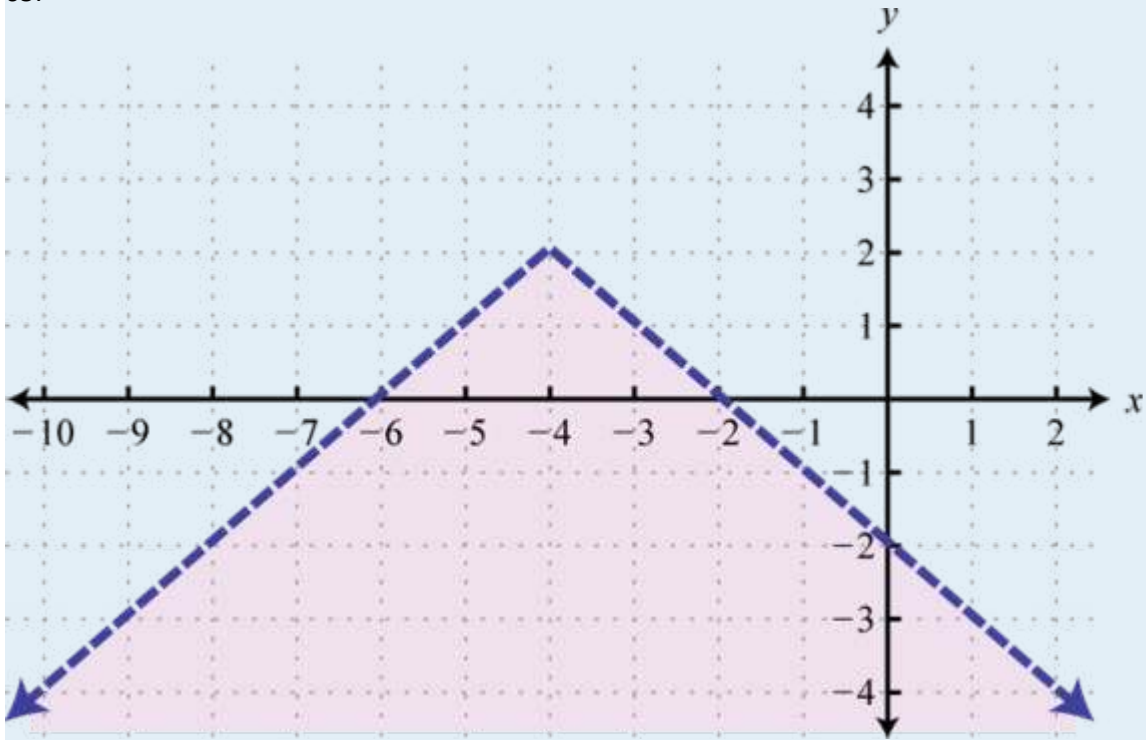


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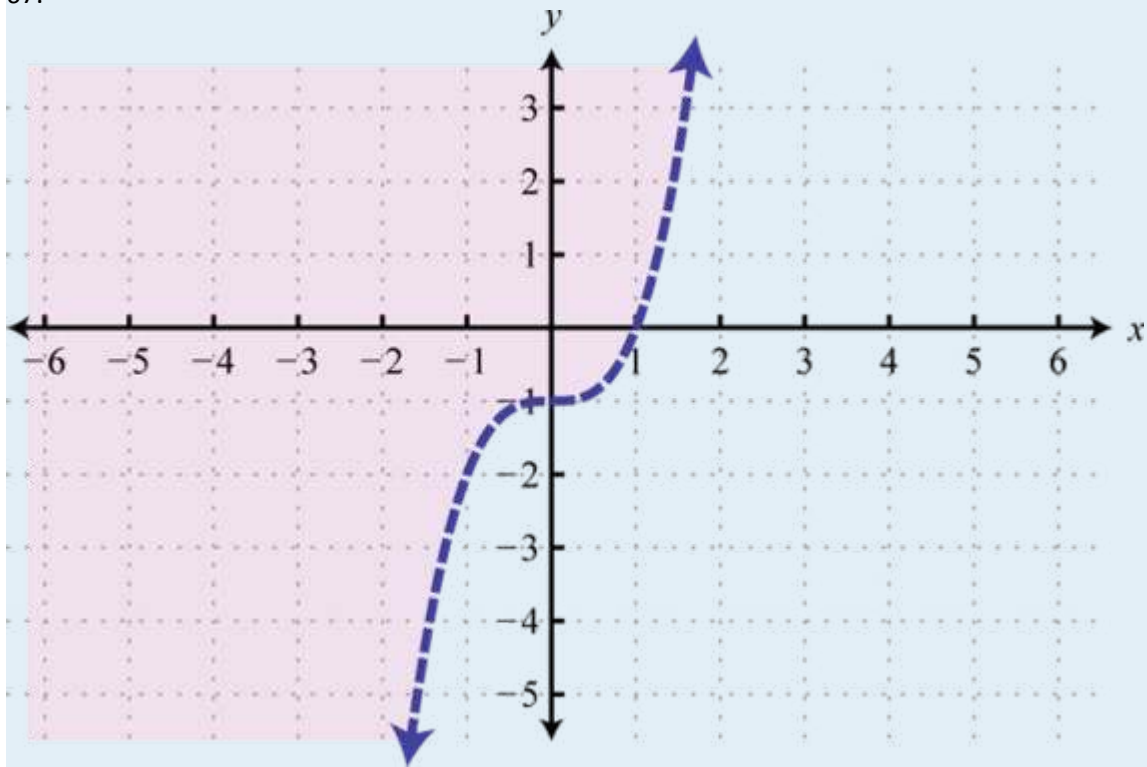


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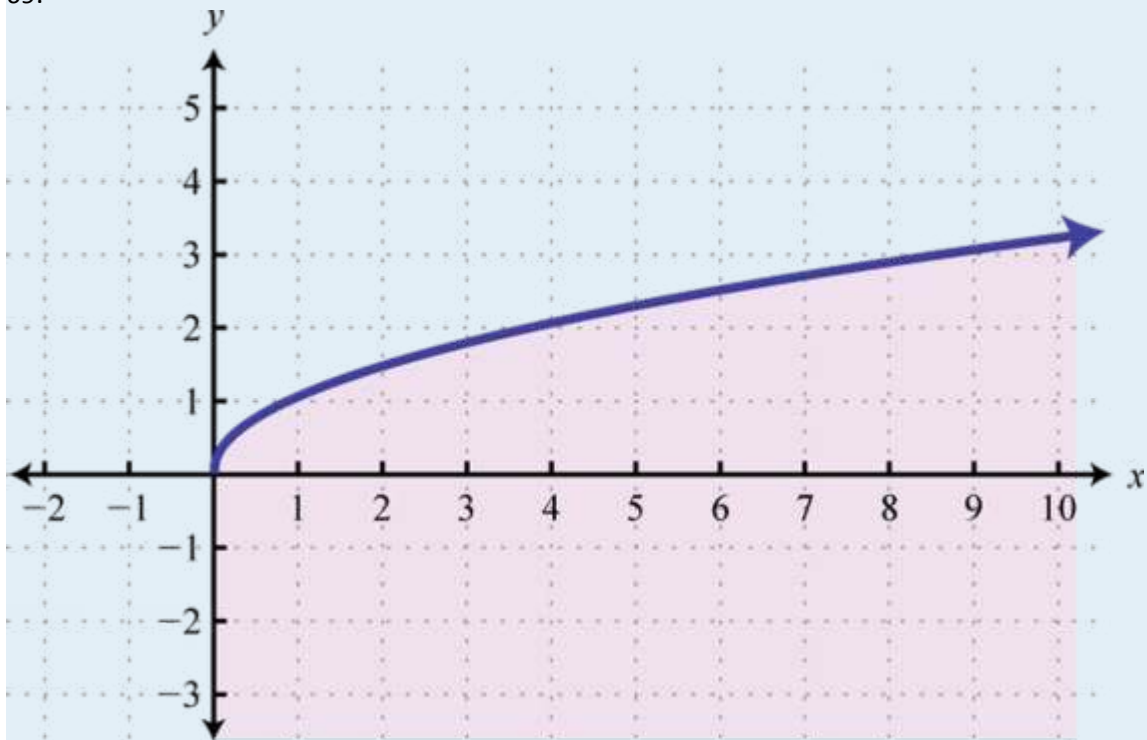


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69.

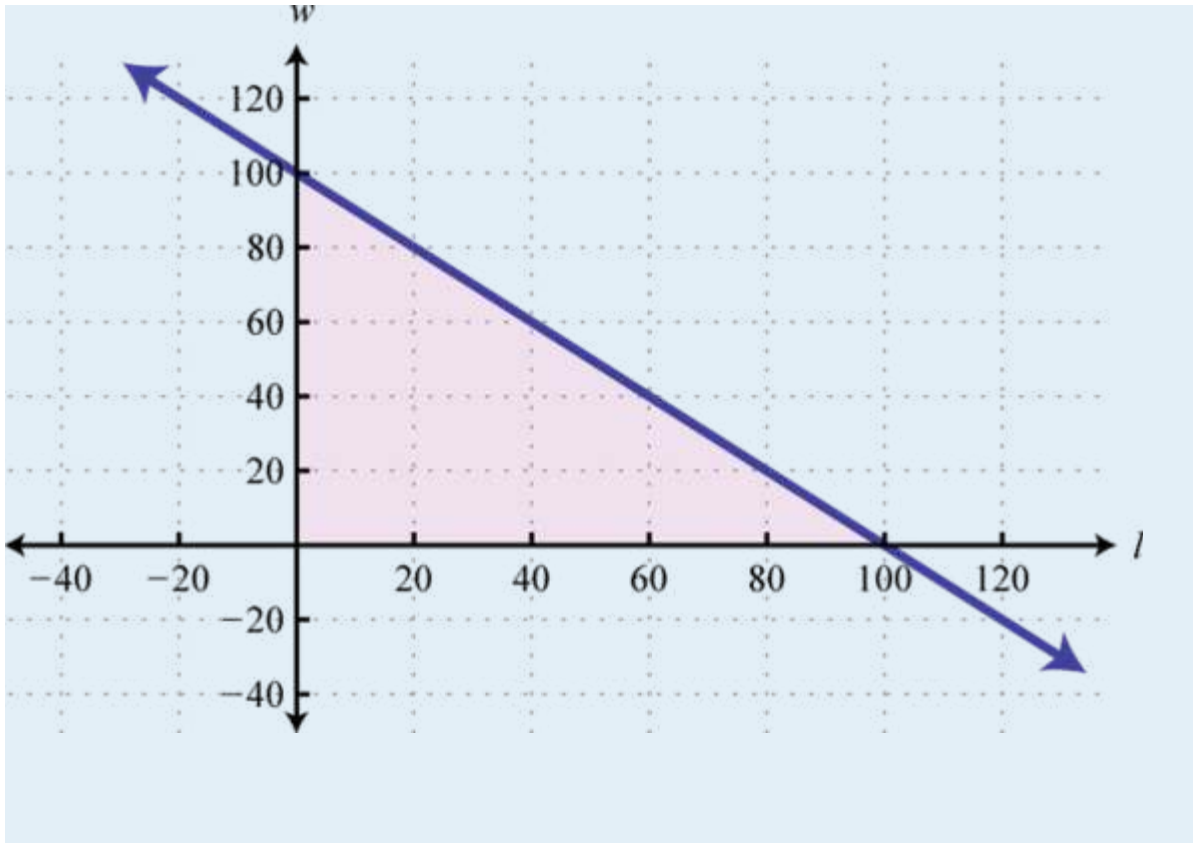


70.

71.  $l + w \leq 100$ ;







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## 2.8 Review Exercises and Sample Exam

### REVIEW EXERCISES RELATIONS, GRAPHS, AND FUNCTIONS

Determine the domain and range and state whether the function is a relation or not.

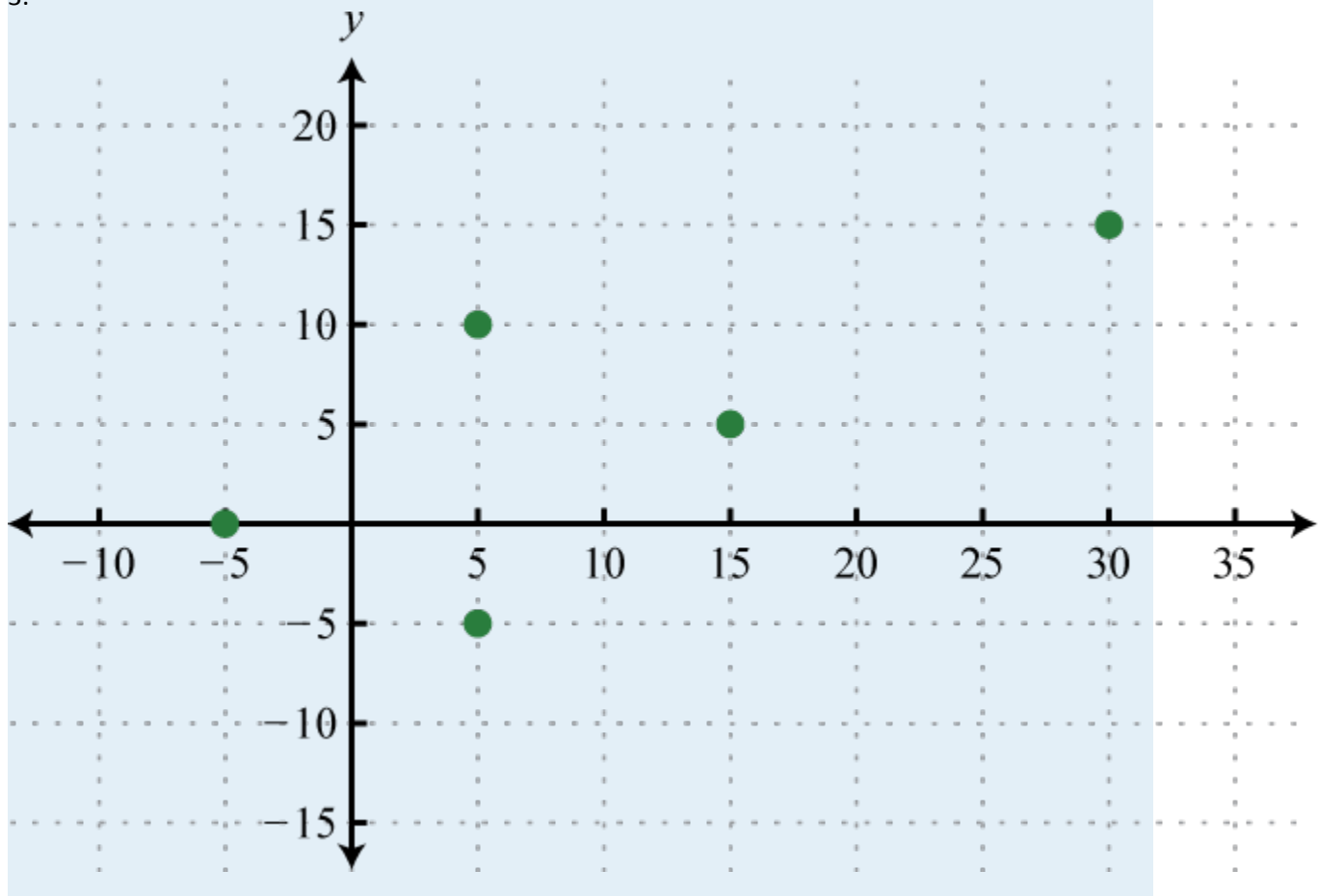
1.  $\{(-4, -1), (-5, 3), (10, 3), (11, 2), (15, 1)\}$
2.  $\{(-3, 0), (-2, 1), (1, 3), (2, 7), (2, 5)\}$

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3.

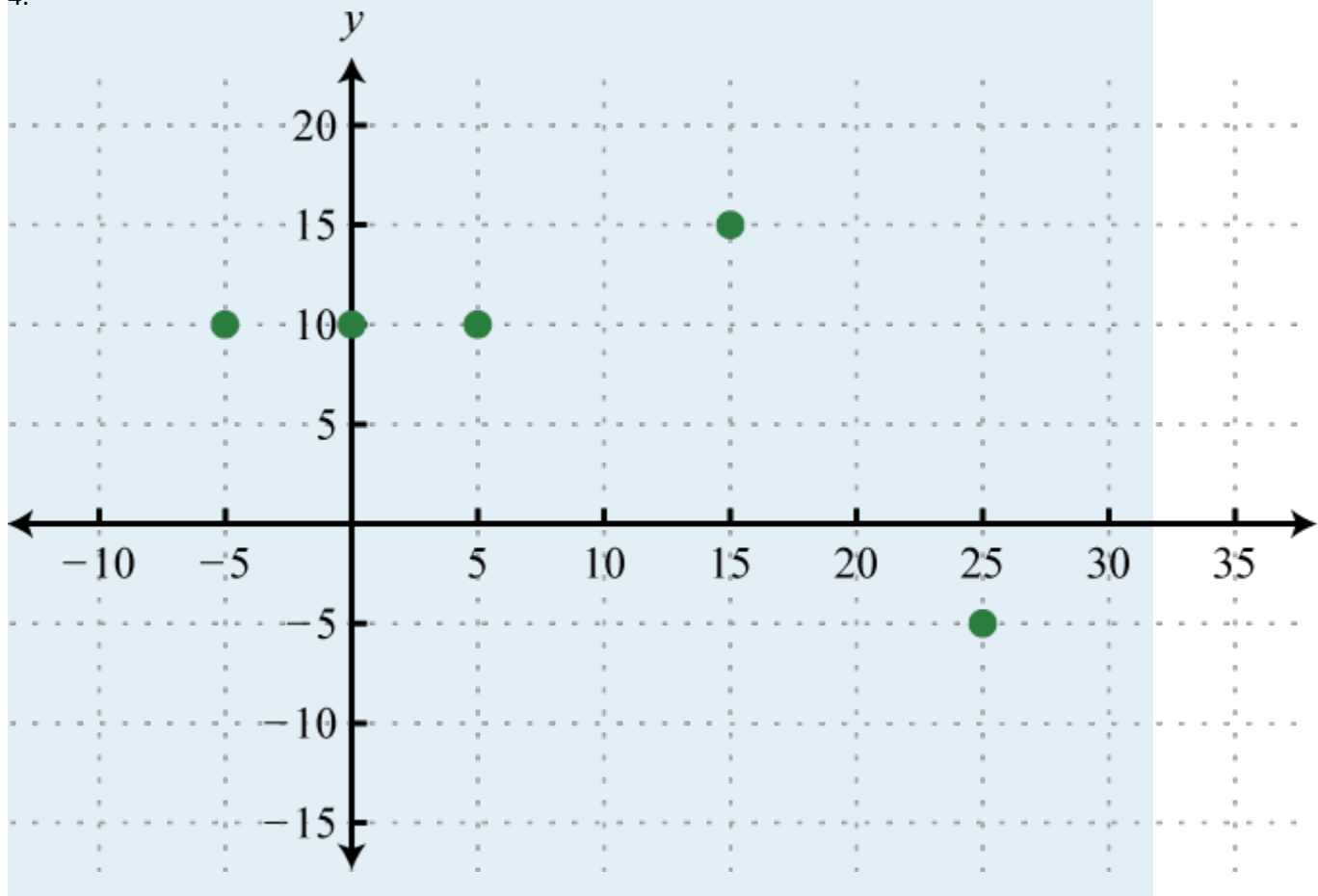


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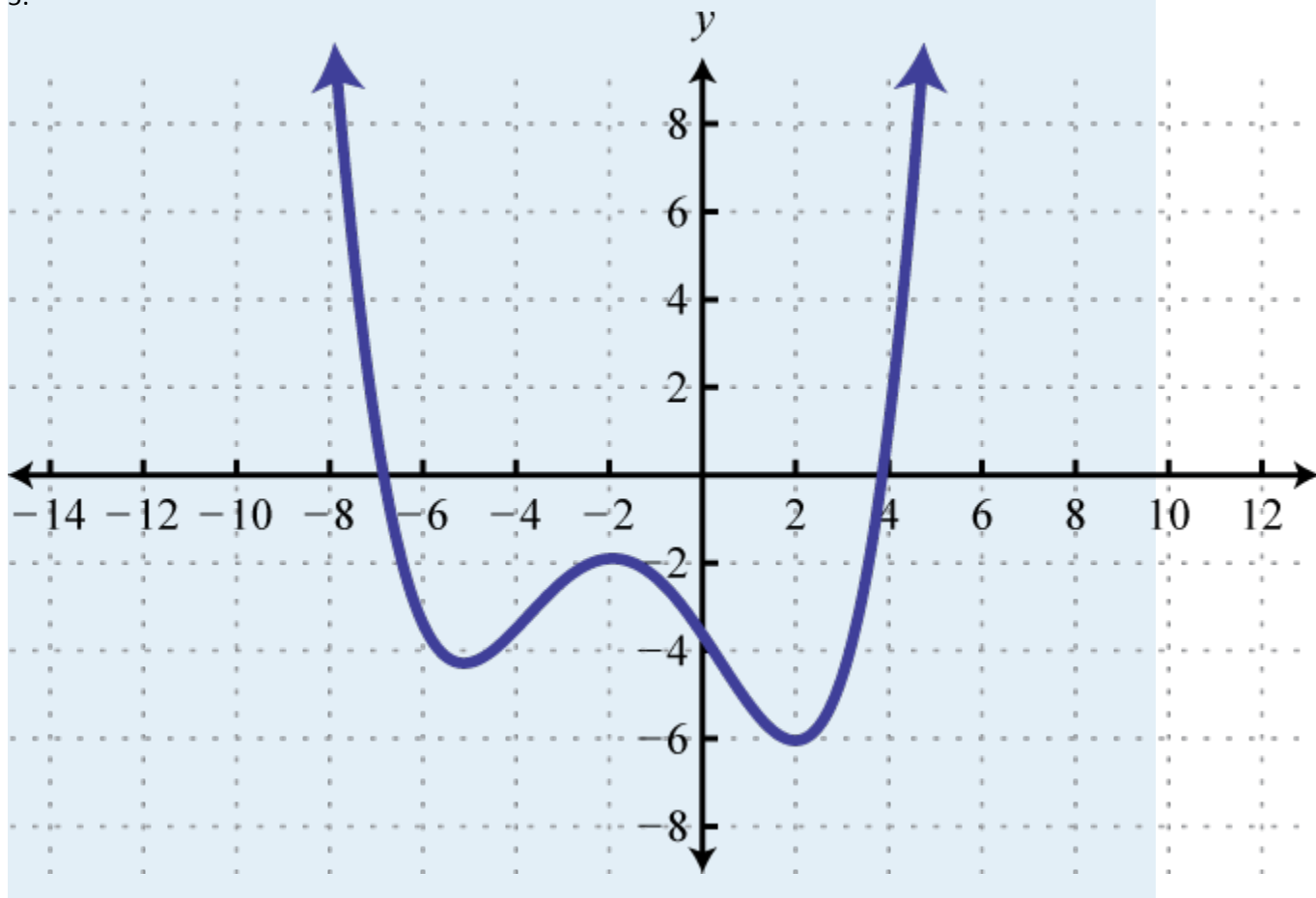
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5.

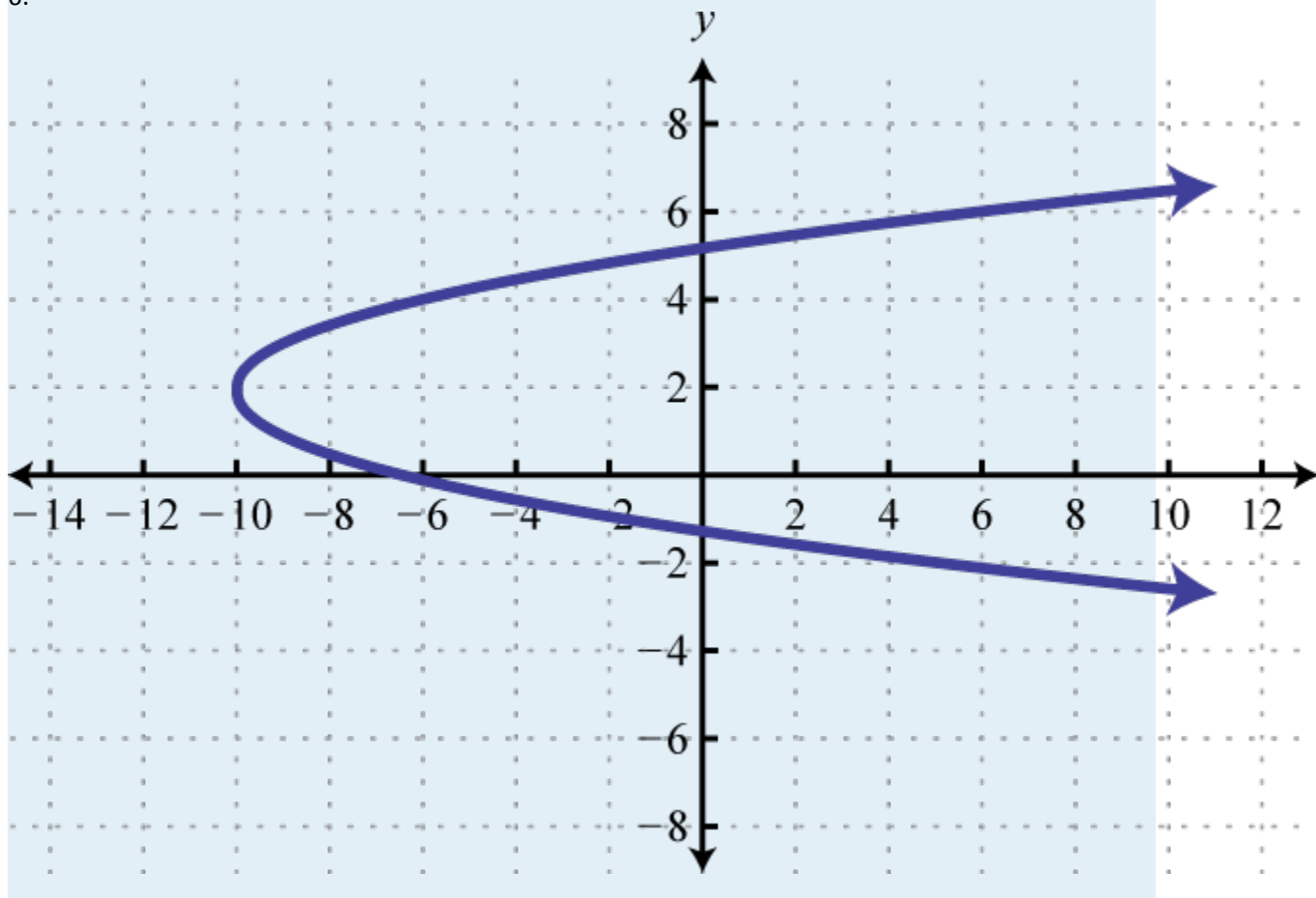


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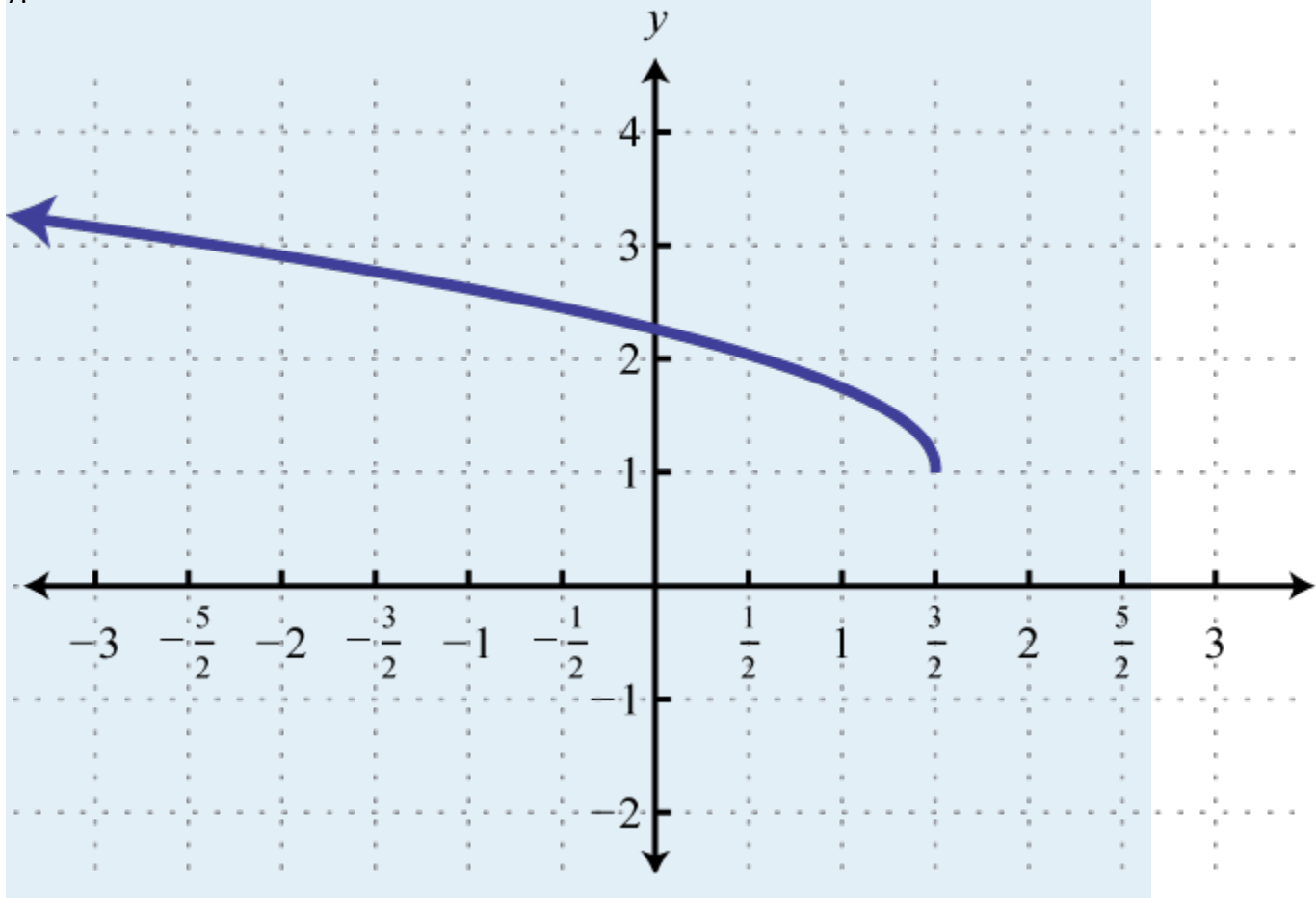
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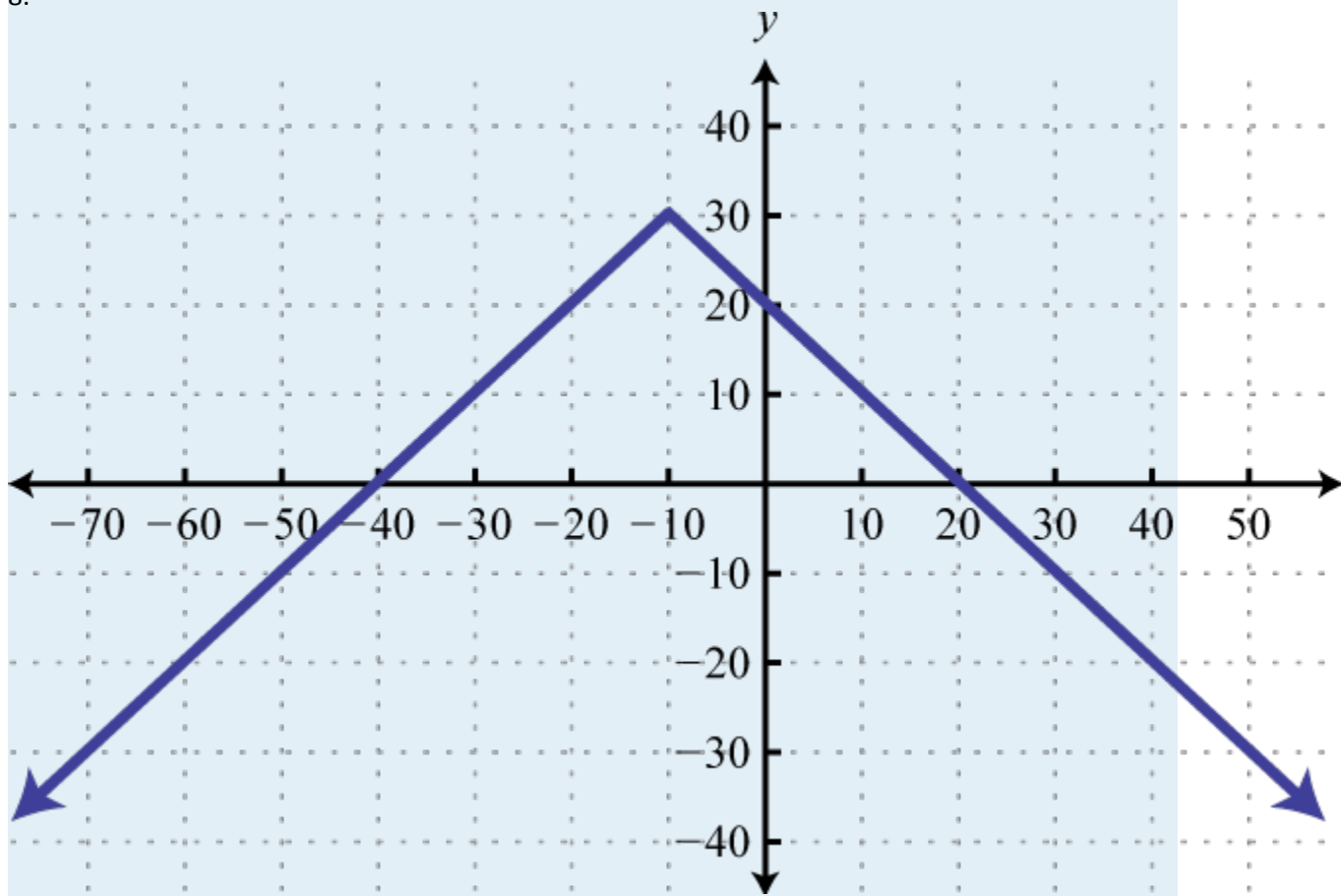


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8.



Evaluate.

1. ;  $h(-8)$ ,  $h(3)$ , and  $h(4a + 1)$
2.  $p(x) = 4 - x$ ;  $p(-10)$ ,  $p(0)$ , and  $p(5a - 1)$
3. ; find  $f(-5)$ ,  $f(0)$ , and  $f(x + h)$
4. ; find  $f(-3)$ ,  $f(2)$ , and  $f(x + h)$
5. ; find  $g(5)$ ,  $g(1)$ ,  $g(13)$
6. ; find  $h(-7)$ ,  $h(-6)$ , and  $h(21)$
7.  $f(x) = 8x + 3$ ; find  $x$  where  $f(x) = 10$ .
8.  $g(x) = 5 - 3x$ ; find  $x$  where  $g(x) = -4$ .

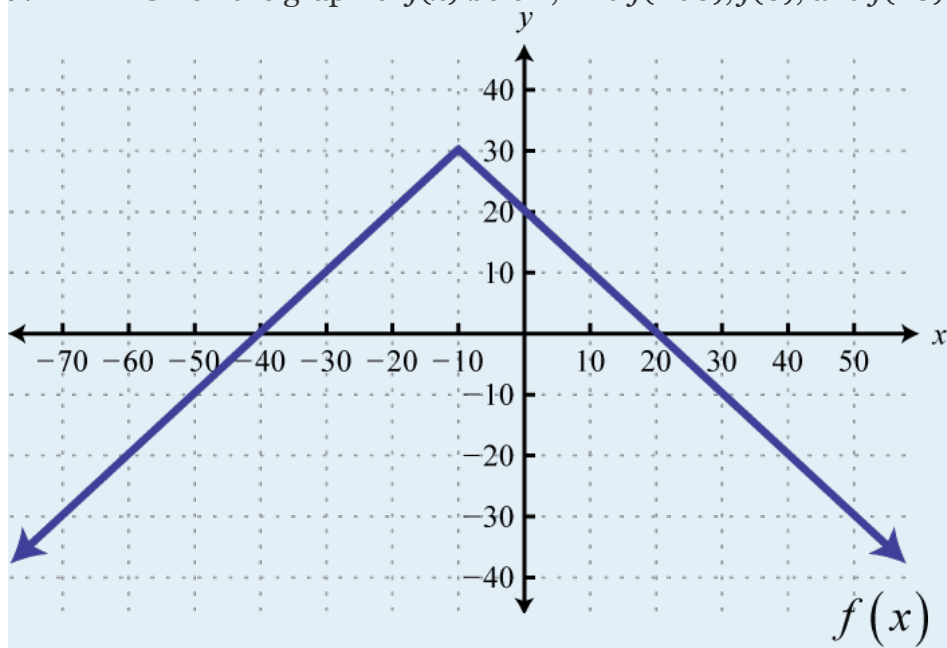
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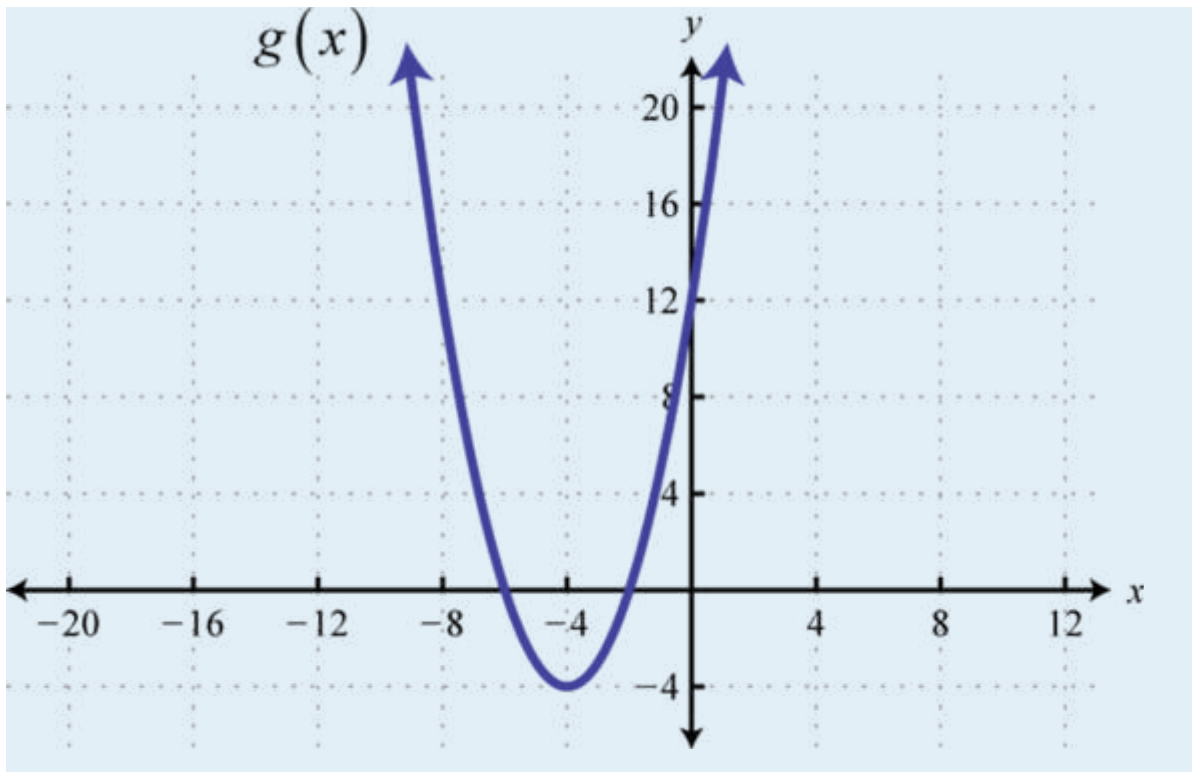


9. Given the graph of  $f(x)$  below, find  $f(-60)$ ,  $f(0)$ , and  $f(20)$ .



10. Given the graph of  $g(x)$  below, find  $x$  where  $g(x) = -4$  and  $g(x) = 12$ .





## LINEAR FUNCTIONS AND THEIR GRAPHS

Graph and label the intercepts.

19.  $4x - 8y = 12$

20.  $9x + 4y = 6$

21.

22.

Graph the linear function and label the  $x$ -intercept.

19.

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20.

21.

22.  $f(x) = 3x - 5$

23.

24.  $h(x) = -6$

Find the slope of the line passing through the given points.

19.  $(-5, 3)$  and  $(-4, 1)$

20.  $(7, -8)$  and  $(-9, -2)$

21. and

22. and

23.  $(-14, 7)$  and  $(-10, 7)$

24.  $(6, -5)$  and  $(6, -2)$

Graph  $f$  and  $g$  on the same rectangular coordinate plane. Use the graph to find all values of  $x$  for which the given relation is true. Verify your answer algebraically.

19.  $\therefore f(x) = g(x)$

20.  $f(x) = 5x - 2$ ,  $g(x) = 3$ ;  $f(x) \geq g(x)$

21.  $f(x) = -4x + 3$ ,  $g(x) = -x + 6$ ;  $f(x) < g(x)$

22.  $\therefore f(x) \leq g(x)$

## MODELING LINEAR FUNCTIONS

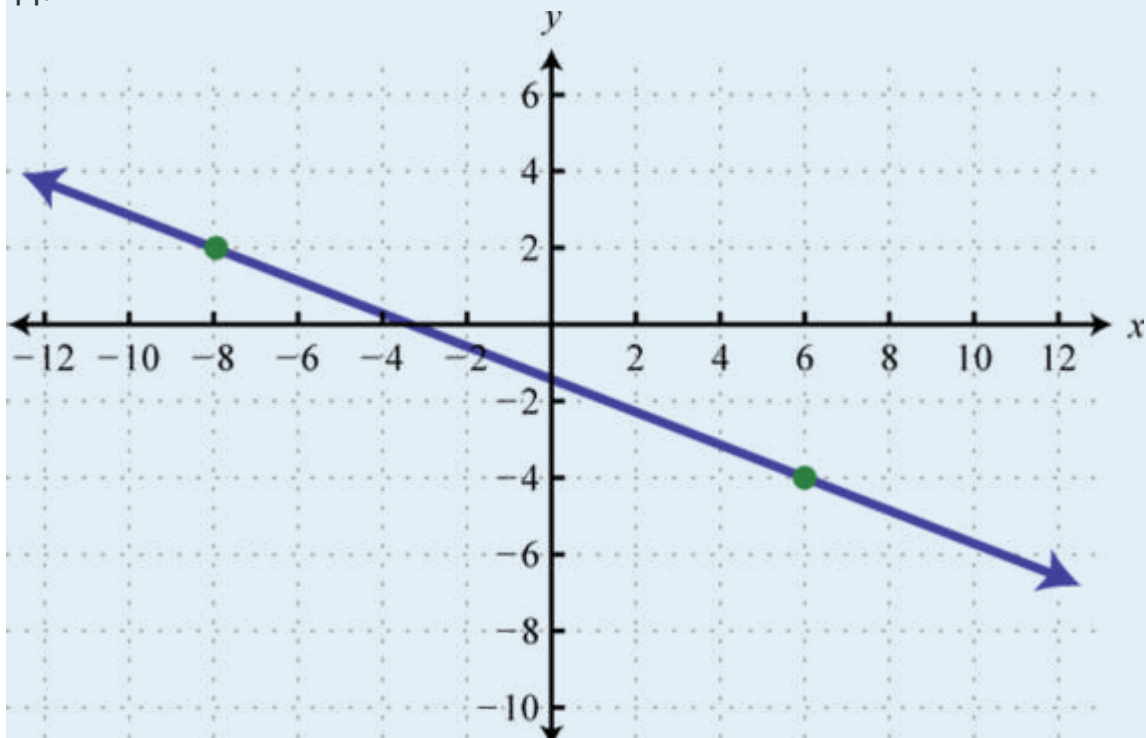
Find the linear function passing through the given points.

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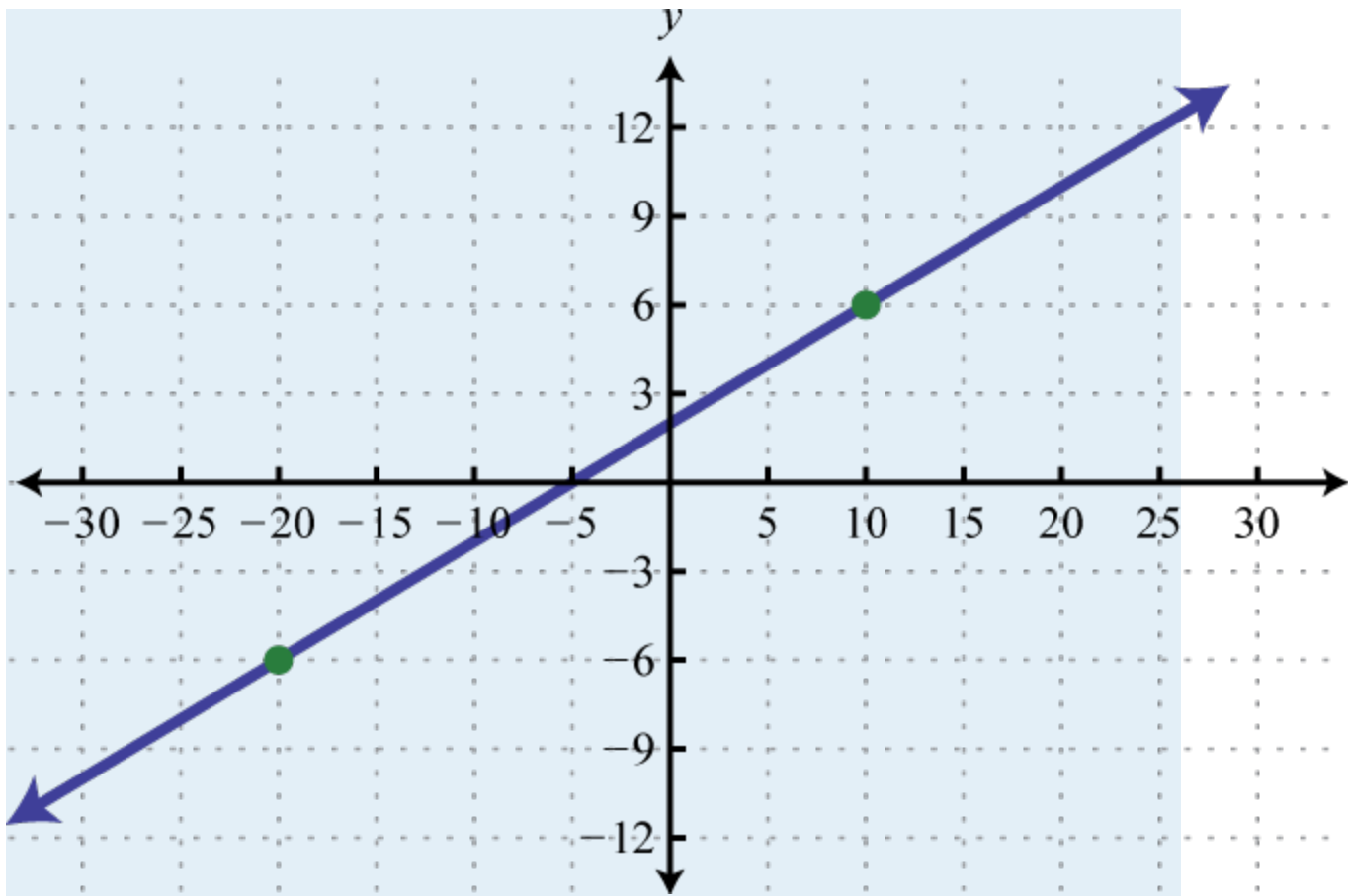


39.  $(1, -5)$  and  
40. and  $(-2, 8)$   
41.  $(7, -6)$  and  $(5, -7)$   
42.  $(-5, -6)$  and  $(-3, -9)$   
43. Find the equation of the given linear function:  
44.



45. Find the equation of the given linear function:





Find the equation of the line:

39. Parallel to  $8x - 3y = 24$  and passing through  $(-9, 4)$ .
40. Parallel to  $6x + 2y = 24$  and passing through
41. Parallel to and passing through  $(-4, -1)$ .
42. Perpendicular to  $14x + 7y = 10$  and passing through  $(8, -3)$ .
43. Perpendicular to  $15x - 3y = 6$  and passing through  $(-3, 1)$ .
44. Perpendicular to and passing through  $(2, -7)$ .



Use algebra to solve the following.

39. A taxi fare in a certain city includes an initial charge of \$2.50 plus \$2.00 per mile driven. Write a function that gives the cost of a taxi ride in terms of the number of miles driven. Use the function to determine the number of miles driven if the total fare is \$9.70.

40. A salesperson earns a base salary of \$1,800 per month and 4.2% commission on her total sales for that month. Write a function that gives her monthly salary based on her total sales. Use the function to determine the amount of sales for a month in which her salary was \$4,824.

41. A certain automobile sold for \$1,200 in 1980, after which it began to be considered a collector's item. In 1994, the same automobile sold at auction for \$5,750. Write a linear function that models the value of the automobile in terms of the number of years since 1980. Use it to estimate the value of the automobile in the year 2000.

42. A specialized industrial robot was purchased new for \$62,400. It has a lifespan of 12 years, after which it will be considered worthless. Write a linear function that models the value of the robot. Use the function to determine its value after 8 years of operation.

43. In 1950, the U.S. Census Bureau estimated the population of Detroit, MI to be 1.8 million people. In 1990, the population was estimated to have decreased to 1 million. Write a linear function that gives the population of Detroit in millions of people, in terms of years since 1950. Use the function to estimate the year in which the population decreased to 700,000 people.

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44. Online sales of a particular product are related to the number of clicks on its advertisement. It was found that 100 clicks in a week result in \$112 of online sales, and that 500 clicks result in \$160 of online sales. Write a linear function that models the online sales of the product based on the number of clicks on its advertisement. How many clicks are needed to result in \$250 of weekly online sales from this product?

45. The cost in dollars of producing  $n$  bicycles is given by the formula  $C(n) = 80n + 3,380$ . If each bicycle can be sold for \$132, write a function that gives the profit generated by producing and selling  $n$  bicycles. Use the formula to determine the number of bicycles that must be produced and sold to profit at least \$10,000.

46. Determine the breakeven point from the previous exercise.

## BASIC FUNCTIONS

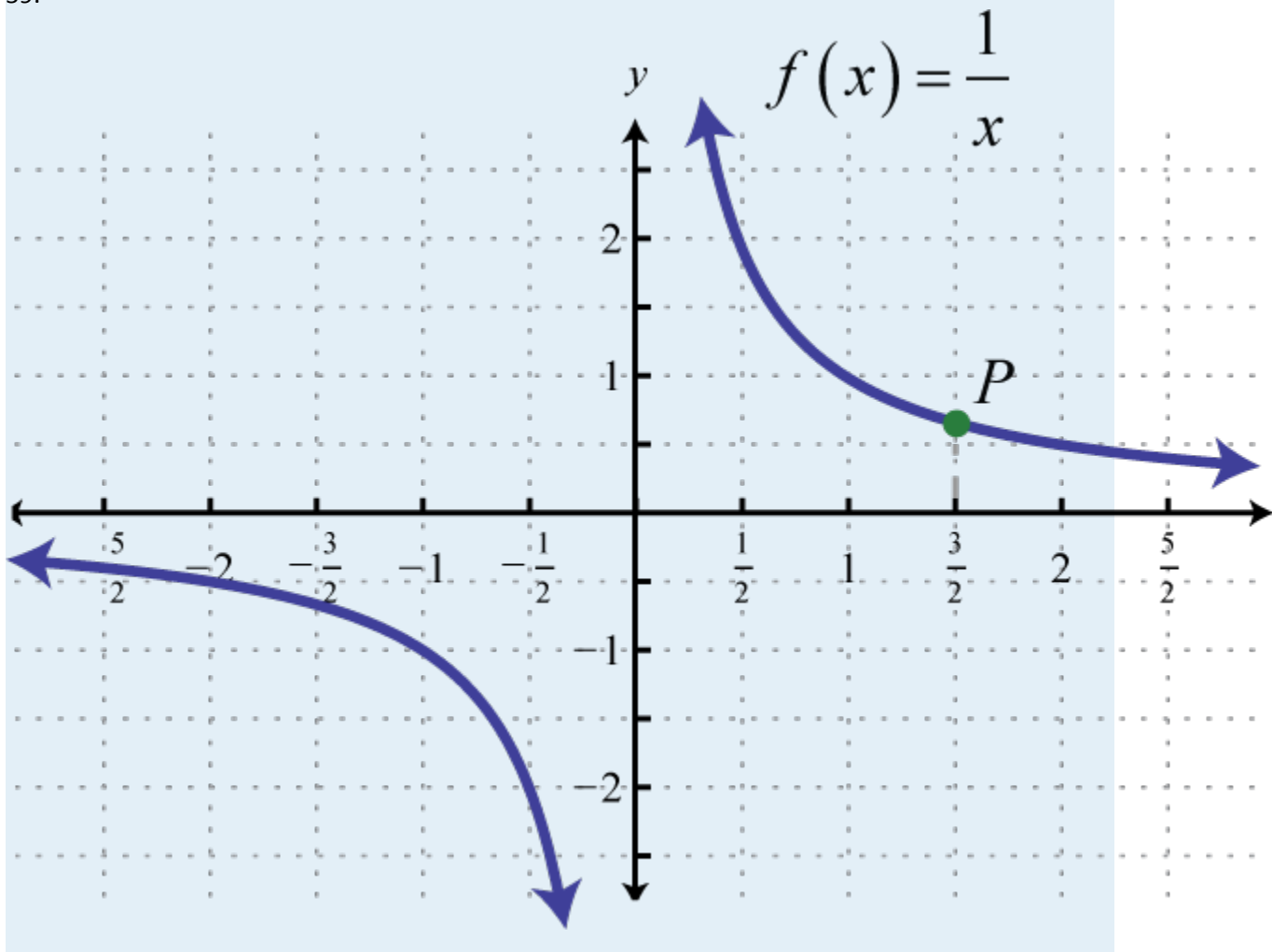
Find the ordered pair that specifies the point  $P$ .

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59.

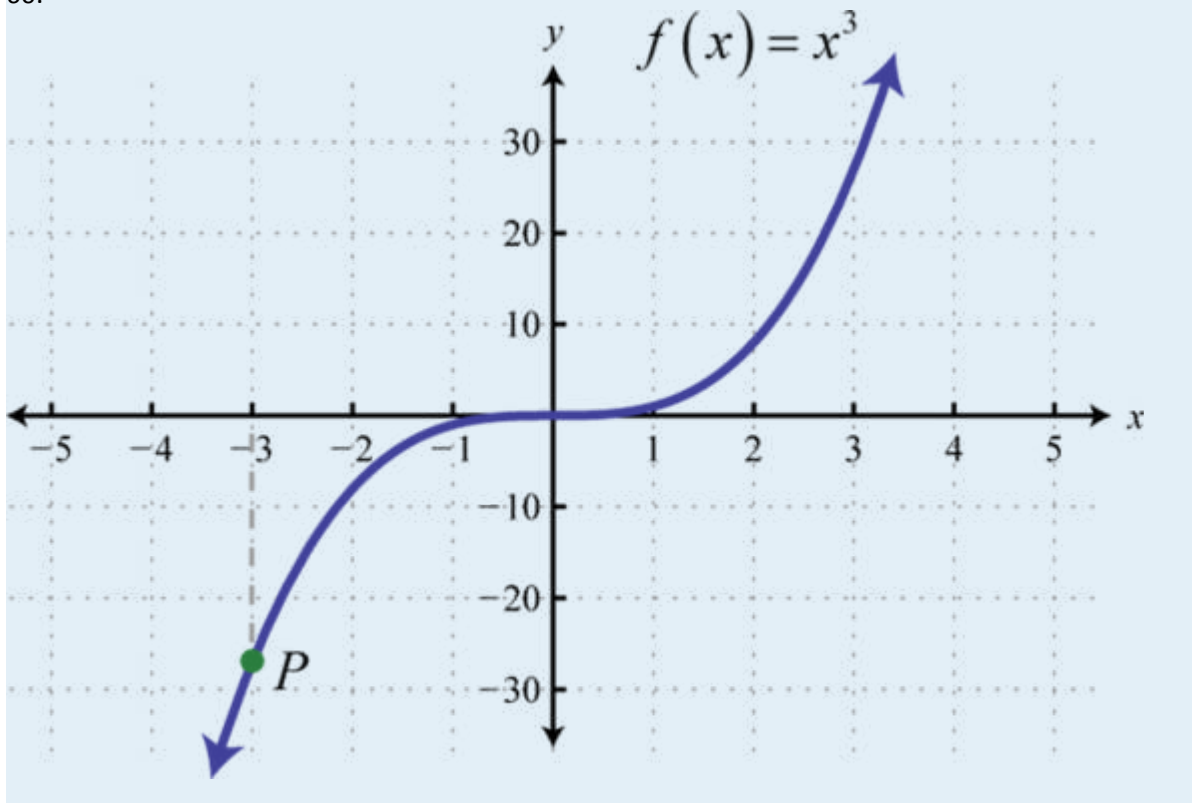


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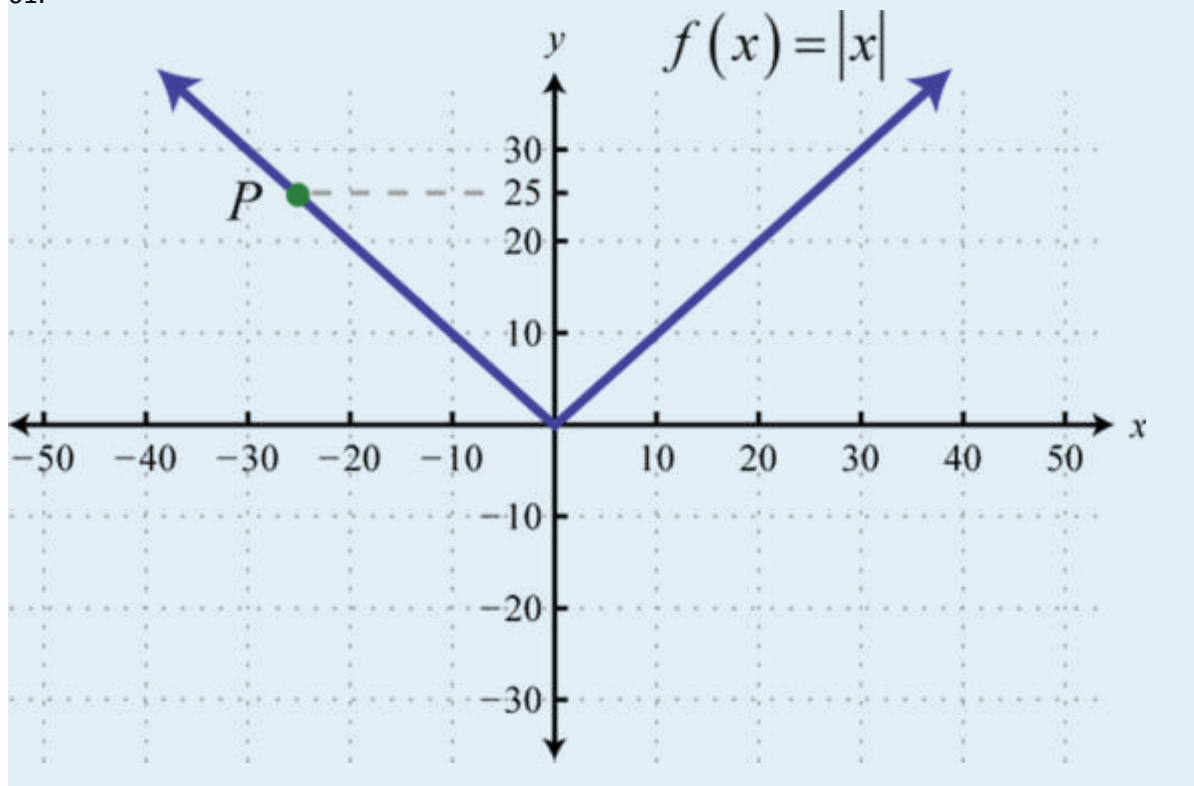
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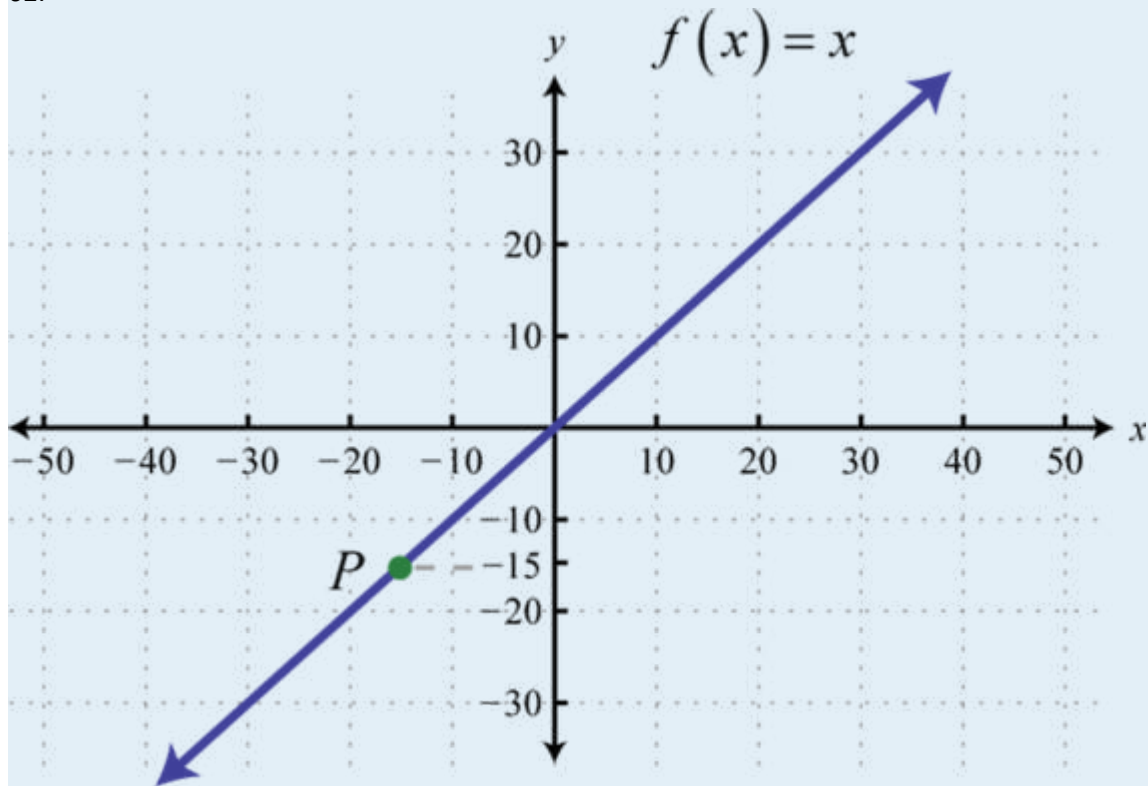


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62.



Graph the piecewise defined functions.

59.

60.

61.

62.

63.

64.

65.

66.  $g(x) = \lceil \lceil x \rceil \rceil + 2$



Evaluate.

59.  
Find  $f(-10)$ ,  $f(-6)$ , and  $f(0)$ .

60.  
Find  $h(-1)$ ,  $h(0)$ , and  $h(12)$ .

61.  
Find  $g(-10)$ ,  $g(0)$  and  $g(8)$ .

62.  
Find  $q(-53)$ ,  $q(1)$  and  $q(16)$ .

## TRANSFORMATIONS

Sketch the graph of the given function.

75.

76.

77.  $p(x) = x - 9$

78.

79.  $f(x) = |x - 20| - 40$

80.

81.

82.

83.

84.

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85.  $g(x) = -|x + 4| - 8$

86.

87.

88.

89.  $g(x) = -2|x + 10| + 8$

90.

91.

92.

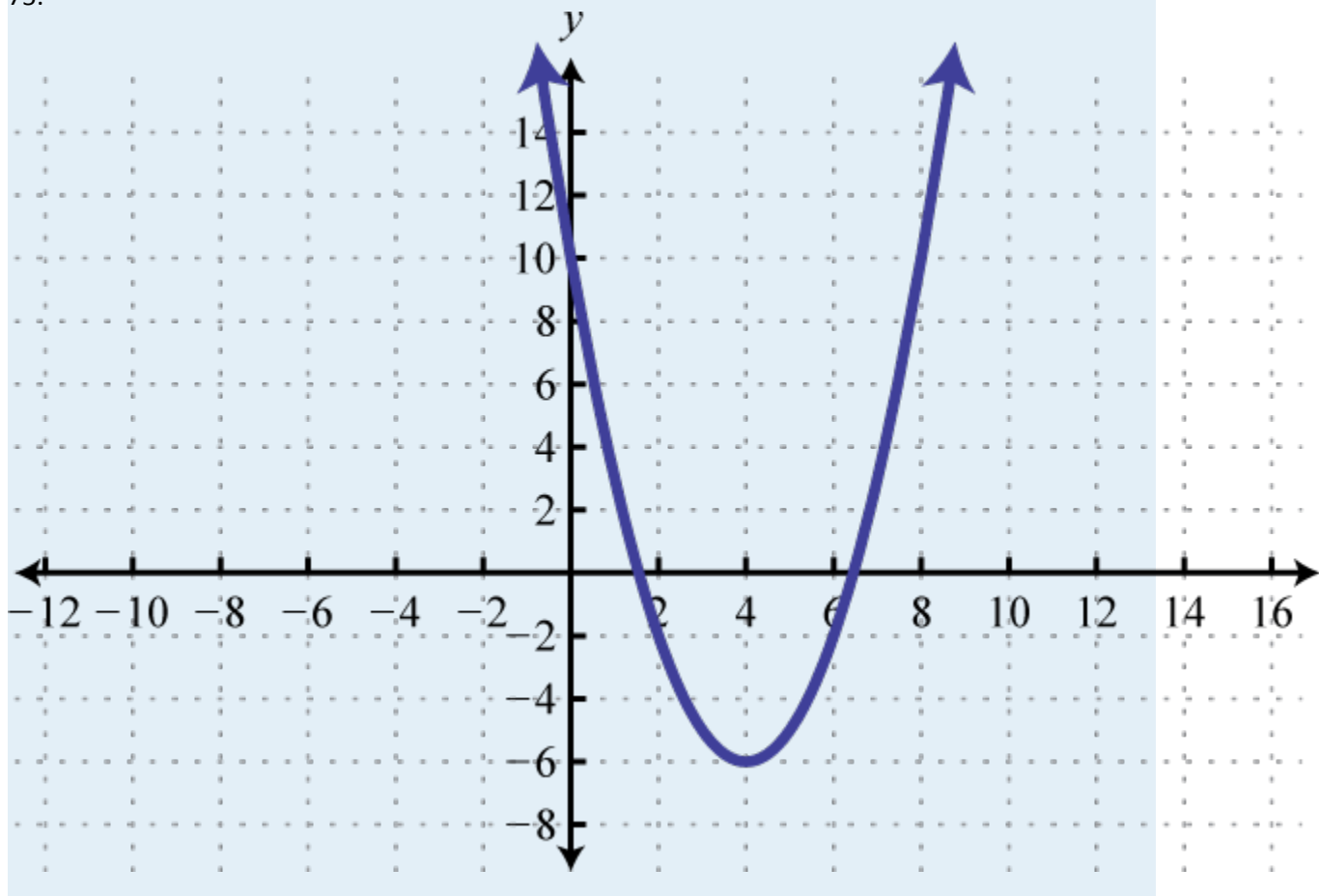
Write an equation that represents the function whose graph is given.

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75.

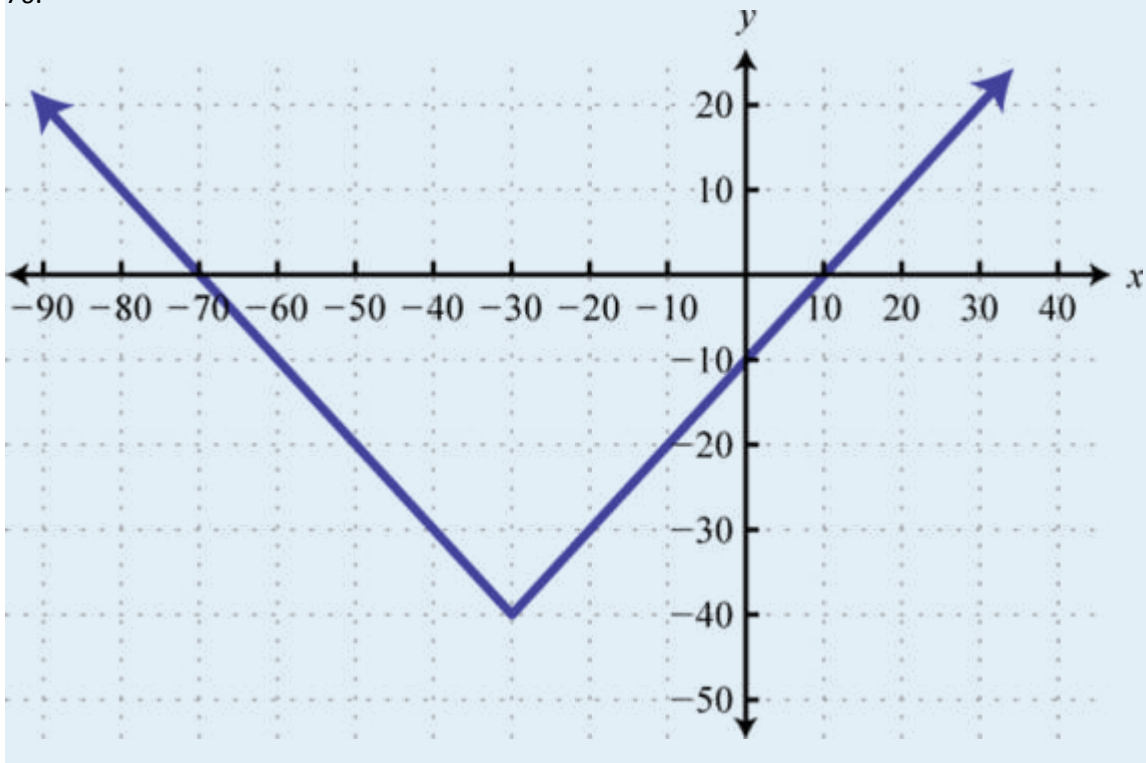


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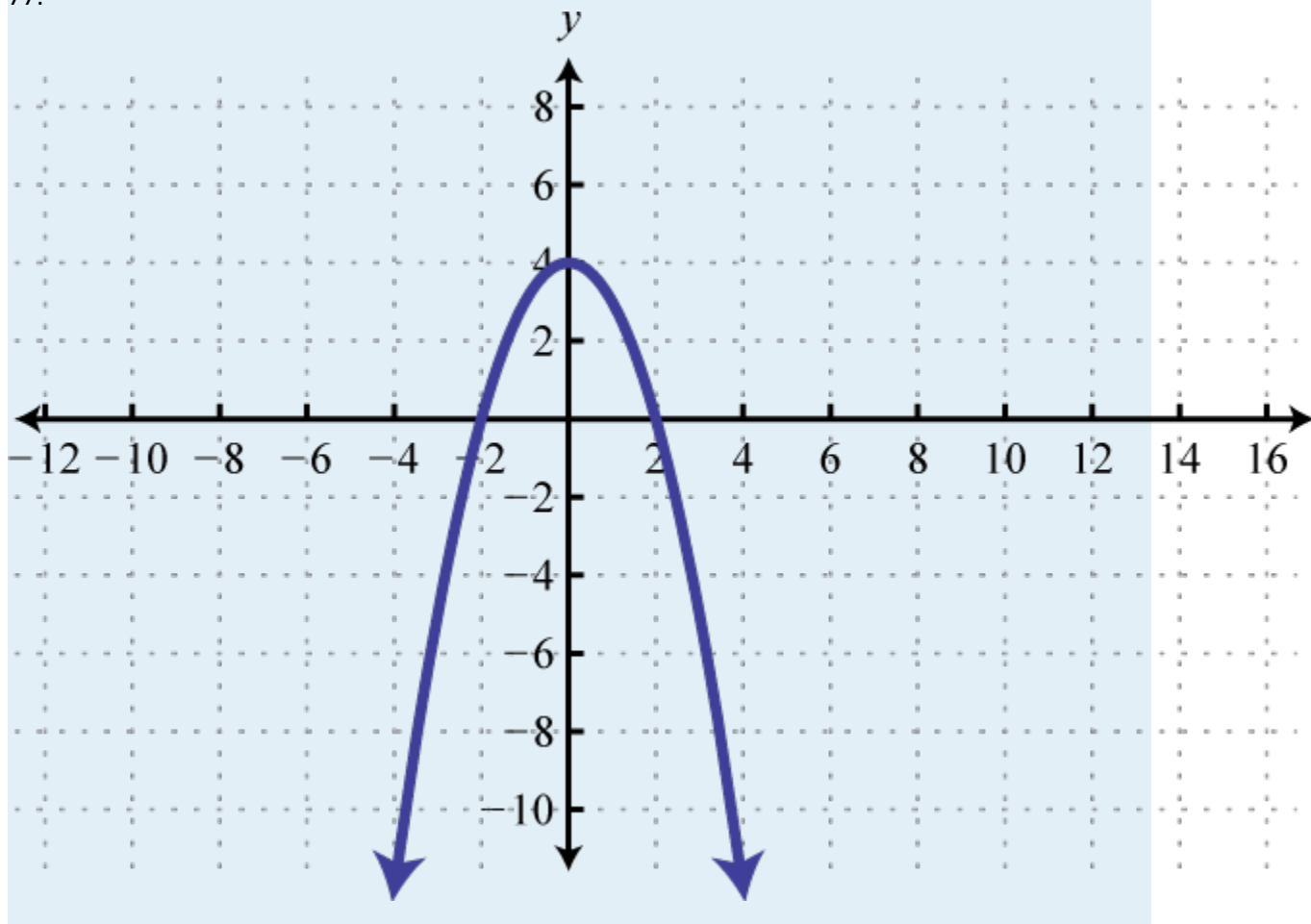


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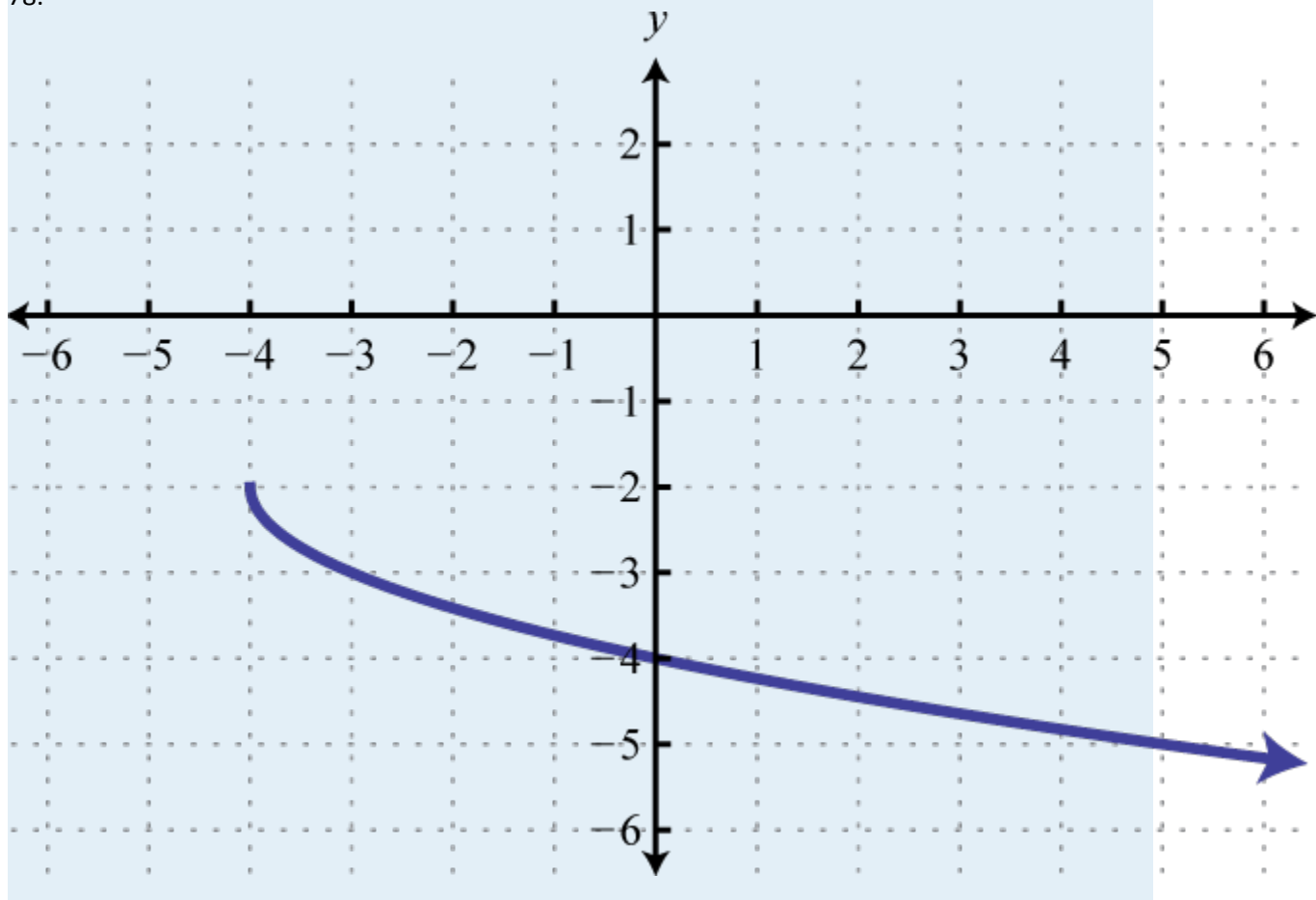
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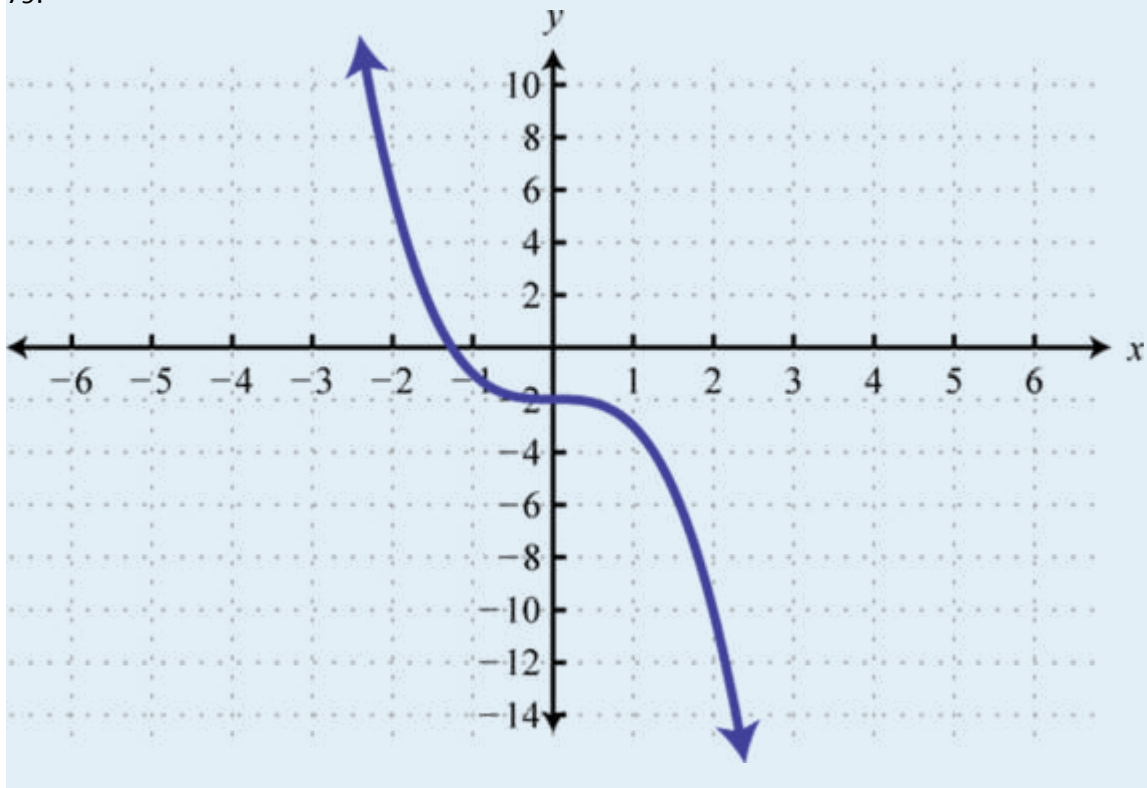


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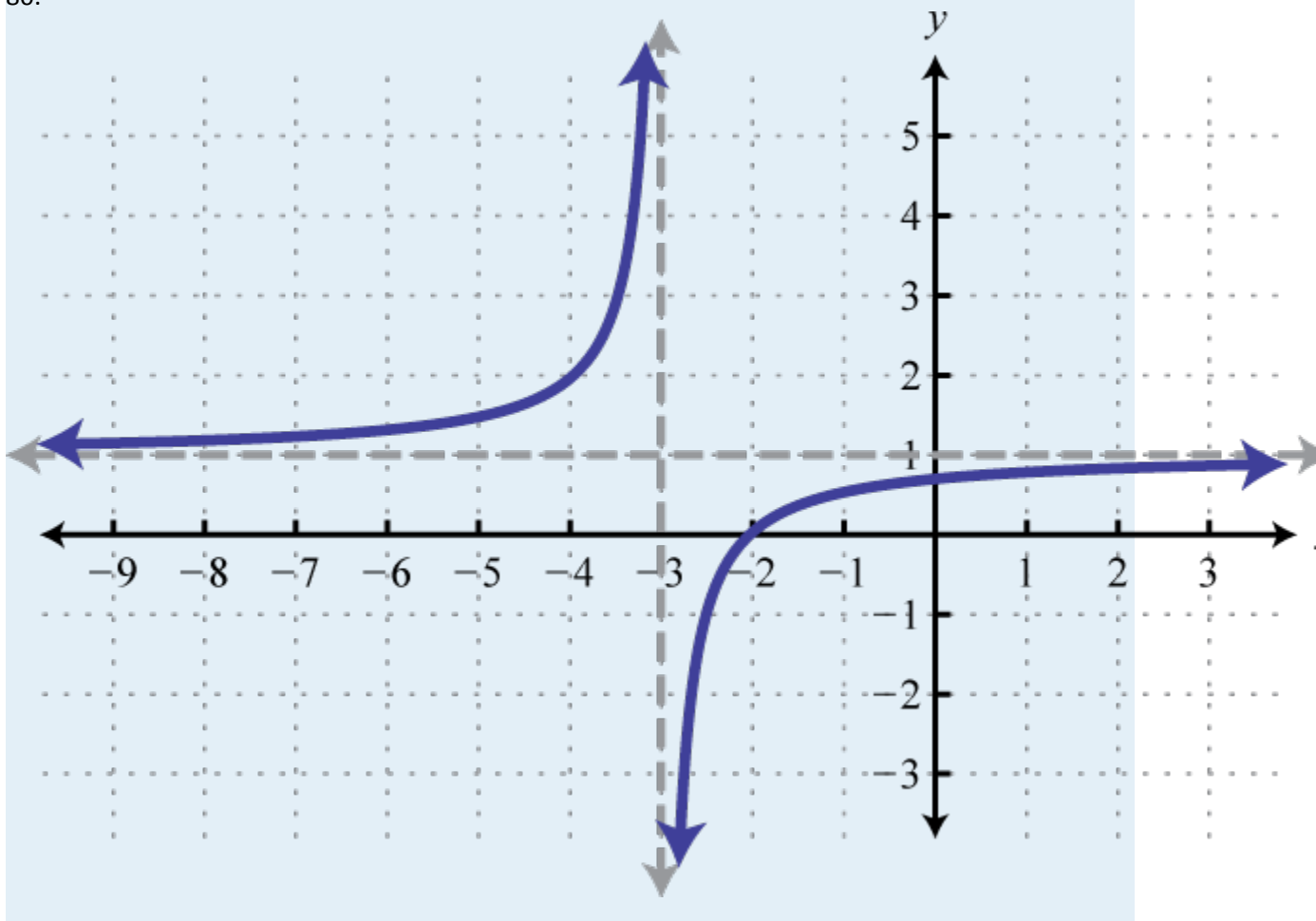


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80.

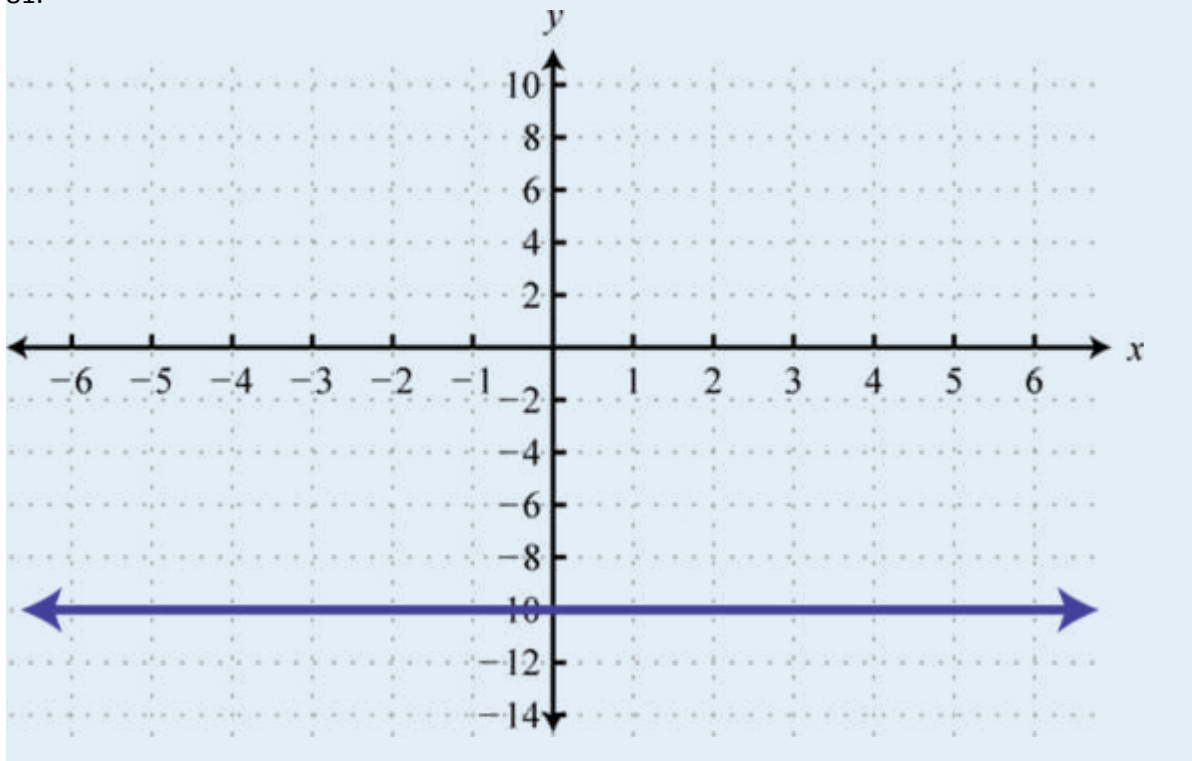


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81.

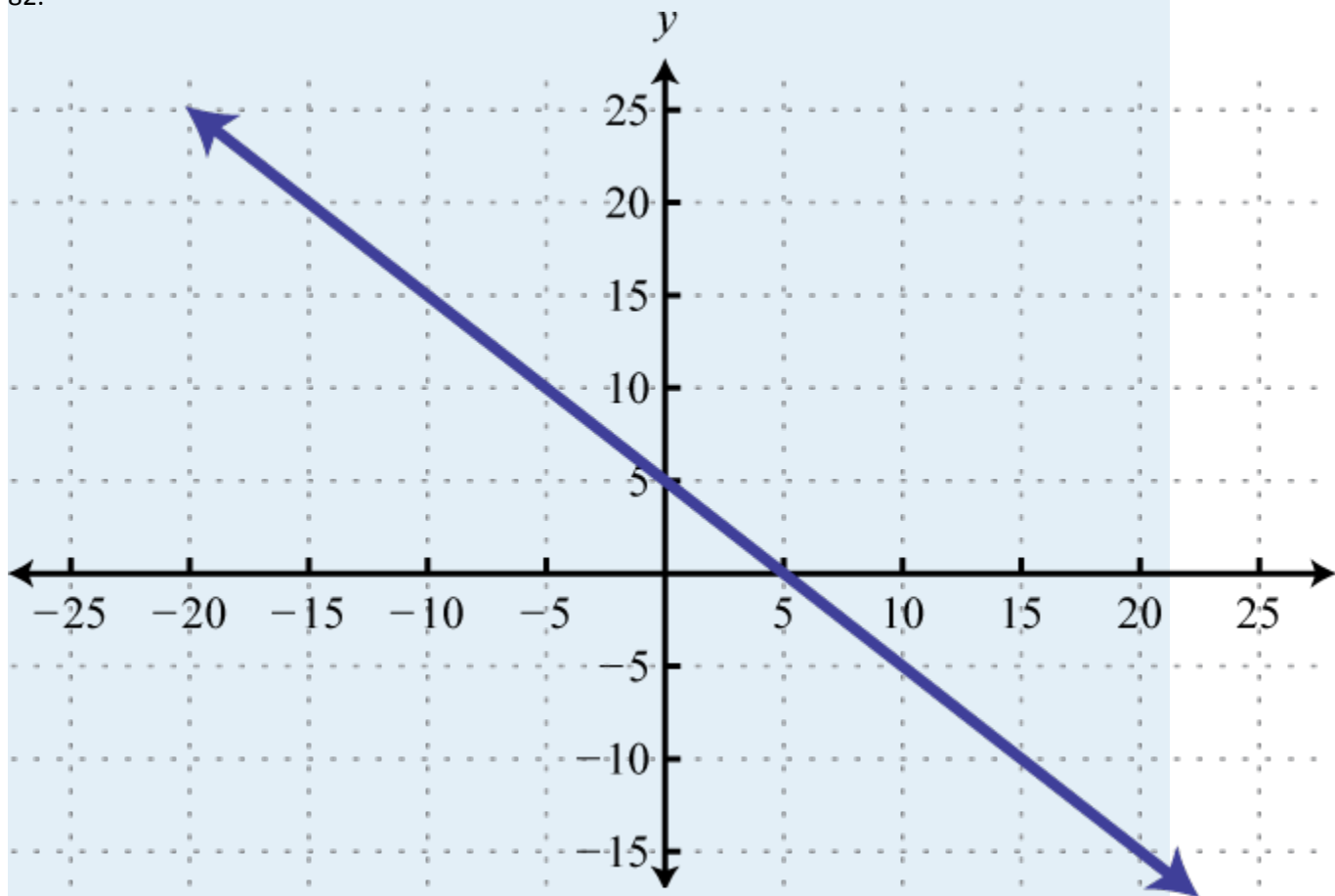


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82.



## SOLVING ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

Solve.

101.  $|5x-4| = 14$

102.  $|4-3x| = 4$

103.  $9-5|x-4| = 4$

104.  $6 + 2|x + 10| = 12$

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105.  $|3x-6| + 5 = 5$   
106.  $0.2|x-1.8| = 4.6$   
107.  
108.  
109.  $|3x-9| = |4x + 3|$   
110.  $|9x-7| = |3 + 8x|$

Solve. Graph the solutions on a number line and give the corresponding interval notation.

101.  $|2x + 3| < 1$   
102.  $|10x-15| \leq 25$   
103.  $|6x-1| \leq 11$   
104.  $|x-12| > 7$   
105.  
106.  $5-|x + 6| \geq 4$   
107.  $|3x + 1| + 7 \leq 4$   
108.  $2|x-3| + 6 > 4$   
109.  
110.  $6.4-3.2|x + 1.6| > 0$

## INEQUALITIES WITH TWO VARIABLES

Is the ordered pair a solution to the given inequality?

121.  $9x-2y < -1$ ;  $(-1, -3)$

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122. ; (1, -12)
123. ;
124.  $x - y \leq -6$ ; (-1, 7)
125. ; (-3, 5)
126.  $y > |x - 6| + 10$ ; (-4, 12)
127. ; (-1, 0)
128. ; (-3, 4)

Graph the solution set.

121.  $x + y < 6$
122.  $2x - 3y \geq 9$
123.  $3x - y \leq 6$
124.  $y + 4 > 0$
125.  $x - 6 \geq 0$
- 126.
- 127.
- 128.
129.  $y < -|x| + 9$
130.  $y > |x - 12| + 3$
- 131.
- 132.

## ANSWERS

1. Domain:  $\{-5, -4, 10, 11, 15\}$ ; range:  $\{-1, 1, 2, 3\}$ ; function: yes

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2.

3. Domain:  $\{-5, 5, 15, 30\}$ ; range:  $\{-5, 0, 5, 10, 15\}$ ; function: no

4.

5. Domain:  $(-\infty, \infty)$ ; range:  $[-6, \infty)$ ; function: yes

6.

7. Domain: ; range:  $[1, \infty)$ ; function: yes

8.

9.  $h(-8) = -7$ , , and

10.

11.  $f(-5) = 58, f(0) = 3$ , and

12.

13.  $g(5) = 3, g(1) = 1, g(13) = 5$

14.

15.

16.

17.  $f(-60) = -20, f(0) = 20, f(20) = 0$

18.

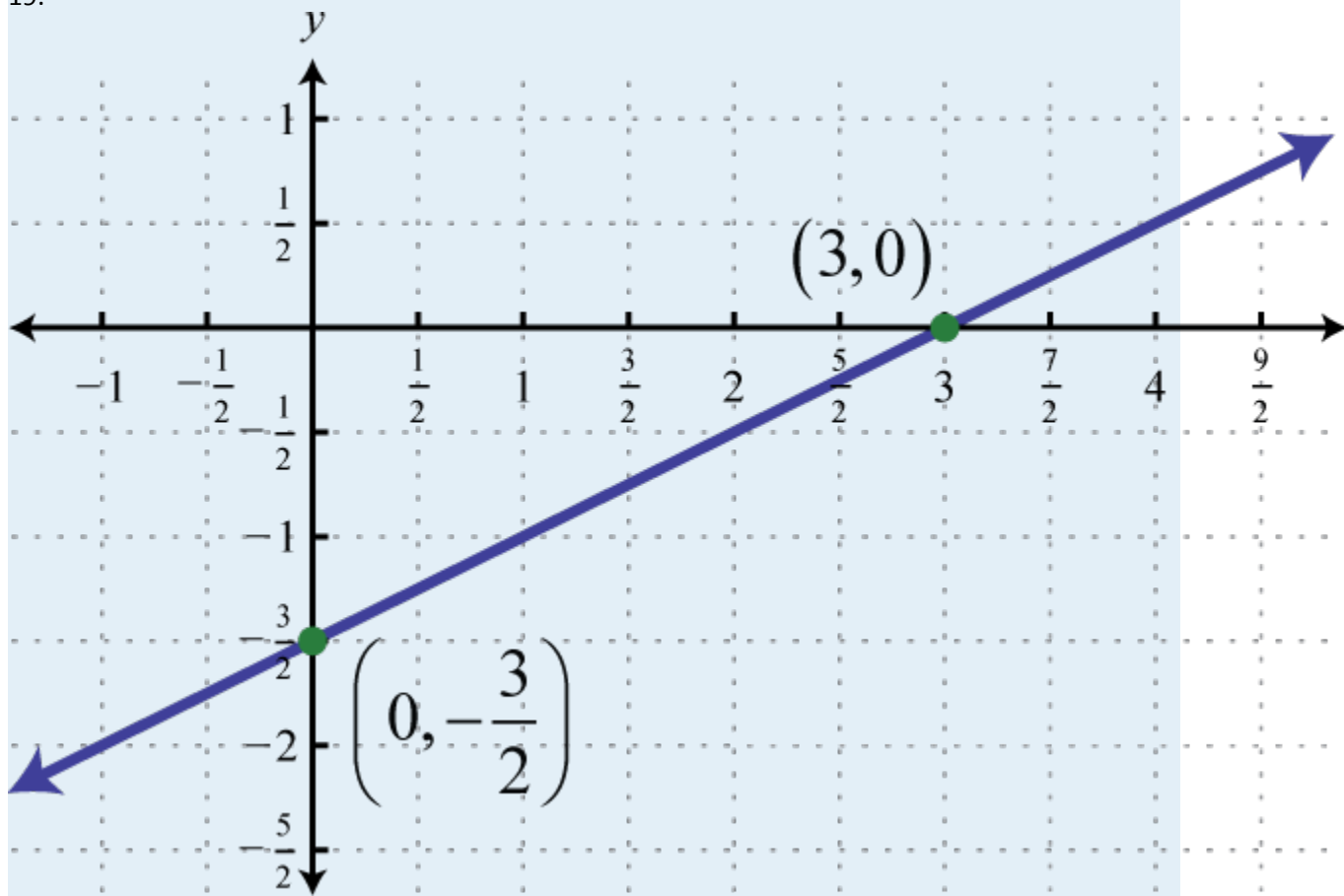
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19.



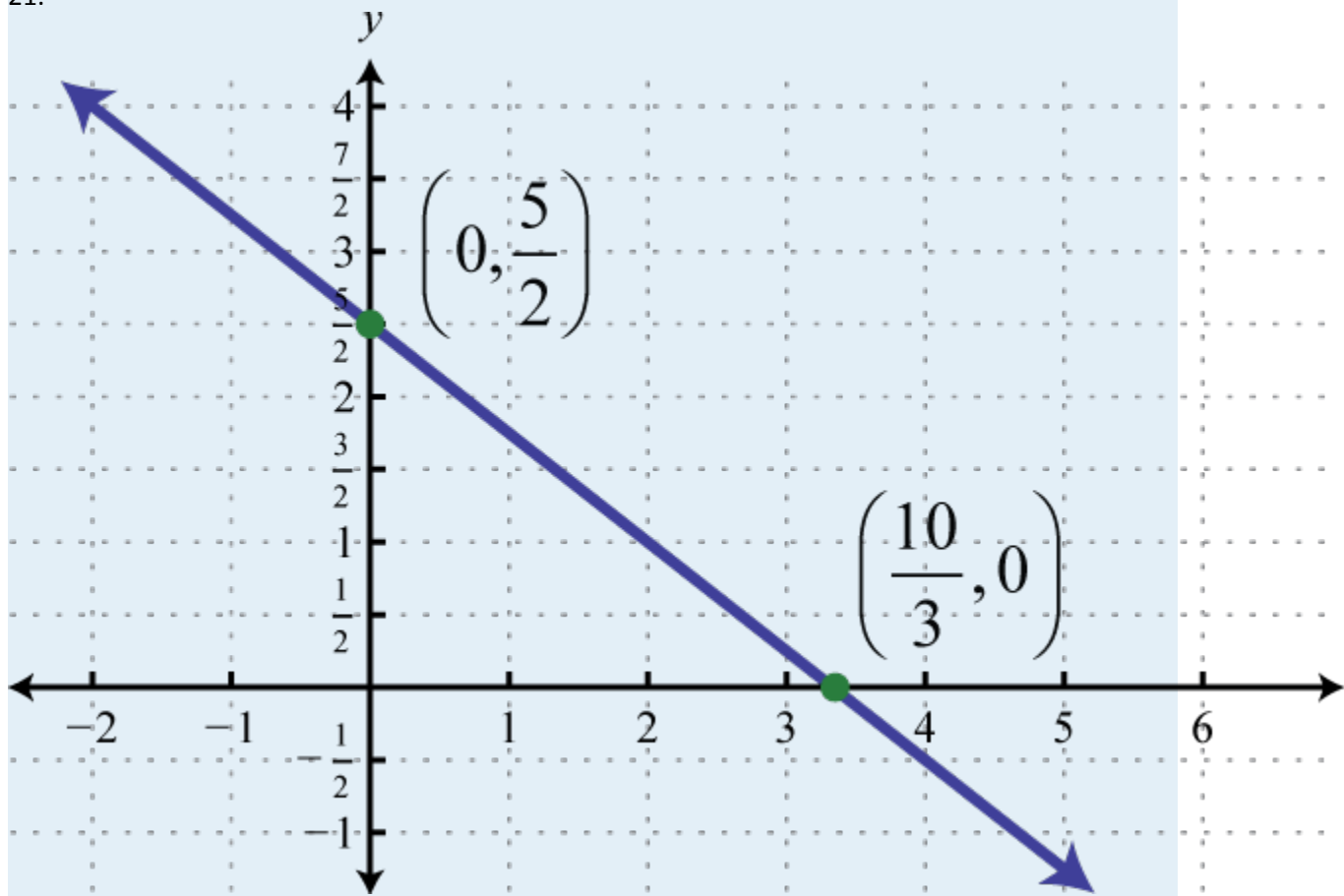
20.

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21.

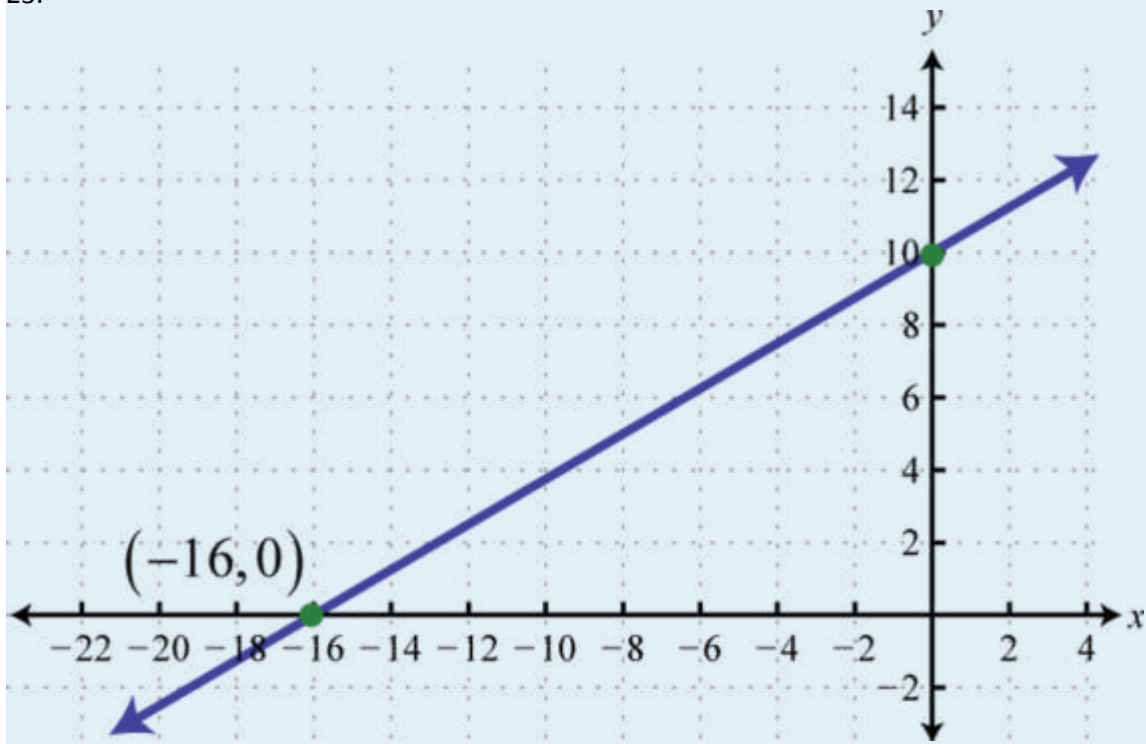


22.

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23.

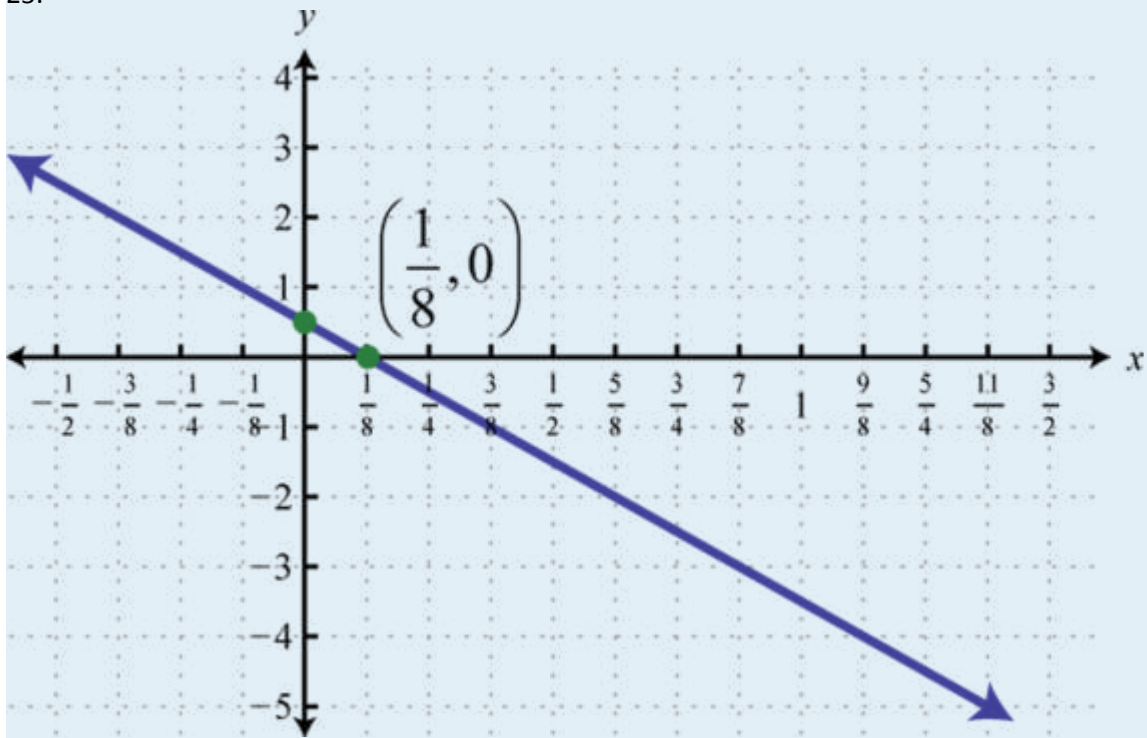


24.

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25.

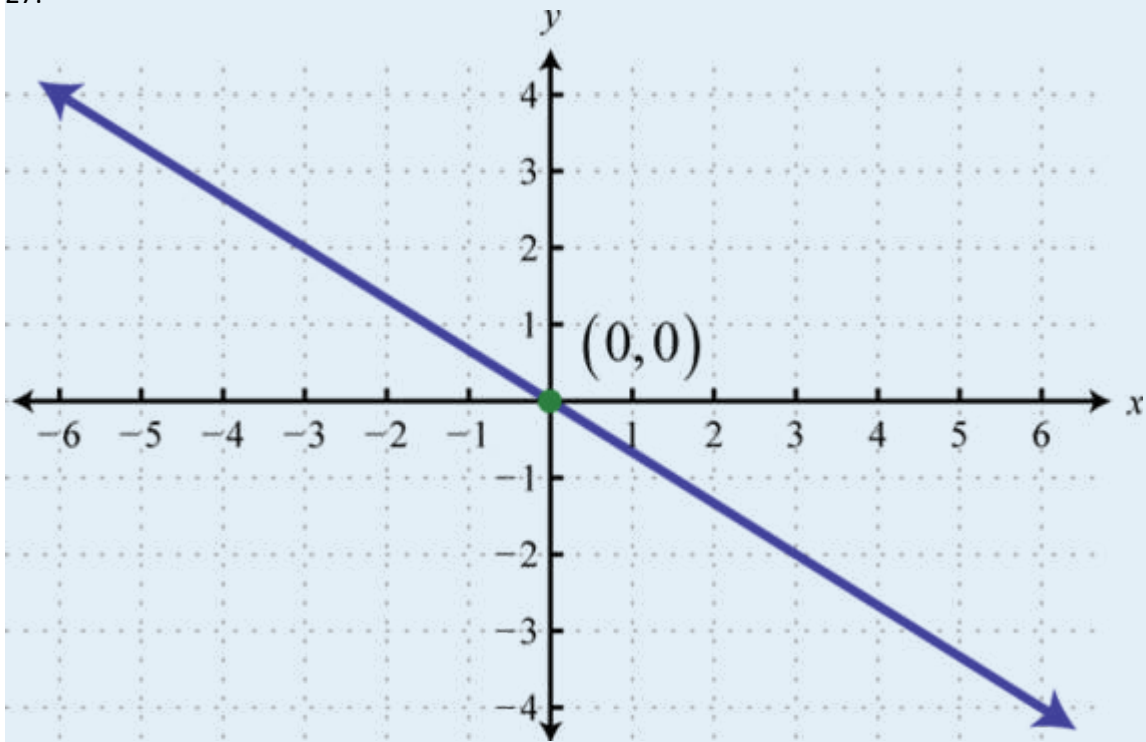


26.

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27.



28.

29.  $m = -2$

30.

31.

32.

33.  $m = 0$

34.

35.  $x = 2$

36.

37.  $(-1, \infty)$

38.



39.  $f(x) = -2x - 3$

40.

41.

42.

43.

44.

45.

46.

47.

48.

49.

50.

51.  $C(x) = 2x + 2.5$ ; 3.6 miles

52.

53.  $V(t) = 325t + 1,200$ ; \$7,700

54.

55.  $p(x) = -0.02x + 1.8$ ; 2005

56.

57.  $P(n) = 52n - 3$ ; 380; 258 bicycles

58.

59.

60.

61.  $(-25, 25)$

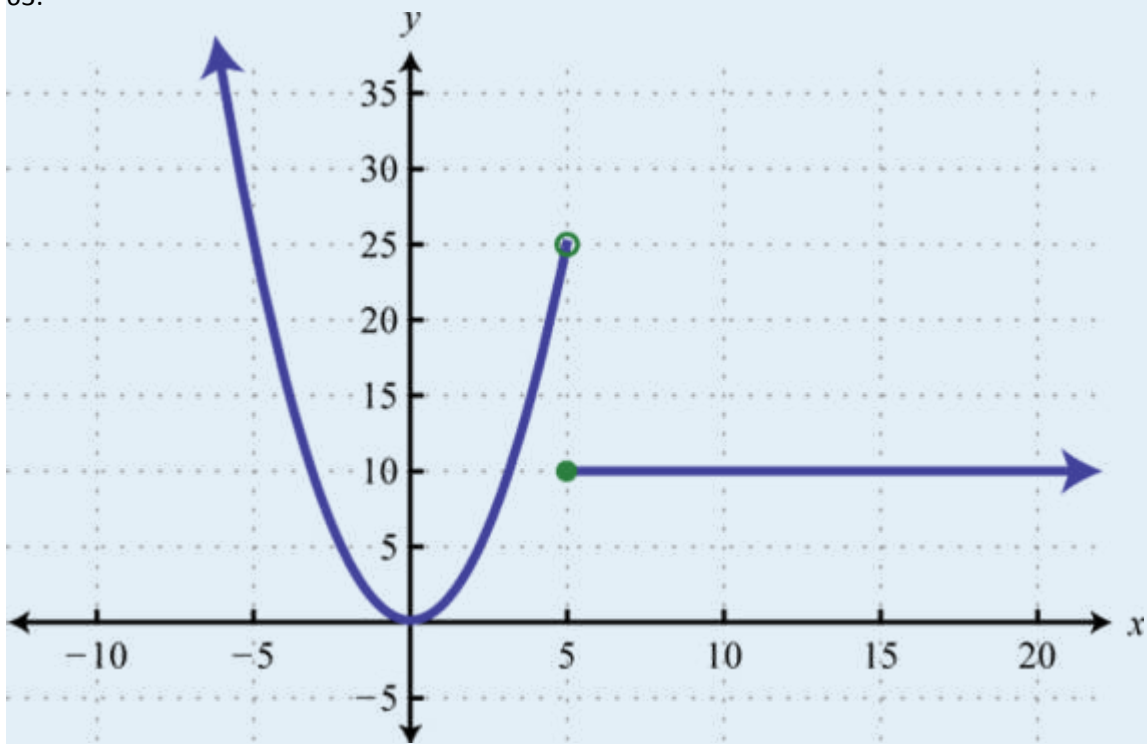
62.

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63.

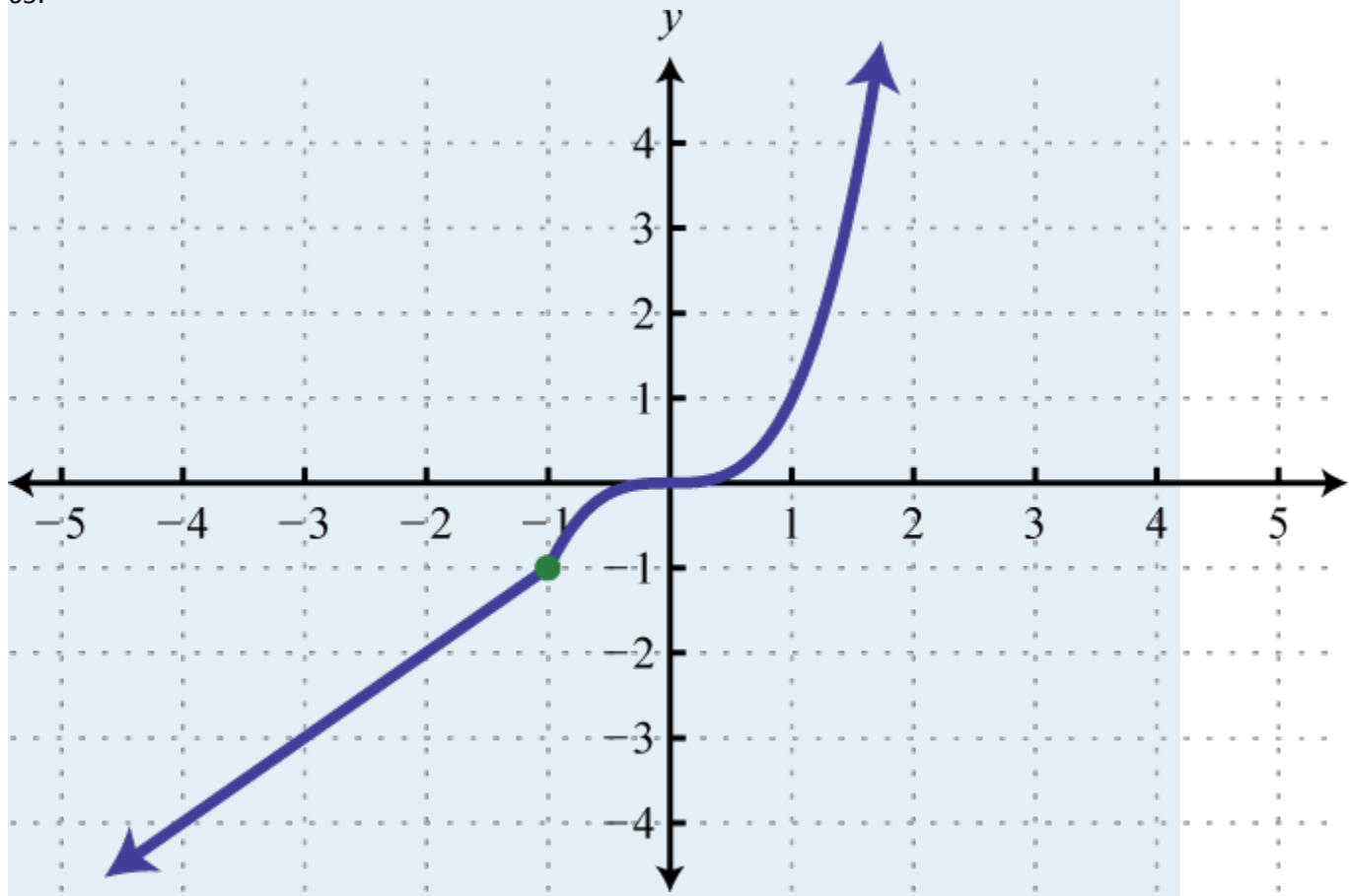


64.

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Saylor URL: <http://www.saylor.org/books/>



65.



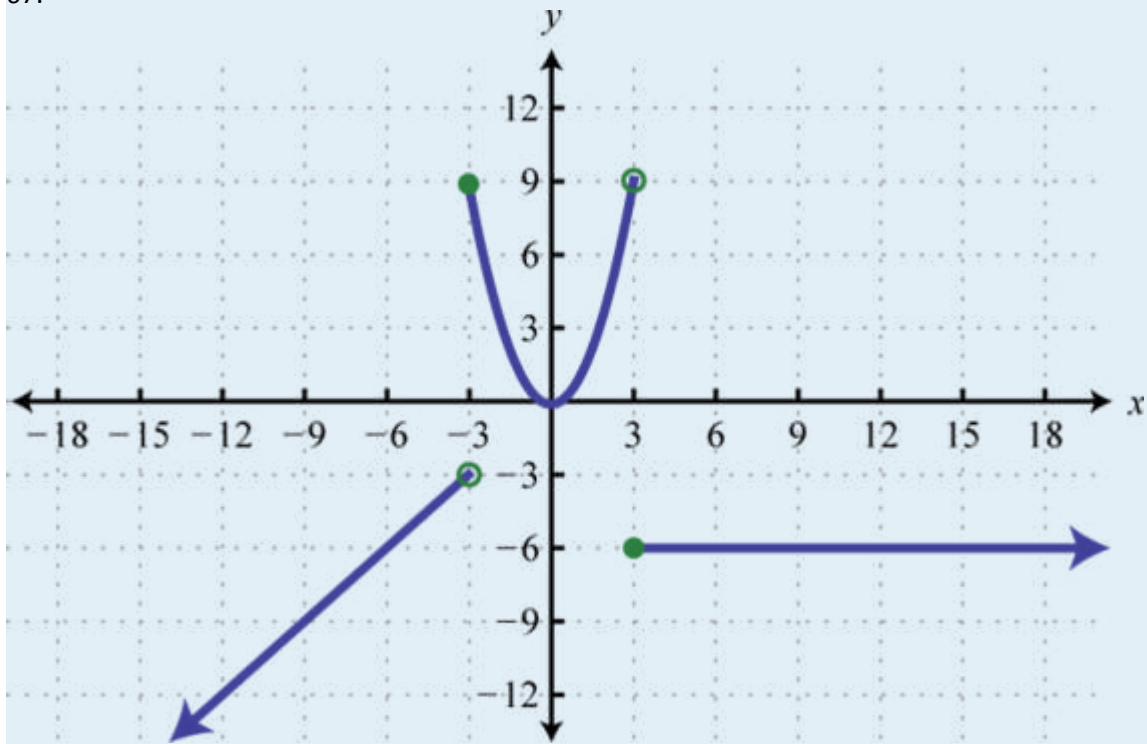
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Saylor URL: <http://www.saylor.org/books/>





67.

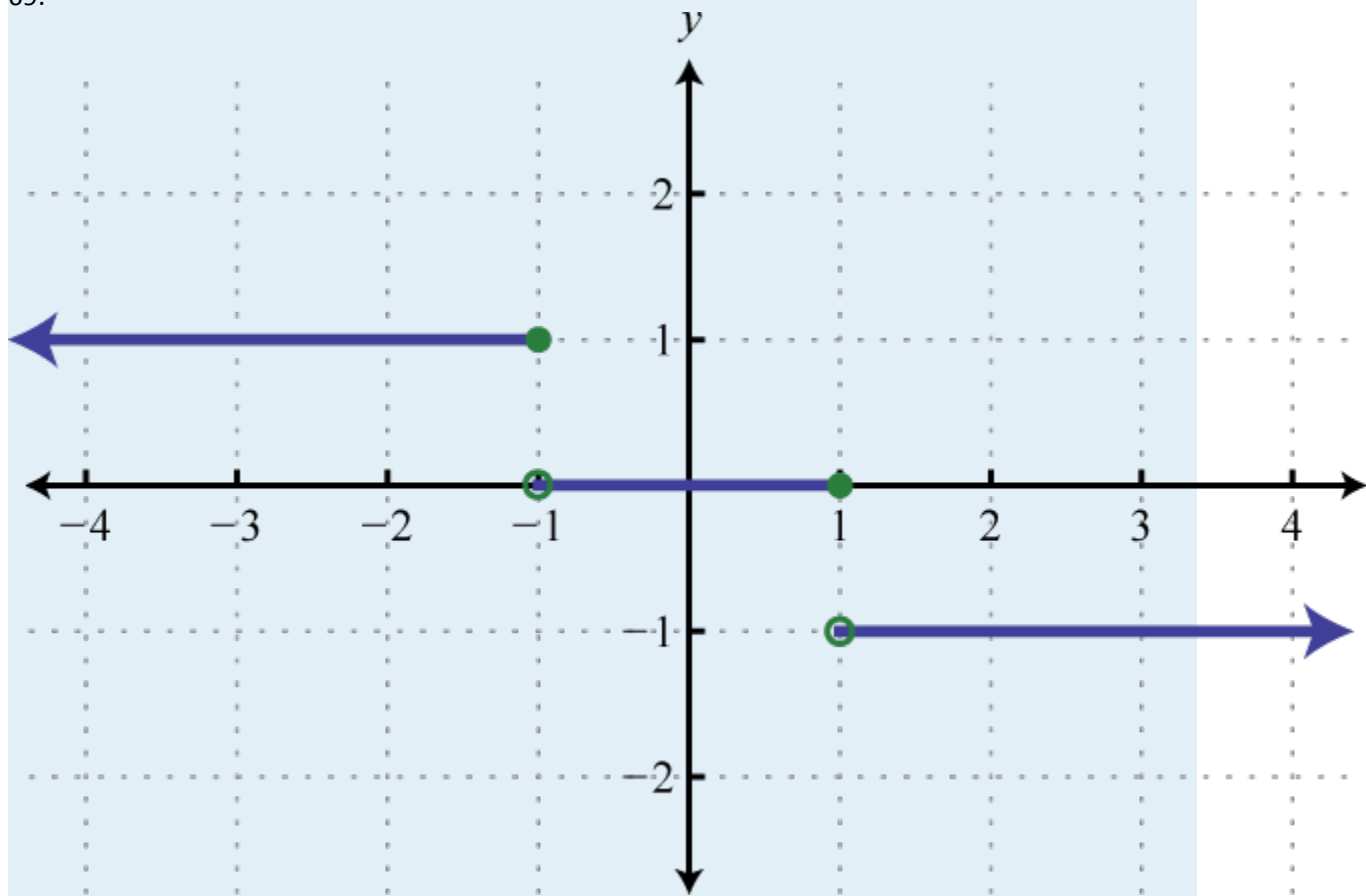


68.

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69.



70.

71.  $f(-10) = -52, f(-6) = 36, f(0) = 0$

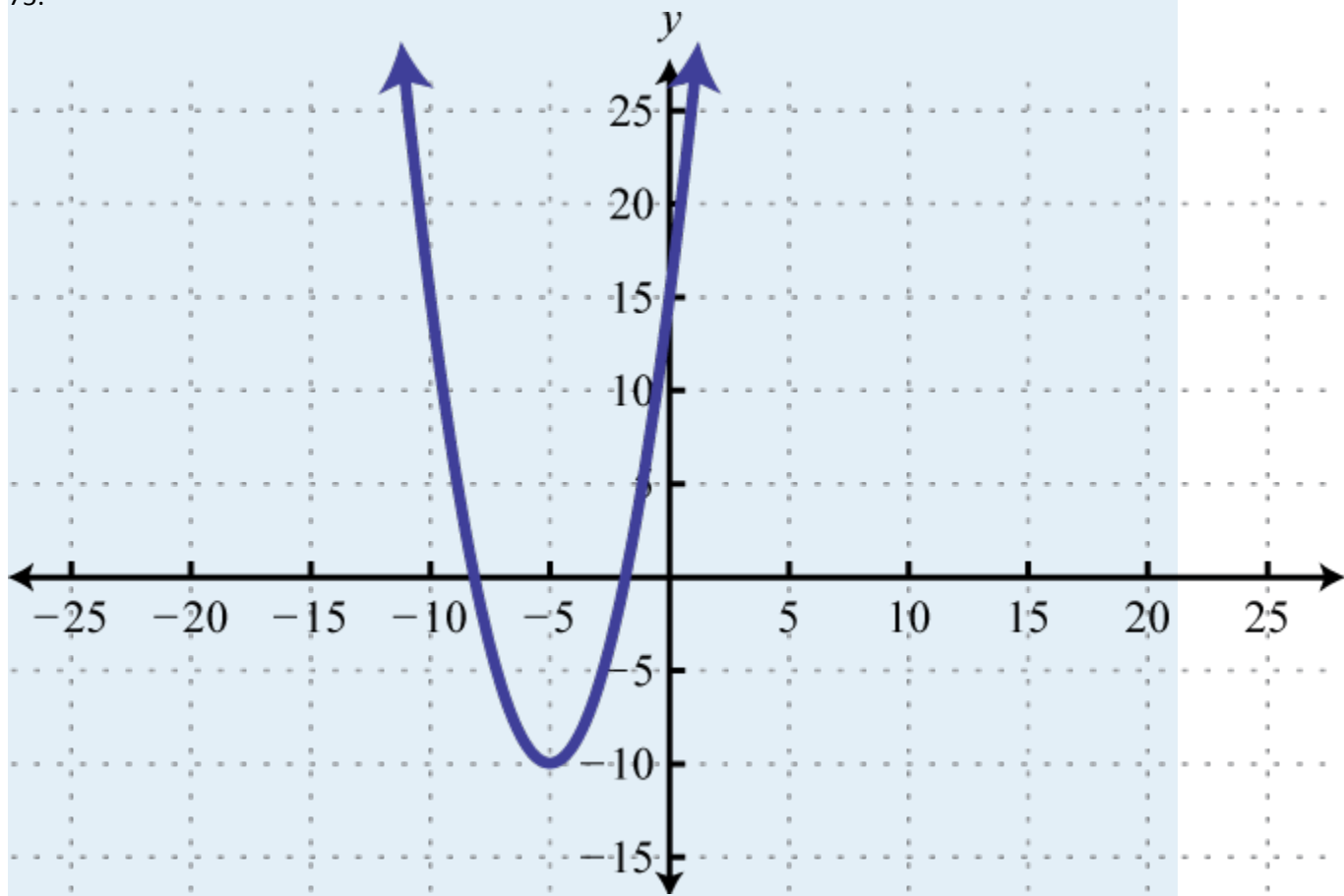
72.

73.  $g(-10) = -5, g(-4) = -13,$

74.



75.



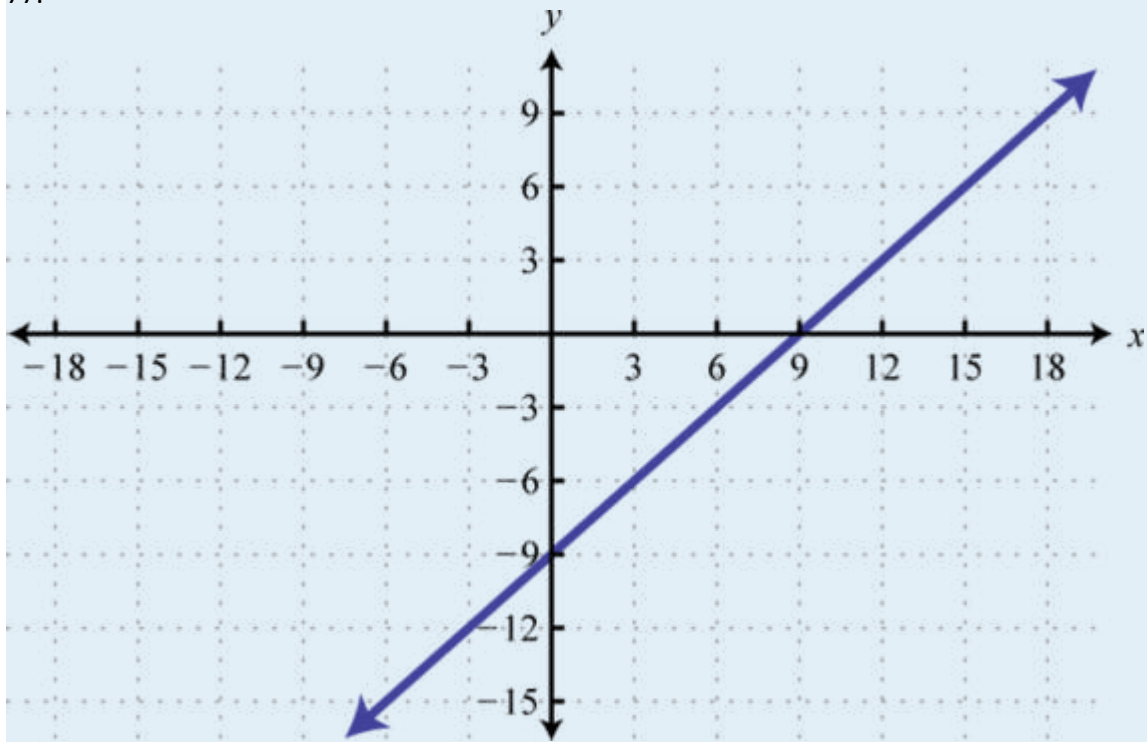
76.

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77.



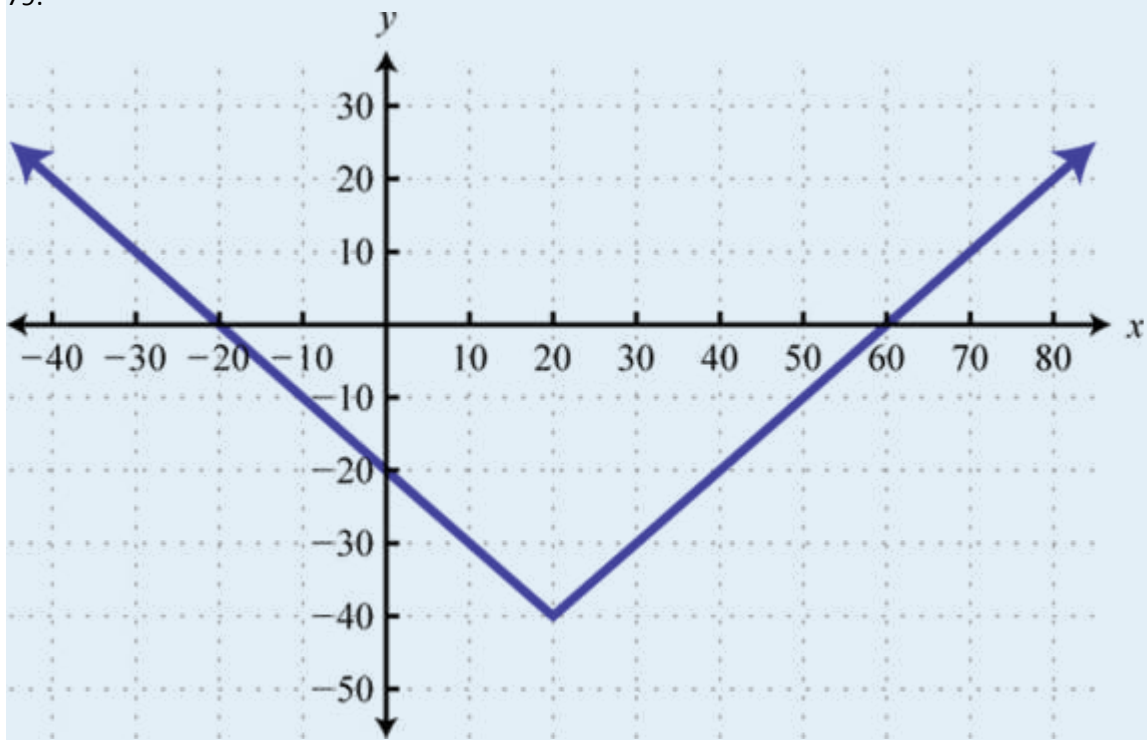
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79.

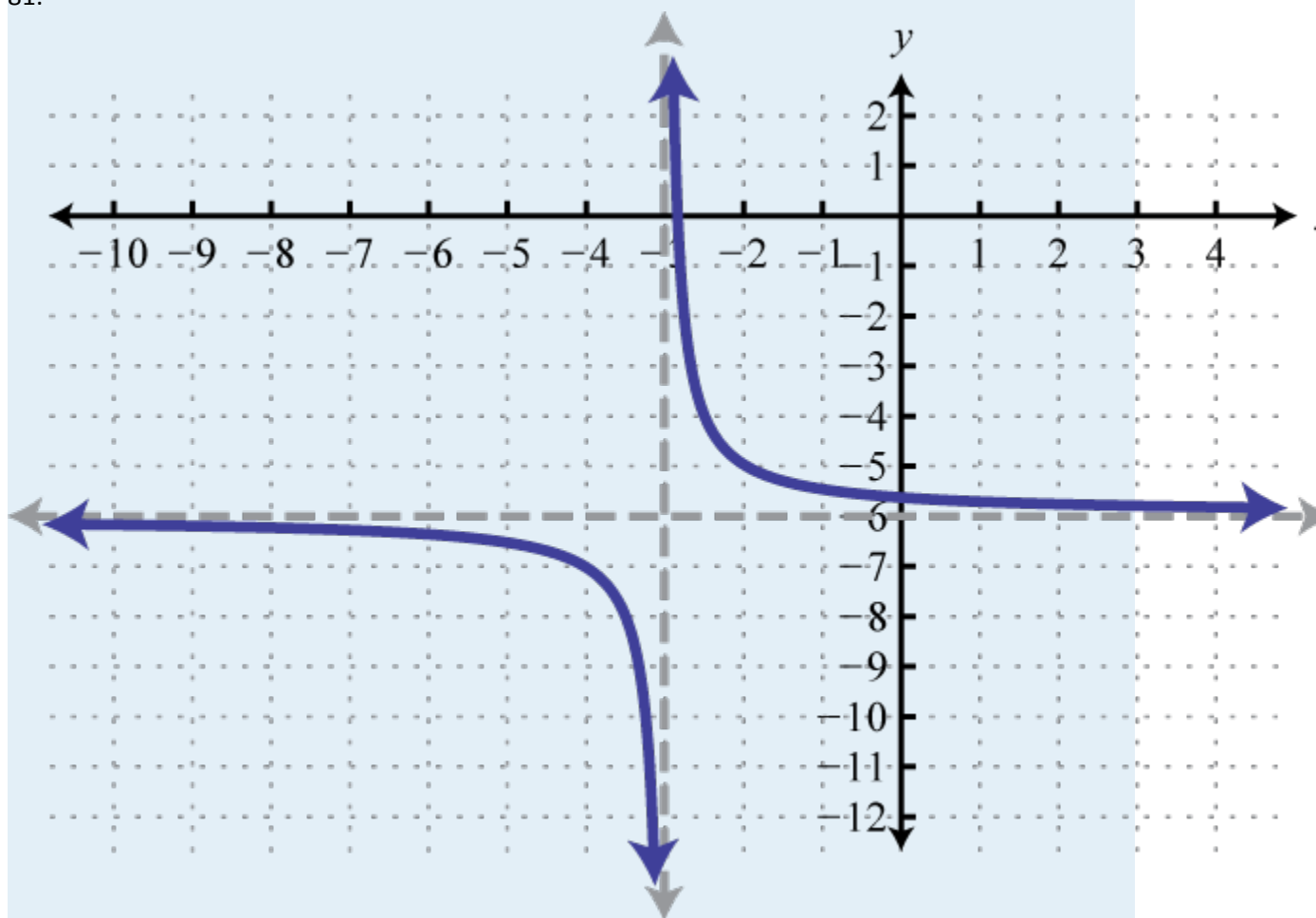


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Saylor URL: <http://www.saylor.org/books/>



81.

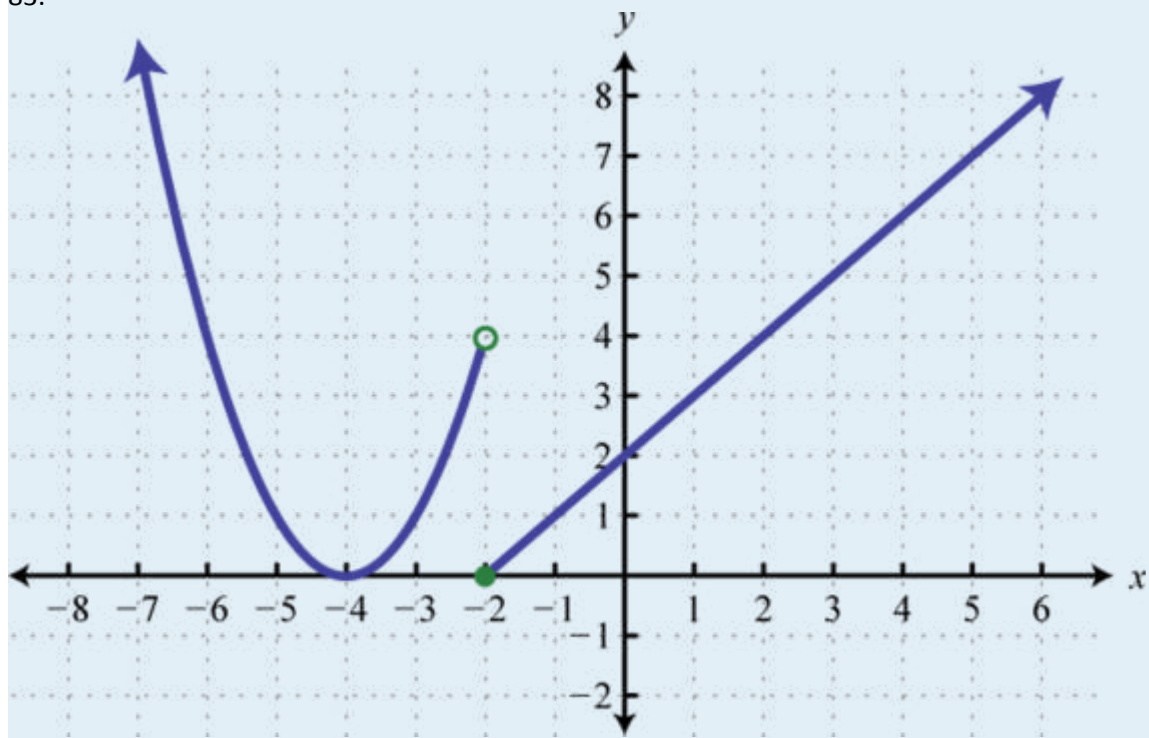


82.

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83.

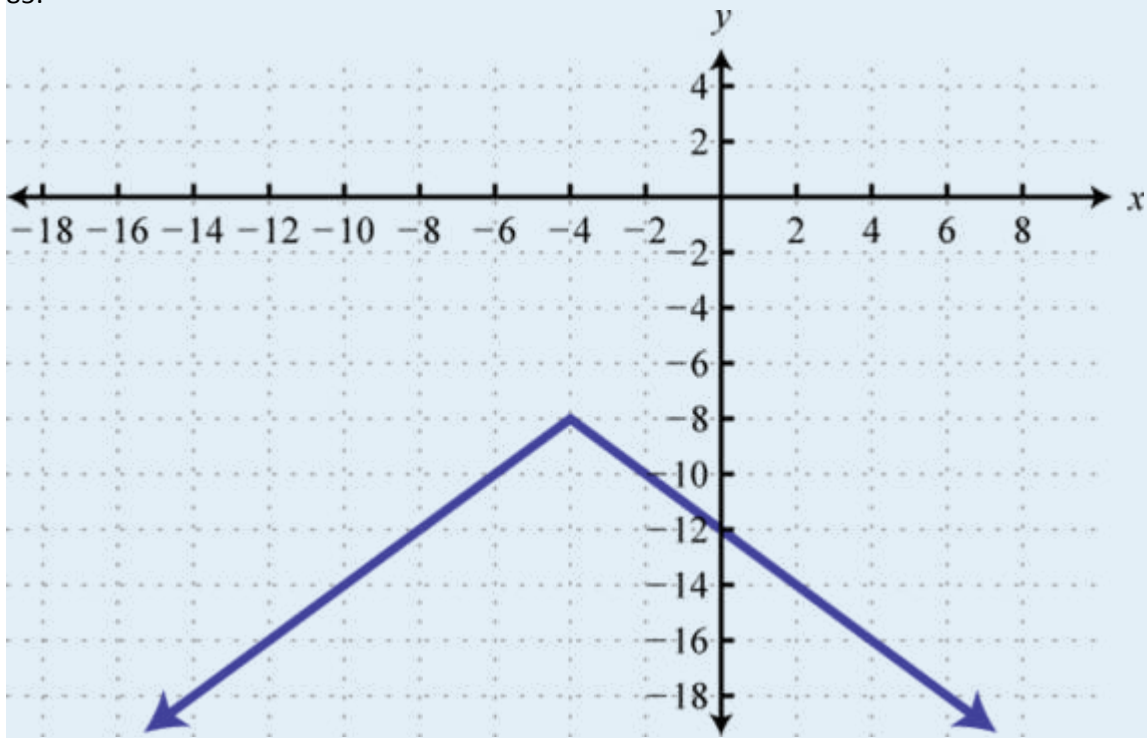


84.

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85.



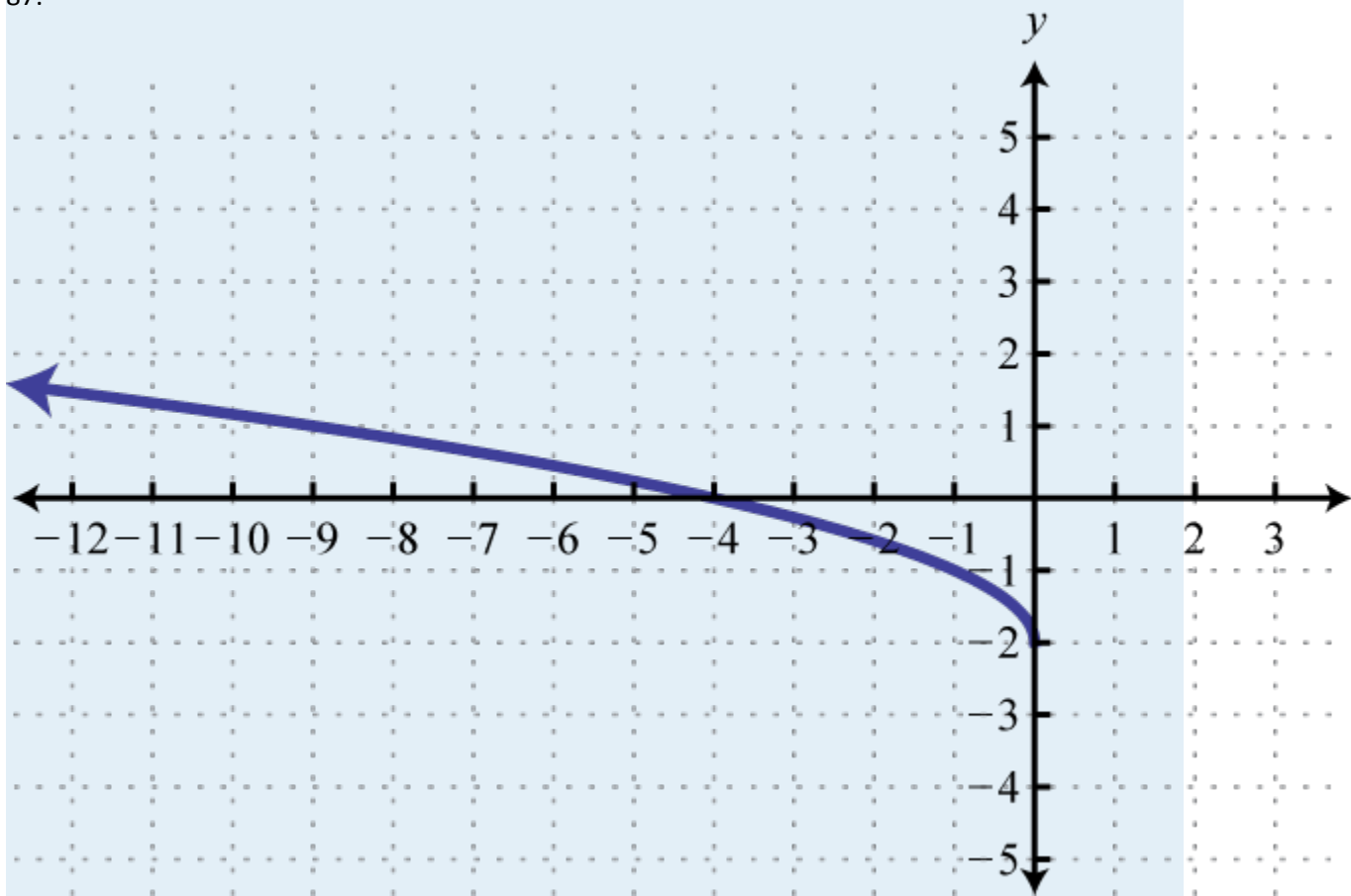
86.

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87.

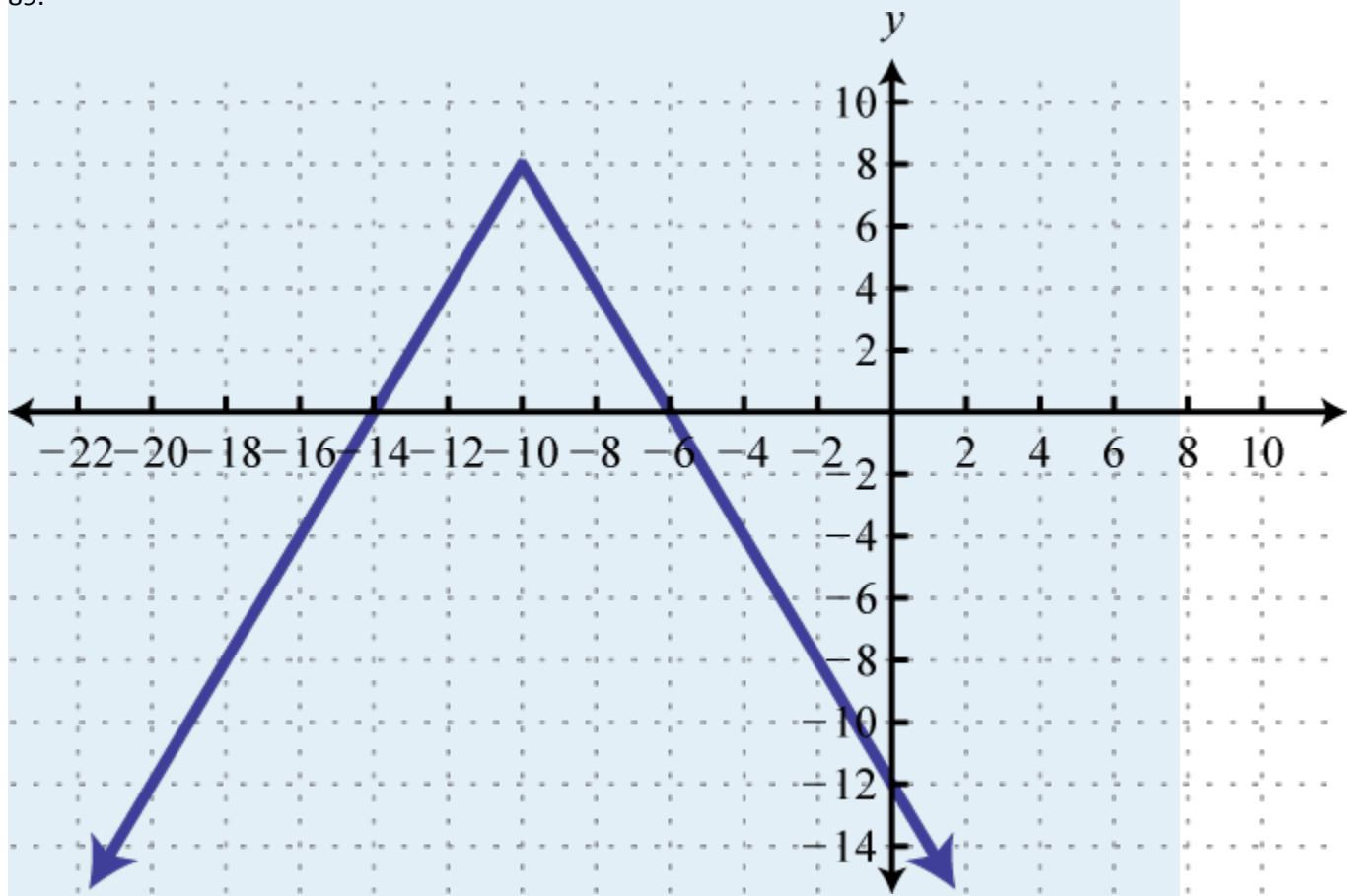


88.

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89.

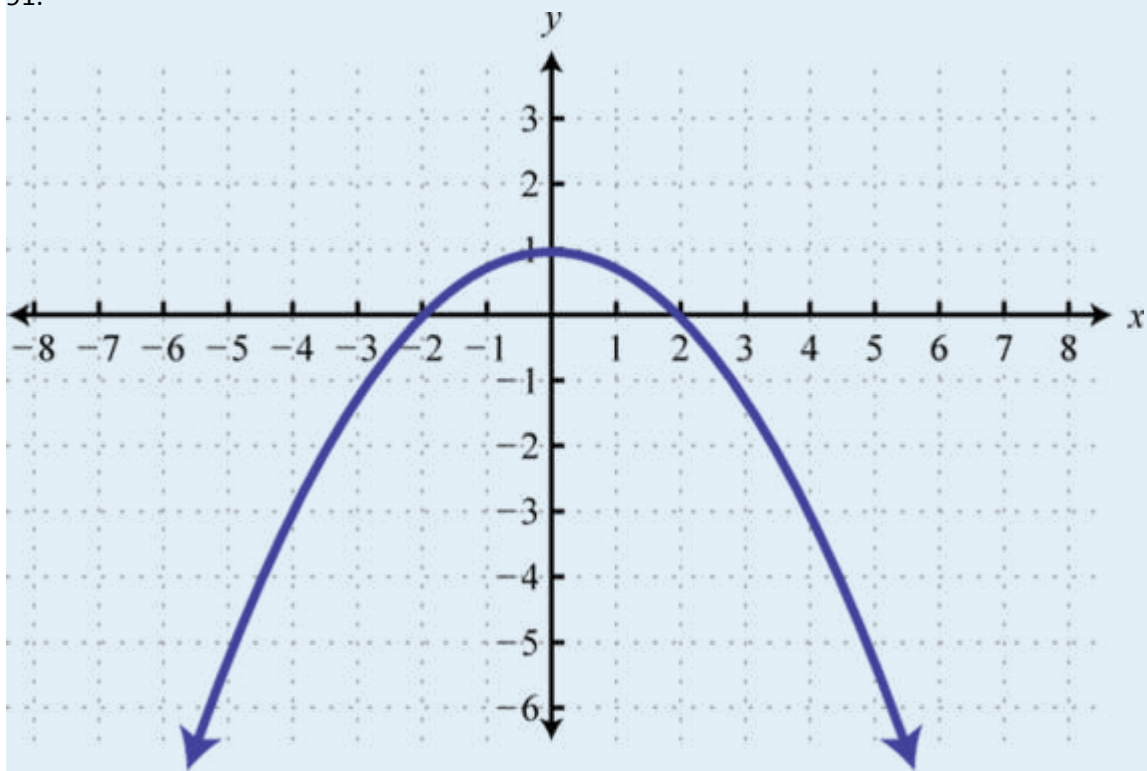


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91.



92.

93.

94.

95.

96.

97.

98.

99.  $f(x) = -10$

100.

101. -2,

102.



103. 3, 5

104.

105. 2

106.

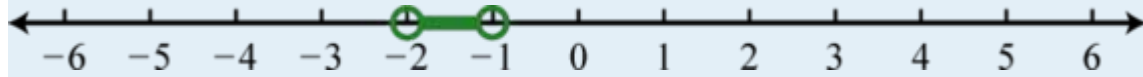
107. -1,

108.

109. -12,

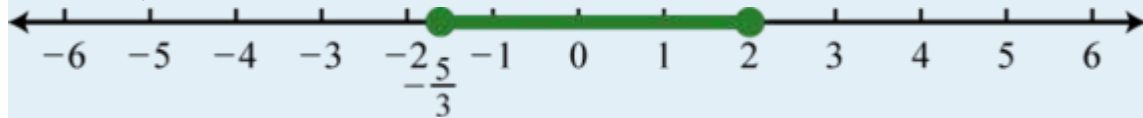
110.

111.  $(-2, -1)$ ;



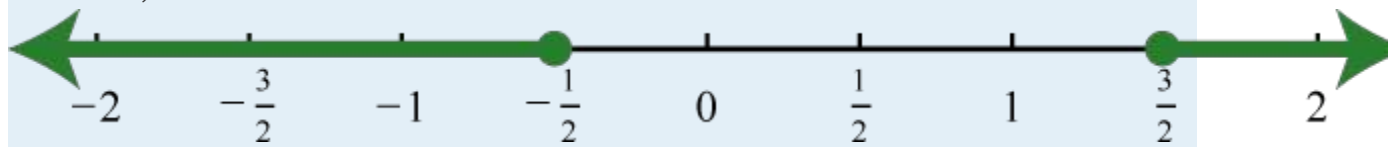
112.

113. ;



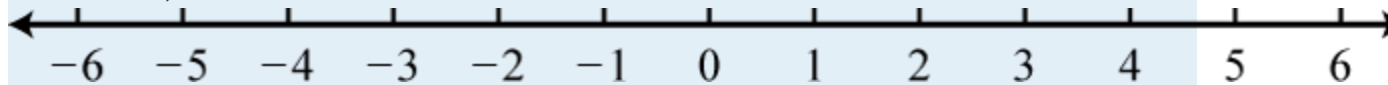
114.

115. ;



116.

117.  $\emptyset$ ;



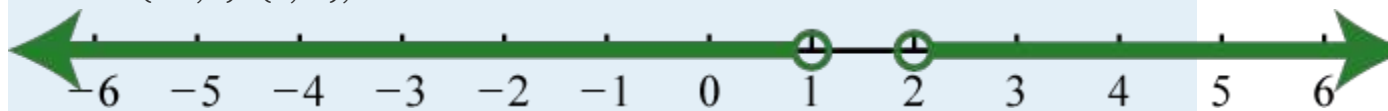
Attributed to John Redden

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118.

119.  $(-\infty, 1) \cup (2, \infty)$ ;



120.

121. Yes

122.

123. Yes

124.

125. Yes

126.

127. No

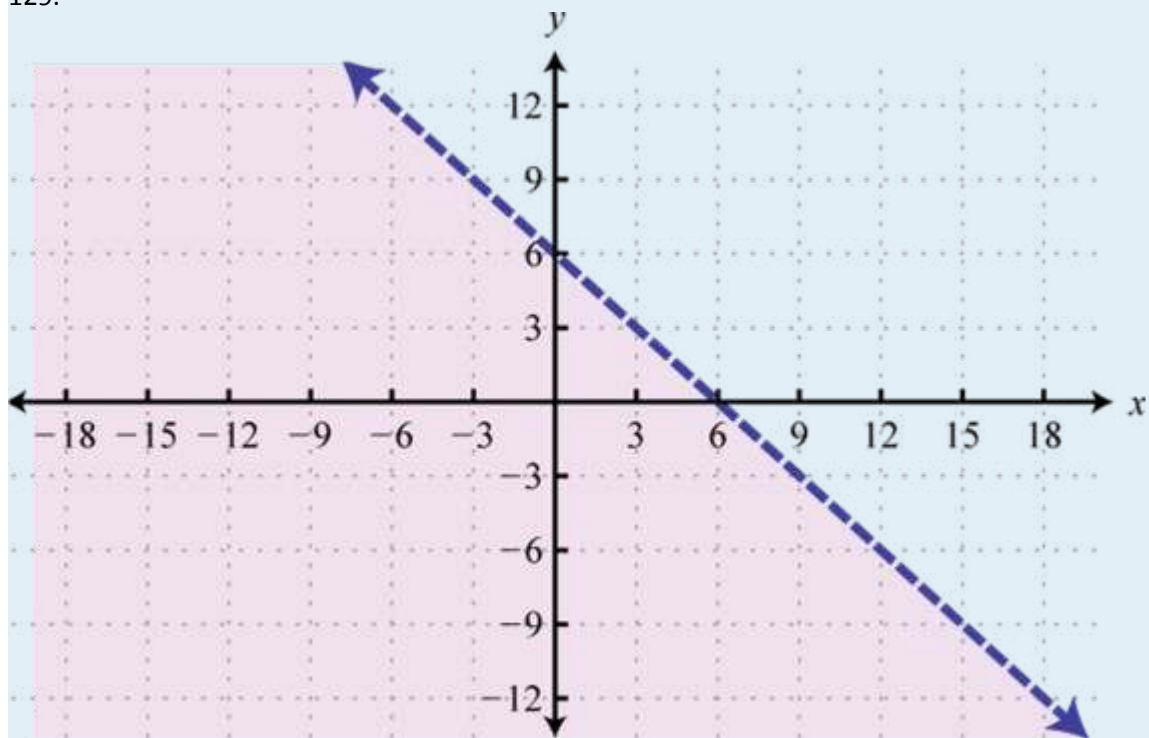
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129.

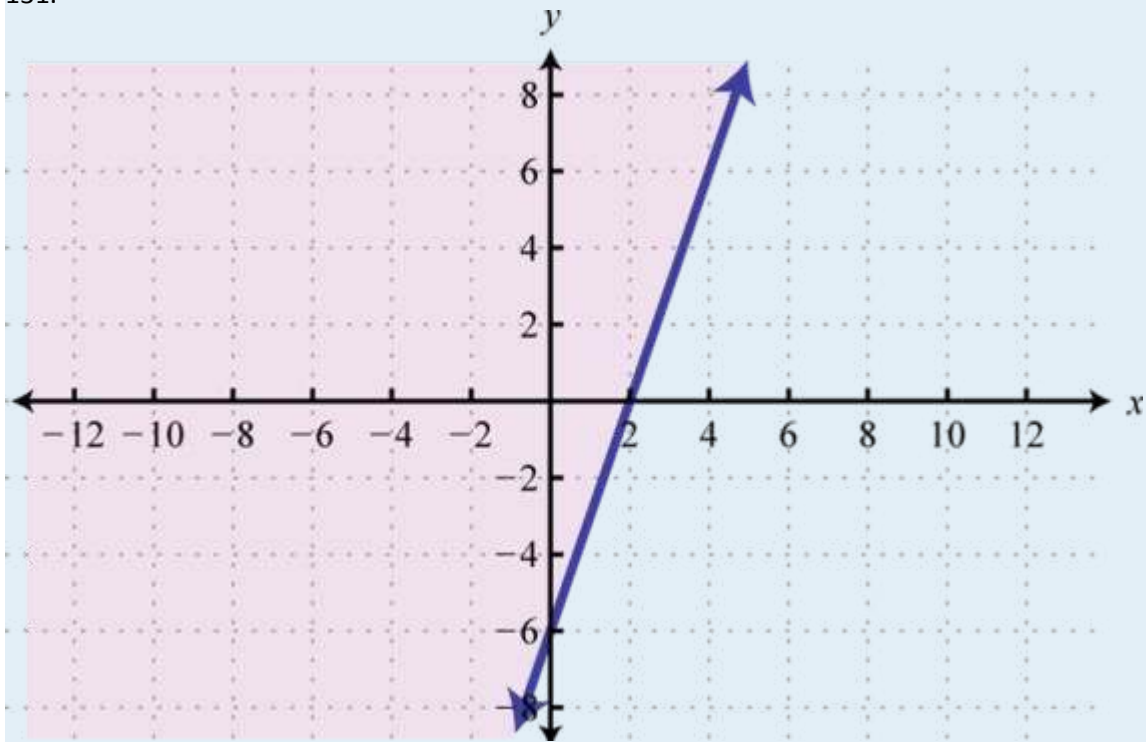


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131.

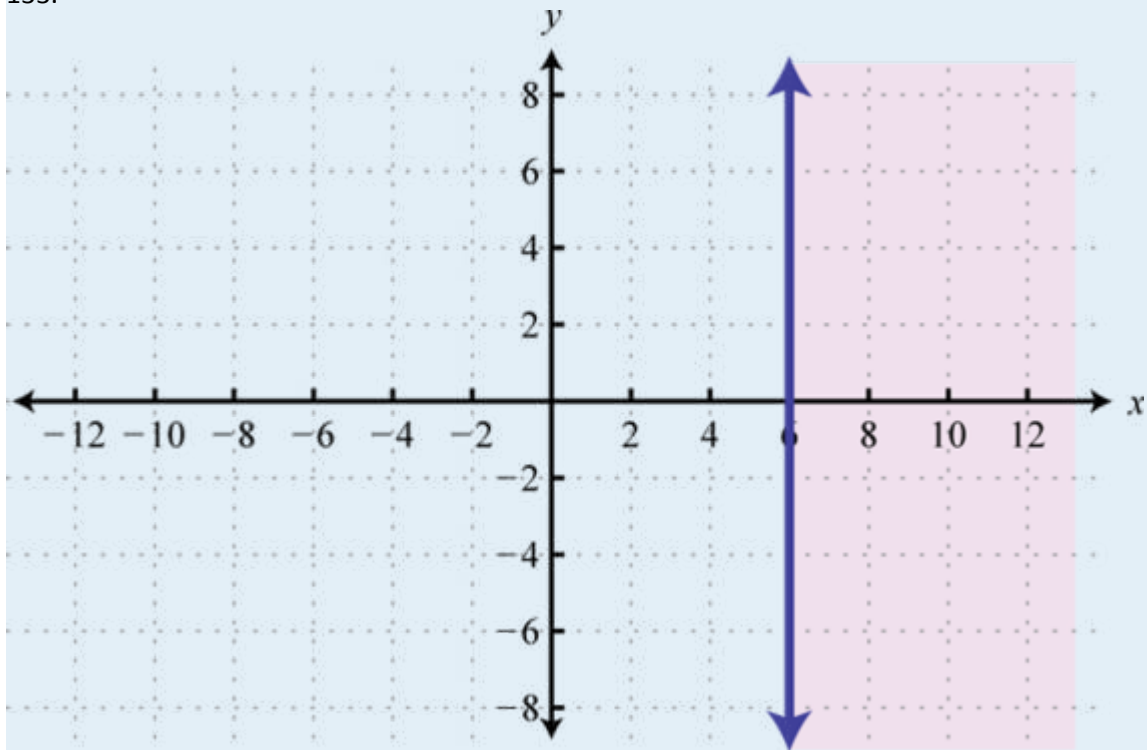


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133.



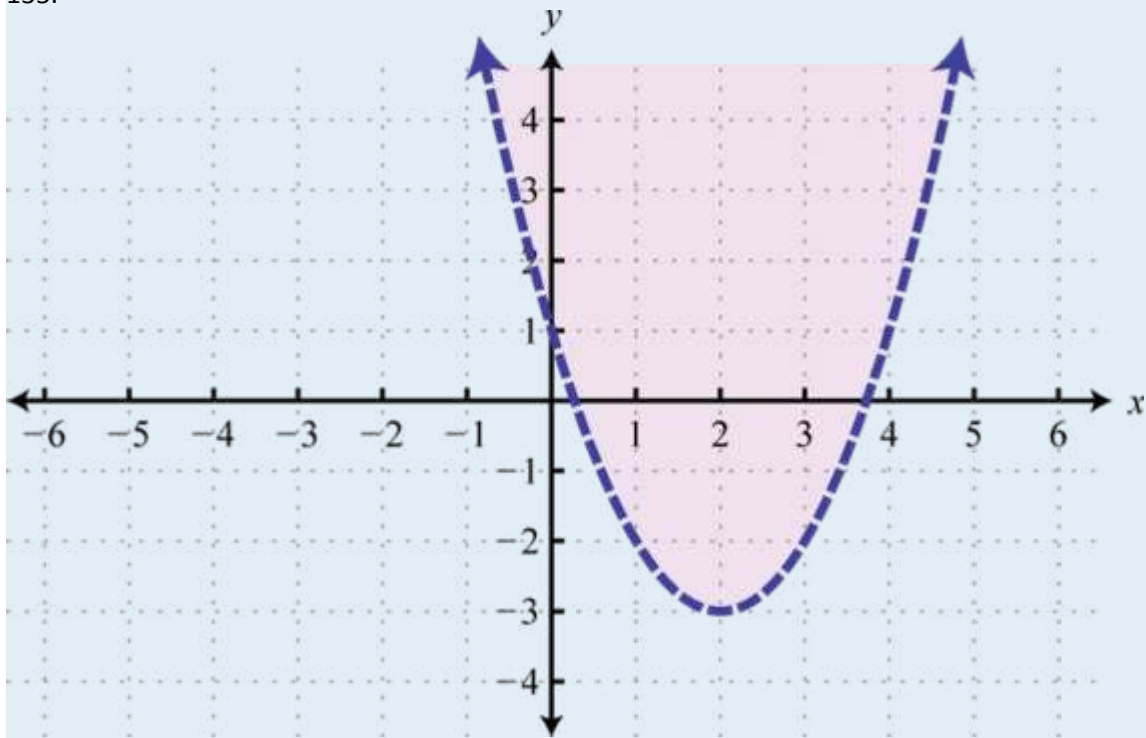
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135.

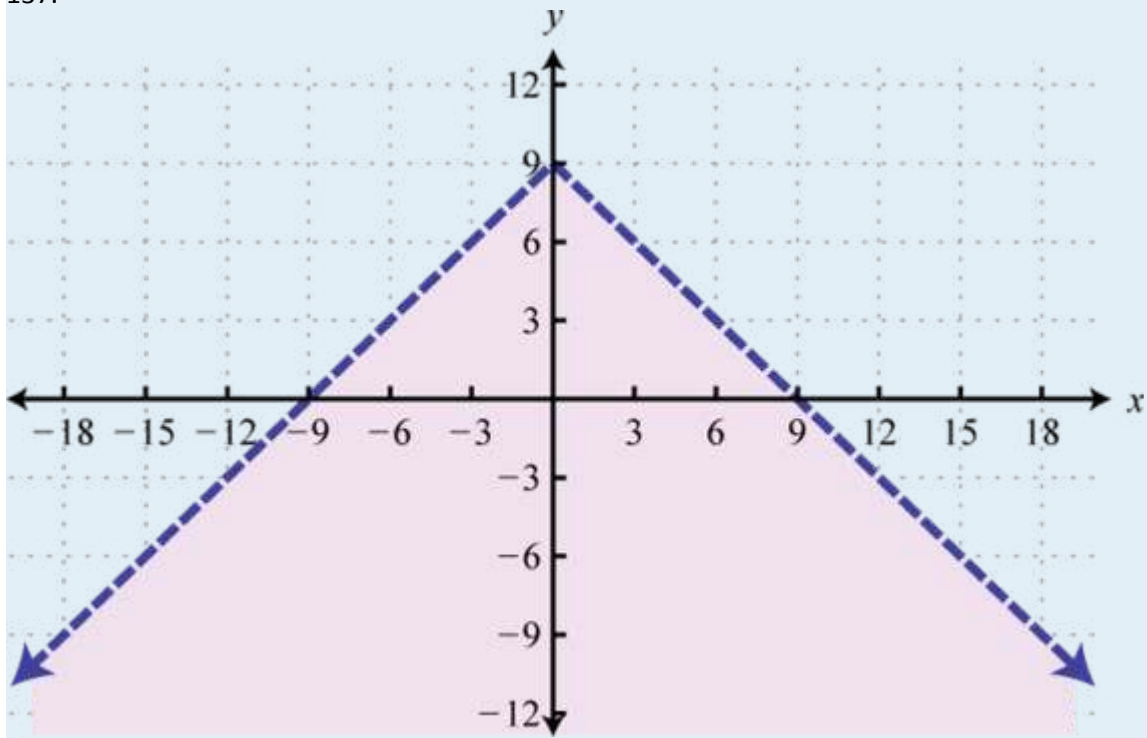


136.

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137.

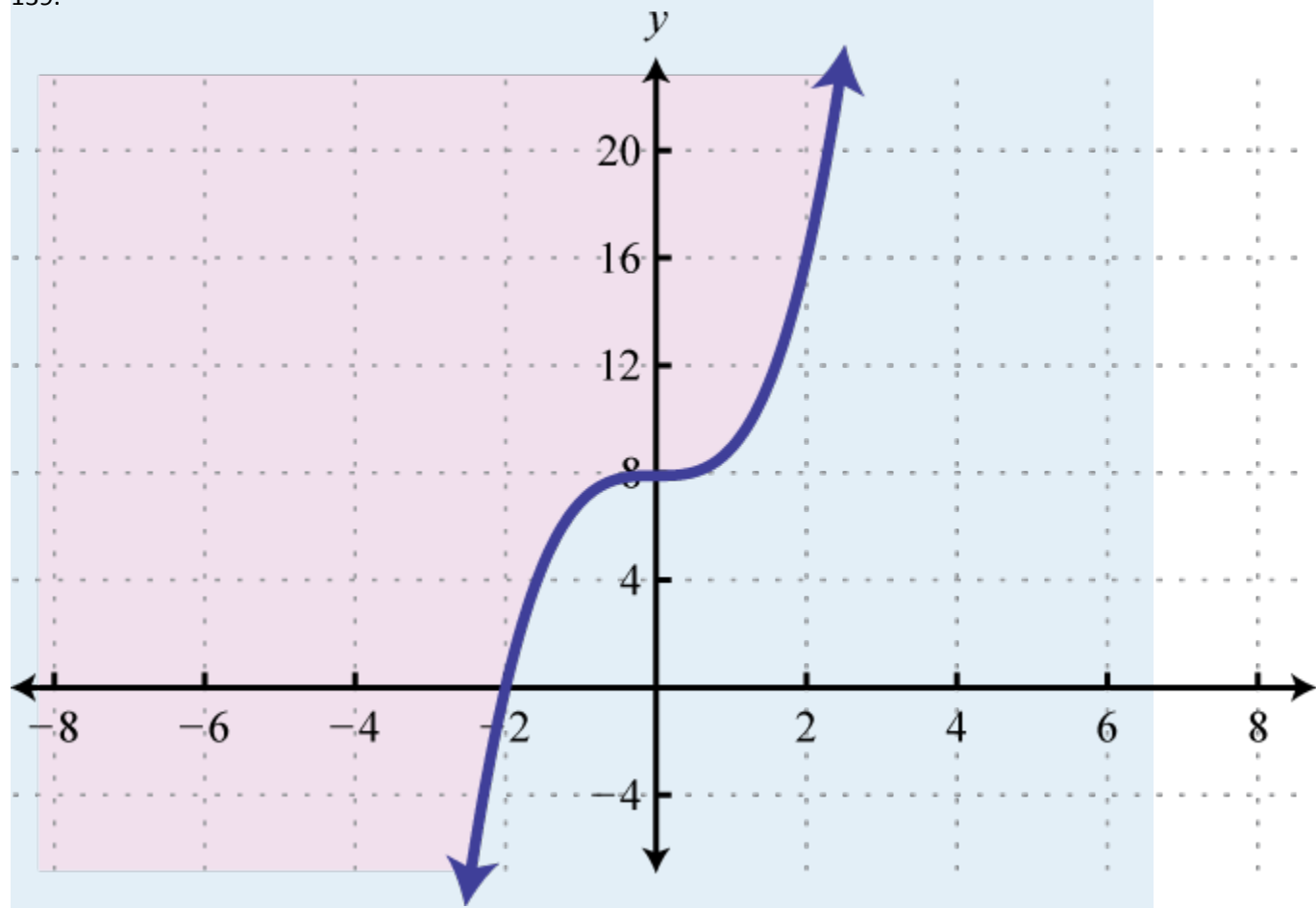


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139.

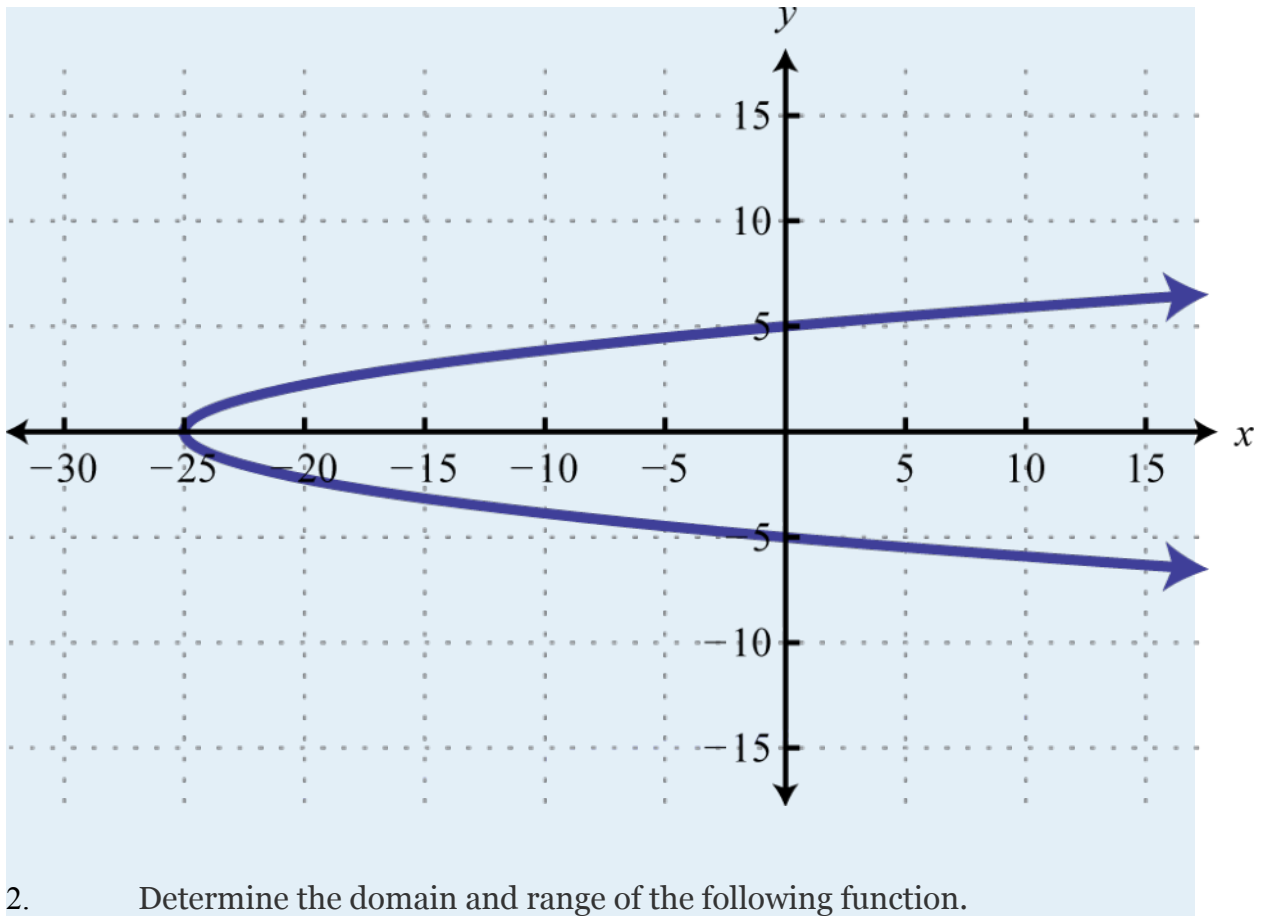


140.

## SAMPLE EXAM

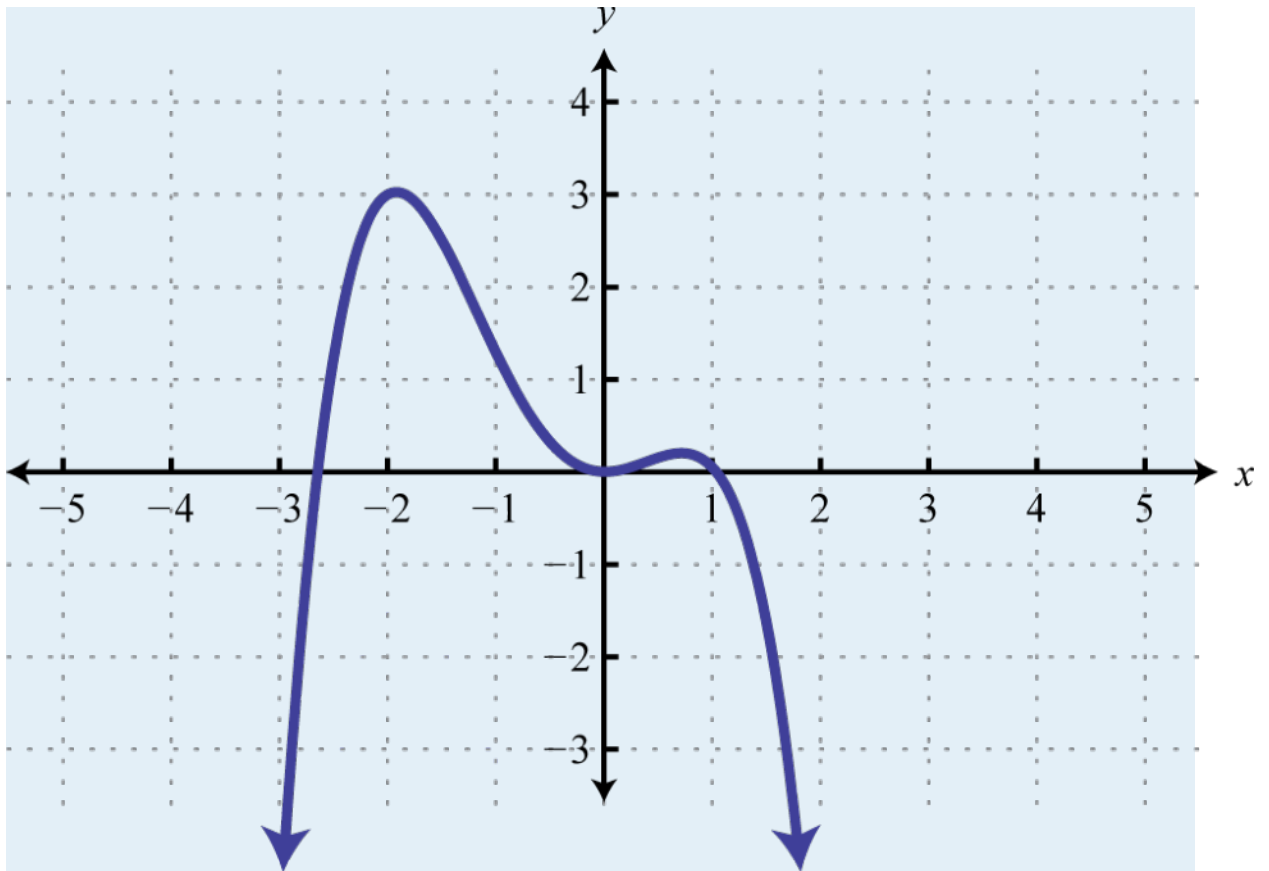
1. Determine whether or not the following graph represents a function or not. Explain.





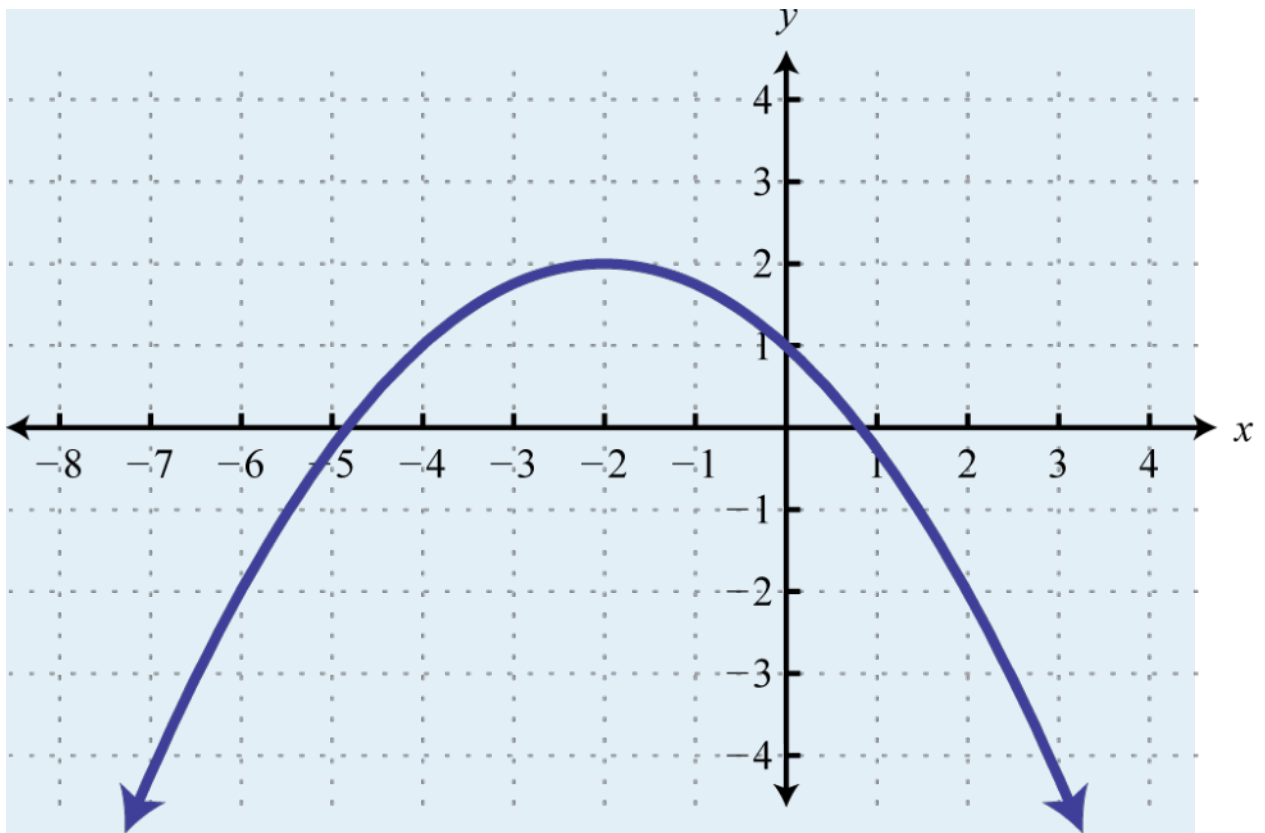
Attributed to John Redden  
Saylor URL: <http://www.saylor.org/books/>





3. Given  $g$ , find  $g(-1)$ ,  $g(0)$ , and  $g(x + h)$ .
4. Given the graph of a function  $f$ :





- 4.a. Find  $f(-6)$ ,  $f(0)$ , and  $f(2)$ .
- 4.b. Find  $x$  where  $f(x) = 2$ .
5. Graph and label the  $x$ -intercept.
6. Find a linear function passing through and  $(2, -2)$ .
7. Find the equation of the line parallel to  $2x - 6y = 3$  and passing through  $(-1, -2)$ .
8. Find the equation of the line perpendicular to  $3x - 4y = 12$  and passing through  $(6, 1)$ .
9. The annual revenue of a new web-services company in dollars is given by  $R(n) = 125n$ , where  $n$  represents the number of users the company has



registered. The annual maintenance cost of the company's registered user base in dollars is given by the formula  $C(n) = 85n + 22,480$  where  $n$  represents the users.

9.a. Write a function that models the annual profit based on the number of registered users.

9.b. Determine the number of registered users needed to break even.

10. A particular search engine assigns a ranking to a webpage based on the number of links that direct users to the webpage. If no links are found, the webpage is assigned a ranking of 1. If 40 links are found directing users to the webpage, the search engine assigns a page ranking of 5.

10.a. Find a linear function that gives the webpage ranking based on the number of links that direct users to it.

10.b. How many links will be needed to obtain a page ranking of 7?

Use the transformations to sketch the graph of the following functions and state the domain and range.

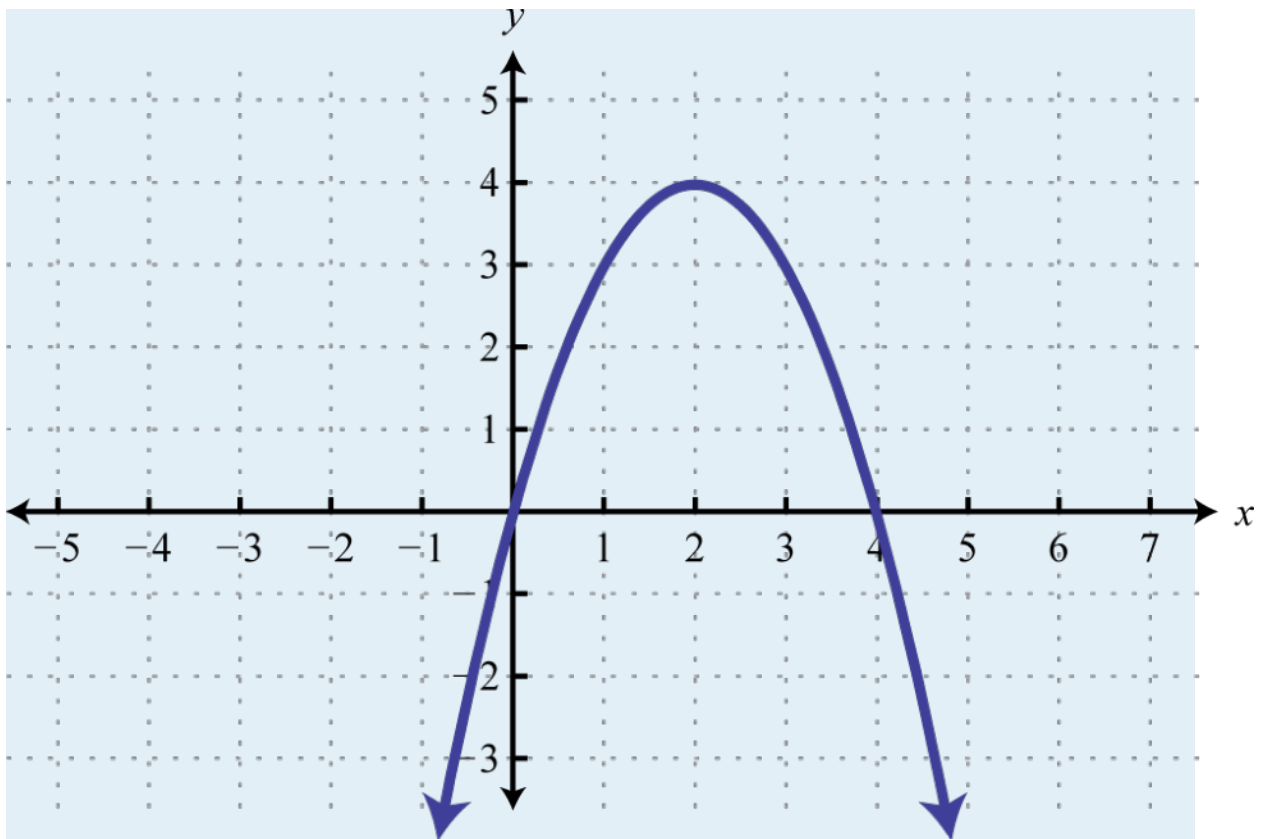
1.  $g(x) = |x + 4| - 5$

2.

3.

4. Given the graph, determine the function definition and its domain and range:





5. Sketch the graph:  $h(x) =$

6. Sketch the graph:

Solve.

1.  $|2x-1| + 2 = 7$

2.  $10-5|2x-3| = 0$

3.  $|7x + 4| = |9x-1|$

Solve and graph the solution set.





1.  $|2x-4|-5 < 7$
2.  $6 + |3x-5| \geq 13$
3.  $5-3|x-4| \geq -10$
4.  $3|7x-1| + 5 \leq 2$

Graph the solution set.

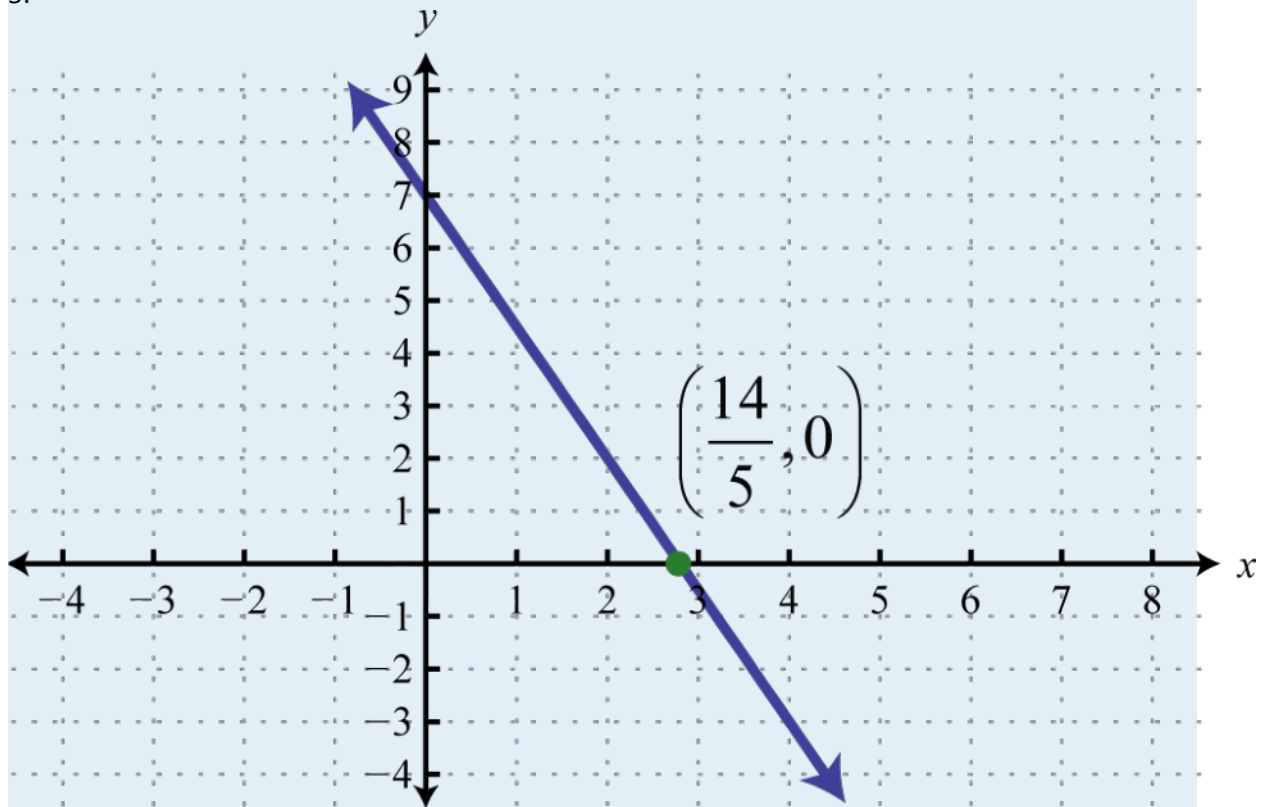
- 1.
2.  $y > -(x-2)^2 + 4$

## ANSWERS

1. The graph is not a function; it fails the vertical line test.
- 2.
3.  $g(-1) = 7$ ,  $g(0) = 1$ , and
- 4.



5.



6.

7.

8.

9.

a.  $P(n) = 40n - 22,480$ ;

b. 562 users

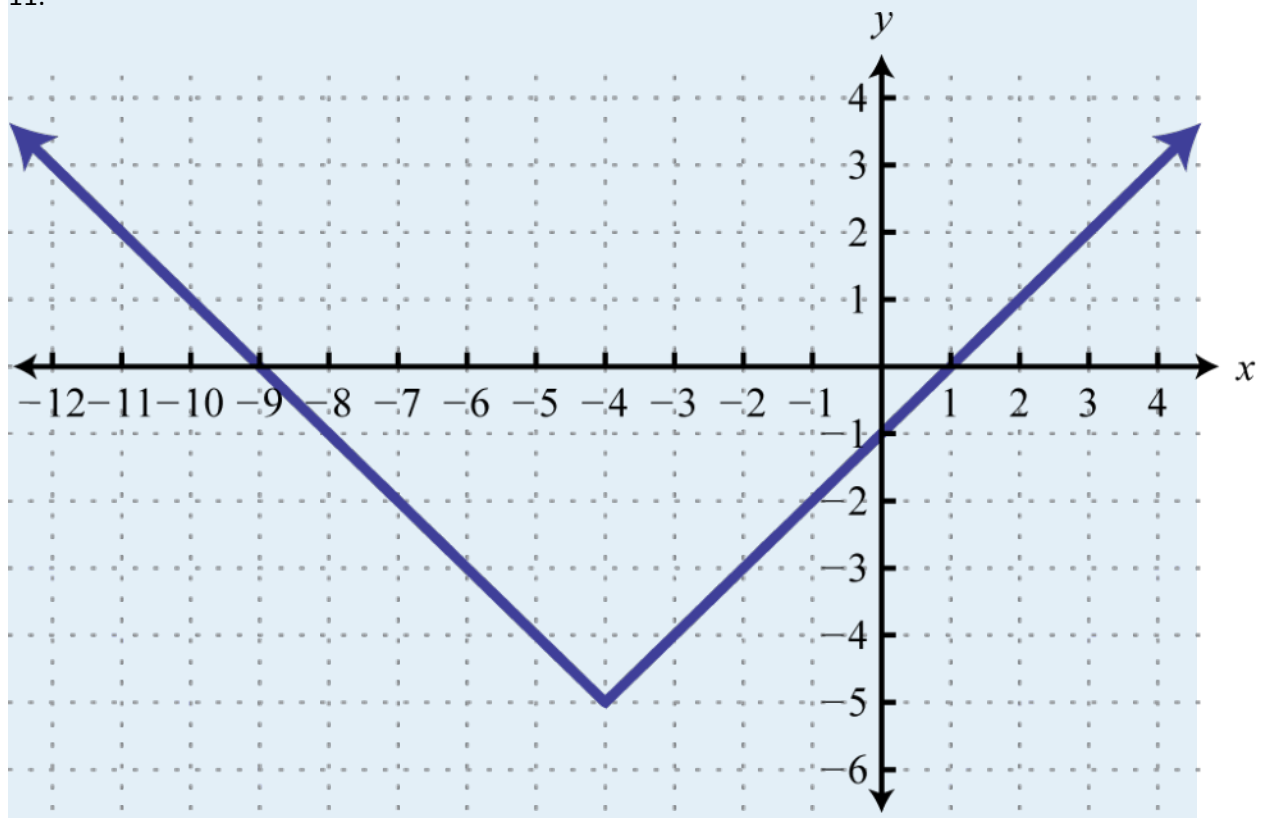
10.

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11.



Domain:  $(-\infty, \infty)$ ; range:  $[-5, \infty)$

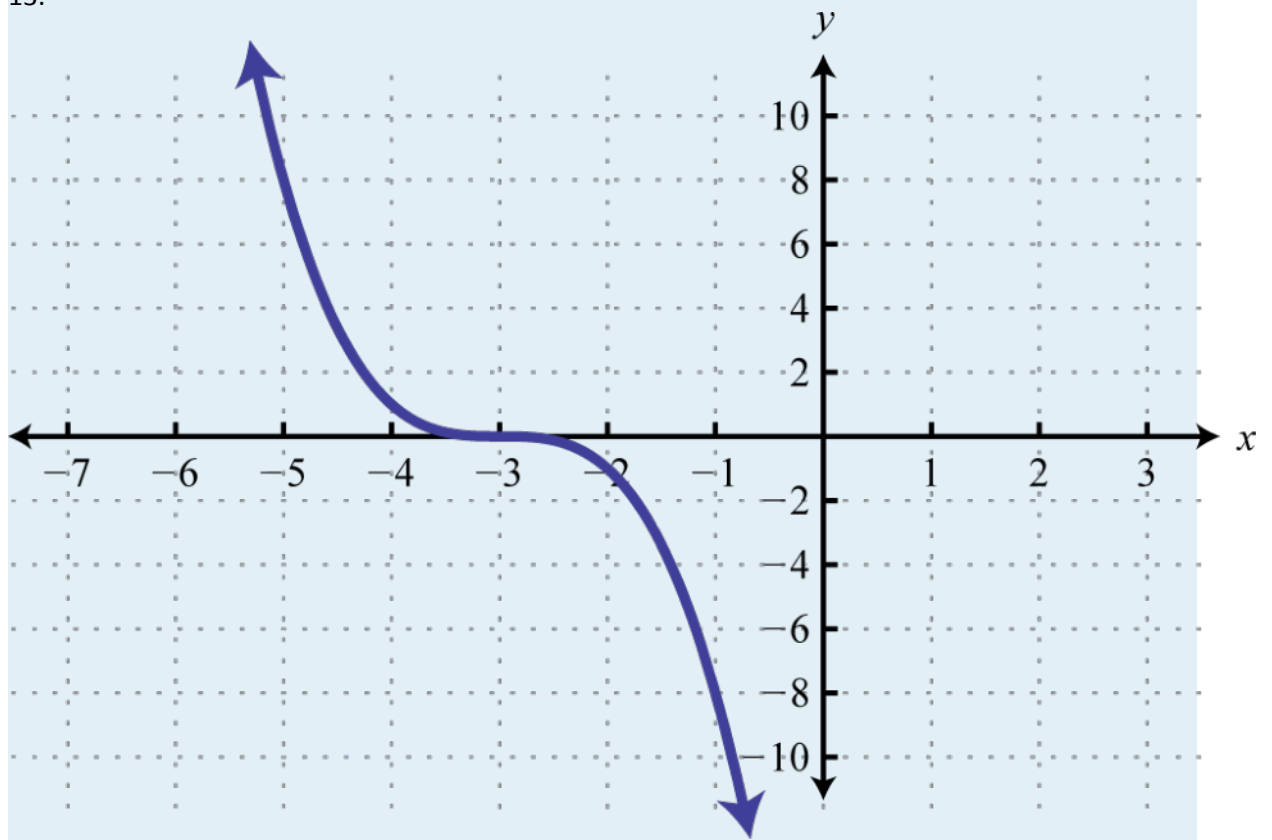
12.

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13.



Domain:  $(-\infty, \infty)$ ; range:  $(-\infty, \infty)$

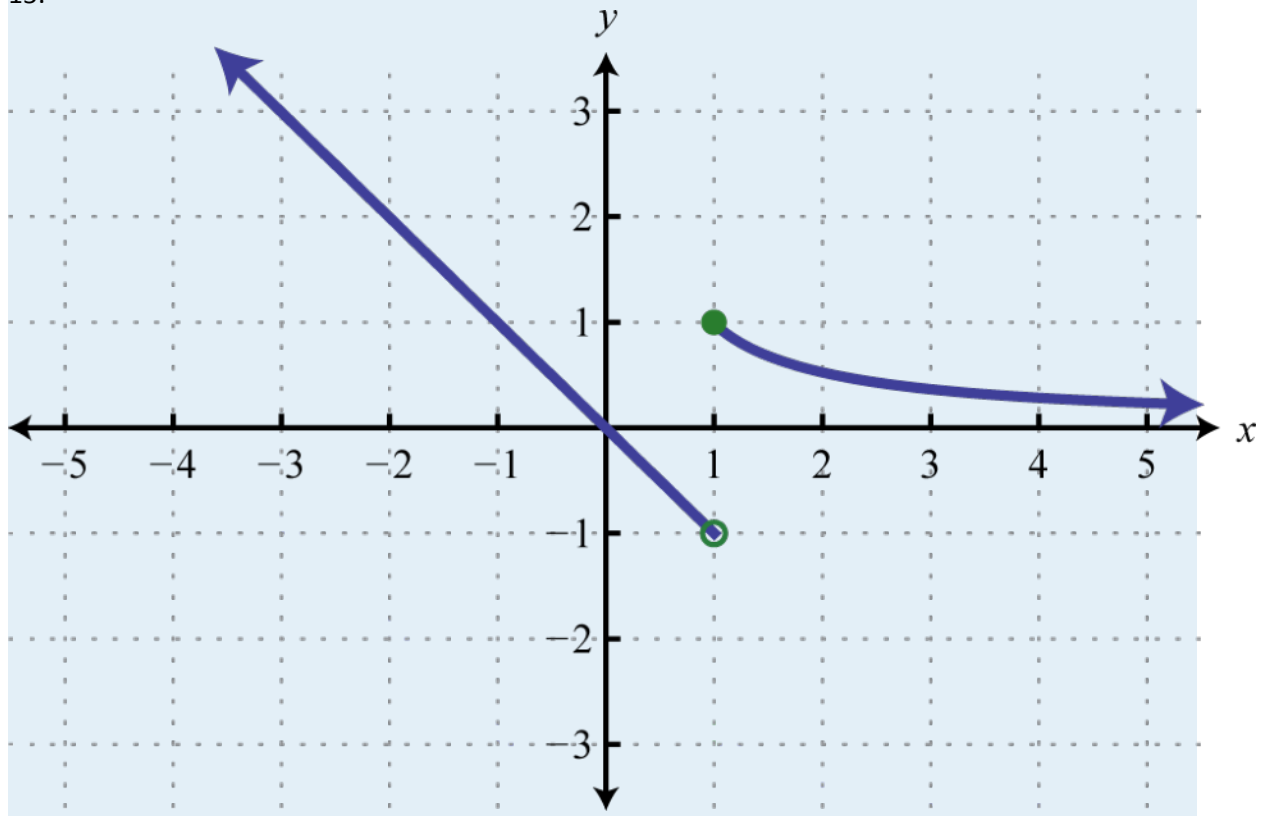
14.

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15.



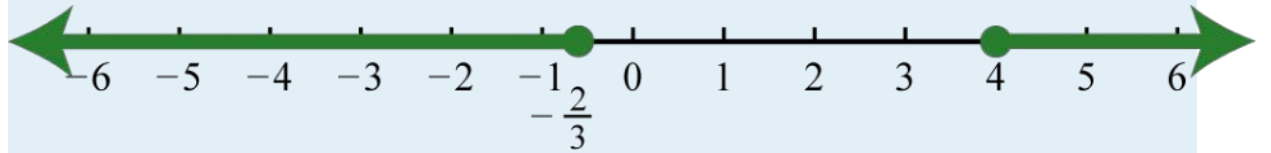
16.

17.  $-2, 3$

18.

19.

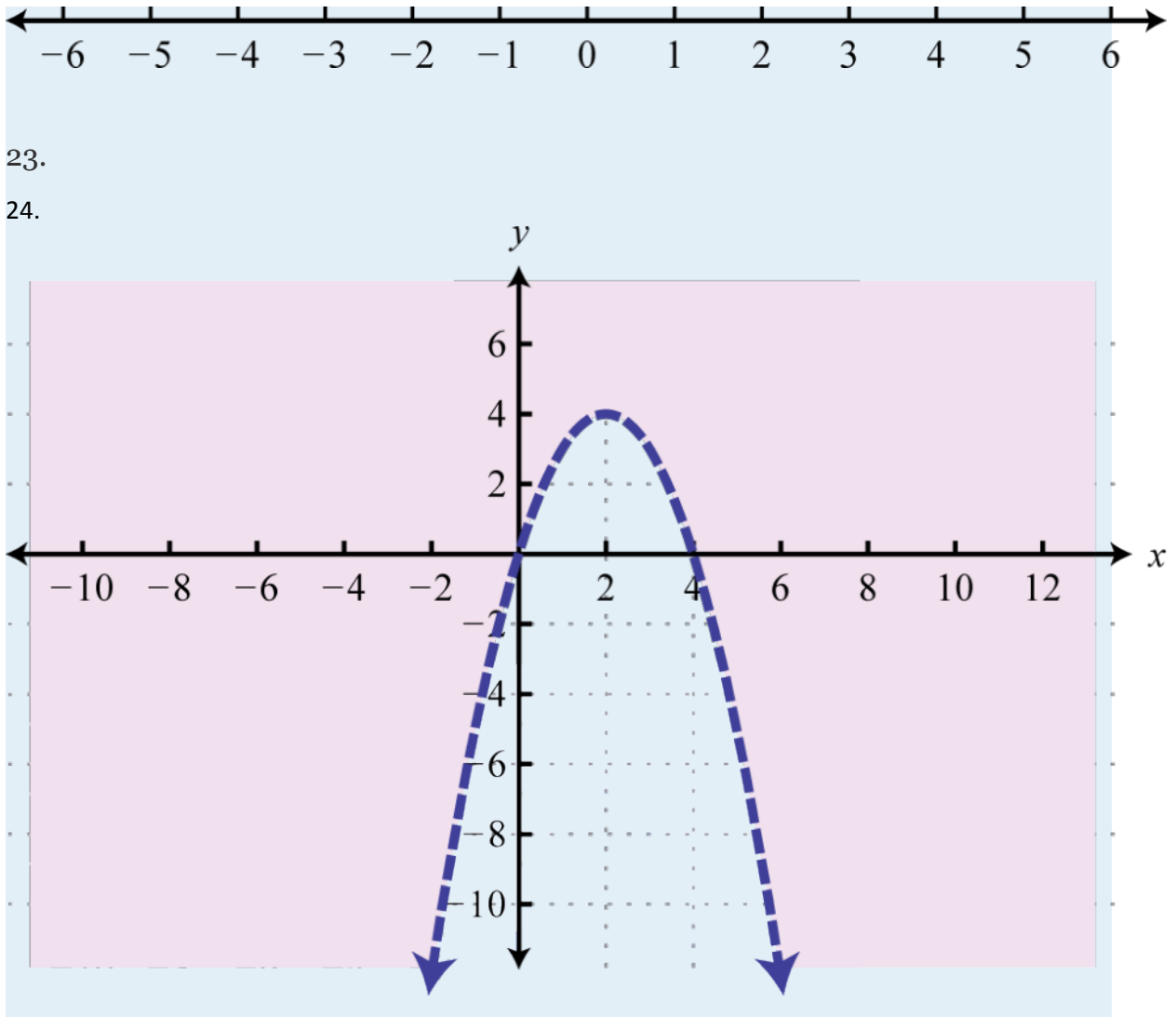
20.



21.

22.  $\emptyset;$





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# Chapter 3

## Solving Linear Systems

### 3.1 Linear Systems with Two Variables and Their Solutions

#### LEARNING OBJECTIVES

1. Solve linear systems using the graphing method.
2. Identify dependent and inconsistent systems.
3. Check solutions to systems of linear equations.

#### Definition of a Linear System with Two Variables

Real-world applications are often modeled using more than one variable and more than one equation. A **system of equations** consists of a set of two or more equations with the same variables. In this section, we will **study linear systems** consisting of two linear equations each with two variables. For example,

A **solution to a linear system**, or **simultaneous solution**, is an ordered pair  $(x, y)$  that solves both of the equations. In this case,  $(3, 2)$  is the only solution. To check that an ordered pair is a solution, substitute the corresponding  $x$ - and  $y$ -values into each equation and then simplify to see if you obtain a true statement for both equations.

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<b><i>Check: (3, 2)</i></b>	
<b><i>Equation 1: <math>2x-3y=0</math></i></b>	<b><i>Equation 2: <math>-4x+2y=-8</math></i></b>
✓	✓

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### Example 1

Determine whether or not  $(1, 0)$  is a solution to the system

Solution:

Substitute the appropriate values into both equations.

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<p><b>Check: (1, 0)</b></p>	
<p><b>Equation 1: <math>x - y = 1</math></b></p>	<p><b>Equation 2: <math>-2x + 3y = 5</math></b></p>
<p>✓</p>	<p>✗</p>

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Answer: Since  $(1, 0)$  does not satisfy *both* equations, it is not a solution.

Try this! Is  $(-2, 4)$  a solution to the system

Answer: Yes

## **Solve by Graphing**

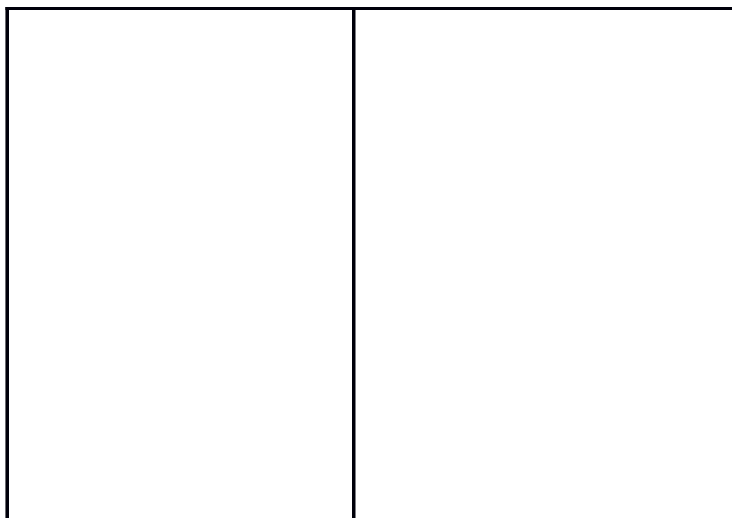
Geometrically, a linear system consists of two lines, where a solution is a point of intersection. To illustrate this, we will graph the following linear system with a solution of  $(3, 2)$ :

First, rewrite the equations in slope-intercept form so that we may easily graph them.

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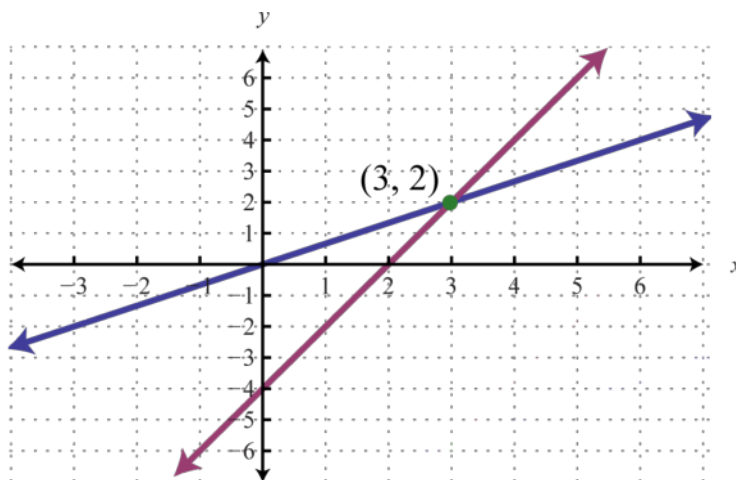
Next, replace these forms of the original equations in the system to obtain what is called an **equivalent system**. Equivalent systems share the same solution set.

If we graph both of the lines on the same set of axes, then we can see that the point of intersection is indeed  $(3, 2)$ , the solution to the system.

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To summarize, linear systems described in this section consist of two linear equations each with two variables. A solution is an ordered pair that corresponds to a point where the two lines intersect in the rectangular coordinate plane. Therefore, one way to solve linear systems is by graphing both lines on the same set of axes and determining the point where they cross. This describes the **graphing method** for solving linear systems.

When graphing the lines, take care to choose a good scale and use a straightedge to draw the line through the points; accuracy is very important here.

## Example 2

Solve by graphing:

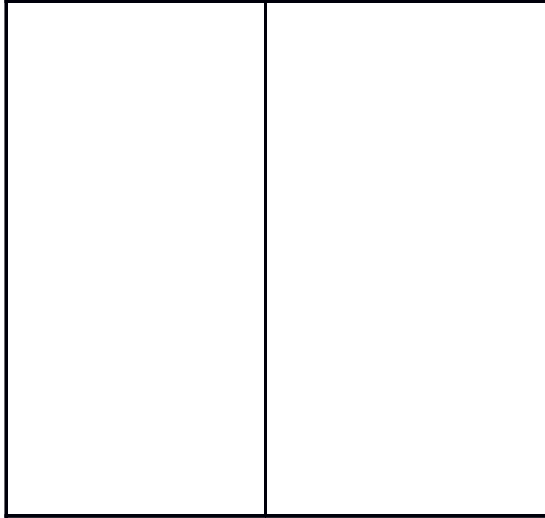
Solution:

Rewrite the linear equations in slope-intercept form.

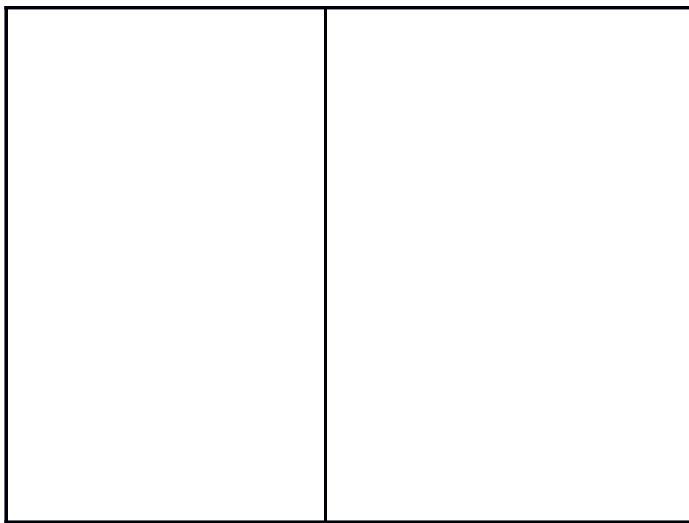
Attributed to John Redden

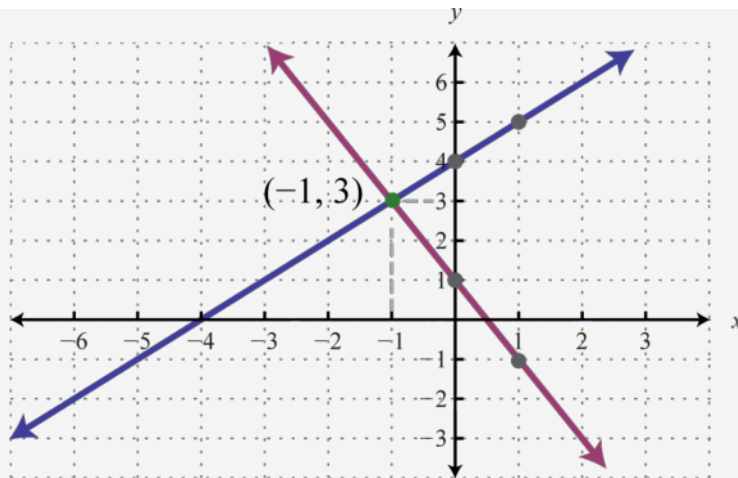
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Write the equivalent system and graph the lines on the same set of axes.





Use the graph to estimate the point where the lines intersect and check to see if it solves the original system. In the above graph, the point of intersection appears to be  $(-1, 3)$ .

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<b><i>Check: (-1, 3)</i></b>	
<b><i>Line 1: <math>x - y = -4</math></i></b>	<b><i>Line 2: <math>2x + y = 1</math></i></b>
✓	✓

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Answer:  $(-1, 3)$

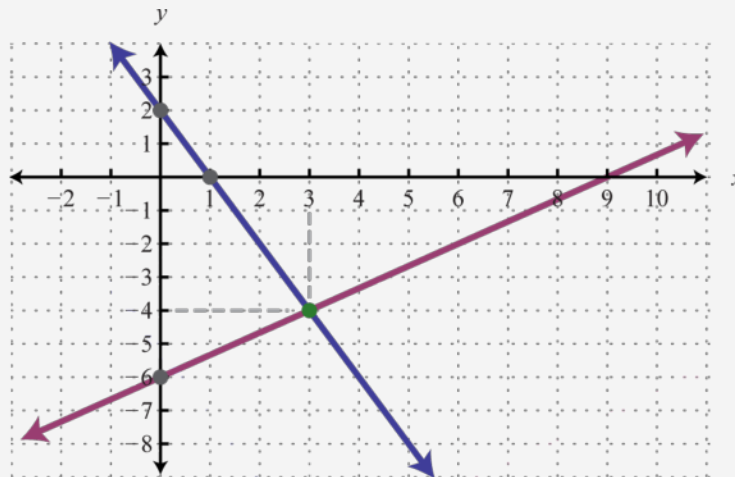
### Example 3

Solve by graphing:

Solution:

We first solve each equation for  $y$  to obtain an equivalent system where the lines are in slope-intercept form.

Graph the lines and determine the point of intersection.



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<b><i>Check: (3, -4)</i></b>	
✓	✓

Answer: (3, -4)

## Example 4

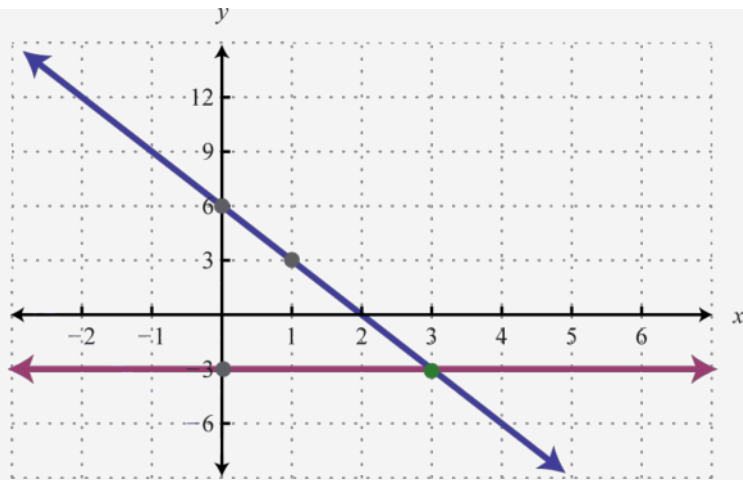
Solve by graphing:

Solution:

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<b><i>Check: (3, -3)</i></b>	
✓	✓

Answer: (3, -3)

The graphing method for solving linear systems is not ideal when a solution consists of coordinates that are not integers. There will be more accurate algebraic methods in sections to come, but for now, the goal is to understand the geometry involved when solving systems. It is important to remember that the solutions to a system correspond to the point, or points, where the graphs of the equations intersect.

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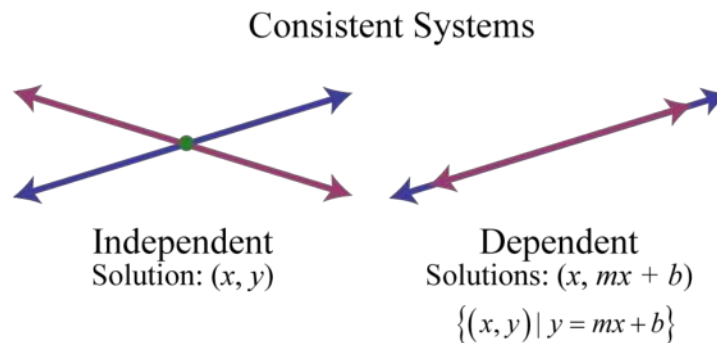


Try this! Solve by graphing:

Answer:  $(-2, 4)$

## Dependent and Inconsistent Systems

A system with at least one solution is called a **consistent system**. Up to this point, all of the examples have been of consistent systems with exactly one ordered pair solution. It turns out that this is not always the case. Sometimes systems consist of two linear equations that are equivalent. If this is the case, the two lines are the same and when graphed will coincide. Hence, the solution set consists of all the points on the line. This is a **dependent system**. Given a consistent linear system with two variables, there are two possible results:



A solution to an **independent system** is an ordered pair  $(x, y)$ . The solution to a dependent system consists of infinitely many ordered pairs  $(x, y)$ . Since any line can be written in slope-intercept form,  $y = mx + b$ , we can express these solutions, dependent on  $x$ , as follows:

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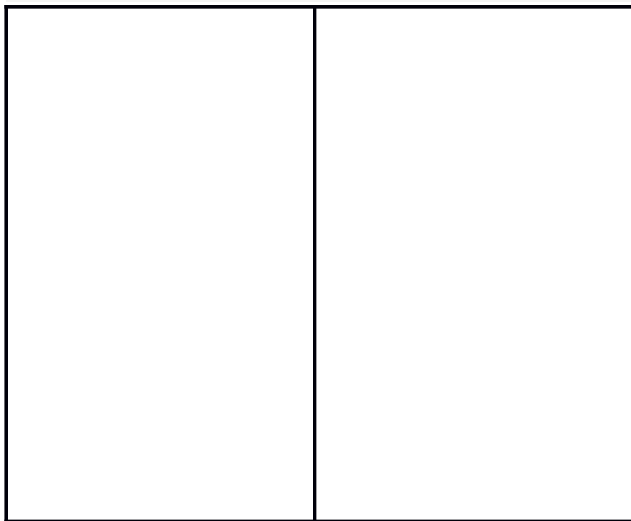
In this text we will express all the ordered pair solutions  $(x, y)$  in the shortened form  $(x, mx + b)$ , where  $x$  is any real number.

## Example 5

Solve by graphing:

Solution:

Determine slope-intercept form for each linear equation in the system.

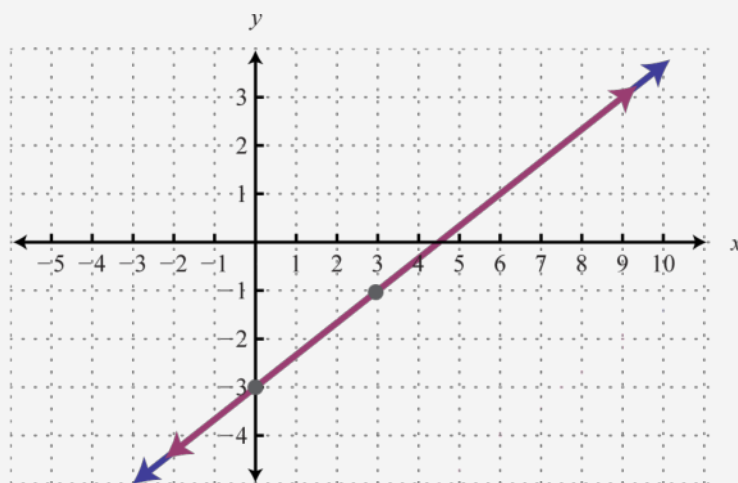


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In slope-intercept form, we can easily see that the system consists of two lines with the same slope and same  $y$ -intercept. They are, in fact, the same line. And the system is dependent.



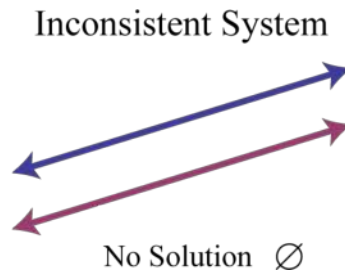
Answer:

In this example, it is important to notice that the two lines have the same slope and same  $y$ -intercept. This tells us that the two equations are equivalent and that the simultaneous solutions are all the points on the line. This is a dependent system, and the infinitely many solutions are expressed using the form  $(x, mx+b)$ . Other resources may express this set using set notation,  $\{(x, mx+b) \mid x \in \mathbb{R}\}$ , which reads “the set of all ordered pairs  $(x, y)$  such that .”

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Sometimes the lines do not cross and there is no point of intersection. Such a system has no solution,  $\emptyset$ , and is called an **inconsistent system**.



## Example 6

Solve by graphing:

Solution:

Determine slope-intercept form for each linear equation.

--	--

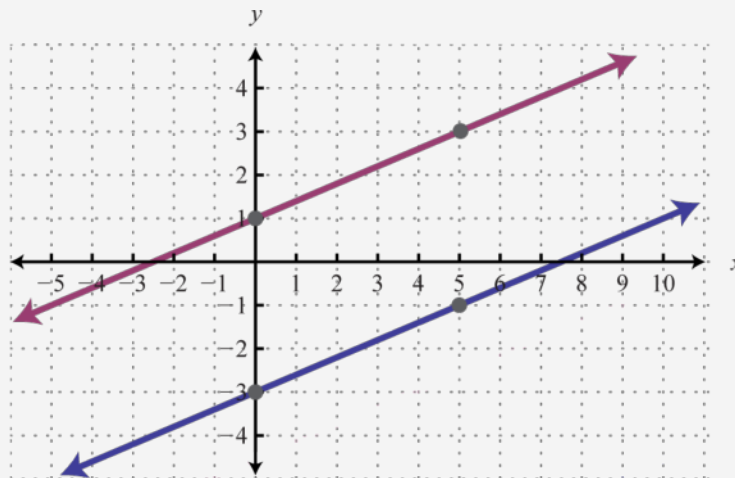
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In slope-intercept form, we can easily see that the system consists of two lines with the same slope and different  $y$ -intercepts. Therefore, the lines are parallel and will never intersect.



Answer: There is no simultaneous solution,  $\emptyset$ .

Try this! Solve by graphing:

Answer:  $(x, -x-1)$

## KEY TAKEAWAYS

In this section, we limit our study to systems of two linear equations with two variables. Solutions to such systems, if they exist, consist of ordered pairs that

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satisfy both equations. Geometrically, solutions are the points where the graphs intersect.

The graphing method for solving linear systems requires us to graph both of the lines on the same set of axes as a means to determine where they intersect.

The graphing method is not the most accurate method for determining solutions, particularly when a solution has coordinates that are not integers. It is a good practice to always check your solutions.

Some linear systems have no simultaneous solution. These systems consist of equations that represent parallel lines with different  $y$ -intercepts and do not intersect in the plane. They are called inconsistent systems and the solution set is the empty set,  $\emptyset$ .

Some linear systems have infinitely many simultaneous solutions. These systems consist of equations that are equivalent and represent the same line. They are called dependent systems and their solutions are expressed using the notation  $(x, mx+b)$ , where  $x$  is any real number.

## TOPIC EXERCISES

### PART A: DEFINITIONS

Determine whether or not the given ordered pair is a solution to the given system.

1.  $(3, -2)$ ;
2.  $(-5, 0)$ ;



3.  $(-2, -6);$

4.  $(2, -7);$

5.  $(0, -3);$

6.  $;$

7.  $;$

8.  $(-3, 4);$

9.  $(-5, -3);$

10.  $(4, 2);$

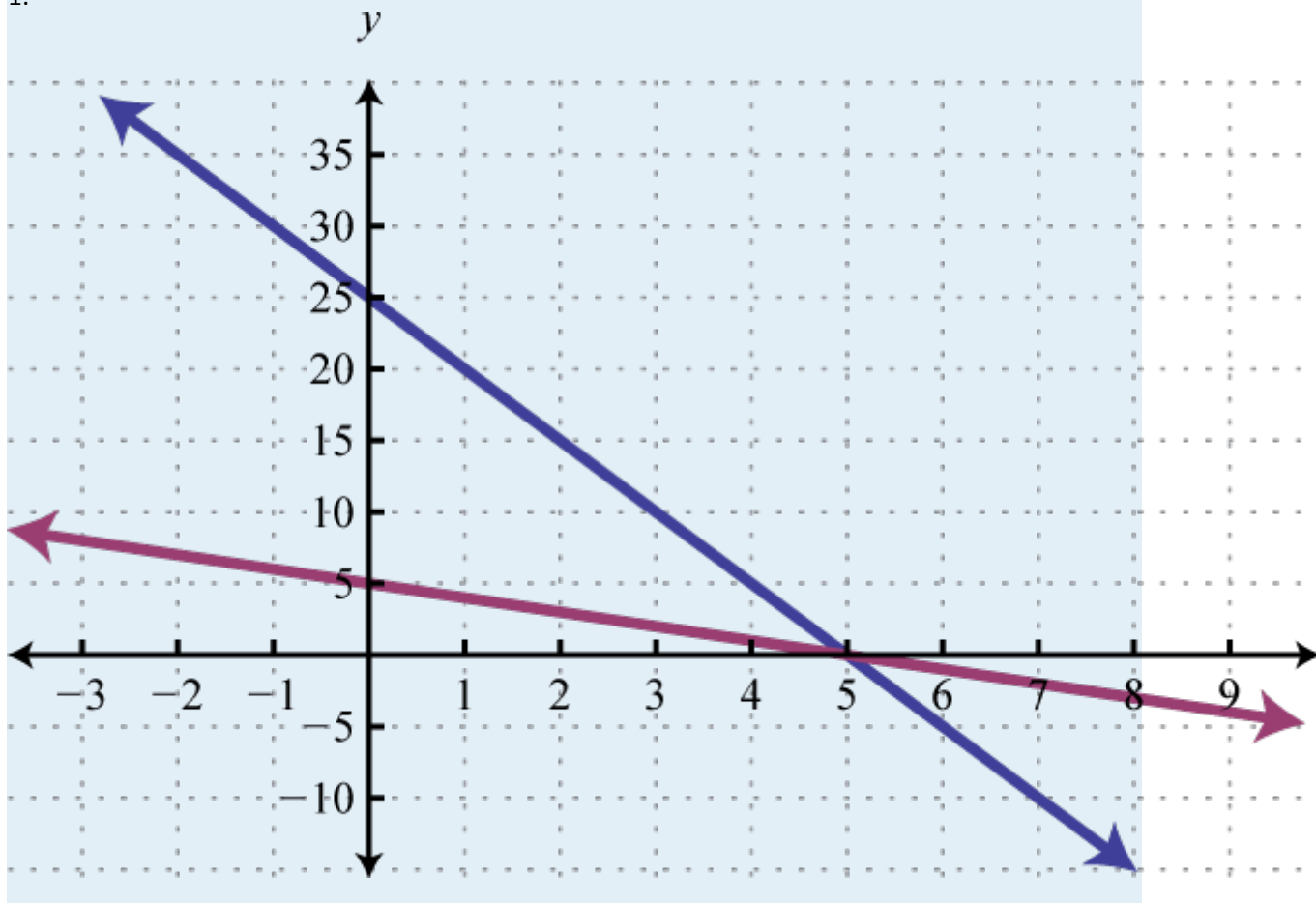
Given the graphs, determine the simultaneous solution.

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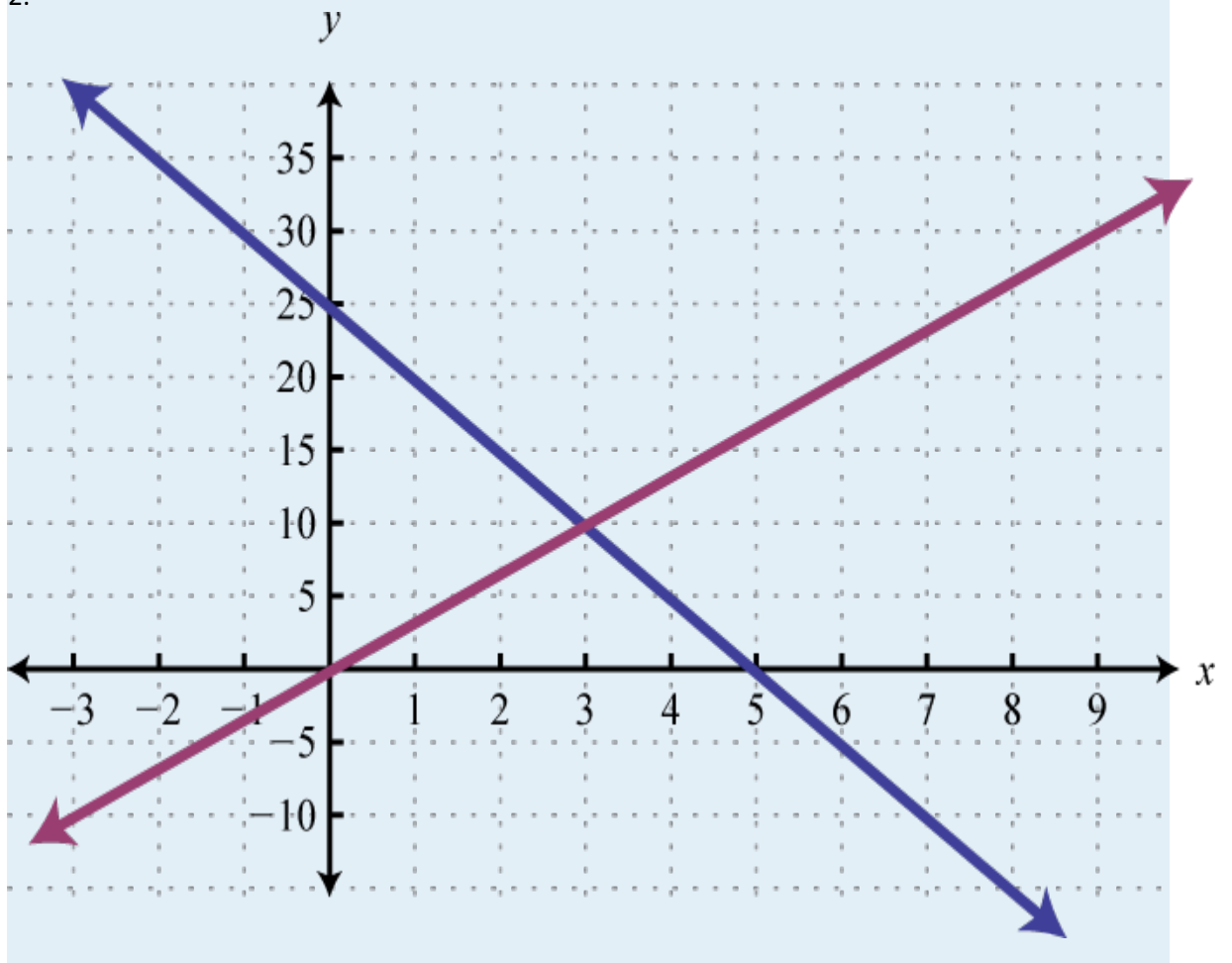
1.



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2.

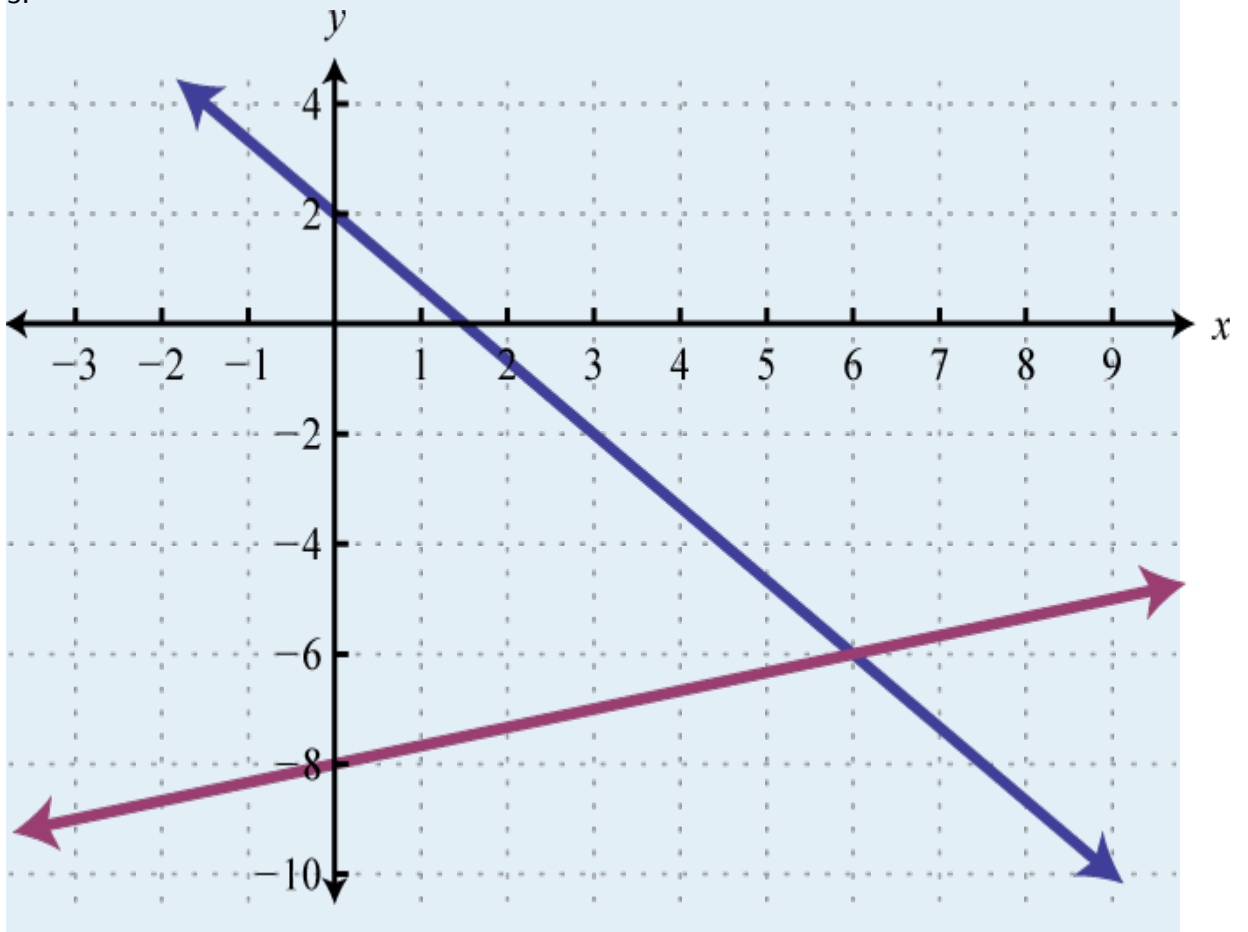


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3.

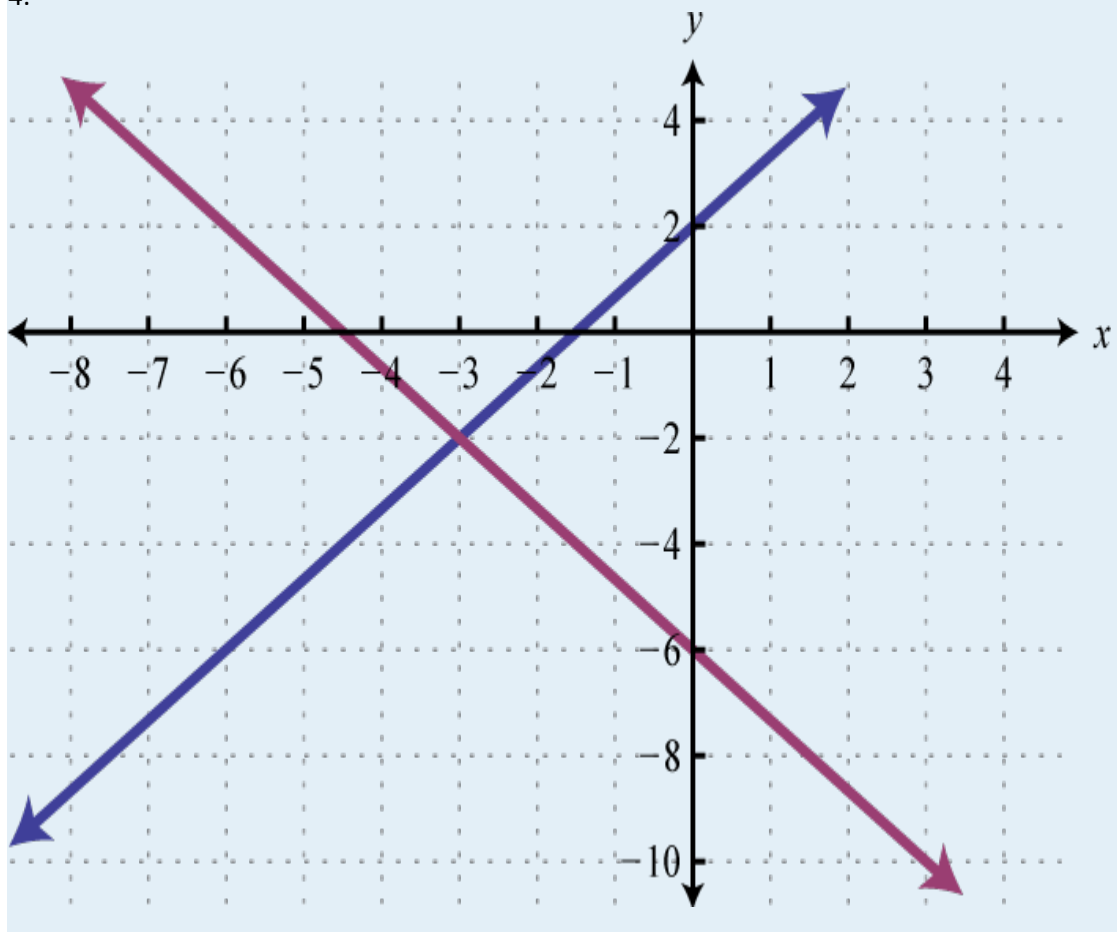


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4.

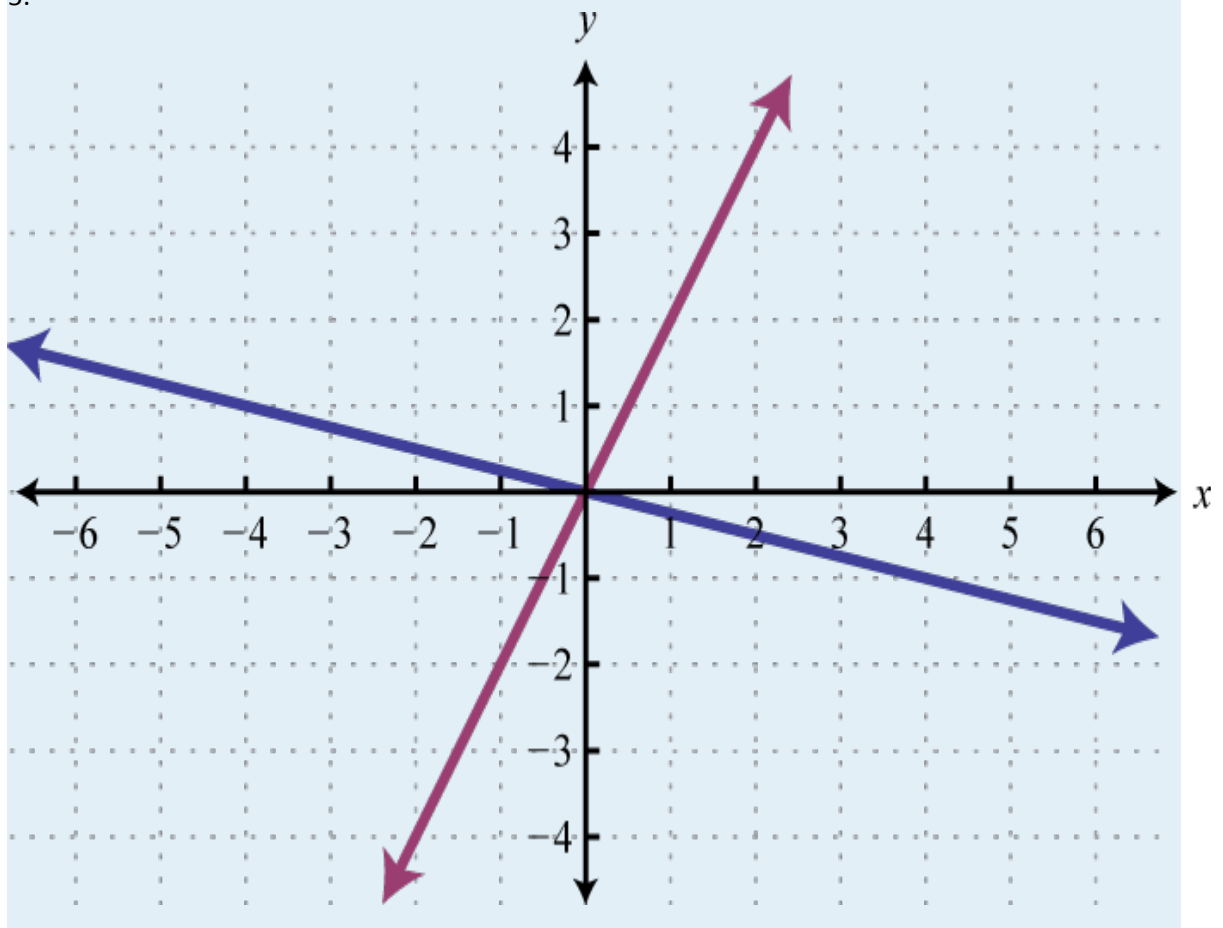


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5.



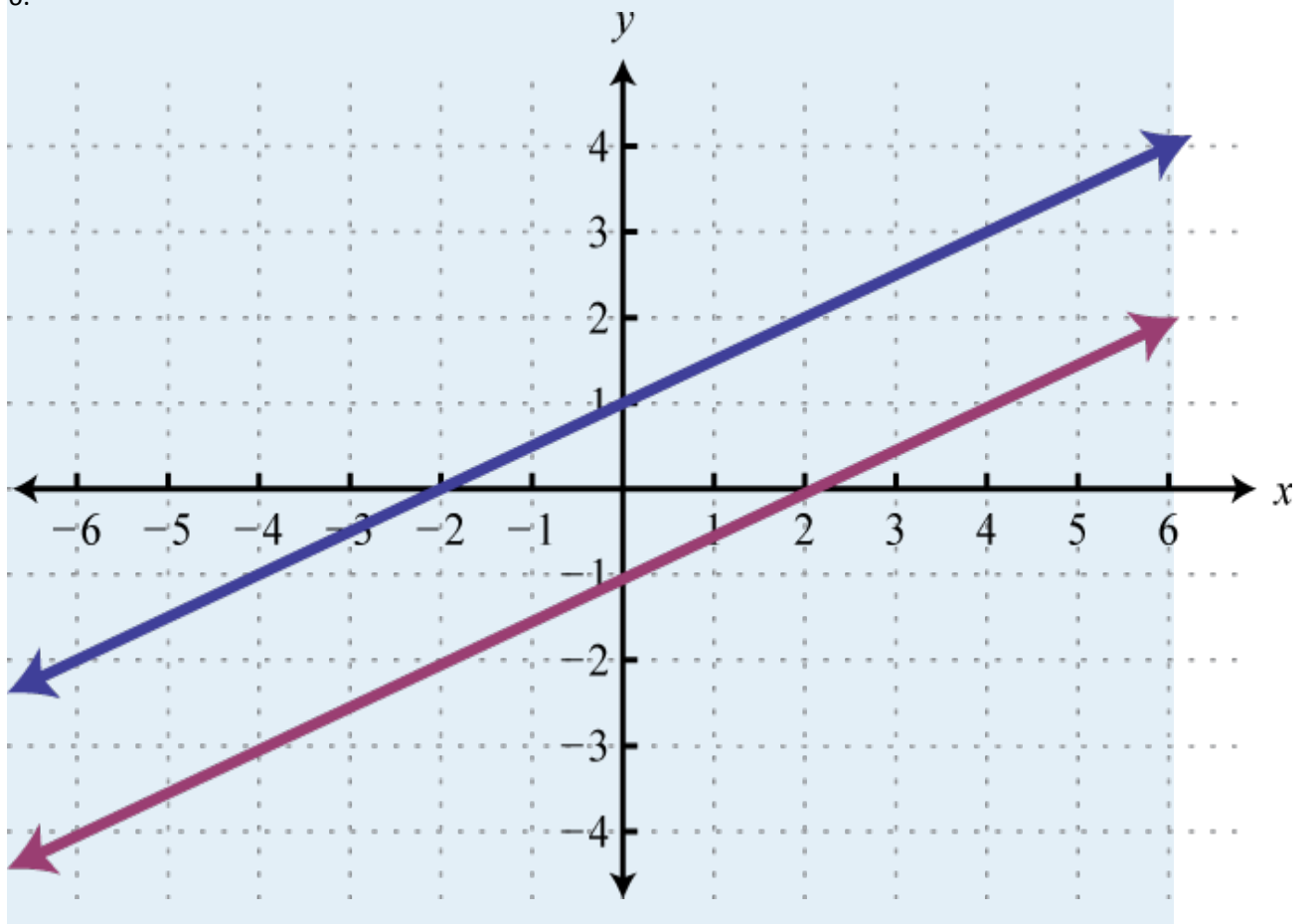
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6.

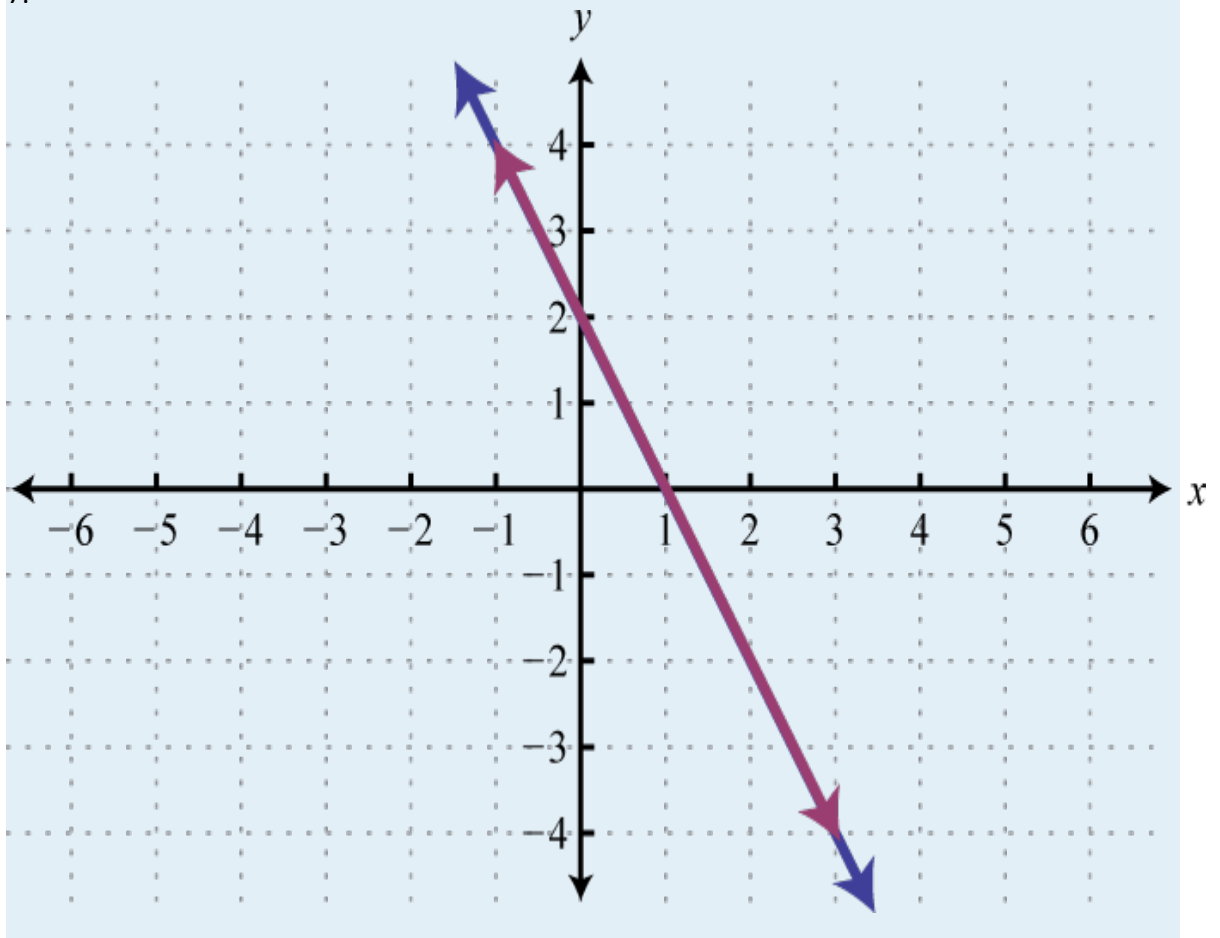


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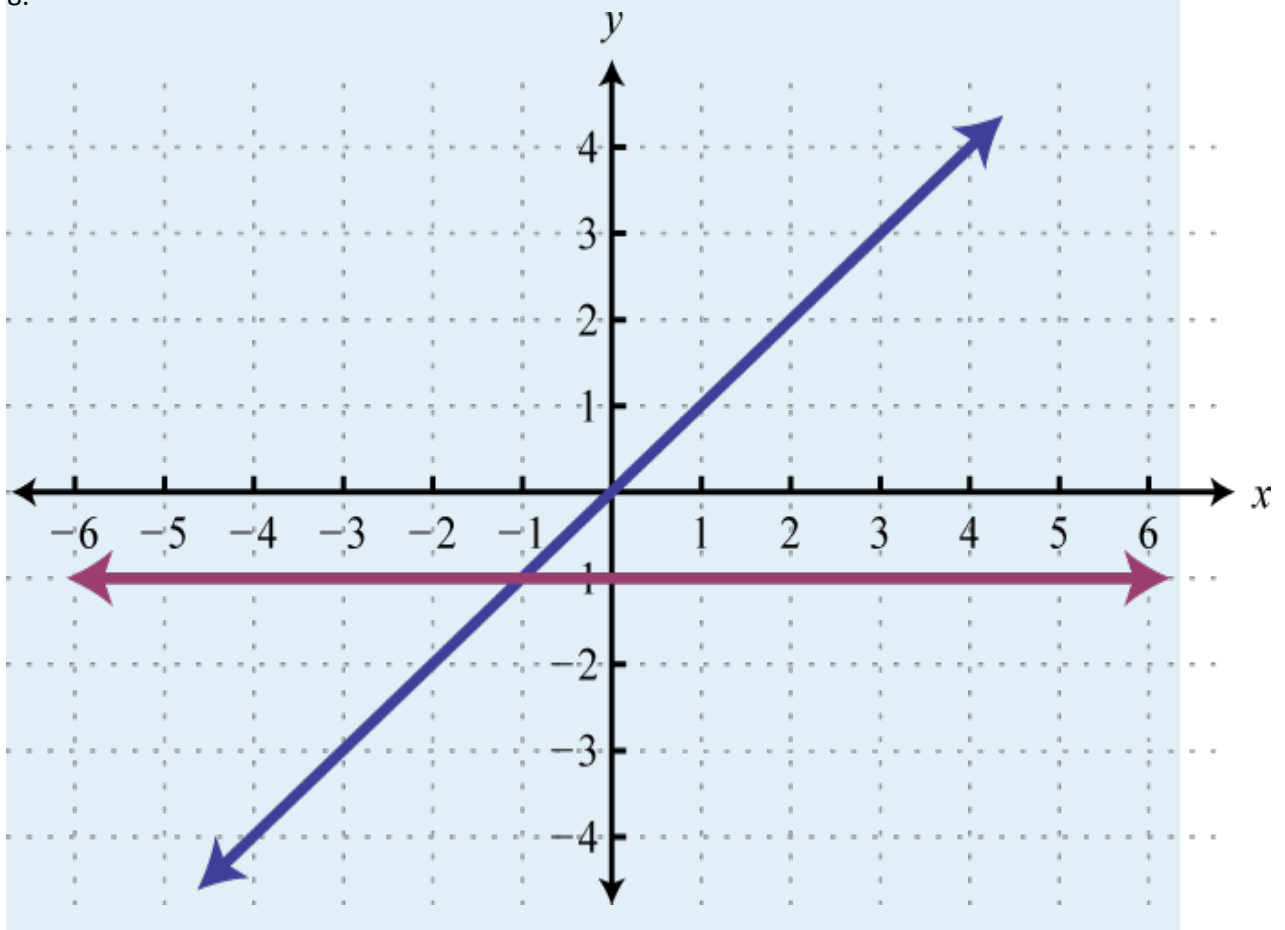


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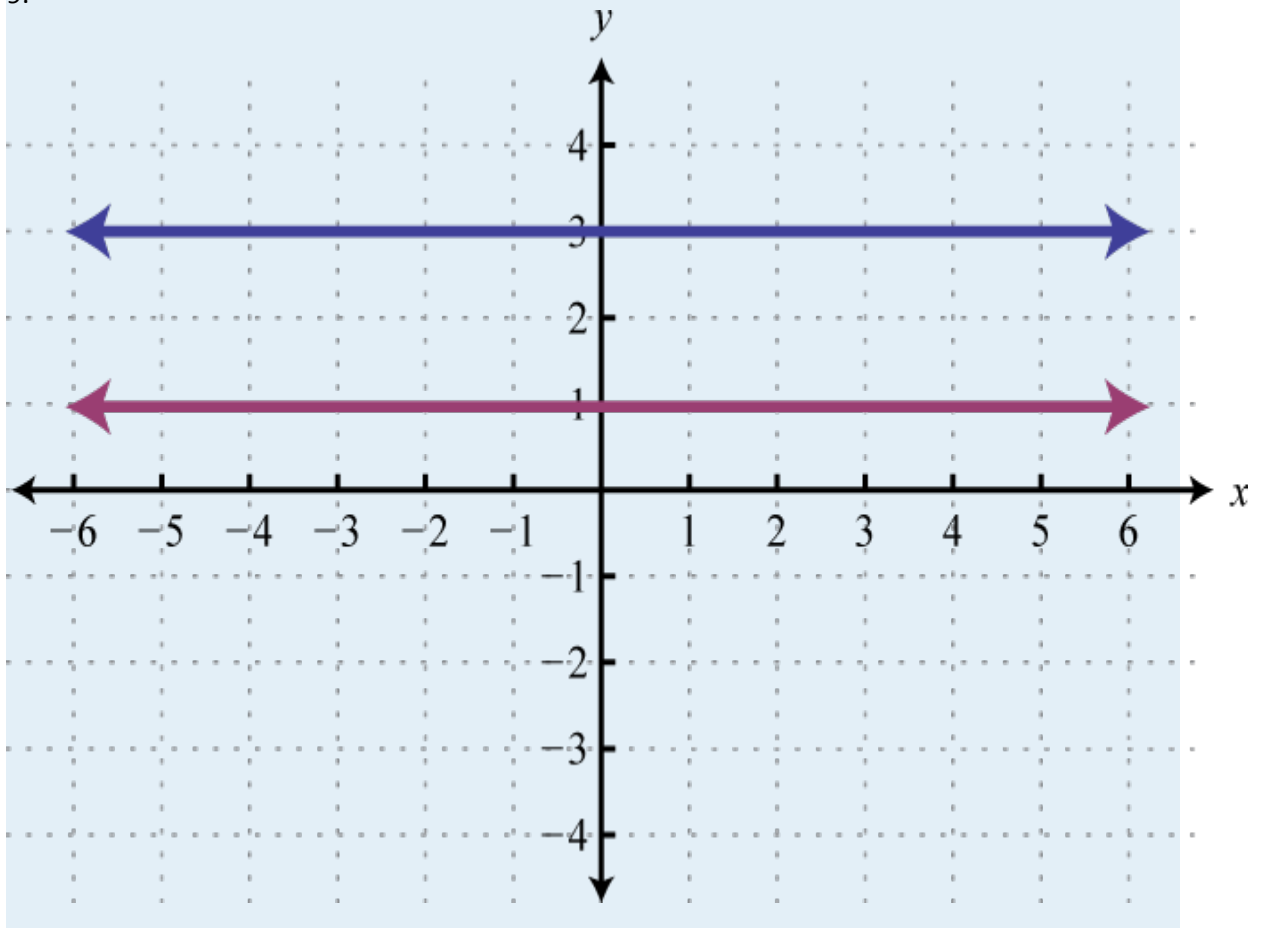
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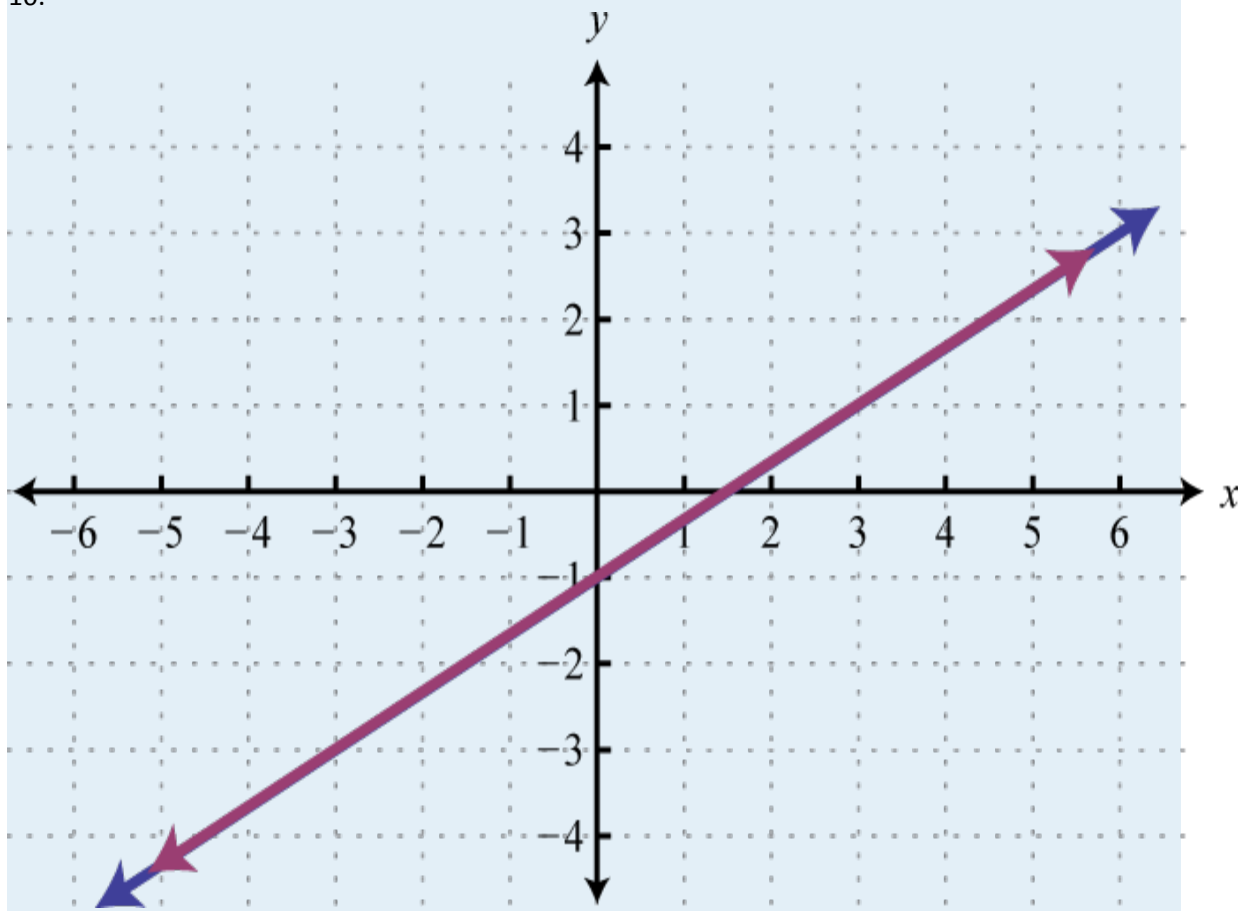


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10.



## PART B: SOLVE BY GRAPHING

Solve by graphing.

21.

22.

23.

24.

25.

26.

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- 28.
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- 54.
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- 59.
- 60.
61. Assuming  $m$  is nonzero solve the system:
62. Assuming  $b$  is nonzero solve the system:
63. Find the equation of the line perpendicular to  $y = -2x + 4$  and passing through  $(3, 3)$ . Graph this line and the given line on the same set of axes and determine where they intersect.
64. Find the equation of the line perpendicular to  $y - x = 2$  and passing through  $(-5, 1)$ . Graph this line and the given line on the same set of axes and determine where they intersect.
65. Find the equation of the line perpendicular to  $y = -5$  and passing through  $(2, -5)$ . Graph both lines on the same set of axes.
66. Find the equation of the line perpendicular to the  $y$ -axis and passing through the origin.
67. Use the graph of  $f$  to determine the  $x$ -value where  $y = -3$ . Verify your answer using algebra.
68. Use the graph of  $f$  to determine the  $x$ -value where  $y = 5$ . Verify your answer using algebra.

## PART C: DISCUSSION BOARD TOPICS

69. Discuss the weaknesses of the graphing method for solving systems.

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70. Explain why the solution set to a dependent linear system is denoted by  $(x, mx+b)$ .
71. Draw a picture of a dependent linear system as well as a picture of an inconsistent linear system. What would you need to determine the equations of the lines that you have drawn?

## ANSWERS

1. No
- 2.
3. No
- 4.
5. Yes
- 6.
7. No
- 8.
9. Yes
- 10.
11.  $(5, 0)$
- 12.
13.  $(6, -6)$
- 14.
15.  $(0, 0)$
- 16.
17.  $(x, -2x+2)$
- 18.

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19.  $\emptyset$
- 20.
21.  $(-2, 3)$
- 22.
23.  $(3, -1)$
- 24.
25.  $(5, 3)$
- 26.
27.  $(-3, -2)$
- 28.
29.  $(10, 0)$
- 30.
31.  $\emptyset$
- 32.
33.  $(3, 4)$
- 34.
35.  $(-3, -5)$
- 36.
37.  $(6, 2)$
- 38.
39.  $\emptyset$
- 40.
41.  $(x, x)$
- 42.

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43.  $(2, -3)$

44.

45.  $(10, 5)$

46.

47.  $(-9, 6)$

48.

49.

50.

51.  $(-5, 4)$

52.

53.  $(0, 0)$

54.

55.  $\emptyset$

56.

57.  $\emptyset$

58.

59.

60.

61.  $(0, b)$

62.

63.  $;$   $(1, 2)$

64.

65.  $x=2$

66.

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67.  $x=9$
- 68.
69. Answer may vary
- 70.
71. Answer may vary

## 3.2 Solving Linear Systems with Two Variables

### LEARNING OBJECTIVES

1. Solve linear systems using the substitution method.
2. Solve linear systems using the elimination method.
3. Identify the strengths and weaknesses of each method.

### The Substitution Method

In this section, we review a completely algebraic technique for solving systems, the **substitution method**. The idea is to solve one equation for one of the variables and substitute the result into the other equation. After performing this substitution step, we are left with a single equation with one variable, which can be solved using algebra.

### Example 1

Solve by substitution:


Solution:

Solve for either variable in either equation. If you choose the first equation, you can isolate  $y$  in one step.

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Substitute the expression  $-2x-3$  for the variable  $y$  in the *other* equation.

$$\begin{cases} 2x + y = -3 & \Rightarrow y = -2x - 3 \\ 3x - 2y = -8 \end{cases}$$


$$3x - 2(-2x - 3) = -8$$

This leaves us with an equivalent equation with one variable, which can be solved using the techniques learned up to this point. Solve for the remaining variable.

**Back substitute** to find the other coordinate. Substitute  $x = -2$  into either of the original equations or their equivalents. Typically, we use the equivalent equation that we found when isolating a variable in the first step.

Remember to present the solution as an ordered pair:  $(-2, 1)$ . Verify that these coordinates solve both equations of the original system:

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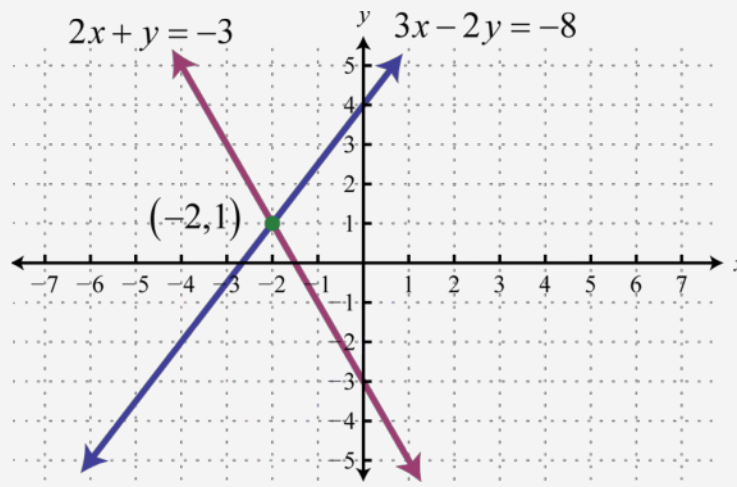


<b><i>Check: (-2, 1)</i></b>	
<b><i>Equation 1</i></b>	<b><i>Equation 2</i></b>
✓	✓

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The graph of this linear system follows:



The substitution method for solving systems is a completely algebraic method. Thus graphing the lines is not required.

Answer:  $(-2, 1)$

## Example 2

Solve by substitution:

Solution:

It does not matter which variable we choose to isolate first. In this case, begin by solving for  $x$  in the first equation.

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Next, substitute into the second equation and solve for  $y$ .

Back substitute into the equation used in the substitution step:

Answer:

Try this! Solve by substitution:

Answer:

As we know, not all linear systems have only one ordered pair solution. Next, we explore what happens when using the substitution method to solve a dependent system.

### Example 3

Solve by substitution:

Solution:

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Since the first equation has a term with coefficient 1, we choose to solve for that first.

Next, substitute this expression in for  $y$  in the second equation.

This process led to a true statement; hence the equation is an identity and any real number is a solution. This indicates that the system is dependent. The simultaneous solutions take the form  $(x, mx + b)$ , or in this case,  $(x, 5x - 1)$ , where  $x$  is any real number.

Answer:  $(x, 5x - 1)$

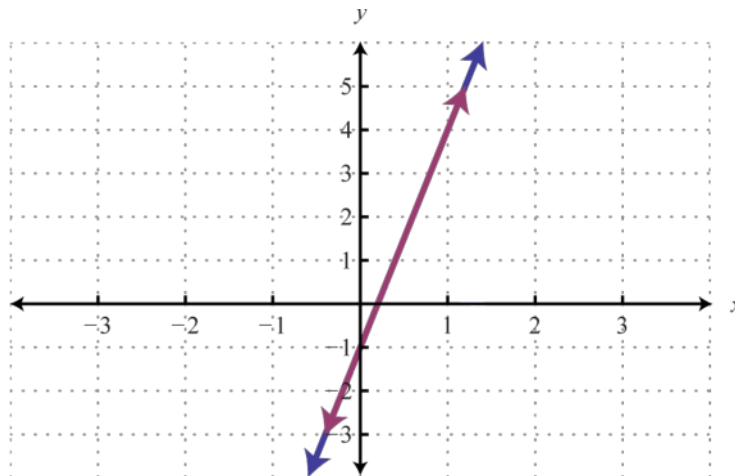
To have a better understanding of the previous example, rewrite both equations in slope-intercept form and graph them on the same set of axes.

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We can see that both equations represent the same line, and thus the system is dependent. Now explore what happens when solving an inconsistent system using the substitution method.

## Example 4

Solve by substitution:

Solution:

Solve for  $y$  in the first equation.

Substitute into the second equation and solve.

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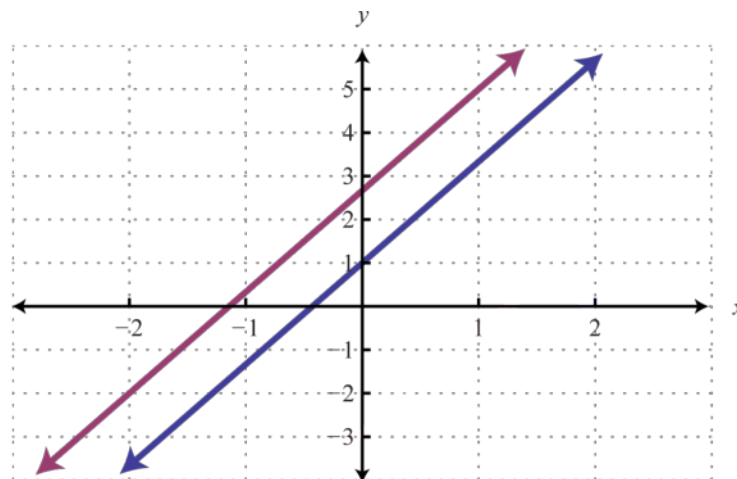
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Solving leads to a false statement. This indicates that the equation is a contradiction. There is no solution for  $x$  and hence no solution to the system.

Answer:  $\emptyset$

A false statement indicates that the system is inconsistent, or in geometric terms, that the lines are parallel and do not intersect. To illustrate this, determine the slope-intercept form of each line and graph them on the same set of axes.



In slope-intercept form, it is easy to see that the two lines have the same slope but different  $y$ -intercepts.

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Try this! Solve by substitution:

Answer:

## The Elimination Method

In this section, the goal is to review another completely algebraic method for solving a system of linear equations called the **elimination method** or **addition method**. This method depends on the **addition property of equations**: given algebraic expressions  $A$ ,  $B$ ,  $C$ , and  $D$  we have

$$\text{If } A=B \text{ and } C=D, \text{ then } A+C=B+D$$

Consider the following system:

We can add the equations together to eliminate the variable  $y$ .

This leaves us with a linear equation with one variable that can be easily solved:

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At this point, we have the  $x$ -coordinate of the simultaneous solution, so all that is left to do is back substitute to find the corresponding  $y$ -value.

The solution to the system is  $(3, 2)$ . Of course, the variable is not always so easily eliminated. Typically, we have to find an equivalent system by applying the multiplication property of equality to one or both of the equations as a means to line up one of the variables to eliminate. The goal is to arrange that either the  $x$  terms or the  $y$  terms are opposites, so that when the equations are added, the terms eliminate.

## Example 5

Solve by elimination:

Solution:

We choose to eliminate the terms with variable  $y$  because the coefficients have different signs. To do this, we first determine the least common multiple of the coefficients; in this case, the LCM  $(3, 2)$  is 6. Therefore, multiply both sides of both equations by the appropriate values to obtain coefficients of  $-6$  and  $6$ . This results in the following equivalent system:

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The terms involving  $y$  are now lined up to eliminate. Add the equations together and solve for  $x$ .

Back substitute.

Therefore the simultaneous solution is  $(1, 2)$ . The check follows.

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<b><i>Check: (1, 2)</i></b>	
<b><i>Equation 1:</i></b>	<b><i>Equation 2:</i></b>
✓	✓

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Answer: (1, 2)

Sometimes linear systems are not given in standard form  $ax+by=c$ . When this is the case, it is best to rearrange the equations before beginning the steps to solve by elimination. Also, we can eliminate either variable. The goal is to obtain a solution for one of the variables and then back substitute to find a solution for the other.

## Example 6

Solve by elimination:

Solution:

First, rewrite the second equation in standard form.

This results in an equivalent system in standard form, where like terms are aligned in columns.

We can eliminate the term with variable  $x$  if we multiply the second equation by  $-4$ .

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$$\begin{cases} 12x + 5y = 11 \\ 3x - 4y = 1 \end{cases} \xrightarrow{\times(-4)} \begin{cases} 12x + 5y = 11 \\ -12x + 16y = -4 \end{cases}$$

Next, we add the equations together,

Back substitute.

Answer: (79, 13)

Try this! Solve by elimination:

Answer: (-5, 3)

At this point, we explore what happens when solving dependent and inconsistent systems using the elimination method.

## Example 7

Solve by elimination:

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Solution:

To eliminate the variable  $x$ , we could multiply the first equation by  $-2$ .

$$\begin{cases} 3x - y = 7 \\ 6x - 2y = 14 \end{cases} \xrightarrow{\times(-2)} \begin{cases} -6x + 2y = -14 \\ 6x - 2y = 14 \end{cases}$$

Now adding the equations we have

A true statement indicates that this is a dependent system. The lines coincide, and we need  $y$  in terms of  $x$  to present the solution set in the form  $(x, mx+b)$ . Choose one of the original equations and solve for  $y$ . Since the equations are equivalent, it does not matter which one we choose.

Answer:  $(x, 3x-7)$

Try this! Solve by elimination:

Answer: No solution,  $\emptyset$

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Given a linear system where the equations have fractional coefficients, it is usually best to clear the fractions before beginning the elimination method.

## Example 8

Solve:

Solution:

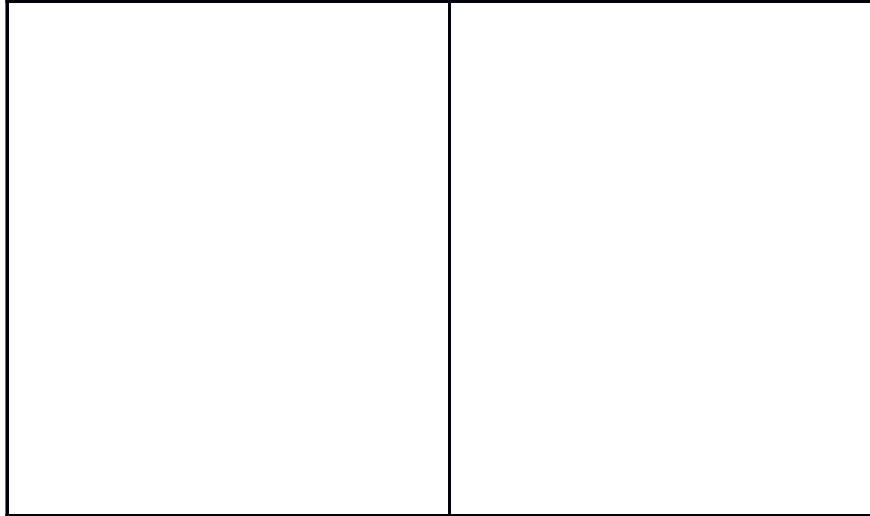
Recall that we can clear fractions by multiplying both sides of an equation by the least common multiple of the denominators (LCD). Take care to distribute and then simplify.

<i>Equation 1</i>	<i>Equation 2</i>
-------------------	-------------------

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This results in an equivalent system where the equations have integer coefficients,

Solve using the elimination method.

$$\begin{cases} -x + 5y = 8 \\ 3x + 7y = -2 \end{cases} \xrightarrow{\times 3} \begin{cases} -3x + 15y = 24 \\ 3x + 7y = -2 \end{cases}$$

Back substitute.

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Answer:  $(-3, 1)$

We can use a similar technique to clear decimals before solving.

Try this! Solve using elimination:

Answer:  $(5, -2)$

## Summary of the Methods for Solving Linear Systems

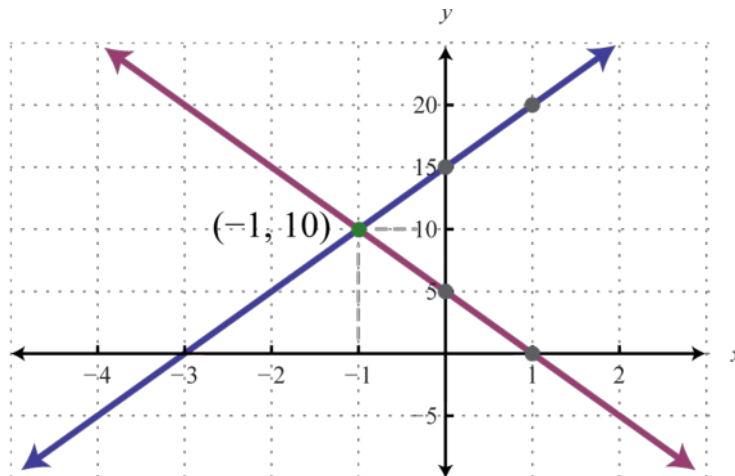
We have reviewed three methods for solving linear systems of two equations with two variables. Each method is valid and can produce the same correct result. In this section, we summarize the strengths and weaknesses of each method.

The graphing method is useful for understanding what a system of equations is and what the solutions must look like. When the equations of a system are graphed on the same set of axes, we can see that the solution is the point where the graphs intersect. The graphing is made easy when the equations are in slope-intercept form. For example,

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The simultaneous solution  $(-1, 10)$  corresponds to the point of intersection. One drawback of this method is that it is very inaccurate. When the coordinates of the solution are not integers, the method is practically unusable. If we have a choice, we typically avoid this method in favor of the more accurate algebraic techniques.

The substitution method, on the other hand, is a completely algebraic method. It requires you to solve for one of the variables and substitute the result into the other equation. The resulting equation has one variable for which you can solve. This method is particularly useful when there is a variable within the system with coefficient of 1. For example,

In this case, it is easy to solve for  $y$  in the first equation and then substitute the result into the other equation. One drawback of this method is that it often leads to equivalent



equations with fractional coefficients, which are tedious to work with. If there is not a coefficient of 1, then it usually is best to choose the elimination method.

The elimination method is a completely algebraic method which makes use of the addition property of equations. We multiply one or both of the equations to obtain equivalent equations where one of the variables is eliminated if we add them together. For example,

To eliminate the terms involving  $x$ , we would multiply both sides of the first equation by 5 and both sides of the second equation by  $-2$ . This results in an equivalent system where the variable  $x$  is eliminated when we add the equations together. Of course, there are other combinations of numbers that achieve the same result. We could even choose to eliminate the variable  $y$ . No matter which variable is eliminated first, the solution will be the same. Note that the substitution method, in this case, would require tedious calculations with fractional coefficients. One weakness of the elimination method, as we will see later in our study of algebra, is that it does not always work for nonlinear systems.

## KEY TAKEAWAYS

The substitution method requires that we solve for one of the variables and then substitute the result into the other equation. After performing the substitution step, the resulting equation has one variable and can be solved using the techniques learned up to this point.

The elimination method is another completely algebraic method for solving a system of equations. Multiply one or both of the equations in a system by certain

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numbers to obtain an equivalent system where at least one variable in both equations have opposite coefficients. Adding these equivalent equations together eliminates that variable, and the resulting equation has one variable for which you can solve.

It is a good practice to first rewrite the equations in standard form before beginning the elimination method.

Solutions to systems of two linear equations with two variables, if they exist, are ordered pairs  $(x, y)$ .

If the process of solving a system of equations leads to a false statement, then the system is inconsistent and there is no solution,  $\emptyset$ .

If the process of solving a system of equations leads to an identity, then the system is dependent and there are infinitely many solutions that can be expressed using the form  $(x, mx + b)$ .

## TOPIC EXERCISES

### PART A: SUBSTITUTION METHOD

Solve by substitution.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.

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- 10.
- 11.
- 12.
- 13.
- 14.
- 15.
- 16.
- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.
- 25.
- 26.
- 27.
- 28.
- 29.
- 30.
- 31.
- 32.
- 33.
- 34.
- 35.

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36.

37.

38.

## PART B: ELIMINATION METHOD

Solve by elimination.

39.

40.

41.

42.

43.

44.

45.

46.

47.

48.

49.

50.

51.

52.

53.

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- 60.
- 61.
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
- 69.
- 70.
- 71.
- 72.
- 73.
- 74.
- 75.
- 76.

## PART C: MIXED PRACTICE

Solve using any method.

- 77.
- 78.
- 79.
- 80.

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- 81.
- 82.
- 83.
- 84.
- 85.
- 86.
- 87.
- 88.
- 89.
- 90.
- 91.
- 92.
- 93.
- 94.
- 95.
- 96.
- 97.
- 98.
- 99.
- 100.

## PART D: DISCUSSION BOARD

101. Explain to a beginning algebra student how to choose a method for solving a system of two linear equations. Also, explain what solutions look like and why.
102. Make up your own linear system with two variables and solve it using all three methods. Explain which method was preferable in your exercise.\

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## ANSWERS

1.  $(-2, 11)$
- 2.
3.  $(2, 2)$
- 4.
5.  $\emptyset$
- 6.
7.  $(x, 2x+3)$
- 8.
9.  $(4, -2)$
- 10.
11.  $(3, -2)$
- 12.
13.  $(3, -4)$
- 14.
- 15.
- 16.
- 17.
- 18.
19.  $\emptyset$
- 20.
21.  $(2, -3)$
- 22.

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23.

24.

25.  $\emptyset$

26.

27.  $(1, 1)$

28.

29.

30.

31.

32.

33.  $(0, 0)$

34.

35.  $(1, 2)$

36.

37.  $\emptyset$

38.

39.

40.

41.  $(-4, 2)$

42.

43.  $(-8, -1)$

44.

45.

46.

47.  $\emptyset$

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- 48.
- 49.
- 50.
- 51.  $(-1, -2)$
- 52.
- 53.  $(-28, 0)$
- 54.
- 55.
- 56.
- 57.  $(1, 2)$
- 58.
- 59.  $(-1, -4)$
- 60.
- 61.  $(-5, 2)$
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
- 69.  $(120, 77)$
- 70.
- 71.
- 72.



73.  $\emptyset$
- 74.
75.  $(0.5, -3)$
- 76.
- 77.
- 78.
- 79.
- 80.
- 81.
- 82.
83.  $(-2, -10)$
- 84.
85.  $(3, -1)$
- 86.
87.  $(32, 0)$
- 88.
89.  $(x, -2x+6)$
- 90.
91.  $(-4, 3)$
- 92.
93.  $\emptyset$
- 94.
- 95.
- 96.
97.  $(0.8, -5.4)$

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98.

99.  $\emptyset$

100.

101. Answer may vary

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## 3.3 Applications of Linear Systems with Two Variables

### LEARNING OBJECTIVES

1. Set up and solve applications involving relationships between two variables.
2. Set up and solve mixture problems.
3. Set up and solve uniform motion problems (distance problems).

### Problems Involving Relationships between Two Variables

If we translate an application to a mathematical setup using two variables, then we need to form a linear system with two equations. Setting up word problems with two variables often simplifies the entire process, particularly when the relationships between the variables are not so clear.

#### Example 1

The sum of 4 times a larger integer and 5 times a smaller integer is 7. When twice the smaller integer is subtracted from 3 times the larger, the result is 11. Find the integers.

Solution:

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Begin by assigning variables to the larger and smaller integer.

Let  $x$  represent the larger integer.

Let  $y$  represent the smaller integer.

When using two variables, we need to set up two equations. The first sentence describes a sum and the second sentence describes a difference.

$$\begin{array}{r} \text{3 times the larger} \\ \underbrace{\hspace{1.5cm}} \\ 3x \end{array} - \begin{array}{r} \text{twice the smaller} \\ \underbrace{\hspace{1.5cm}} \\ 2y \end{array} = 11$$

$$\begin{array}{r} \text{3 times the larger} \\ \underbrace{\hspace{1.5cm}} \\ 3x \end{array} - \begin{array}{r} \text{twice the smaller} \\ \underbrace{\hspace{1.5cm}} \\ 2y \end{array} = 11$$

This leads to the following system:

Solve using the elimination method. To eliminate the variable  $y$  multiply the first equation by 2 and the second by 5.

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Add the equations in the equivalent system and solve for  $x$ .

Back substitute to find  $y$ .

Answer: The larger integer is 3 and the smaller integer is  $-1$ .

Try this! An integer is 1 less than twice that of another. If their sum is 20, find the integers.

Answer: The two integers are 7 and 13.

Next consider applications involving simple interest and money.

## Example 2

A total of \$12,800 was invested in two accounts. Part was invested in a CD at a 3.18% annual interest rate and part was invested in a money market fund at a 4.5% annual interest rate. If the total simple interest for one year was \$465, then how much was invested in each account?

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Solution:

Begin by identifying two variables.

Let  $x$  represent the amount invested at

Let  $y$  represent the amount invested at

The total amount in both accounts can be expressed as

$$x+y=12,800$$

To set up a second equation, use the fact that the total interest was \$465. Recall that the interest for one year is the interest rate times the principal ( $I=prt=pr \cdot 1=pr$ ). Use this to add the interest in both accounts. Be sure to use the decimal equivalents for the interest rates given as percentages.

These two equations together form the following linear system:

Eliminate  $x$  by multiplying the first equation by  $-0.03125$ .

$$\begin{cases} x + y = 12,800 \\ 0.03125x + 0.0475y = 465 \end{cases} \xrightarrow{\times(-0.03125)} \begin{cases} -0.03125x - 0.03125y = -400 \\ 0.03125x + 0.0475y = 465 \end{cases}$$

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Next, add the resulting equations.

Back substitute to find  $x$ .

Answer: \$4,000 was invested at and \$8,800 was invested at

### Example 3

A jar consisting of only nickels and dimes contains 58 coins. If the total value is \$4.20, how many of each coin is in the jar?

Solution:

Let  $n$  represent the number of nickels in the jar.

Let  $d$  represent the number of dimes in the jar.

The total number of coins in the jar can be expressed using the following equation:

$$n + d = 58$$

Next, use the value of each coin to determine the total value \$4.20.

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This leads us to following linear system:

Here we will solve using the substitution method. In the first equation, we can solve for  $n$ .

$$\begin{cases} n + d = 58 \\ 0.05n + 0.10d = 4.20 \end{cases} \Rightarrow n = 58 - d$$

Substitute  $n=58-d$  into the second equation and solve for  $d$ .

Now back substitute to find the number of nickels.

Answer: There are 32 nickels and 26 dimes in the jar.

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Try this! Joey has a jar full of 40 coins consisting of only quarters and nickels. If the total value is \$5.00, how many of each coin does Joey have?

Answer: Joey has 15 quarters and 25 nickels.

## Mixture Problems

Mixture problems often include a percentage and some total amount. It is important to make a distinction between these two types of quantities. For example, if a problem states that a 20-ounce container is filled with a 2% saline (salt) solution, then this means that the container is filled with a mixture of salt and water as follows:

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	Percentage	Amount
Salt	$2\% = 0.02$	$0.02(20 \text{ ounces}) = 0.4 \text{ ounces}$
Water	$98\% = 0.98$	$0.98(20 \text{ ounces}) = 19.6 \text{ ounces}$

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In other words, we multiply the percentage times the total to get the amount of each part of the mixture.

## Example 4

A 1.8% saline solution is to be combined and mixed with a 3.2% saline solution to produce 35 ounces of a 2.2% saline solution. How much of each is needed?

Solution:

Let  $x$  represent the amount of 1.8% saline solution needed.

Let  $y$  represent the amount of 3.2% saline solution needed.

The total amount of saline solution needed is 35 ounces. This leads to one equation,

$$x+y=35$$

The second equation adds up the amount of salt in the correct percentages. The amount of salt is obtained by multiplying the percentage times the amount, where the variables  $x$  and  $y$  represent the amounts of the solutions. The amount of salt in the end solution is 2.2% of the 35 ounces, or  $.022(35)$ .

The algebraic setup consists of both equations presented as a system:

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Solve.

$$\begin{cases} x + y = 35 \\ 0.018x + 0.032y = 0.022(35) \end{cases} \xrightarrow{\times(-0.018)} \Rightarrow \begin{cases} -0.018x - 0.018y = -0.63 \\ 0.018x + 0.032y = 0.77 \end{cases}$$

Add the resulting equations together

Back substitute to find  $x$ .

Answer: We need 25 ounces of the 1.8% saline solution and 10 ounces of the 3.2% saline solution.

## Example 5

An 80% antifreeze concentrate is to be mixed with water to produce a 48-liter mixture containing 25% antifreeze. How much water and antifreeze concentrate is needed?

Solution:

Let  $x$  represent the amount of 80% antifreeze concentrate needed.

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Let  $y$  represent the amount of water needed.

The total amount of the mixture must be 48 liters.

$$x+y=48$$

The second equation adds up the amount of antifreeze from each solution in the correct percentages. The amount of antifreeze in the end result is 25% of 48 liters, or  $0.25(48)$ .

Now we can form a system of two linear equations and two variables as follows:

Use the second equation to find  $x$ :

Back substitute to find  $y$ .

Answer: We need to mix 33 liters of water with 15 liters of antifreeze concentrate.

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Try this! A chemist wishes to create 100 ml of a solution with 12% acid content. He uses two types of stock solutions, one with 30% acid content and another with 10% acid content. How much of each does he need?

Answer: The chemist will need to mix 10 ml of the 30% acid solution with 90 ml of the 10% acid solution.

## Uniform Motion Problems (Distance Problems)

Recall that the distance traveled is equal to the average rate times the time traveled at that rate,  $D=r \cdot t$ . These uniform motion problems usually have a lot of data, so it helps to first organize that data in a chart and then set up a linear system. In this section, you are encouraged to use two variables.

### Example 6

An executive traveled a total of 4 hours and 875 miles by car and by plane. Driving to the airport by car, she averaged 50 miles per hour. In the air, the plane averaged 320 miles per hour. How long did it take her to drive to the airport?

Solution:

We are asked to find the time it takes her to drive to the airport; this indicates that time is the unknown quantity.

Let  $x$  represent the time it took to drive to the airport.

Let  $y$  represent the time spent in the air.

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Fill in the chart with the given information.

	<i>Distance = Rate × Time</i>		
<i>Travel by car</i>		50 mph	$x$
<i>Travel by air</i>		320 mph	$y$
<b>Total</b>	875 mi		4 hours

Use the formula  $D=r \cdot t$  to fill in the unknown distances.

	<i>Distance = Rate × Time</i>		
<i>Travel by car</i>	$50x$	50 mph	$x$
<i>Travel by air</i>	$320y$	320 mph	$y$
<b>Total</b>	875 mi		4 hours

The distance column and the time column of the chart help us to set up the following linear system.

	<i>Distance = Rate × Time</i>		
<i>Travel by car</i>	$50x$	50 mph	$x$
<i>Travel by air</i>	$320y$	320 mph	$y$
<b>Total</b>	875 mi		4 hours

$50x + 320y = 875$                        $x + y = 4$

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Solve.

$$\begin{cases} x + y = 4 \\ 50x + 320y = 875 \end{cases} \xrightarrow{\times(-50)} \begin{cases} -50x - 50y = -200 \\ 50x + 320y = 875 \end{cases}$$

Now back substitute to find the time  $x$  it took to drive to the airport:

Answer: It took her 112 hours to drive to the airport.

It is not always the case that time is the unknown quantity. Read the problem carefully and identify what you are asked to find; this defines your variables.

## Example 7

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Flying with the wind, a light aircraft traveled 240 miles in 2 hours. The aircraft then turned against the wind and traveled another 135 miles in 1.5 hours. Find the speed of the airplane and the speed of the wind.

Solution:

Begin by identifying variables.

Let  $x$  represent the speed of the airplane.

Let  $w$  represent the speed of the wind.

Use the following chart to organize the data:

	<i>Distance = Rate × Time</i>		
<i>Flight with wind</i>	240 mi		2 hrs
<i>Flight against wind</i>	135 mi		1.5 hrs
<b>Total</b>			

With the wind, the airplane's total speed is  $x + w$ . Flying against the wind, the total speed is  $x - w$ .

	<i>Distance = Rate × Time</i>		
<i>Flight with wind</i>	240 mi	$x + w$	2 hrs
<i>Flight against wind</i>	135 mi	$x - w$	1.5 hrs
<b>Total</b>			

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Use the rows of the chart along with the formula  $D=r \cdot t$  to construct a linear system that models this problem. Take care to group the quantities that represent the rate in parentheses.

<i>Distance = Rate × Time</i>			
<i>Flight with wind</i>	240 mi	$x + w$	2 hrs
<i>Flight against wind</i>	135 mi	$x - w$	1.5 hrs
<i>Total</i>			

$240 = (x + w) \cdot 2$

$135 = (x - w) \cdot 1.5$

If we divide both sides of the first equation by 2 and both sides of the second equation by 1.5, then we obtain the following equivalent system:

Here  $w$  is lined up to eliminate.

Back substitute.





Answer: The speed of the airplane is 105 miles per hour and the speed of the wind is 15 miles per hour.

Try this! A boat traveled 27 miles downstream in 2 hours. On the return trip, which was against the current, the boat was only able to travel 21 miles in 2 hours. What were the speeds of the boat and of the current?

Answer: The speed of the boat was 12 miles per hour and the speed of the current was 1.5 miles per hour.

## KEY TAKEAWAYS

Use two variables as a means to simplify the algebraic setup of applications where the relationship between unknowns is unclear.

Carefully read the problem several times. If two variables are used, then remember that you need to set up two linear equations in order to solve the problem.

Be sure to answer the question in sentence form and include the correct units for the answer.

## TOPIC EXERCISES

### PART A: APPLICATIONS INVOLVING TWO VARIABLES

Set up a linear system and solve.

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1. The sum of two integers is 45. The larger integer is 3 less than twice the smaller. Find the two integers.
2. The sum of two integers is 126. The larger is 18 less than 5 times the smaller. Find the two integers.
3. The sum of two integers is 41. When 3 times the smaller is subtracted from the larger the result is 17. Find the two integers.
4. The sum of two integers is 46. When the larger is subtracted from twice the smaller the result is 2. Find the two integers.
5. The difference of two integers is 11. When twice the larger is subtracted from 3 times the smaller, the result is 3. Find the integers.
6. The difference of two integers is 6. The sum of twice the smaller and the larger is 72. Find the integers.
7. The sum of 3 times a larger integer and 2 times a smaller is 15. When 3 times the smaller integer is subtracted from twice the larger, the result is 23. Find the integers.
8. The sum of twice a larger integer and 3 times a smaller is 10. When the 4 times the smaller integer is added to the larger, the result is 0. Find the integers.
9. The difference of twice a smaller integer and 7 times a larger is 4. When 5 times the larger integer is subtracted from 3 times the smaller, the result is  $-5$ . Find the integers.
10. The difference of a smaller integer and twice a larger is 0. When 3 times the larger integer is subtracted from 2 times the smaller, the result is  $-5$ . Find the integers.
11. The length of a rectangle is 5 more than twice its width. If the perimeter measures 46 meters, then find the dimensions of the rectangle.

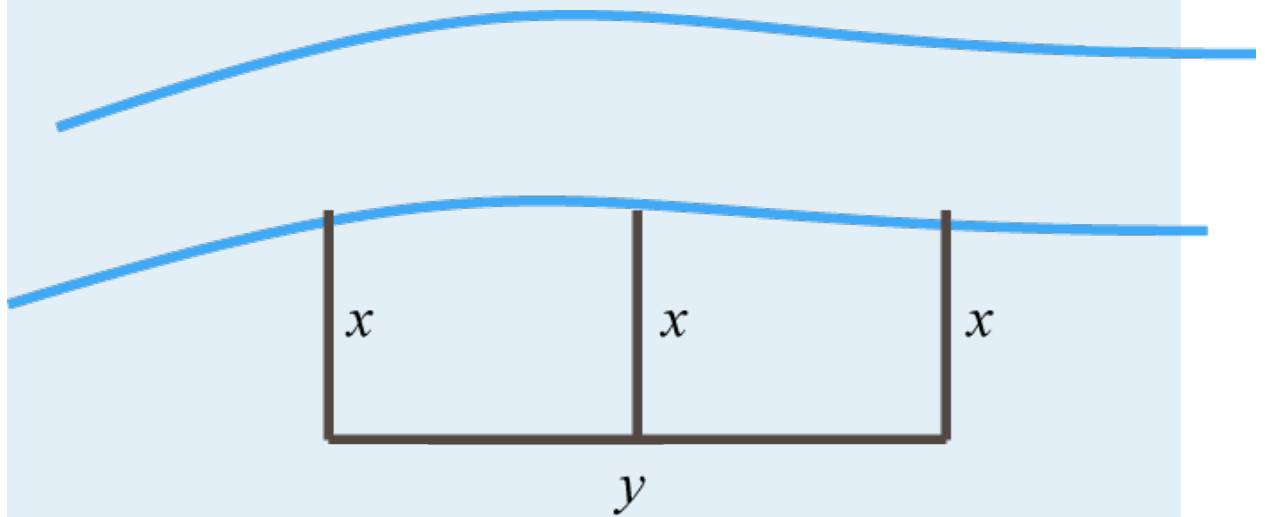
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12. The width of a rectangle is 2 centimeters less than one-half its length. If the perimeter measures 62 centimeters, then find the dimensions of the rectangle.

13. A partitioned rectangular pen next to a river is constructed with a total 136 feet of fencing (see illustration). If the outer fencing measures 114 feet, then find the dimensions of the pen.

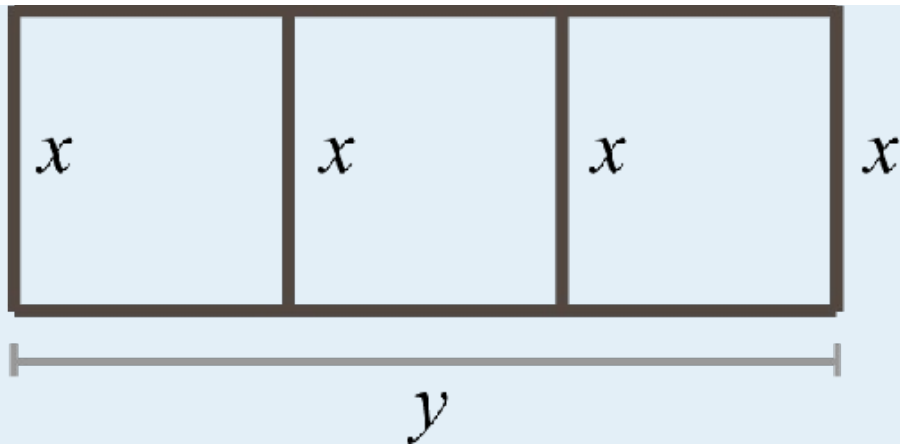


14. A partitioned rectangular pen is constructed with a total 168 feet of fencing (see illustration). If the perimeter measures 138 feet, then find the dimensions of the pen.

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15. Find  $a$  and  $b$  such that the system has solution  $(2,1)$ . (Hint: Substitute the given  $x$ - and  $y$ -values and solve the resulting linear system in terms of  $a$  and  $b$ .)
16. Find  $a$  and  $b$  such that the system has solution  $(3,-1)$ .
17. A line passes through two points  $(5,-9)$  and  $(-3,7)$ . Use these points and  $y=mx+b$  to construct a system of two linear equations in terms of  $m$  and  $b$  and solve it.
18. A line passes through two points  $(2,7)$  and  $(-1,3)$ . Use these points and  $y=mx+b$  to construct a system of two linear equations in terms of  $m$  and  $b$  and solve it.
19. A \$5,200 principal is invested in two accounts, one earning 3% interest and another earning 6% interest. If the total interest for the year is \$210, then how much is invested in each account?
20. Harry's \$2,200 savings is in two accounts. One account earns 2% annual interest and the other earns 4%. His total interest for the year is \$69. How much does he have in each account?

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21. Janine has two savings accounts totaling \$6,500. One account earns 234% annual interest and the other earns 312%. If her total interest for the year is \$211, then how much is in each account?
22. Margaret has her total savings of \$24,200 in two different CD accounts. One CD earns 4.6% interest and another earns 3.4% interest. If her total interest for the year is \$1,007.60, then how much does she have in each CD account?
23. Last year Mandy earned twice as much interest in her Money Market fund as she did in her regular savings account. The total interest from the two accounts was \$246. How much interest did she earn in each account?
24. A small business invested \$120,000 in two accounts. The account earning 4% annual interest yielded twice as much interest as the account earning 3% annual interest. How much was invested in each account?
25. Sally earns \$1,000 per month plus a commission of 2% of sales. Jane earns \$200 per month plus 6% of her sales. At what monthly sales figure will both Sally and Jane earn the same amount of pay?
26. The cost of producing specialty book shelves includes an initial set-up fee of \$1,200 plus an additional \$20 per unit produced. Each shelf can be sold for \$60 per unit. Find the number of units that must be produced and sold where the costs equal the revenue generated.
27. Jim was able to purchase a pizza for \$12.35 with quarters and dimes. If he uses 71 coins to buy the pizza, then how many of each did he have?
28. A cash register contains \$5 bills and \$10 bills with a total value of \$350. If there are 46 bills total, then how many of each does the register contain?
29. Two families bought tickets for the home basketball game. One family ordered 2 adult tickets and 4 children's tickets for a total of \$36.00. Another family

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ordered 3 adult tickets and 2 children's tickets for a total of \$32.00. How much did each ticket cost?

30. Two friends found shirts and shorts on sale at a flea market. One bought 4 shirts and 2 shorts for a total of \$28.00. The other bought 3 shirts and 3 shorts for a total of \$30.75. How much was each shirt and each pair of shorts?

31. A community theater sold 140 tickets to the evening musical for a total of \$1,540. Each adult ticket was sold for \$12 and each child ticket was sold for \$8. How many adult tickets were sold?

32. The campus bookstore sells graphing calculators for \$110 and scientific calculators for \$16. On the first day of classes 50 calculators were sold for a total of \$1,646. How many of each were sold?

33. A jar consisting of only nickels and quarters contains 70 coins. If the total value is \$9.10, how many of each coin are in the jar?

34. Jill has \$9.20 worth of dimes and quarters. If there are 68 coins in total, how many of each does she have?

## PART B: MIXTURE PROBLEMS

Set up a linear system and solve.

35. A 17% acid solution is to be mixed with a 9% acid solution to produce 8 gallons of a 10% acid solution. How much of each is needed?

36. A nurse wishes to obtain 28 ounces of a 1.5% saline solution. How much of a 1% saline solution must she mix with a 4.5% saline solution to achieve the desired mixture?

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37. A customer ordered 4 pounds of a mixed peanut product containing 12% cashews. The inventory consists of only two mixes containing 10% and 26% cashews. How much of each type must be mixed to fill the order?
38. One alcohol solution contains 10% alcohol and another contains 25% alcohol. How much of each should be mixed together to obtain 2 gallons of a 13.75% alcohol solution?
39. How much cleaning fluid concentrate, with 60% alcohol content, must be mixed with water to obtain a 24-ounce mixture with 15% alcohol content?
40. How many pounds of pure peanuts must be combined with a 20% peanut mix to produce 2 pounds of a 50% peanut mix?
41. A 50% fruit juice concentrate can be purchased wholesale. Best taste is achieved when water is mixed with the concentrate in such a way as to obtain a 15% fruit juice mixture. How much water and concentrate is needed to make a 60-ounce fruit juice drink?
42. Pure sugar is to be mixed with a fruit salad containing 10% sugar to produce 65 ounces of a salad containing 18% sugar. How much pure sugar is required?
43. A custom aluminum alloy is created by mixing 150 grams of a 15% aluminum alloy and 350 grams of a 55% aluminum alloy. What percentage of aluminum is in the resulting mixture?
44. A research assistant mixed 500 milliliters of a solution that contained a 12% acid with 300 milliliters of water. What percentage of acid is in the resulting solution?

## PART C: UNIFORM MOTION PROBLEMS

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Set up a linear system and solve.

45. The two legs of a 432-mile trip took 8 hours. The average speed for the first leg of the trip was 52 miles per hour and the average speed for the second leg of the trip was 60 miles per hour. How long did each leg of the trip take?
46. Jerry took two buses on the 265-mile trip from Los Angeles to Las Vegas. The first bus averaged 55 miles per hour and the second bus was able to average 50 miles per hour. If the total trip took 5 hours, then how long was spent in each bus?
47. An executive was able to average 48 miles per hour to the airport in her car and then board an airplane that averaged 210 miles per hour. The 549-mile business trip took 3 hours. How long did it take her to drive to the airport?
48. Joe spends 1 hour each morning exercising by jogging and then cycling for a total of 15 miles. He is able to average 6 miles per hour jogging and 18 miles per hour cycling. How long does he spend jogging each morning?
49. Swimming with the current Jack can swim 2.5 miles in  $t$  hour. Swimming back, against the same current, he can only swim 2 miles in the same amount of time. How fast is the current?
50. A light aircraft flying with the wind can travel 180 miles in 1 hour. The aircraft can fly the same distance against the wind in 2 hours. Find the speed of the wind.
51. A light airplane flying with the wind can travel 600 miles in 4 hours. On the return trip, against the wind, it will take 5 hours. What are the speeds of the airplane and of the wind?

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52. A boat can travel 15 miles with the current downstream in 1 hours. Returning upstream against the current, the boat can only travel 8 miles in the same amount of time. Find the speed of the current.
53. Mary jogged the trail from her car to the cabin at the rate of 6 miles per hour. She then walked back to her car at a rate of 4 miles per hour. If the entire trip took 1 hour, then how long did it take her to walk back to her car?
54. Two trains leave the station traveling in opposite directions. One train is 8 miles per hour faster than the other and in 2 hours they are 230 miles apart. Determine the average speed of each train.
55. Two trains leave the station traveling in opposite directions. One train is 12 miles per hour faster than the other and in 3 hours they are 300 miles apart. Determine the average speed of each train.
56. A jogger can sustain an average running rate of 8 miles per hour to his destination and 6 miles an hour on the return trip. Find the total distance the jogger ran if the total time running was 1 hour.

## PART E: DISCUSSION BOARD

57. Compose a number or money problem of your own and share it on the discussion board.
58. Compose a mixture problem of your own and share it on the discussion board.
59. Compose a uniform motion problem of your own and share it on the discussion board.

## ANSWERS

1. The integers are 16 and 29.
- 2.

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3. The integers are 6 and 35.
- 4.
5. The integers are 25 and 36.
- 6.
7. The integers are  $-3$  and 7.
- 8.
9. The integers are  $-5$  and  $-2$ .
- 10.
11. Length: 17 meters; width: 6 meters
- 12.
13. Width: 22 feet; length: 70 feet
- 14.
15.  $a=3, b=2$
- 16.
17.  $m=-2, b=1$
- 18.
19. \$3,400 at 3% and \$1,800 at 6%
- 20.
21. \$2,200 at 2% and \$4,300 at 3%
- 22.
23. Savings: \$82; Money Market: \$164.
- 24.
25. \$20,000
- 26.

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27. 35 quarters and 36 dimes
- 28.
29. Adults \$7.00 each and children \$5.50 each.
- 30.
31. 105 adult tickets were sold.
- 32.
33. The jar contains 42 nickels and 28 quarters.
- 34.
35. 7 gallons of the 9% acid solution and 1 gallon of the 17% acid solution
- 36.
37. 3.5 pounds of the 10% cashew mix and 0.5 pounds of the 26% cashew mix
- 38.
39. 6 ounces of cleaning fluid concentrate
- 40.
41. 18 ounces of fruit juice concentrate and 42 ounces of water
- 42.
43. 43%
- 44.
45. The first leg of the trip took 6 hours and the second leg took 2 hours.
- 46.
47. It took her hour to drive to the airport.
- 48.
49. 0.5 miles per hour.
- 50.

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51. Airplane: 135 miles per hour; wind: 15 miles per hour
- 52.
53. hour
- 54.
55. One train averaged 44 miles per hour and the other averaged 56 miles per hour.
- 56.
57. Answer may vary
- 58.
59. Answer may vary

## 3.4 Solving Linear Systems with Three Variables

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## LEARNING OBJECTIVES

1. Check solutions to linear systems with three variables.
2. Solve linear systems with three variables by elimination.
3. Identify dependent and inconsistent systems.
4. Solve applications involving three unknowns.

### Solutions to Linear Systems with Three Variables

Real-world applications are often modeled using more than one variable and more than one equation. In this section, we will study linear systems consisting of three linear equations each with three variables. For example,

A solution to such a linear system is an **ordered triple**  $(x, y, z)$  that solves all of the equations. In this case,  $(-2, 1, 3)$  is the only solution. To check that an ordered triple is a solution, substitute in the corresponding  $x$ -,  $y$ -, and  $z$ -values and then simplify to see if you obtain a true statement from all three equations.

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<p><i>Check:</i> <math>(-2,1,3)</math></p>		
✓	✓	✓

Because the ordered triple satisfies all three equations we conclude that it is indeed a solution.

## Example 1

Determine whether or not  $(-2, 1, 3)$  is a solution to the following linear system:

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Solution:

<i>Check:</i>		
✓	✓	✗

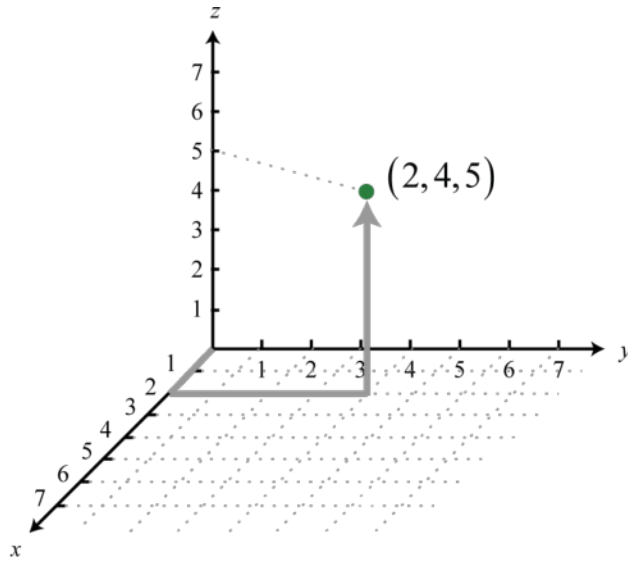
Answer: The point does not satisfy all of the equations and thus is not a solution.

An ordered triple such as  $(2,4,5)$  can be graphed in three-dimensional space as follows:

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The ordered triple indicates position relative to the origin (0, 0, 0), in this case, 2 units along the  $x$ -axis, 4 units parallel to the  $y$ -axis, and 5 units parallel to the  $z$ -axis. **A**

**linear equation with three variables** is in standard form if

$$ax+by+cz=d$$

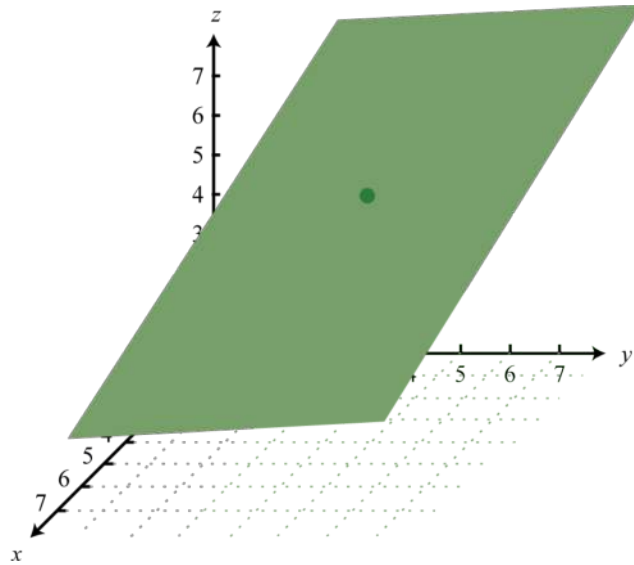
where  $a$ ,  $b$ ,  $c$ , and  $d$  are real numbers. For example,  $6x+y+2z=26$  is in standard form. Solving for  $z$ , we obtain and can consider both  $x$  and  $y$  to be the independent variables. When graphed in three-dimensional space, its graph will form a straight flat surface called a **plane**.

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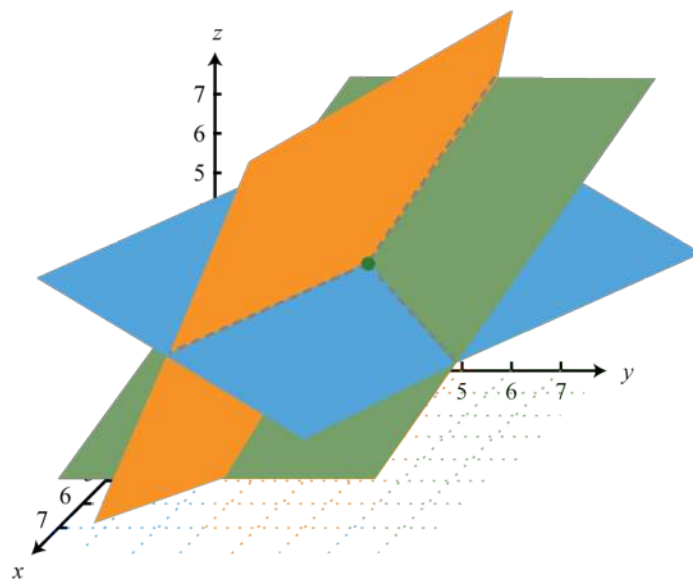
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Therefore, the graph of a system of three linear equations and three unknowns will consist of three planes in space. If there is a simultaneous solution, the system is consistent and the solution corresponds to a point where the three planes intersect.



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Graphing planes in three-dimensional space is not within the scope of this textbook.

However, it is always important to understand the geometric interpretation.

Try this! Determine whether or not  $(3, -1, 2)$  a solution to the system:

Answer: Yes, it is a solution.

## Solve Linear Systems with Three Variables by Elimination

In this section, the elimination method is used to solve systems of three linear equations with three variables. The idea is to eliminate one of the variables and resolve the original system into a system of two linear equations, after which we can then solve as usual. The steps are outlined in the following example.

### Example 2

Solve:

Solution:

All three equations are in standard form. If this were not the case, it would be a best practice to rewrite the equations in standard form before beginning this process.

Step 1: Choose any two of the equations and eliminate a variable. In this case, we can line up the variable  $z$  to eliminate if we group 3 times the first equation with the second equation.

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$$\begin{array}{l} (1) \\ (2) \end{array} \begin{cases} 3x + 2y - z = -7 \\ 6x - y + 3z = -4 \end{cases} \xrightarrow{\times 3} \begin{cases} 9x + 6y - 3z = -21 \\ 6x - y + 3z = -4 \end{cases}$$

Next, add the equations together.



Step 2: Choose any other two equations and eliminate the same variable. We can line up  $z$  to eliminate again if we group  $-2$  times the first equation with the third equation.

$$\begin{array}{l} (1) \\ (3) \end{array} \begin{cases} 3x + 2y - z = -7 \\ x + 10y - 2z = 2 \end{cases} \xrightarrow{\times(-2)} \begin{cases} -6x - 4y + 2z = 14 \\ x + 10y - 2z = 2 \end{cases}$$

And then add,



Step 3: Solve the resulting system of two equations with two unknowns. Here we solve by elimination. Multiply the second equation by 3 to line up the variable  $x$  to eliminate.

$$\begin{cases} 15x + 5y = -25 \\ -5x + 6y = 16 \end{cases} \xrightarrow{\times 3} \begin{cases} 15x + 5y = -25 \\ -15x + 18y = 48 \end{cases}$$

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Next, add the equations together.

Step 4: Back substitute and determine all of the coordinates. To find  $x$  use the following,

Now choose one of the original equations to find  $z$ ,

Hence the solution, presented as an ordered triple  $(x, y, z)$ , is  $(-2, 1, 3)$ . This is the same system that we checked in the beginning of this section.

Answer:  $(-2, 1, 3)$

It does not matter which variable we initially choose to eliminate, as long as we eliminate it twice with two different sets of equations.

### Example 3

Solve:

Solution:

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Because  $y$  has coefficient 1 in the second equation, choose to eliminate this variable. Use equations 1 and 2 to eliminate  $y$ .

$$\begin{array}{l} (1) \\ (2) \end{array} \left\{ \begin{array}{l} -6x - 3y + 3z = -14 \\ 9x + y - 6z = 5 \end{array} \right. \xRightarrow{\times 3} \left\{ \begin{array}{l} -6x - 3y + 3z = -14 \\ 27x + 3y - 18z = 15 \end{array} \right. \\ \hline 21x \quad -15z = 1 \quad \checkmark$$

Next use equations 2 and 3 to eliminate  $y$  again.

$$\begin{array}{l} (2) \\ (3) \end{array} \left\{ \begin{array}{l} 9x + y - 6z = 5 \\ 3x + 2y - 7z = 15 \end{array} \right. \xRightarrow{\times(-2)} \left\{ \begin{array}{l} -18x - 2y + 12z = -10 \\ 3x + 2y - 7z = 15 \end{array} \right. \\ \hline -15x \quad + 5z = 5 \quad \checkmark$$

This leaves a system of two equations with two variables  $x$  and  $z$ ,

Multiply the second equation by 3 and eliminate the variable  $z$ .

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$$\begin{cases} 21x - 15z = 1 \\ -15x + 5z = 5 \end{cases} \xrightarrow{\times 3} \begin{cases} 21x - 15z = 1 \\ -45x + 15z = 15 \end{cases}$$

$$\begin{array}{r} -24x \quad = 16 \\ x = -\frac{16}{24} \\ x = -\frac{2}{3} \end{array}$$

Now back substitute to find  $z$ .

Finally, choose one of the original equations to find  $y$ .

Answer:  $(-23, 5, -1)$

## Example 4

Solve:

Solution:

In this example, there is no obvious choice of variable to eliminate. We choose to eliminate  $x$ .



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Next use equations 2 and 3 to eliminate  $x$  again.



This leaves a system of two equations with two variables  $y$  and  $z$ ,

Multiply the first equation by  $-1$  as a means to eliminate the variable  $y$ .

$$\begin{array}{l} \left\{ \begin{array}{l} 10y + 31z = 36 \\ 10y + 16z = 21 \end{array} \right. \xrightarrow{\times(-1)} \left\{ \begin{array}{l} -10y - 31z = -36 \\ 10y + 16z = 21 \end{array} \right. \\ \hline -15z = -15 \\ z = 1 \end{array}$$

Now back substitute to find  $y$ .

Choose any one of the original equations to find  $x$ .

Answer:  $(-3, 12, 1)$

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Try this! Solve:  $\begin{cases} 2x-3y-z=7 \\ 3x+5y-3z=21 \\ 4x-y+2z=17 \end{cases}$

Answer:

## Dependent and Inconsistent Systems

Just as with linear systems with two variables, not all linear systems with three variables have a single solution. Sometimes there are no simultaneous solutions.

### Example 5

Solve the system:

Solution:

In this case we choose to eliminate the variable  $y$ .



Next use equations 2 and 3 to eliminate  $y$  again.

$$\begin{array}{l} (2) \quad \begin{cases} 21x - 4y + 18z = 7 \\ -9x + y - 9z = -8 \end{cases} \\ (3) \quad \begin{cases} 21x - 4y + 18z = 7 \\ -36x + 4y - 36z = -32 \end{cases} \end{array} \xrightarrow{\times 4} \begin{array}{l} \begin{cases} 21x - 4y + 18z = 7 \\ -36x + 4y - 36z = -32 \end{cases} \\ \hline -15x \quad -18z = -25 \end{array} \quad \checkmark$$

This leaves a system of two equations with two variables  $x$  and  $z$ ,

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Multiply the first equation by  $-3$  and eliminate the variable  $z$ .

$$\begin{array}{l} \left\{ \begin{array}{l} -5x - 6z = -3 \\ -15x - 18z = -25 \end{array} \right. \xrightarrow{\times(-3)} \left\{ \begin{array}{l} 15x + 18z = 9 \\ -15x - 18z = -25 \end{array} \right. \\ \hline 0 = -16 \quad \times \end{array}$$

Adding the resulting equations together leads to a false statement, which indicates that the system is inconsistent. There is no simultaneous solution.

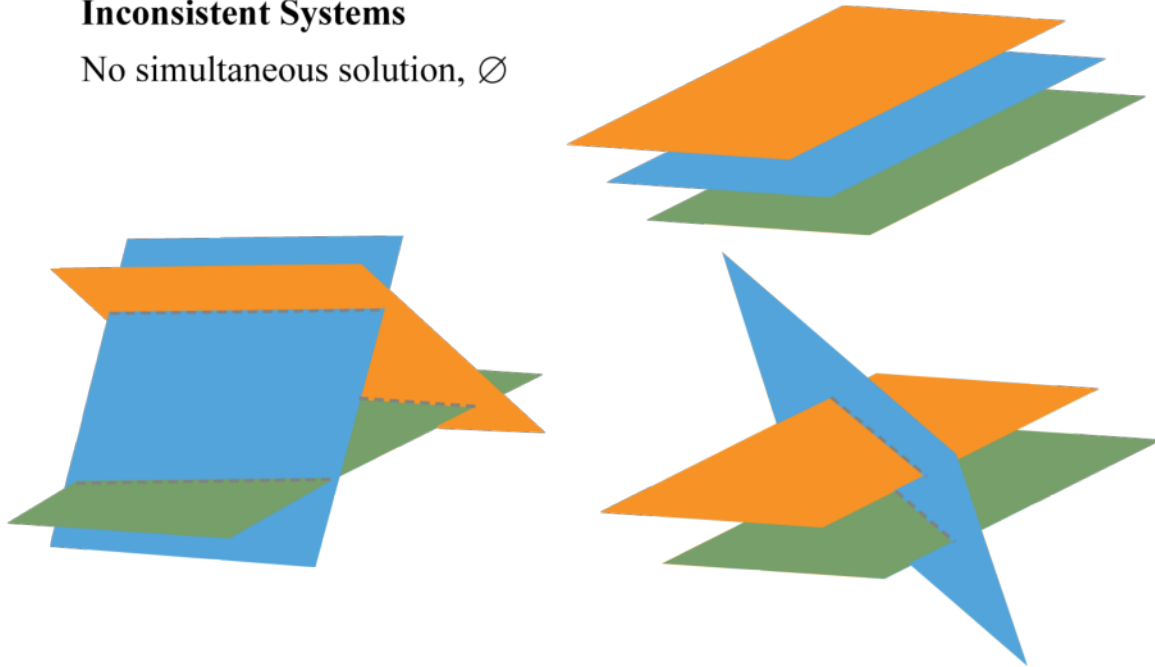
Answer:  $\emptyset$

A system with no solutions is an inconsistent system. Given three planes, no simultaneous solution can occur in a number of ways.



## Inconsistent Systems

No simultaneous solution,  $\emptyset$



Just as with linear systems with two variables, some linear systems with three variables have infinitely many solutions. Such systems are called dependent systems.

### Example 6

Solve the system:

Solution:

Eliminate  $z$  by adding the first and second equations together.



Next use equations 1 and 3 to eliminate  $z$  again.

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$$\begin{array}{l}
 (1) \quad \left\{ \begin{array}{l} 7x - 4y + z = -15 \\ 5x + 12y - 5z = -5 \end{array} \right. \xrightarrow{\times 5} \left\{ \begin{array}{l} 35x - 20y + 5z = -75 \\ 5x + 12y - 5z = -5 \end{array} \right. \\
 (3) \quad \left\{ \begin{array}{l} 7x - 4y + z = -15 \\ 5x + 12y - 5z = -5 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 35x - 20y + 5z = -75 \\ 5x + 12y - 5z = -5 \end{array} \right. \\
 \hline
 \qquad \qquad \qquad 40x - 8y \qquad = -80 \quad \checkmark
 \end{array}$$

This leaves a system of two equations with two variables  $x$  and  $y$ ,

Line up the variable  $y$  to eliminate by dividing the first equation by 2 and the second equation by  $-8$ .

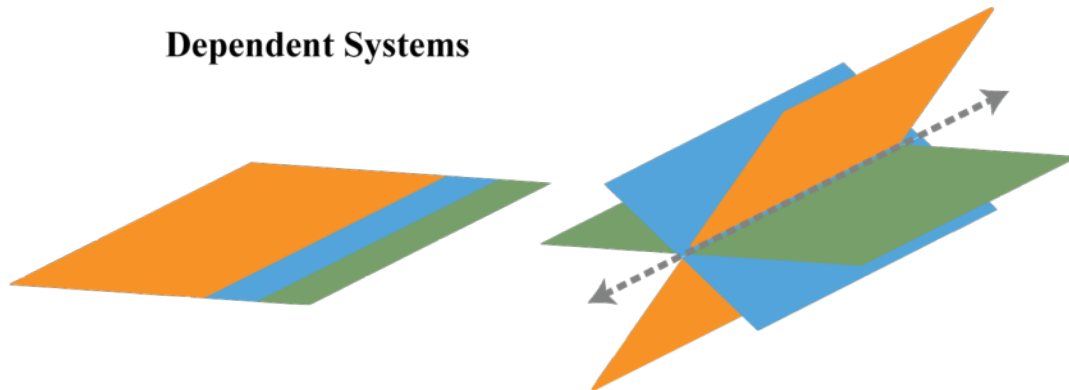
A true statement indicates that the system is dependent. To express the infinite number of solutions  $(x, y, z)$  in terms of one variable, we solve for  $y$  and  $z$  both in terms of  $x$ .

Once we have  $y$  in terms of  $x$ , we can solve for  $z$  in terms of  $x$  by back substituting into one of the original equations.

Answer:  $(x, 5x+10, 13x+25)$ .



A consistent system with infinitely many solutions is a dependent system. Given three planes, infinitely many simultaneous solutions can occur in a number of ways.



Try this! Solve:

Answer:

## Applications Involving Three Unknowns

Many real-world applications involve more than two unknowns. When an application requires three variables, we look for relationships between the variables that allow us to write three equations.

### Example 7

A community theater sold 63 tickets to the afternoon performance for a total of \$444. An adult ticket cost \$8, a child ticket cost \$4, and a senior ticket cost \$6. If twice as

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many tickets were sold to adults as to children and seniors combined, how many of each ticket were sold?

Solution:

Begin by identifying three variables.

Let  $x$  represent the number of adult tickets sold.

Let  $y$  represent the number of child tickets sold.

Let  $z$  represent the number of senior tickets sold.

The first equation comes from the statement that 63 tickets were sold.

$$(1) \quad x+y+z=63$$

The second equation comes from total ticket sales.

$$(2) \quad 8x+4y+6z=444$$

The third equation comes from the statement that twice as many adult tickets were sold as child and senior tickets combined.

Therefore, the problem is modeled by the following linear system.

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Solving this system is left as an exercise. The solution is  $(42, 9, 12)$ .

Answer: The theater sold 42 adult tickets, 9 child tickets, and 12 senior tickets.

## KEY TAKEAWAYS

A simultaneous solution to a linear system with three equations and three variables is an ordered triple  $(x, y, z)$  that satisfies all of the equations. If it does not solve each equation, then it is not a solution.

We can solve systems of three linear equations with three unknowns by elimination. Choose any two of the equations and eliminate a variable. Next choose any other two equations and eliminate the same variable. This will result in a system of two equations with two variables that can be solved by any method learned previously.

If the process of solving a system leads to a false statement, then the system is inconsistent and has no solution.

If the process of solving a system leads to a true statement, then the system is dependent and has infinitely many solutions.

To solve applications that require three variables, look for relationships between the variables that allow you to write three linear equations.

## TOPIC EXERCISES

### PART A: LINEAR SYSTEMS WITH THREE VARIABLES

Determine whether or not the given ordered triple is a solution to the given system.

1.  $(3, -2, -1)$ ;

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2.  $(-8,-1,5);$

3.  $(1,-9,2);$

4.  $(-4,1,-3);$

5.  $;$

6.  $;$

7.  $(3,-2,1);$

8.  $;$

9.  $;$

10.  $(-1,5,-7);$

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## PART B: SOLVING LINEAR SYSTEMS WITH THREE VARIABLES

Solve.

11.

12.

13.

14.

15.

16.

17.

18.

19.

20.

21.

22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

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34.

35.

36.

37.

38.

39.

40.

41.

42.

43.

44.

## PART C: APPLICATIONS

Set up a system of equations and use it to solve the following.

45. The sum of three integers is 38. Two less than 4 times the smaller integer is equal to the sum of the others. The sum of the smaller and larger integer is equal to 2 more than twice that of the other. Find the integers.

46. The sum of three integers is 40. Three times the smaller integer is equal to the sum of the others. Twice the larger is equal to 8 more than the sum of the others. Find the integers.

47. The sum of the angles  $A$ ,  $B$ , and  $C$  of a triangle is  $180^\circ$ . The larger angle  $C$  is equal to twice the sum of the other two. Four times the smallest angle  $A$  is equal to the difference of angle  $C$  and  $B$ . Find the angles.

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48. The sum of the angles  $A$ ,  $B$ , and  $C$  of a triangle is  $180^\circ$ . Angle  $C$  is equal to the sum of the other two angles. Five times angle  $A$  is equal to the sum of angle  $C$  and  $B$ . Find the angles.
49. A total of \$12,000 was invested in three interest earning accounts. The interest rates were 2%, 4%, and 5%. If the total simple interest for one year was \$400 and the amount invested at 2% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account?
50. Joe invested his \$6,000 bonus in three accounts earning 412% interest. He invested twice as much in the account earning 412% as he did in the other two accounts combined. If the total simple interest for the year was \$234, how much did Joe invest in each account?
51. A jar contains nickels, dimes, and quarters. There are 105 coins with a total value of \$8.40. If there are 3 more than twice as many dimes as quarters, find how many of each coin are in the jar.
52. A billfold holds one-dollar, five-dollar, and ten-dollar bills and has a value of \$210. There are 50 bills total where the number of one-dollar bills is one less than twice the number of five-dollar bills. How many of each bill are there?
53. A nurse wishes to prepare a 15-ounce topical antiseptic solution containing 3% hydrogen peroxide. To obtain this mixture, purified water is to be added to the existing 1.5% and 10% hydrogen peroxide products. If only 3 ounces of the 10% hydrogen peroxide solution is available, how much of the 1.5% hydrogen peroxide solution and water is needed?
54. A chemist needs to produce a 32-ounce solution consisting of 834% acid. He has three concentrates with 5%, 10%, and 40% acid. If he is to use twice as much of the 5% acid solution as the 10% solution, then how many ounces of the 40% solution will he need?

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55. A community theater sold 128 tickets to the evening performance for a total of \$1,132. An adult ticket cost \$10, a child ticket cost \$5, and a senior ticket cost \$6. If three times as many tickets were sold to adults as to children and seniors combined, how many of each ticket were sold?

56. James sold 82 items at the swap meet for a total of \$504. He sold packages of socks for \$6, printed t-shirts for \$12, and hats for \$5. If he sold 5 times as many hats as he did t-shirts, how many of each item did he sell?

57. A parabola passes through three points  $(-1,7)$ ,  $(1,-1)$  and  $(2,-2)$ . Use these points and to construct a system of three linear equations in terms of  $a$ ,  $b$ , and  $c$  and then solve the system.

58. A parabola passes through three points  $(-2,11)$ ,  $(-1,4)$  and  $(1,2)$ . Use these points and to construct a system of three linear equations in terms of  $a$ ,  $b$ , and  $c$  and solve it.

## PART D: DISCUSSION BOARD

59. On a note card, write down the steps for solving a system of three linear equations with three variables using elimination. Use your notes to explain to a friend how to solve one of the exercises in this section.

60. Research and discuss curve fitting. Why is curve fitting an important topic?

## ANSWERS

1. No
- 2.
3. Yes
- 4.

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5. Yes
- 6.
7. No
- 8.
9. No
- 10.
11.  $(2, -1, -3)$
- 12.
13.  $(4, 1, -3)$
- 14.
15.  $(1, -1, 3)$
- 16.
17.  $\emptyset$
- 18.
19.  $(5, -10, -6)$
- 20.
- 21.
- 22.
- 23.
- 24.
- 25.
- 26.
27.  $(1, -2, 6)$
- 28.
29.  $(-1, 2, -2)$

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- 30.
31.  $(1, -3, 5)$
- 32.
33.  $(1, 1, 0)$
- 34.
35.  $(0, 1, -2)$
- 36.
37.  $(5, 0, 6)$
- 38.
39.  $\emptyset$
- 40.
41.  $(x, 2x-1, 3x+2)$
- 42.
43.  $(1, 2, 3)$
- 44.
45. 8, 12, 18
- 46.
47.  $A = 20^\circ$ ,  $B = 40^\circ$ , and  $C = 120^\circ$
- 48.
49. The amount invested at 2% was \$6,000, the amount invested at 4% was \$2,000, and the amount invested at 5% was \$4,000.
- 50.
51. 72 nickels, 23 dimes, and 10 quarters
- 52.

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53. 10 ounces of the 1.5% hydrogen peroxide solution and 2 ounces of water
- 54.
55. 96 adult tickets, 20 child tickets, and 12 senior tickets were sold.
- 56.
57.  $a=1$ ,  $b=-4$ , and  $c=2$
- 58.
59. Answer may vary

## 3.5 Matrices and Gaussian Elimination

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## LEARNING OBJECTIVES

1. Use back substitution to solve linear systems in upper triangular form.
2. Convert linear systems to equivalent augmented matrices.
3. Use matrices and Gaussian elimination to solve linear systems.

### Back Substitution

Recall that a linear system of equations consists of a set of two or more linear equations with the same variables. A linear system consisting of three equations in standard form arranged so that the variable  $x$  does not appear in any equation after the first and the variable  $y$  does not appear in any equation after the second is said to be in upper triangular form. For example,

$$\begin{cases} x - 6y + 2z = 16 \\ 3y - 9z = 5 \\ z = -1 \end{cases}$$

Notice that the system forms a triangle where each successive equation contains one less variable. In general,

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If a linear system is in this form, we can easily solve for one of the variables and then back substitute to solve for the remaining variables.

## Example 1

Solve:

Solution:

Recall that solutions to linear systems with two variables, if they exist, are ordered pairs  $(x, y)$ . We can determine the  $y$ -value easily using the second equation.

Next, use the first equation  $3x - y = 7$  and the fact that  $y = -1$  to find  $x$ .

Answer:  $(2, -1)$

## Example 2

Solve:

Solution:

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Recall that solutions to linear systems with three variables, if they exist, are ordered triples  $(x, y, z)$ . Use the second equation  $3y - 9z = 5$  and the fact that  $z = -1$  to find  $y$ .

Next substitute  $y$  and  $z$  into the first equation.

Answer:

Try this! Solve:

Answer:

## Matrices and Gaussian Elimination

In this section the goal is to develop a technique that streamlines the process of solving linear systems. We begin by defining a **matrix**, which is a rectangular array of numbers consisting of rows and columns. Given a linear system in standard form, we create a **coefficient matrix** by writing the coefficients as they appear lined up without the variables or operations as follows.

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The rows represent the coefficients in the equations and the columns represent the coefficients of each variable. Furthermore, if we include a column that represents the constants we obtain what is called **an augmented matrix**. For a linear system with two variables,

And for a linear system with three variables we have

Note: The dashed vertical line provides visual separation between the coefficient matrix and the column of constants. In other algebra resources that you may encounter, this is sometimes omitted.

### Example 3

Construct the augmented matrix that corresponds to:

Solution:

This system consists of two linear equations in standard form; therefore, the coefficients in the matrix appear as they do in the system.

### Example 4

Construct the augmented matrix that corresponds to:

Solution:

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Since the equations are given in standard form, the coefficients appear in the matrix as they do in the system.

A matrix is in upper triangular form if all elements below the leading nonzero element in each successive row are zero. For example,

$$\left[ \begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2 & -1 \end{array} \right]$$

Notice that the elements below the main diagonal are zero and the coefficients above form a triangular shape. In general,

This is important because in this section we outline a process by which certain operations can be made to produce an equivalent linear system in upper triangular form so that it can be solved by using back substitution. An overview of the process is outlined below:



$$\begin{array}{ccc}
 \textit{Linear System} & & \textit{Augmented Matrix} \\
 \left\{ \begin{array}{l} x+2y-4z=5 \\ 2x+y-6z=8 \\ 4x-y-12z=13 \end{array} \right. & \Rightarrow & \left[ \begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 2 & 1 & -6 & 8 \\ 4 & -1 & -12 & 13 \end{array} \right] \\
 & & \Downarrow \\
 \left\{ \begin{array}{l} x+2y-4z=5 \\ -3y+2z=-2 \\ -2z=-1 \end{array} \right. & \Leftarrow & \left[ \begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2 & -1 \end{array} \right] \\
 \textit{Equivalent System} & & \textit{Upper Triangular Form}
 \end{array}$$

Once the system is in upper triangular form, we can use back substitution to easily solve it. It is important to note that the augmented matrices presented here represent linear systems of equations in standard form.

The following **elementary row operations** result in augmented matrices that represent equivalent linear systems:

1. Any two rows may be interchanged.
2. Each element in a row can be multiplied by a nonzero constant.
3. Any row can be replaced by the sum of that row and a multiple of another.

Note: These operations are consistent with the properties used in the elimination method.

To efficiently solve a system of linear equations first construct an augmented matrix. Then apply the appropriate elementary row operations to obtain an augmented matrix



in upper triangular form. In this form, the equivalent linear system can easily be solved using back substitution. This process is called **Gaussian elimination**, named in honor of Carl Friedrich Gauss (1777–1855).

*Figure 3.1*



*Carl Friedrich Gauss (Wikipedia)*

The steps for solving a linear equation with two variables using Gaussian elimination are listed in the following example.

## Example 5

Solve using matrices and Gaussian elimination:

Solution:

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Ensure that the equations in the system are in standard form before beginning this process.

Step 1: Construct the corresponding augmented matrix.

Step 2: Apply the elementary row operations to obtain upper triangular form. In this case, we need only to eliminate the first element of the second row,  $-1$ . To do this, multiply the second row by 9 and add it to the first row.

$$\left[ \begin{array}{cc|c} 9 & -6 & 0 \\ -1 & 2 & 1 \end{array} \right] \xRightarrow{\times 9} \begin{array}{ccc} 9 & -6 & 0 \\ + & -9 & 18 & 9 \\ \hline & 0 & 12 & 9 \end{array}$$

Now use this to replace the second row.

This results in an augmented matrix in upper triangular form.

Step 3: Convert back to a linear system and solve using back substitution. In this example, we have

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Solve the second equation for  $y$ ,

Substitute this value for  $y$  into the first equation to find  $x$ ,

Answer:

The steps for using Gaussian elimination to solve a linear equation with three variables are listed in the following example.

## Example 6

Solve using matrices and Gaussian elimination: .

Solution:

Ensure that the equations in the system are in standard form before beginning this process.

Step 1: Construct the corresponding augmented matrix.

Step 2: Apply the elementary row operations to obtain upper triangular form. We begin by eliminating the first element of the second row, 2 in this case. To do this multiply the first row by  $-2$  and then add it to the second row.

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Use this to replace the second row.

Next, eliminate the first element of the third row, 4 in this case, by multiplying the first row by  $-4$  and adding it to the third row.

Use this to replace the third row.

This results in an augmented matrix where the elements below the first element of the first row are zero. Next eliminate the second element in the third row, in this case  $-9$ . Multiply the second row by  $-3$  and add it to the third row.

$$\left[ \begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & -9 & 4 & -7 \end{array} \right] \xrightarrow{\times(-3)} \begin{array}{cccc} + & 0 & 9 & -6 & 6 \\ & 0 & -9 & 4 & -7 \\ \hline & 0 & 0 & -2 & -1 \end{array}$$

Use this to replace the third row and we can see that we have obtained a matrix in upper triangular form.

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Step 3: Convert back to a linear system and solve using back substitution. In this example, we have

Answer: It is left to the reader to verify that the solution is .

Note: Typically, the work involved in replacing a row by multiplying and adding is done on the side using scratch paper.

## Example 7

Solve using matrices and Gaussian

elimination: .

Solution:

We begin by converting the system to an augmented coefficient matrix.

The elementary row operations are streamlined if the leading nonzero element in a row is 1. For this reason, begin by interchanging row one and two.

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$$\begin{array}{l} (\text{row2}) \rightarrow \\ (\text{row1}) \rightarrow \end{array} \left[ \begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 2 & -9 & 3 & -18 \\ -4 & 23 & 12 & 47 \end{array} \right]$$

Replace row two with the sum of  $-2$  times row one and row two.

$$-2(\text{row1}) + (\text{row2}) \rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 0 & -5 & 9 & -2 \\ -4 & 23 & 12 & 47 \end{array} \right]$$

Replace row three with the sum of  $4$  times row one and row three.

$$4(\text{row1}) + (\text{row3}) \rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 0 & -5 & 9 & -2 \\ 0 & 15 & 0 & 15 \end{array} \right]$$

Next divide row 3 by 15.

$$(\text{row3}) \div 15 \rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 0 & -5 & 9 & -2 \\ 0 & 1 & 0 & 1 \end{array} \right]$$

Interchange row three with row two.

$$\begin{array}{l} (\text{row2}) \rightarrow \\ (\text{row3}) \rightarrow \end{array} \left[ \begin{array}{ccc|c} 1 & -2 & -3 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & -5 & 9 & -2 \end{array} \right]$$

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Next replace row 3 with the sum of 5 times row two and row three.

$$5(\text{row2})+(\text{row3}) \rightarrow \begin{bmatrix} 1 & -2 & -3 & | & -8 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 9 & | & 3 \end{bmatrix}$$

This results in a matrix in upper triangular form. A matrix is in **row echelon form** if it is in upper triangular form where the leading nonzero element of each row is 1. We can obtain this form by replacing row three with the results of dividing it by 9.

$$(\text{row3}) \div 9 \rightarrow \begin{bmatrix} 1 & -2 & -3 & | & -8 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & \frac{1}{3} \end{bmatrix}$$

Convert to a system of linear equations and solve by back substitution.

Here  $y = 1$  and  $z=13$ . Substitute into the first equation to find  $x$ .

Answer: Therefore the solution is .

Technology note: Many modern calculators and computer algebra systems can perform Gaussian elimination. First you will need to find out how to enter a matrix. Then use the

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calculator's functions to find row echelon form. You are encouraged to conduct some web research on this topic for your particular calculator model.

Try this! Solve using Gaussian elimination: .

Answer:  $(6, -4, -1)$

Recall that some consistent linear systems are dependent, that is, they have infinitely many solutions. And some linear systems have no simultaneous solution; they are inconsistent systems.

## Example 8

Solve using matrices and Gaussian elimination: .

Solution:

We begin by converting the system to an augmented coefficient matrix.

Replace row two with  $-2(\text{row } 1) + (\text{row } 2)$  and replace row three with  $-4(\text{row } 1) + (\text{row } 3)$ .

Replace row three with  $-1(\text{row } 2) + (\text{row } 3)$ .

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The last row indicates that this is a dependent system because converting the augmented matrix back to equations we have,

Note that the row of zeros corresponds to the following identity,



In this case, we can express the infinitely many solutions in terms of  $z$ . From the second row we have the following:

And from the first equation,

The solutions take the form  $(x, y, z) = (-5z + 2, -2z - 1, z)$  where  $z$  is any real number.

Answer:  $(-5z + 2, -2z - 1, z)$

Dependent and inconsistent systems can be identified in an augmented coefficient matrix when the coefficients in one row are all zero.

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$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \leftarrow$$

$0 = 0$  ✓  
*Dependent*

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{array} \right] \leftarrow$$

$0 = 2$  ✗  
*Inconsistent*

If a row of zeros has a corresponding constant of zero then the matrix represents a dependent system. If the constant is nonzero then the matrix represents an inconsistent system.

Try this! Solve using matrices and Gaussian elimination: .

Answer:  $\emptyset$

## KEY TAKEAWAYS

A linear system in upper triangular form can easily be solved using back substitution.

The augmented coefficient matrix and Gaussian elimination can be used to streamline the process of solving linear systems.

To solve a system using matrices and Gaussian elimination, first use the coefficients to create an augmented matrix. Apply the elementary row operations as a means to obtain a matrix in upper triangular form. Convert the matrix back to an equivalent linear system and solve it using back substitution.

## TOPIC EXERCISES

### PART A: BACK SUBSTITUTION

Solve using back substitution.

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- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.
- 15.
- 16.
- 17.
- 18.
- 19.
- 20.

## PART B: MATRICES AND GAUSSIAN ELIMINATION

Construct the corresponding augmented matrix (do not solve).

- 21.
- 22.
- 23.

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24.

25.

26.

27.

28.

29.

30.

31.

32.

Solve using matrices and Gaussian elimination.

21.

22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.

35.

36.

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- 37.
- 38.
- 39.
- 40.
- 41.
- 42.
- 43.
- 44.
- 45.
- 46.
- 47.
- 48.
- 49.
- 50.
- 51.
- 52.
- 53.
- 54.
- 55.
- 56.

## PART C: DISCUSSION BOARD

69. Research and discuss the history of Gaussian Elimination. Who is credited for first developing this process? Post something that you found interesting relating to this story.

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70. Research and discuss the history of modern matrix notation. Who is credited for the development? In what fields are they used today? Post your findings on the discussion board.

## ANSWERS

1.

2.

3.

4.

5.  $(-4, 0)$

6.

7.

8.

9.  $(0, 0)$

10.

11.  $(-8, 7, 3)$

12.

13.

14.

15.

16.

17.

18.

19.  $(-3, 5, 9)$

20.

21.

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22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.

35.

36.

37.  $(6, -2)$

38.

39.  $\emptyset$

40.

41.  $(0, 0)$

42.

43.

44.

45.  $(2, 3, 3)$

46.

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- 47. (0, 1, 3)
- 48.
- 49. (1, -1, 2)
- 50.
- 51.
- 52.
- 53.  $\emptyset$
- 54.
- 55. (-7, -13, 7)
- 56.
- 57.
- 58.
- 59.
- 60.
- 61. (-1,2,1)
- 62.
- 63.
- 64.
- 65.  $\emptyset$
- 66.
- 67. (2, -2, 1)
- 68.
- 69. Answer may vary



## 3.6 Determinants and Cramer's Rule

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## LEARNING OBJECTIVES

1. Calculate the determinant of a  $2 \times 2$  matrix.
2. Use Cramer's rule to solve systems of linear equations with two variables.
3. Calculate the determinant of a  $3 \times 3$  matrix.
4. Use Cramer's rule to solve systems of linear equations with three variables.

## Linear Systems of Two Variables and Cramer's Rule

Recall that a matrix is a rectangular array of numbers consisting of rows and columns. We classify matrices by the number of rows  $n$  and the number of columns  $m$ . For example, a  $3 \times 4$  matrix, read "3 by 4 matrix," is one that consists of 3 rows and 4 columns. A **square matrix** is a matrix where the number of rows is the same as the number of columns. In this section we outline another method for solving linear systems using special properties of square matrices. We begin by considering the following  $2 \times 2$  coefficient matrix  $A$ ,

The **determinant** of a  $2 \times 2$  matrix, denoted with vertical lines  $|A|$ , or more compactly as  $\det(A)$ , is defined as follows:

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

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The determinant is a real number that is obtained by subtracting the products of the values on the diagonal.

## Example 1

Calculate: .

Solution:

The vertical line on either side of the matrix indicates that we need to calculate the determinant.

Answer: 4

## Example 2

Calculate: .

Solution:

Notice that the matrix is given in upper triangular form.

Answer:  $-18$

We can solve linear systems with two variables using determinants. We begin with a general  $2 \times 2$  linear system and solve for  $y$ . To eliminate the variable  $x$ , multiply the first equation by  $-$  and the second equation by  $.$

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This results in an equivalent linear system where the variable  $x$  is lined up to eliminate. Now adding the equations we have

$$\begin{array}{r} -a_1a_2x - a_2b_1y = -a_2c_1 \\ + \quad a_1a_2x + a_1b_2y = a_1c_2 \\ \hline (a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1 \\ y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \end{array}$$

Both the numerator and denominator look very much like a determinant of a  $2 \times 2$  matrix. In fact, this is the case. The denominator is the determinant of the coefficient matrix. And the numerator is the determinant of the matrix formed by replacing the column that represents the coefficients of  $y$  with the corresponding column of constants. This special matrix is denoted  $D_y$ .

The value for  $x$  can be derived in a similar manner.

In general, we can form the augmented matrix as follows:

and then determine  $D$ , and  $D_x$  by calculating the following determinants.

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The solution to a system in terms of determinants described above, when  $D \neq 0$ , is called **Cramer's rule**.

This theorem is named in honor of Gabriel Cramer (1704 - 1752).

*Figure 3.2*



*Gabriel Cramer*

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The steps for solving a linear system with two variables using determinants (Cramer's rule) are outlined in the following example.

### Example 3

Solve using Cramer's rule: .

Solution:

Ensure the linear system is in standard form before beginning this process.

Step 1: Construct the augmented matrix and form the matrices used in Cramer's rule.

In the square matrix used to determine  $D$ , replace the first column of the coefficient matrix with the constants. In the square matrix used to determine  $D_x$ , replace the second column with the constants.

Step 2: Calculate the determinants.

Step 3: Use Cramer's rule to calculate  $x$  and  $y$ .

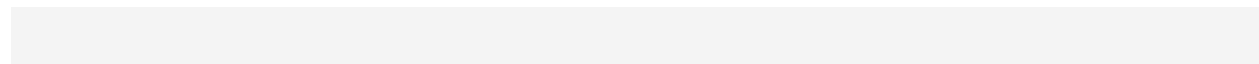
Therefore the simultaneous solution  $(x, y) = (1, 5)$ .

Step 4: The check is optional; however, we do it here for the sake of completeness.

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<i>Check:</i> (1, 5)	
<i>Equation 1</i>	<i>Equation 2</i>
✓	✓

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Answer: (1, 5)

## Example 4

Solve using Cramer's rule:

Solution:

The corresponding augmented coefficient matrix follows.

And we have,

Use Cramer's rule to find the solution.

Answer:

Try this! Solve using Cramer's rule:

Answer: (-1, 23)

When the determinant of the coefficient matrix  $D$  is zero, the formulas of Cramer's rule are undefined. In this case, the system is either dependent or inconsistent depending on

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the values of  $a$  and  $b$ . When  $D=0$  and both  $a=0$  and  $b=0$  the system is dependent. When  $D=0$  and either  $a$  or  $b$  is nonzero then the system is inconsistent.

## Example 5

Solve using Cramer's rule:

Solution:

The corresponding augmented matrix follows.

And we have the following.

If we try to use Cramer's rule we have,

both of which are indeterminate quantities. Because  $D=0$  and both  $a=0$  and  $b=0$  we know this is a dependent system. In fact, we can see that both equations represent the same line if we solve for  $y$ .

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Therefore we can represent all solutions  $(x, -5x+15)$  where  $x$  is a real number.

Answer:  $(x, -5x+15)$

Try this! Solve using Cramer's rule:  $\begin{cases} 3x-2y=10 \\ 6x-4y=12. \end{cases}$

Answer:  $\emptyset$

## Linear Systems of Three Variables and Cramer's Rule

Consider the following  $3 \times 3$  coefficient matrix  $A$ ,

The determinant of this matrix is defined as follows:

Here each  $2 \times 2$  determinant is called the minor of the preceding factor. Notice that the factors are the elements in the first row of the matrix and that they alternate in sign (+ - +).

### Example 6

Calculate: .

Solution:

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To easily determine the minor of each factor in the first row we line out the first row and the corresponding column. The determinant of the matrix of elements that remain determines the corresponding minor.

$$\begin{array}{ccc}
 \begin{array}{|c|c|c|} \hline \textcircled{1} & 3 & 2 \\ \hline 2 & -1 & 3 \\ \hline 0 & 5 & -1 \\ \hline \end{array} & 
 \begin{array}{|c|c|c|} \hline 1 & \textcircled{3} & 2 \\ \hline 2 & -1 & 3 \\ \hline 0 & 5 & -1 \\ \hline \end{array} & 
 \begin{array}{|c|c|c|} \hline 1 & 3 & \textcircled{2} \\ \hline 2 & -1 & 3 \\ \hline 0 & 5 & -1 \\ \hline \end{array}
 \end{array}$$

Take care to alternate the sign of the factors in the first row. The expansion by minors about the first row follows:

Answer: 12

Expansion by minors can be performed about any row or any column. The sign of the coefficients, determined by the chosen row or column, will alternate according to the following sign array.

Therefore, to expand about the second row we will alternate the signs starting with the opposite of the first element. We can expand the previous example about the second row to show that the same answer for the determinant is obtained.

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$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} \quad \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix} \quad \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 5 & -1 \end{vmatrix}$$

And we can write,

Note that we obtain the same answer 12.

## Example 7

Calculate: .

Solution:

The calculations are simplified if we expand about the third column because it contains two zeros.

$$\begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 4 & 3 & 0 \\ 6 & \frac{1}{2} & 2 \\ 4 & 1 & 0 \end{vmatrix}$$

The expansion by minors about the third column follows:

Answer: 16

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It should be noted that there are other techniques used for remembering how to calculate the determinant of a  $3 \times 3$  matrix. In addition, many modern calculators and computer algebra systems can find the determinant of matrices. You are encouraged to research this rich topic.

We can solve linear systems with three variables using determinants. To do this, we begin with the augmented coefficient matrix,

Let  $D$  represent the determinant of the coefficient matrix,

Then determine  $x$ ,  $y$ , and  $z$  by calculating the following determinants.

$$x$$

When  $D \neq 0$ , the solution to the system in terms of the determinants described above can be calculated using Cramer's rule:

Use this to efficiently solve systems with three variables.

## Example 8

Solve using Cramer's rule: .

Solution:

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Begin by determining the corresponding augmented matrix.

Next, calculate the determinant of the coefficient matrix.

Similarly we can calculate  $D_1$ ,  $D_2$ , and  $D_3$ . This is left as an exercise.

Using Cramer's rule we have,

Answer:  $(-4, 0, -3)$

If the determinant of the coefficient matrix  $D = 0$ , then the system is either dependent or inconsistent. This will depend on  $D_1$ ,  $D_2$ , and  $D_3$ . If they are all zero, then the system is dependent. If at least one of these is nonzero, then it is inconsistent.

### Example 9

Solve using Cramer's rule: .

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Solution:

Begin by determining the corresponding augmented matrix.

Next, determine the determinant of the coefficient matrix.

Since  $D = 0$ , the system is either dependent or inconsistent.

However, because is nonzero we conclude the system is inconsistent. There is no simultaneous solution.

Answer:  $\emptyset$

Try this! Solve using Cramer's rule: .

Answer:

## KEY TAKEAWAYS

The determinant of a matrix is a real number.

The determinant of a  $2 \times 2$  matrix is obtained by subtracting the product of the values on the diagonals.

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The determinant of a  $3 \times 3$  matrix is obtained by expanding the matrix using minors about any row or column. When doing this, take care to use the sign array to help determine the sign of the coefficients.

Use Cramer's rule to efficiently determine solutions to linear systems.

When the determinant of the coefficient matrix is 0, Cramer's rule does not apply; the system will either be dependent or inconsistent.

## TOPIC EXERCISES

### PART A: LINEAR SYSTEMS WITH TWO VARIABLES

Calculate the determinant.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.

Solve using Cramer's rule.

- 1.
- 2.

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- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.

## PART B: LINEAR SYSTEMS WITH THREE VARIABLES

Calculate the determinant.

27.  $\begin{vmatrix} 1 & 2 & 1 & 2 & 1 & 3 & 3 & 3 & 2 \end{vmatrix}$
28.  $\begin{vmatrix} 2 & 1 & 3 & 5 & 2 & 2 & 1 & 4 & 3 \end{vmatrix}$
29.  $\begin{vmatrix} -3 & 3 & -2 & 1 & -1 & 5 & -1 & -2 & 1 \end{vmatrix}$
30.  $\begin{vmatrix} 1 & -4 & -1 & -1 & 5 & 2 & 5 & -1 & -3 \end{vmatrix}$
31.  $\begin{vmatrix} 3 & 2 & 5 & -1 & 3 & 2 & 2 & -1 & 1 \end{vmatrix}$
32.  $\begin{vmatrix} 4 & 3 & 0 & 0 & -1 & -5 & -3 & 0 & 2 \end{vmatrix}$
33.  $\begin{vmatrix} 0 & -3 & 0 & -3 & 0 & 2 & 4 & 6 & -3 \end{vmatrix}$
34.  $\begin{vmatrix} 6 & 2 & 8 & -1 & 5 & 4 & -3 & 2 & -1 \end{vmatrix}$
35.  $\begin{vmatrix} 2 & 0 & 0 & 5 & 3 & 0 & 7 & 5 & 4 \end{vmatrix}$
36.  $\begin{vmatrix} 2 & 0 & 0 & 1 & 0 & 3 & 0 & 9 & 1 & 3 & 4 \end{vmatrix}$
37.  $\begin{vmatrix} a & 1 & 0 & 0 & b & 1 & b & 2 & 0 & c & 1 & c & 2 & c & 3 \end{vmatrix}$



38.  $||| a_1 a_2 a_3 ||| b_1 b_2 b_3 ||| c_1 c_2 c_3 |||$

Solve using Cramer's rule.

27.

28.

29.

30.

31.

32.

33.

34.

35.

36.

37.

38.

39.

40.

41.

42.

43.

44.

45.

46.

47.

48.

## PART C: DISCUSSION BOARD

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61. Research and discuss the history of the determinant. Who is credited for first introducing the notation of a determinant?
62. Research other ways in which we can calculate the determinant of a  $3 \times 3$  matrix. Give an example.

## ANSWERS

1. -2
- 2.
3. 11
- 4.
5. 3
- 6.
7. 0
- 8.
9. 4
- 10.
- 11.
- 12.
13. (1, -1)
- 14.
- 15.
- 16.
- 17.
- 18.
19. (0, -2)
- 20.





21.  $\emptyset$
- 22.
- 23.
- 24.
25.  $(a-b, 2b-a)$
- 26.
27. 6
- 28.
29. -39
- 30.
31. 0
- 32.
33. 3
- 34.
35. 24
- 36.
- 37.
- 38.
39.  $(2, 3, -1)$
- 40.
41.  $(-1, 2, -3)$
- 42.
- 43.
- 44.
45.  $\emptyset$

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- 46.
- 47.  $(0, -2, 0)$
- 48.
- 49.
- 50.
- 51.  $(-2, 1, 4)$
- 52.
- 53.
- 54.
- 55.  $\emptyset$
- 56.
- 57.  $(-1, 0, 1)$
- 58.
- 59.  $(a-b, b-c, c)$
- 60.
- 61. Answer may vary

## 3.7 Solving Systems of Inequalities with Two Variables

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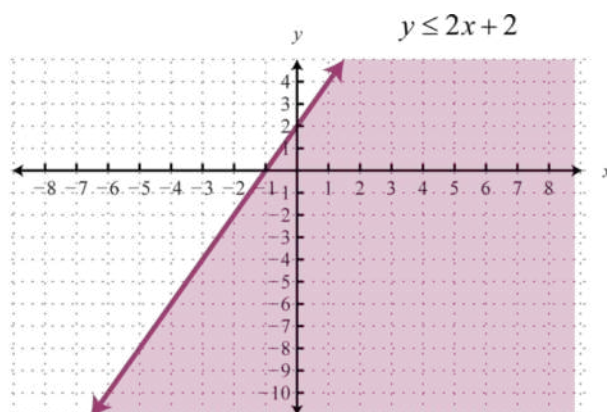
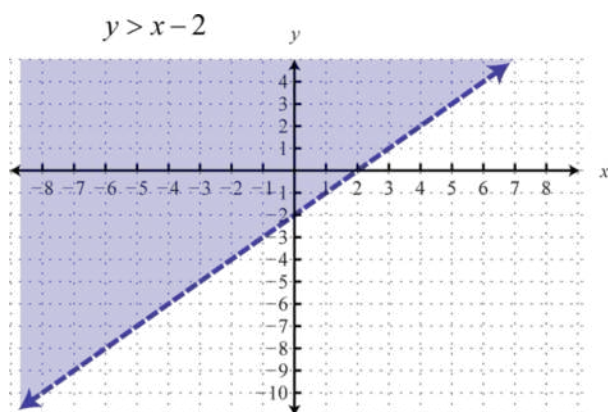
## LEARNING OBJECTIVES

1. Check solutions to systems of inequalities with two variables.
2. Graph solution sets of systems of inequalities.

### Solutions to Systems of Inequalities

A **system of inequalities** consists of a set of two or more inequalities with the same variables. The inequalities define the conditions that are to be considered simultaneously. For example,

We know that each inequality in the set contains infinitely many ordered pair solutions defined by a region in a rectangular coordinate plane. When considering two of these inequalities together, the intersection of these sets will define the set of simultaneous ordered pair solutions. When we graph each of the above inequalities separately we have:

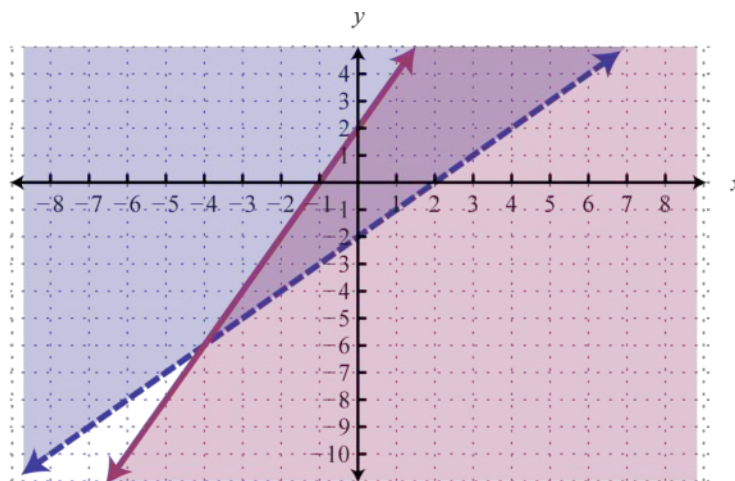


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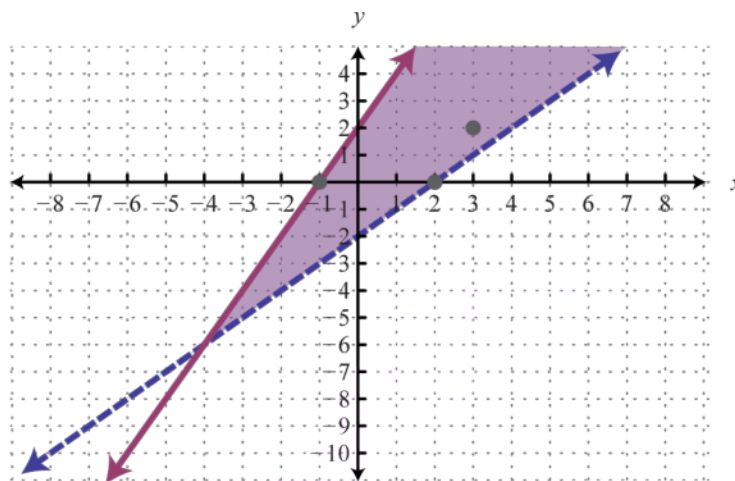
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And when graphed on the same set of axes, the intersection can be determined.



The intersection is shaded darker and the final graph of the solution set will be presented as follows:



The graph suggests that  $(3, 2)$  is a solution because it is in the intersection. To verify this, we can show that it solves both of the original inequalities as follows:

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<i>Check:</i> (3,2)	
✓	✓

Points on the solid boundary are included in the set of simultaneous solutions and points on the dashed boundary are not. Consider the point  $(-1, 0)$  on the solid boundary defined by  $y=2x+2$  and verify that it solves the original system:

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<i>Check:</i> $(-1,0)$	
✓	✓

Notice that this point satisfies both inequalities and thus is included in the solution set. Now consider the point  $(2, 0)$  on the dashed boundary defined by  $y=x-2$  and verify that it does not solve the original system:



<i>Check:</i> (2,0)	
x	✓

This point does not satisfy both inequalities and thus is not included in the solution set.

## Example 1

Determine whether or not  $(-3, 3)$  is a solution to the following system:

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Solution:

Substitute the coordinates of  $(x, y) = (-3, 3)$  into both inequalities.

<i>Check:</i> $(-3, 3)$	
$\times$	$\checkmark$

Answer:  $(-3, 3)$  is not a solution; it does not satisfy both inequalities.

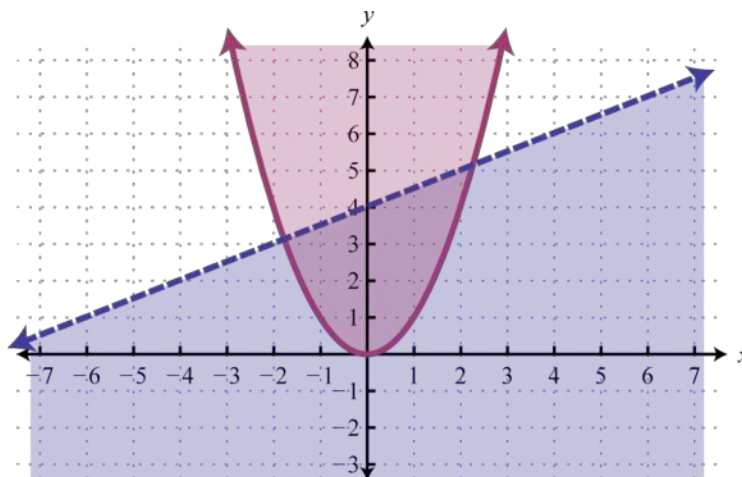
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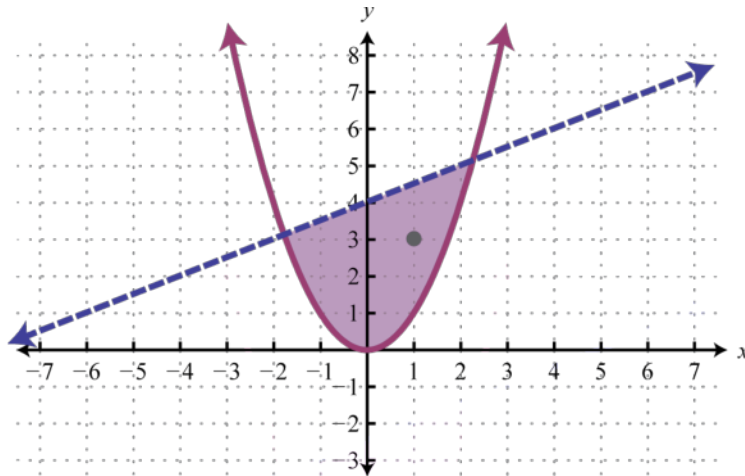
We can graph the solutions of systems that contain nonlinear inequalities in a similar manner. For example, both solution sets of the following inequalities can be graphed on the same set of axes:



And the intersection of both regions contains the region of simultaneous ordered pair solutions.

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From the graph, we expect the ordered pair  $(1, 3)$  to solve both inequalities.

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<i>Check:</i> (1,3)	
✓	✓

## Graphing Solutions to Systems of Inequalities

Solutions to a system of inequalities are the ordered pairs that solve all the inequalities in the system. Therefore, to solve these systems we graph the solution sets of the inequalities on the same set of axes and determine where they intersect. This intersection, or overlap, will define the region of common ordered pair solutions.

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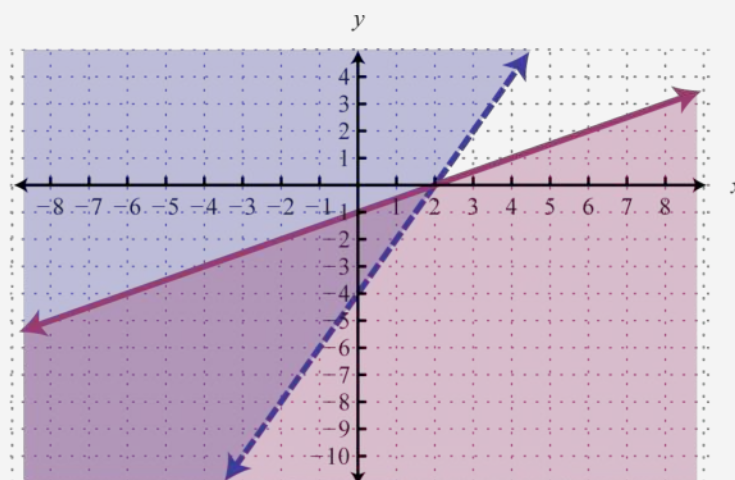
## Example 2

Graph the solution set: .

Solution:

To facilitate the graphing process, we first solve for  $y$ .

For the first inequality, we use a dashed boundary defined by  $y=2x-4$  and shade all points above the line. For the second inequality, we use a solid boundary defined by  $y=x+2$  and shade all points below the line. The intersection is darkened.



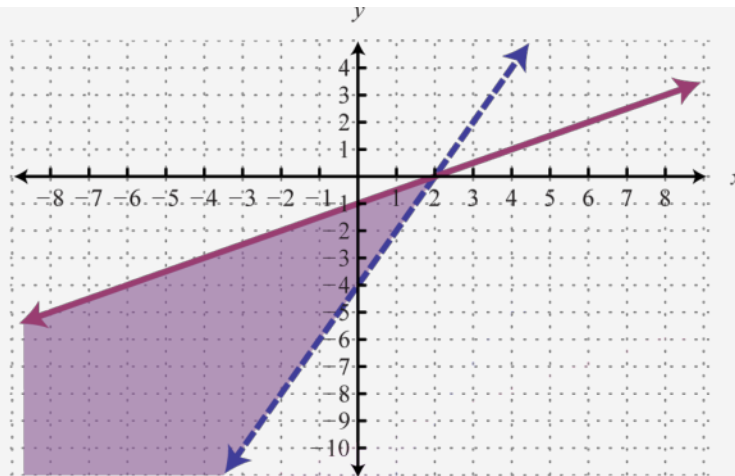
Now we present our solution with only the intersection shaded.

Answer:

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### Example 3

Graph the solution set:

Solution:

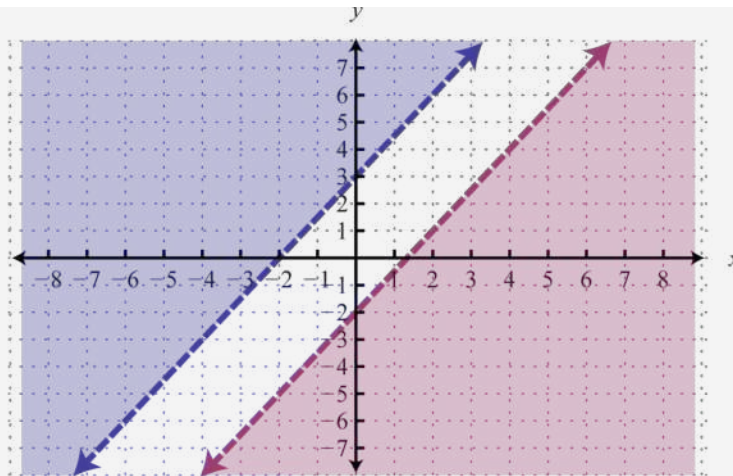
We begin by solving both inequalities for  $y$ .

Because of the strict inequalities, we will use a dashed line for each boundary. For the first inequality shade all points above the boundary and for the second inequality shade all points below the boundary.

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As we can see, there is no intersection of these two shaded regions. Therefore, there are no simultaneous solutions.

Answer:  $\emptyset$

## Example 4

Graph the solution set:

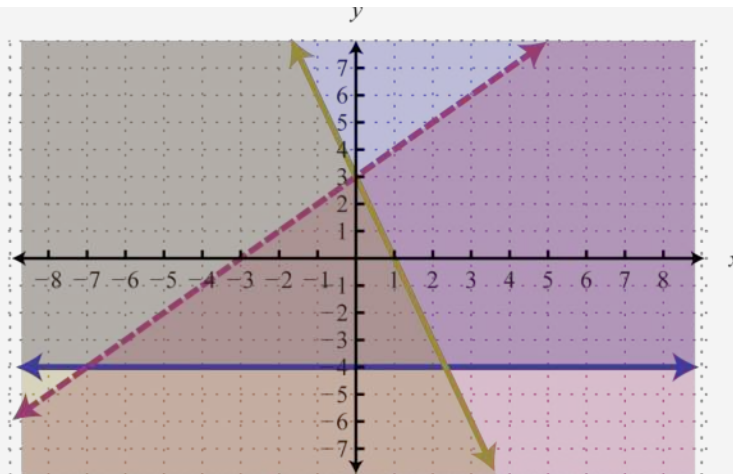
Solution:

Begin by graphing the solution sets to all three inequalities.

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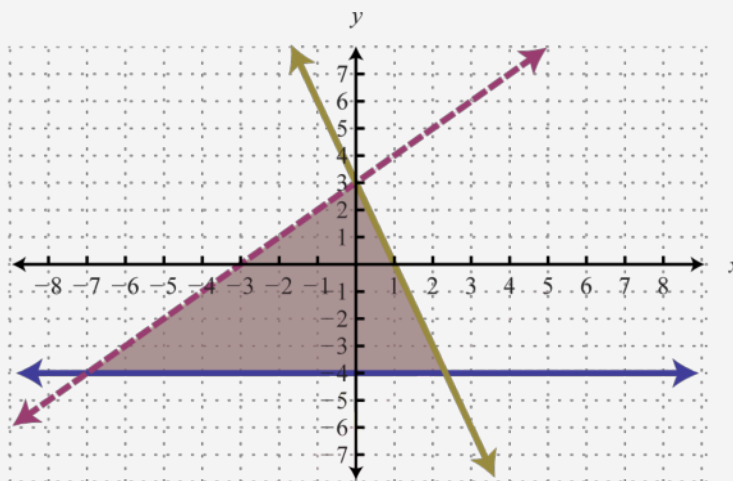
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After graphing all three inequalities on the same set of axes, we determine that the intersection lies in the triangular region pictured below.

Answer:



The graph suggests that  $(-1, 1)$  is a simultaneous solution. As a check, we could substitute that point into the inequalities and verify that it solves all three conditions.

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<p><i>Check:</i> <math>(-1,1)</math></p>		
✓	✓	✓

Use the same technique to graph the solution sets to systems of nonlinear inequalities.

## Example 5

Graph the solution set: .

Solution:

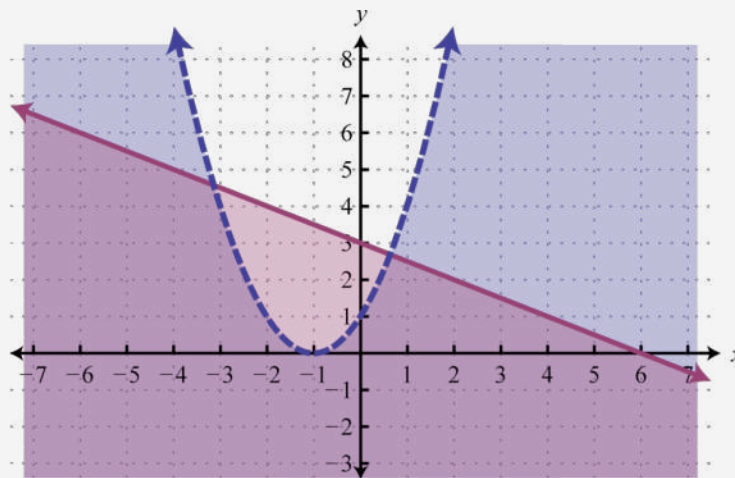
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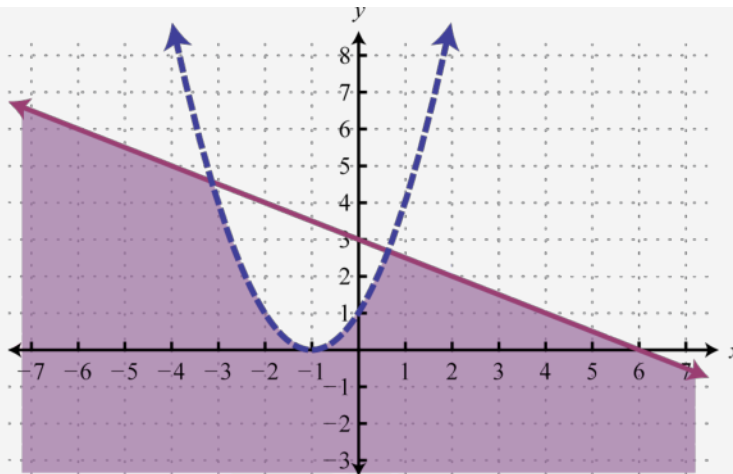
The first inequality has a parabolic boundary. This boundary is a horizontal translation of the basic function to the left 1 unit. Because of the strict inequality, the boundary is dashed, indicating that it is not included in the solution set. The second inequality is linear and will be graphed with a solid boundary. Solution sets to both are graphed below.



After graphing the inequalities on the same set of axes, we determine that the intersection lies in the region pictured below.

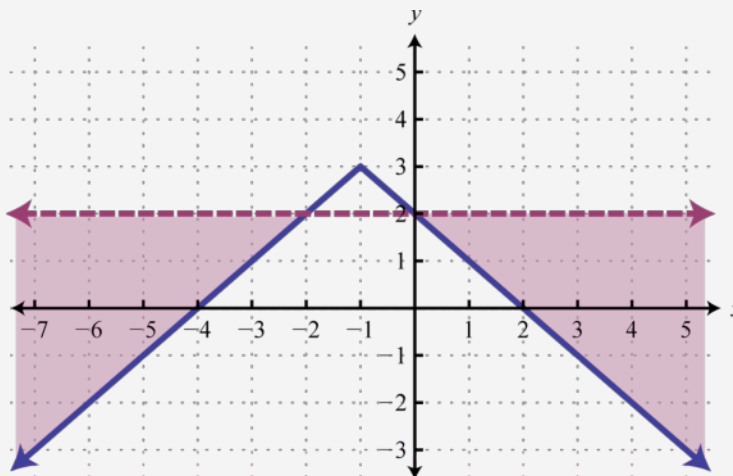
Answer:





Try this! Graph the solution set:

Answer:



## KEY TAKEAWAYS

To graph solutions to systems of inequalities, graph the solution sets of each inequality on the same set of axes and determine where they intersect.

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You can check your answer by choosing a few values inside and out of the shaded region to see if they satisfy the inequalities or not. While this is not a proof, doing so will give a good indication that you have graphed the correct region.

## TOPIC EXERCISES

### PART A: SOLUTIONS TO SYSTEMS OF INEQUALITIES

Determine whether or not the given point is a solution to the given system of inequalities.

1.  $(-2, 1)$ ;
2.  $(-1, -3)$ ;
3.  $(-2, -1)$ ;
4.  $(0, -5)$ ;
5. ;
6. ;
7.  $(-1, -2)$ ;

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8.  $(-5, 2);$

9.  $(0, 3);$

10.  $(1, 1);$

11.  $(-1, 2);$

12.  $(4, 5);$

13.  $(-2, -3);$

14.  $(1, 2);$

15.  $;$

16.  $(-4, 1);$

17.  $;$

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18. ;

19. (4, 2);

20. ;

## PART B: SOLVING SYSTEMS OF INEQUALITIES

Graph the solution set.

21.

22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.

35.

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36.

37.

Construct a system of linear inequalities that describes all points in the first quadrant.

38. Construct a system of linear inequalities that describes all points in the second quadrant.

39. Construct a system of linear inequalities that describes all points in the third quadrant.

40. Construct a system of linear inequalities that describes all points in the fourth quadrant.

Graph the solution set.

21.

22.

23.

24.


25.

26.

27.

28.

29.

30.   $12x + 13y \leq 1$   $y + 4 \geq 0$   $-12x + 13y \leq 1$

31.

32.

33.

34.

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- 35.
- 36.
- 37.
- 38.
- 39.
- 40.
- 41.
- 42.
- 43.
- 44.
- 45.
- 46.
- 47.
- 48.
- 49.
- 50.

## ANSWERS

1. Yes
- 2.
3. Yes
- 4.
5. Yes
- 6.
7. Yes
- 8.

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9. Yes

10.

11. Yes

12.

13. No

14.

15. Yes

16.

17. Yes

18.

19. No

20.

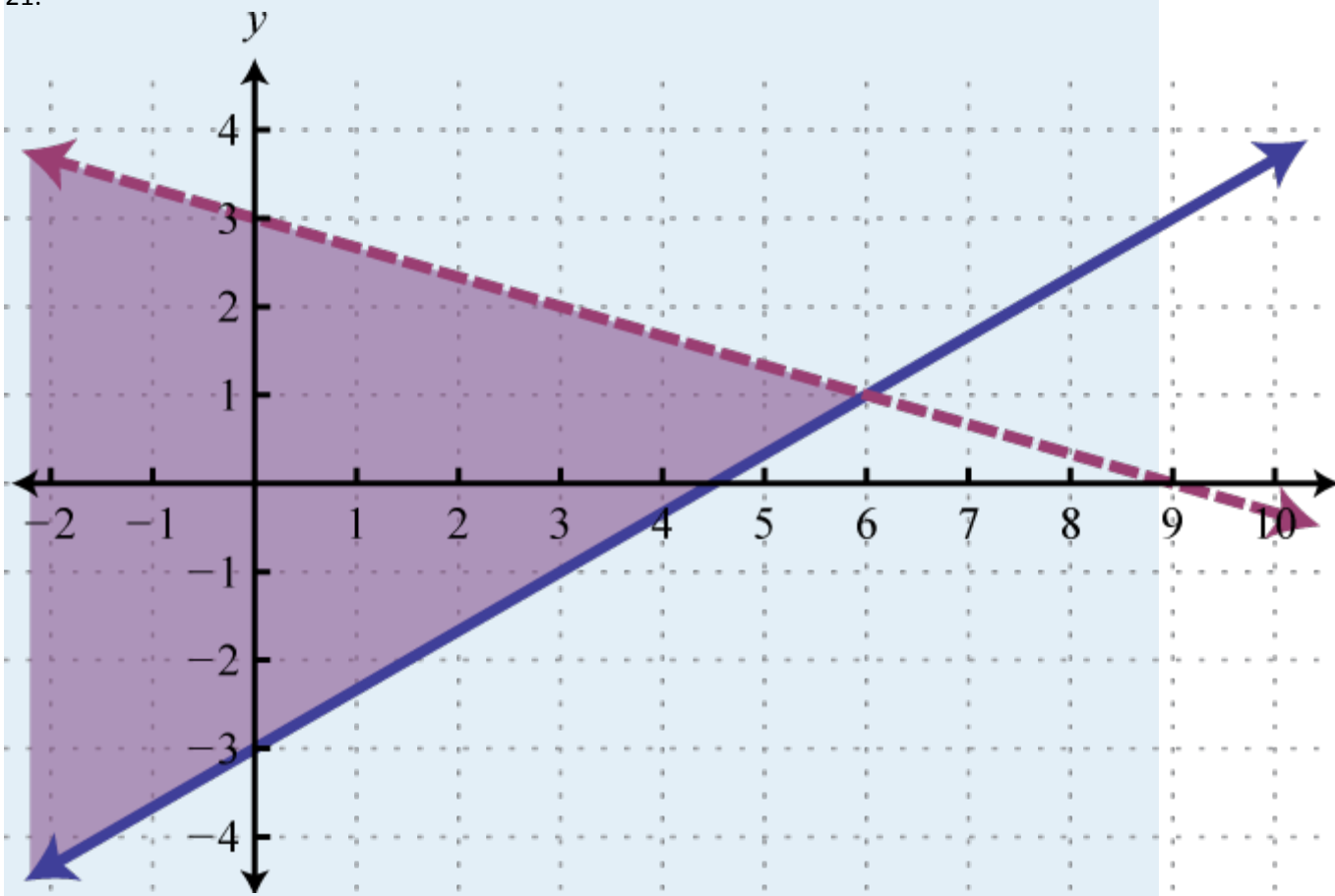
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21.

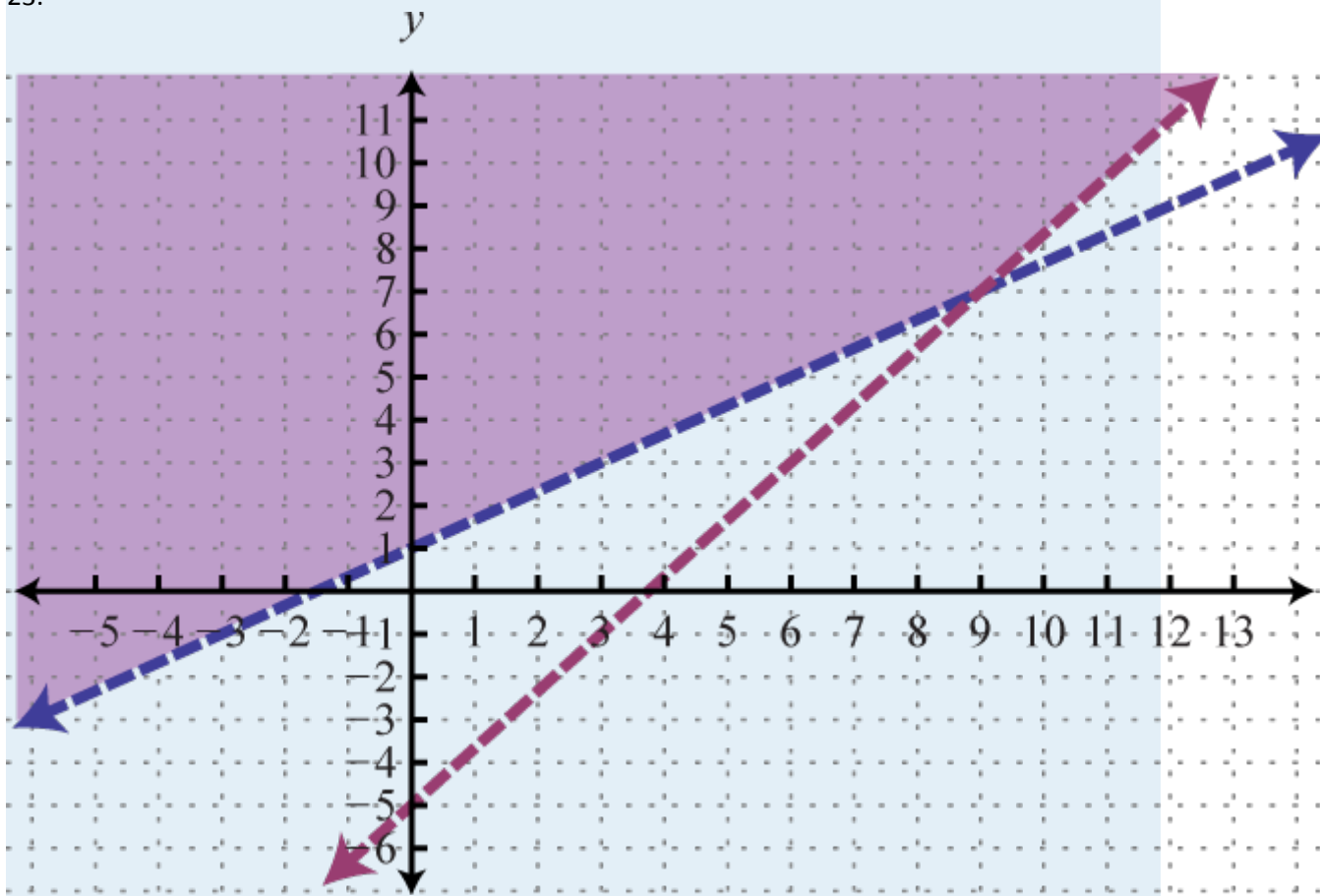


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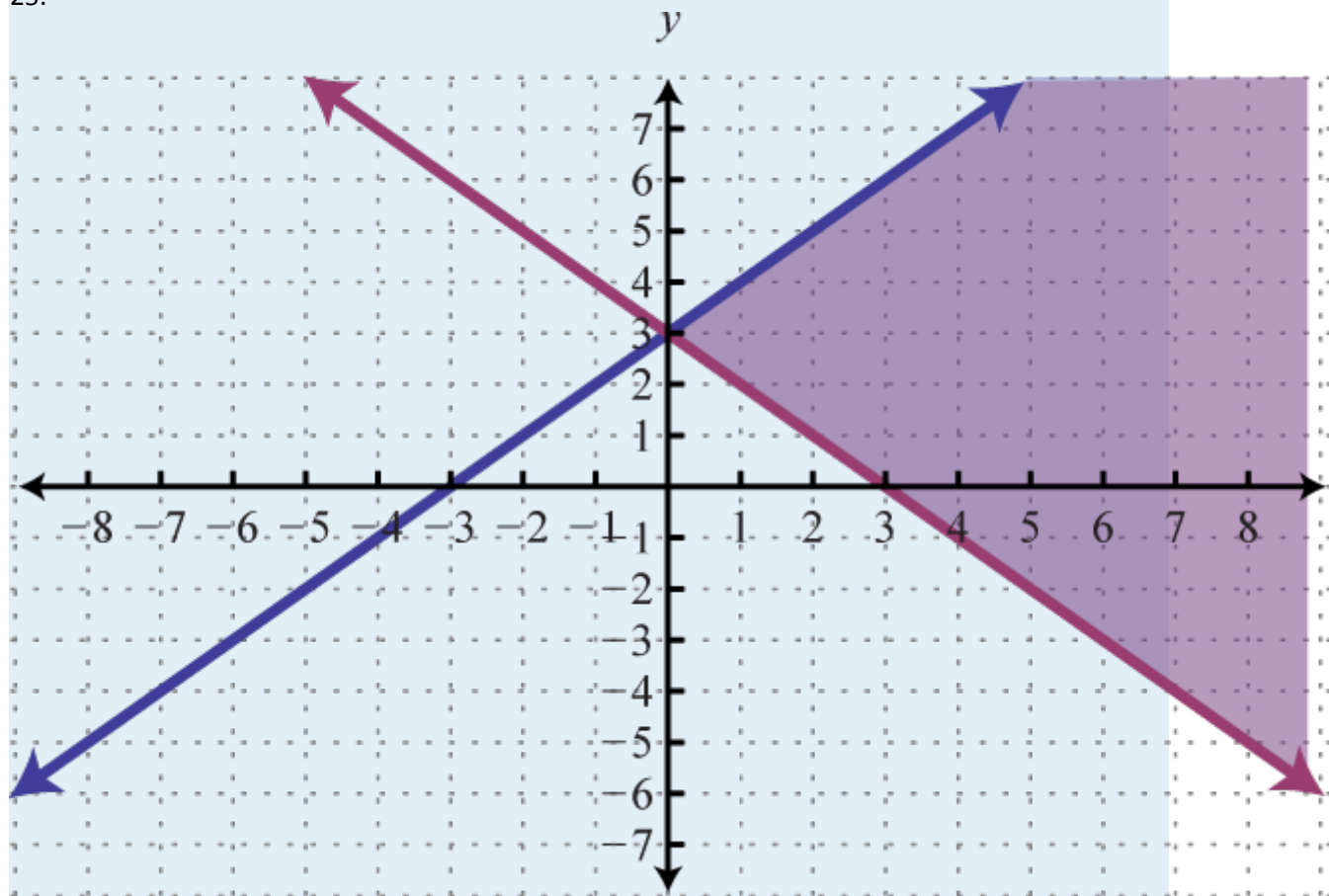


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25.

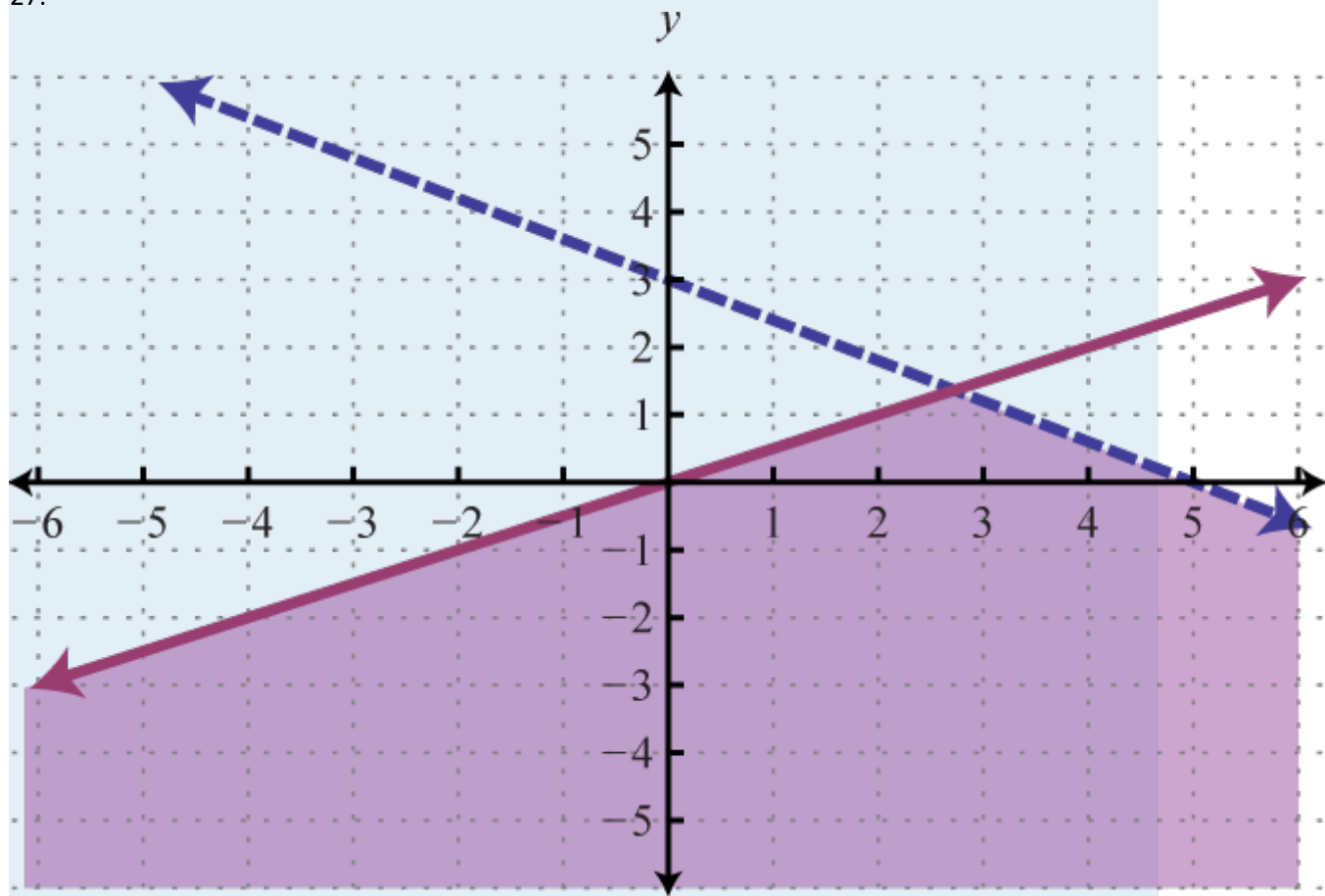


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27.

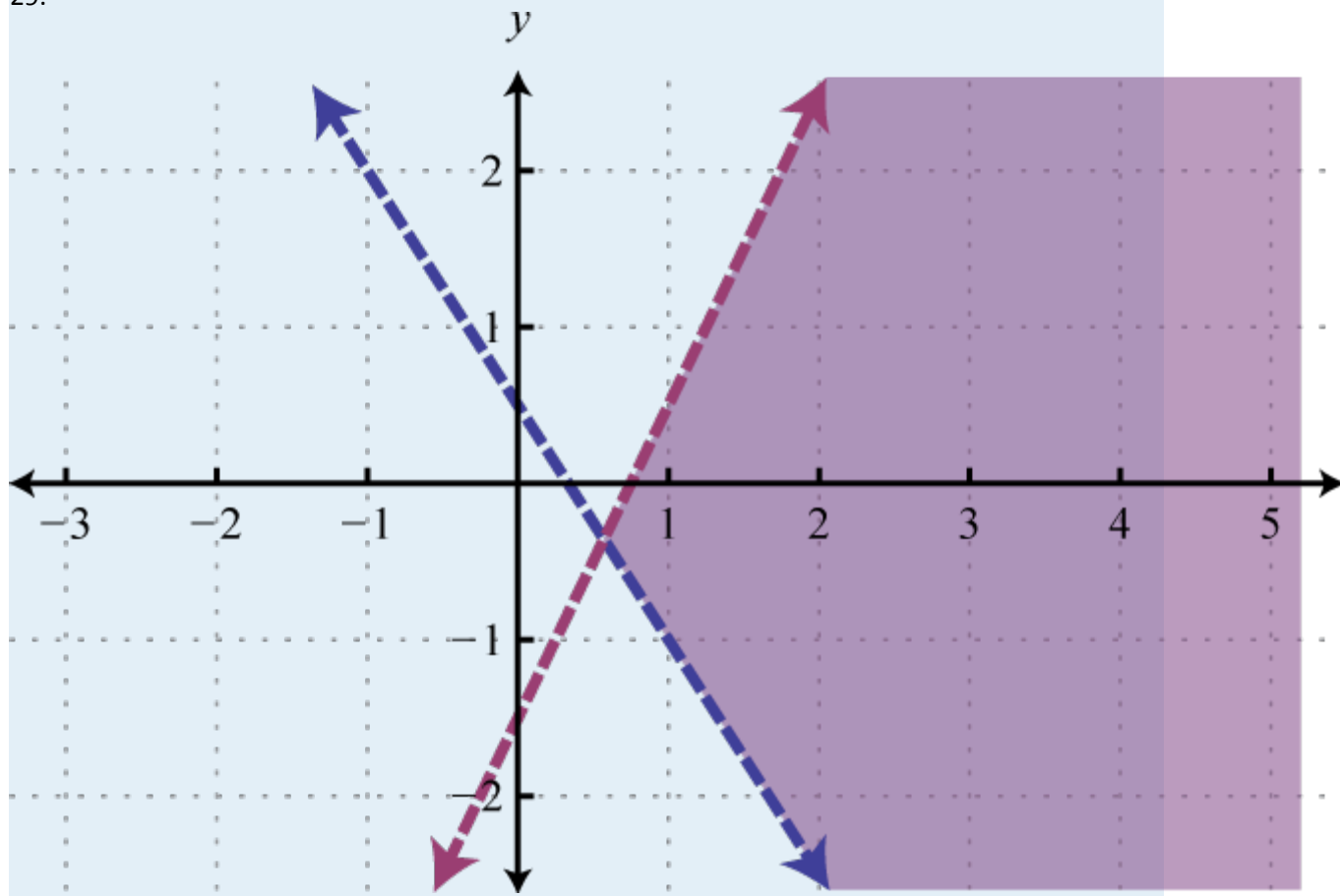


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29.

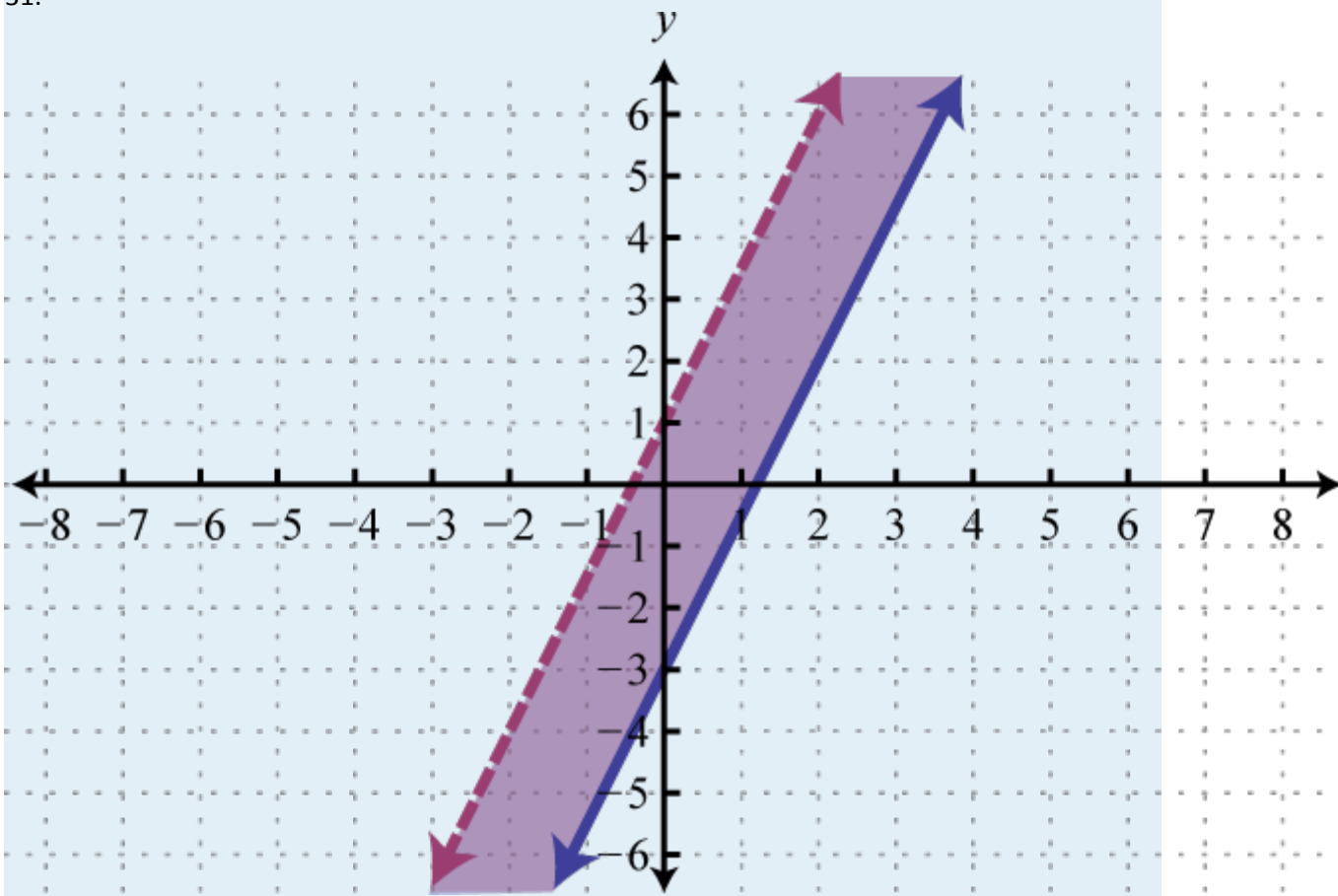


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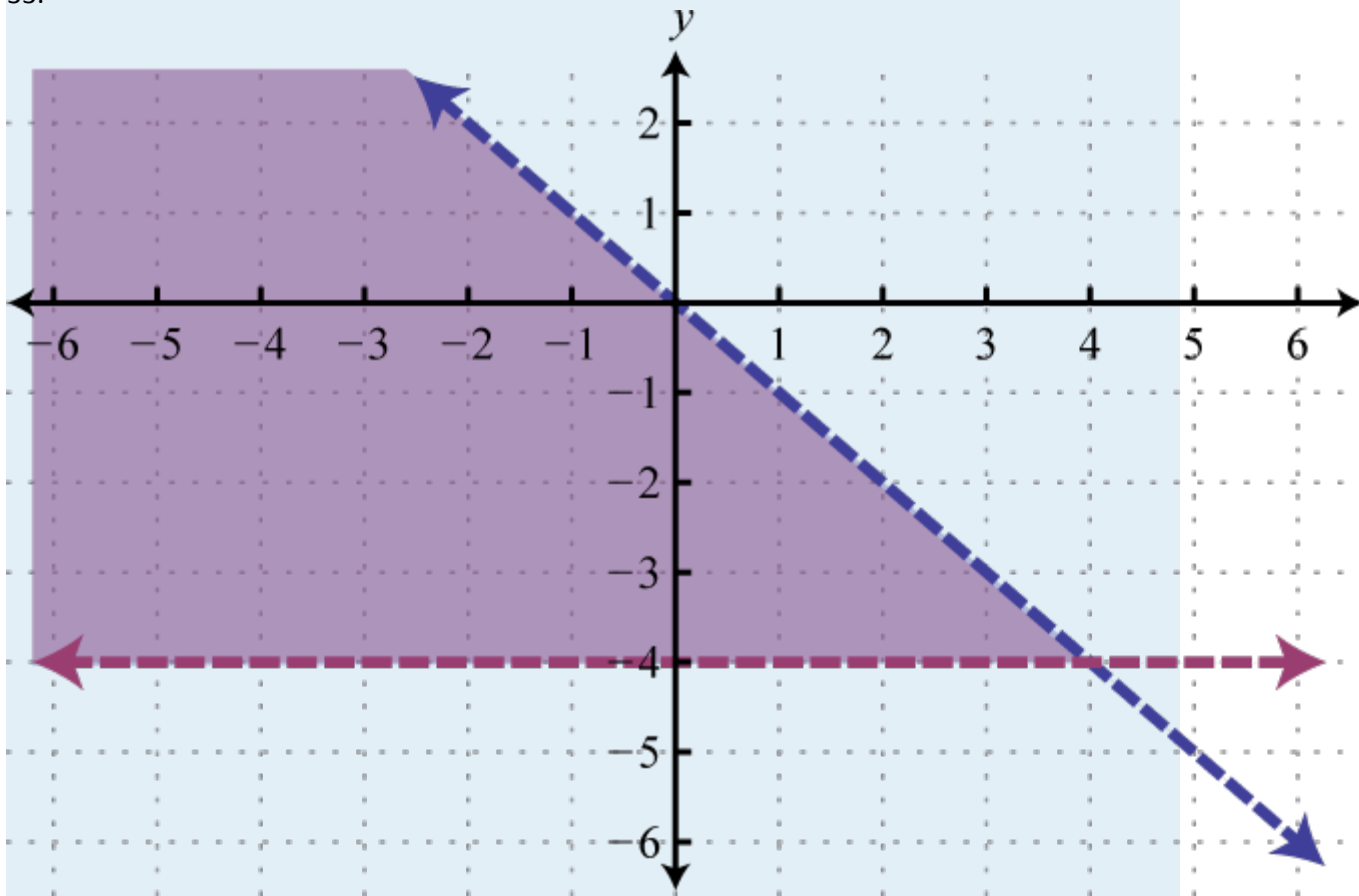


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33.



34.

35.  $\emptyset$

36.

37.

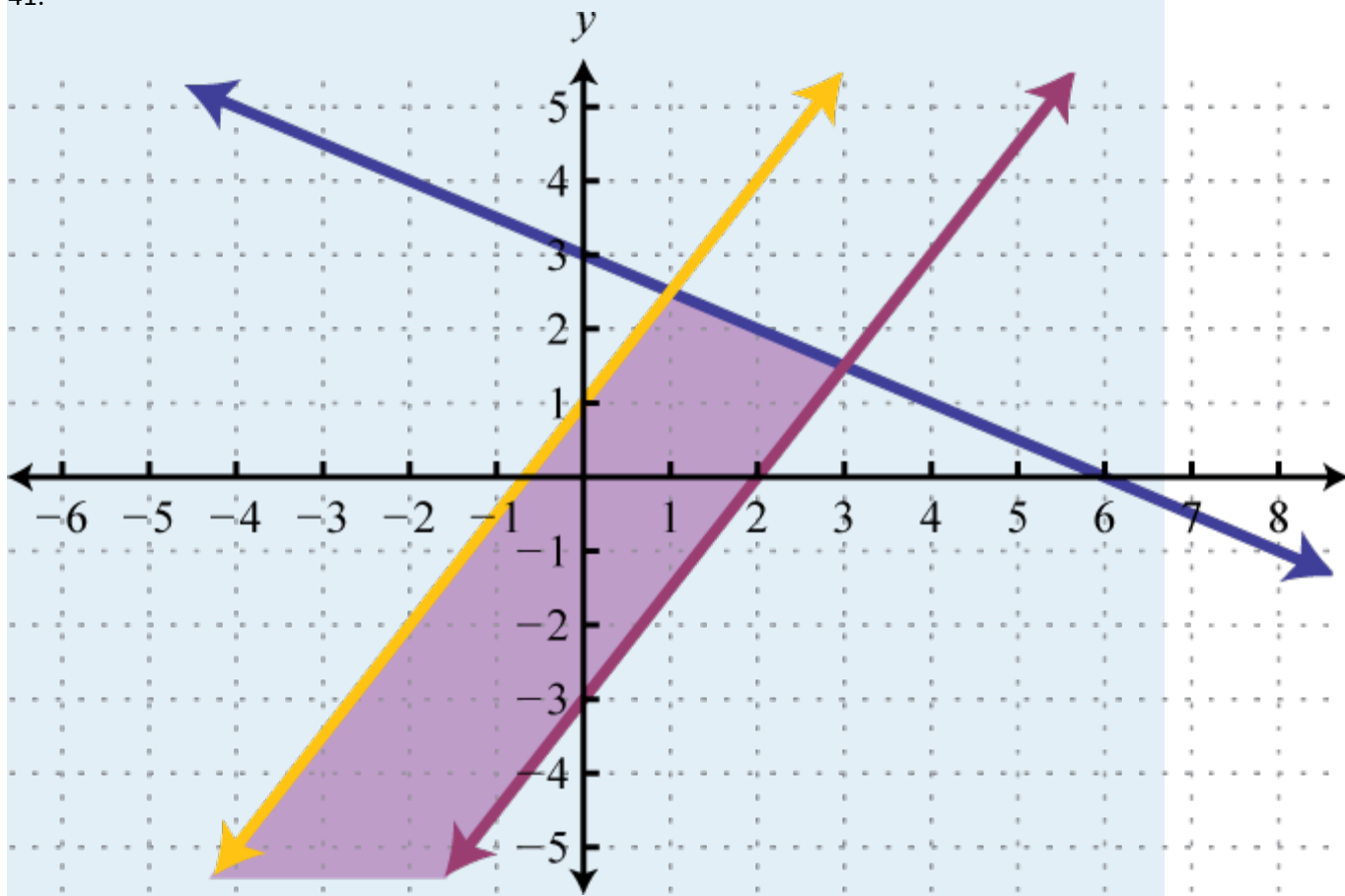
38.

39.

40.



41.



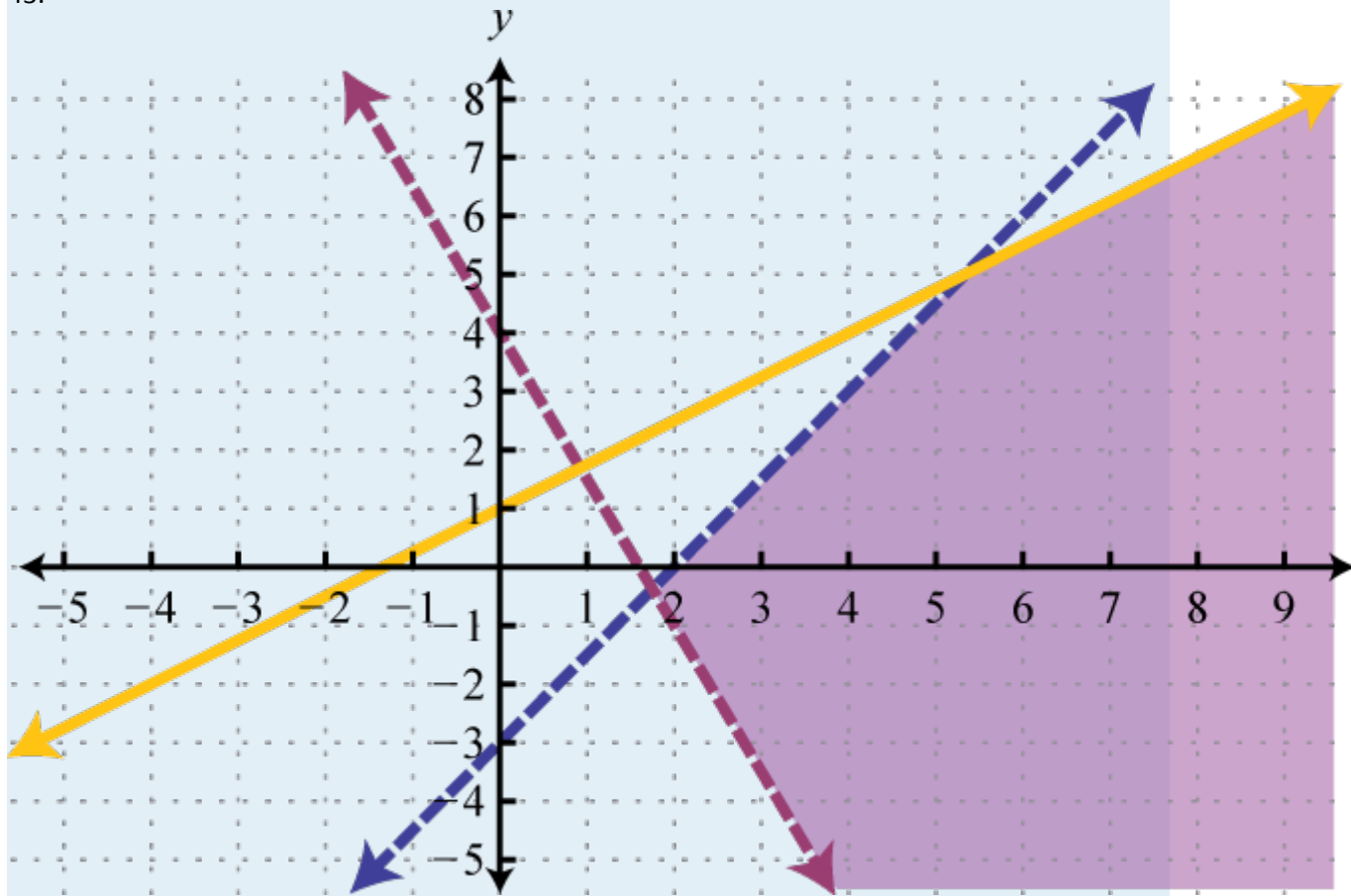
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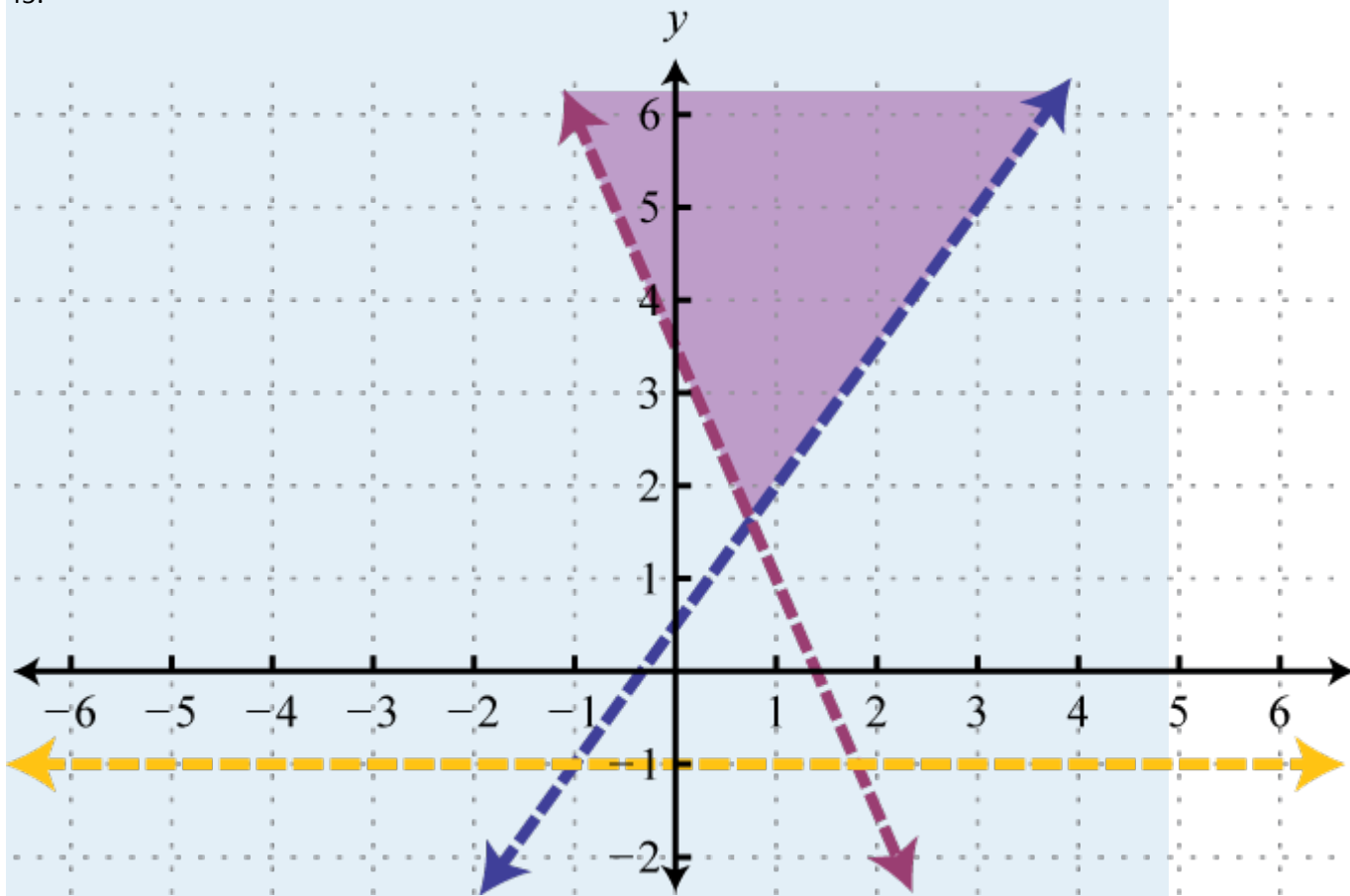


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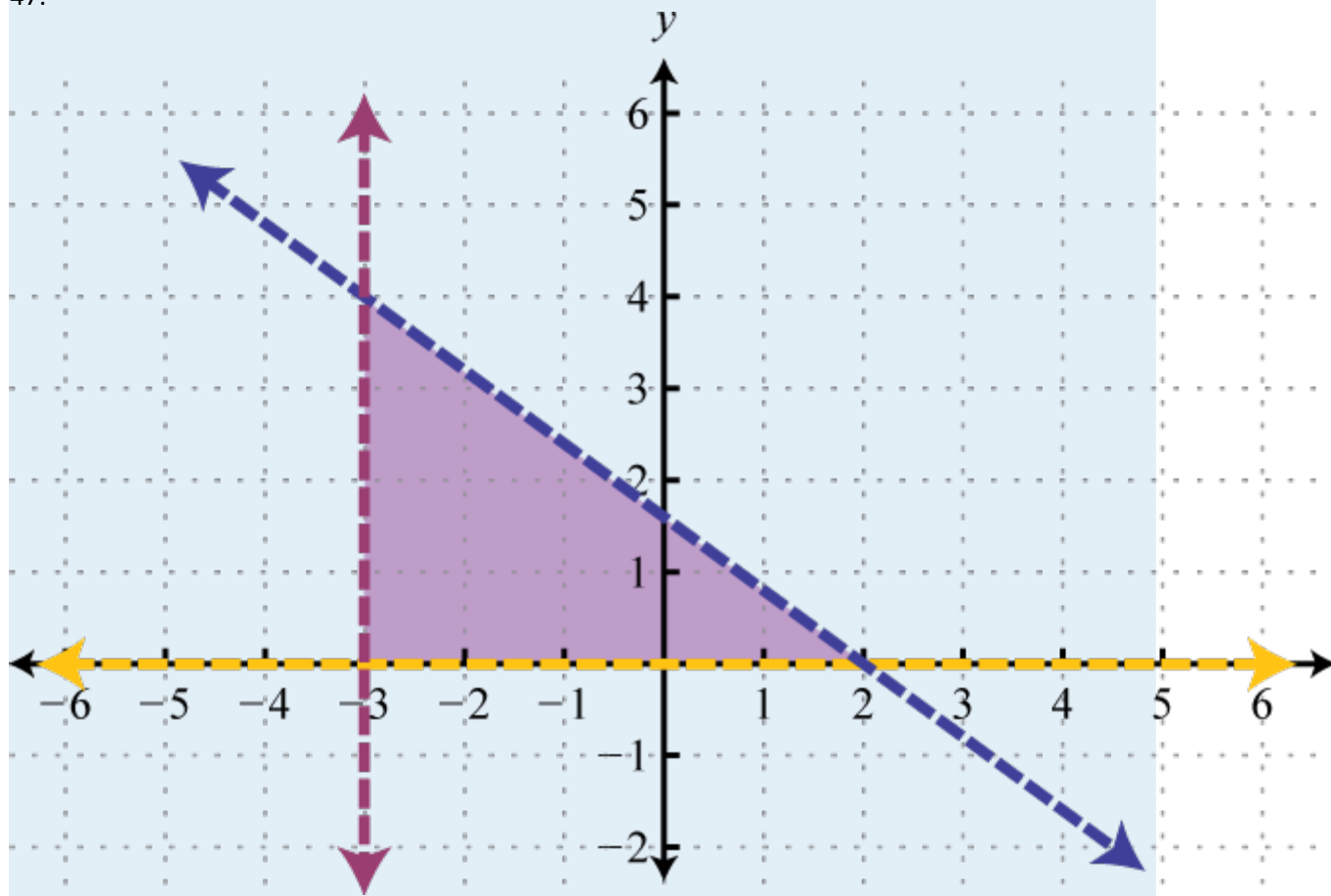


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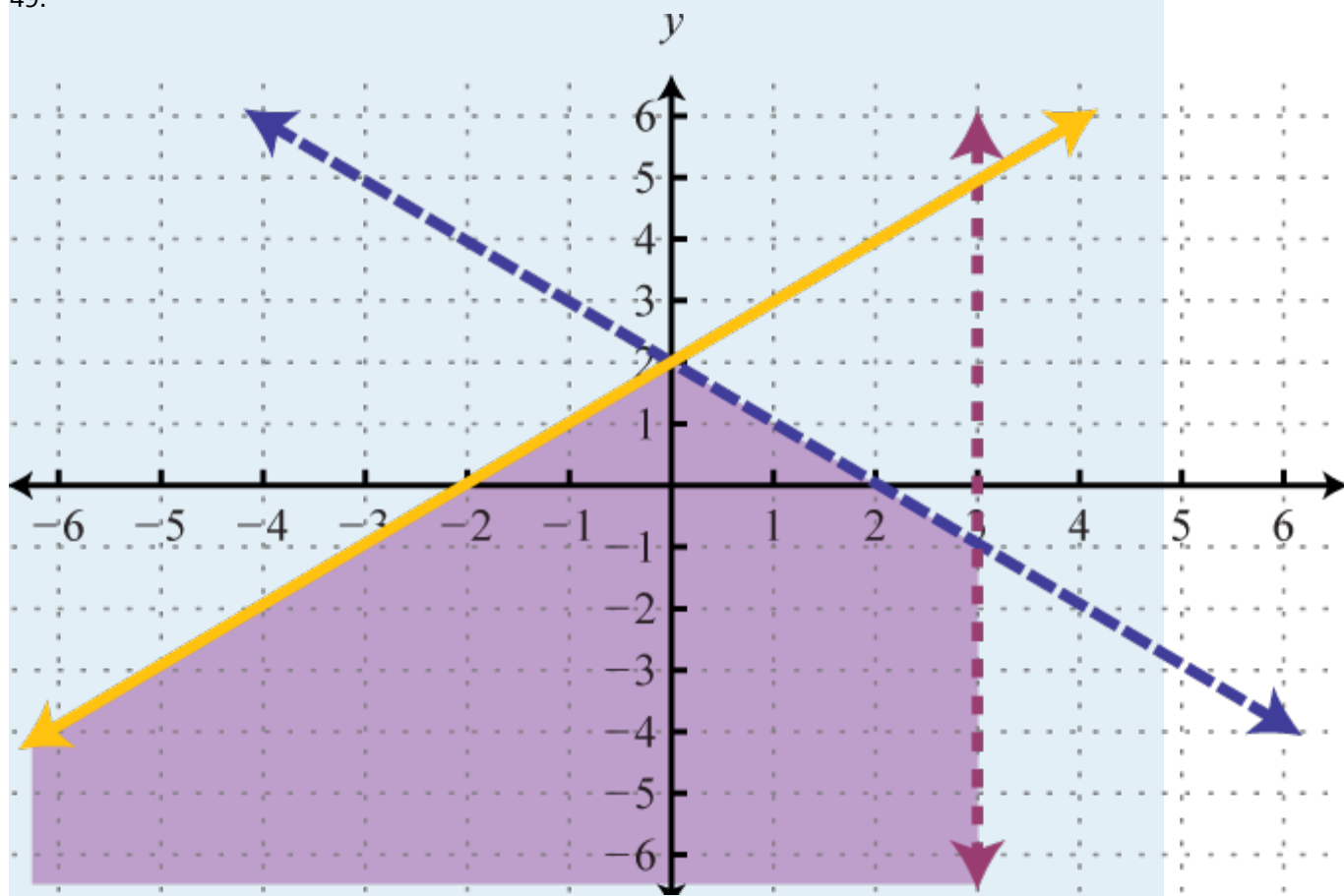


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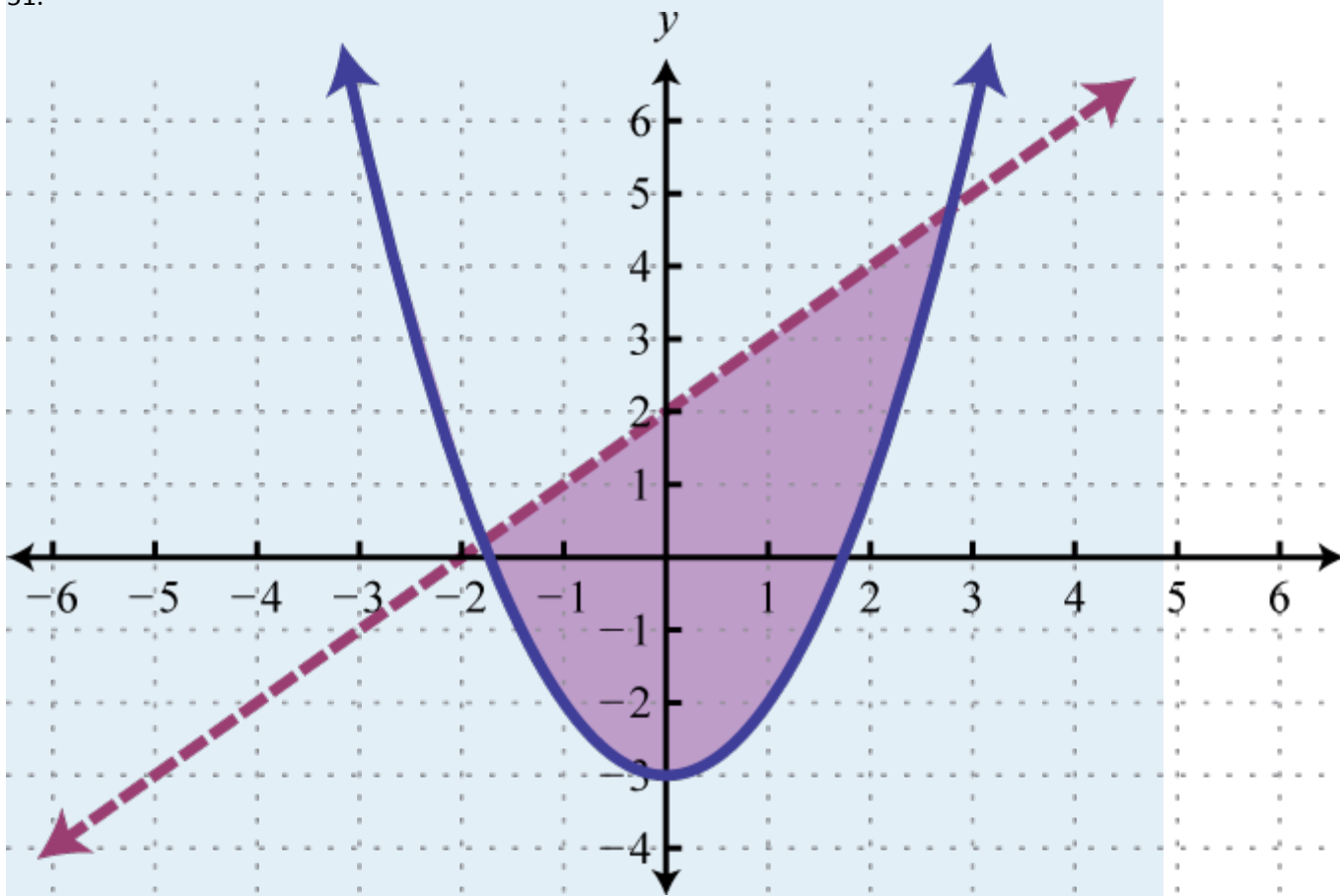


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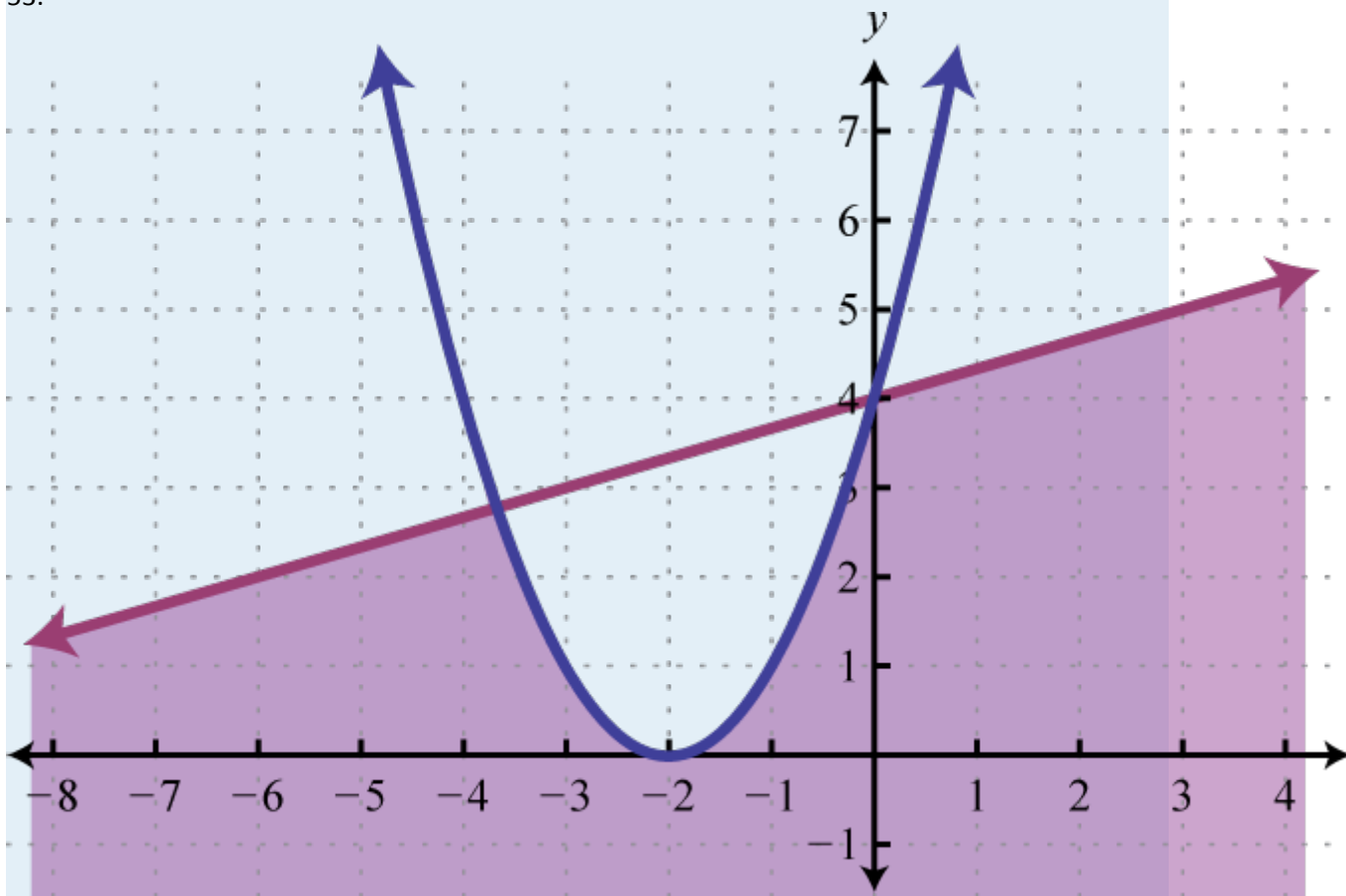


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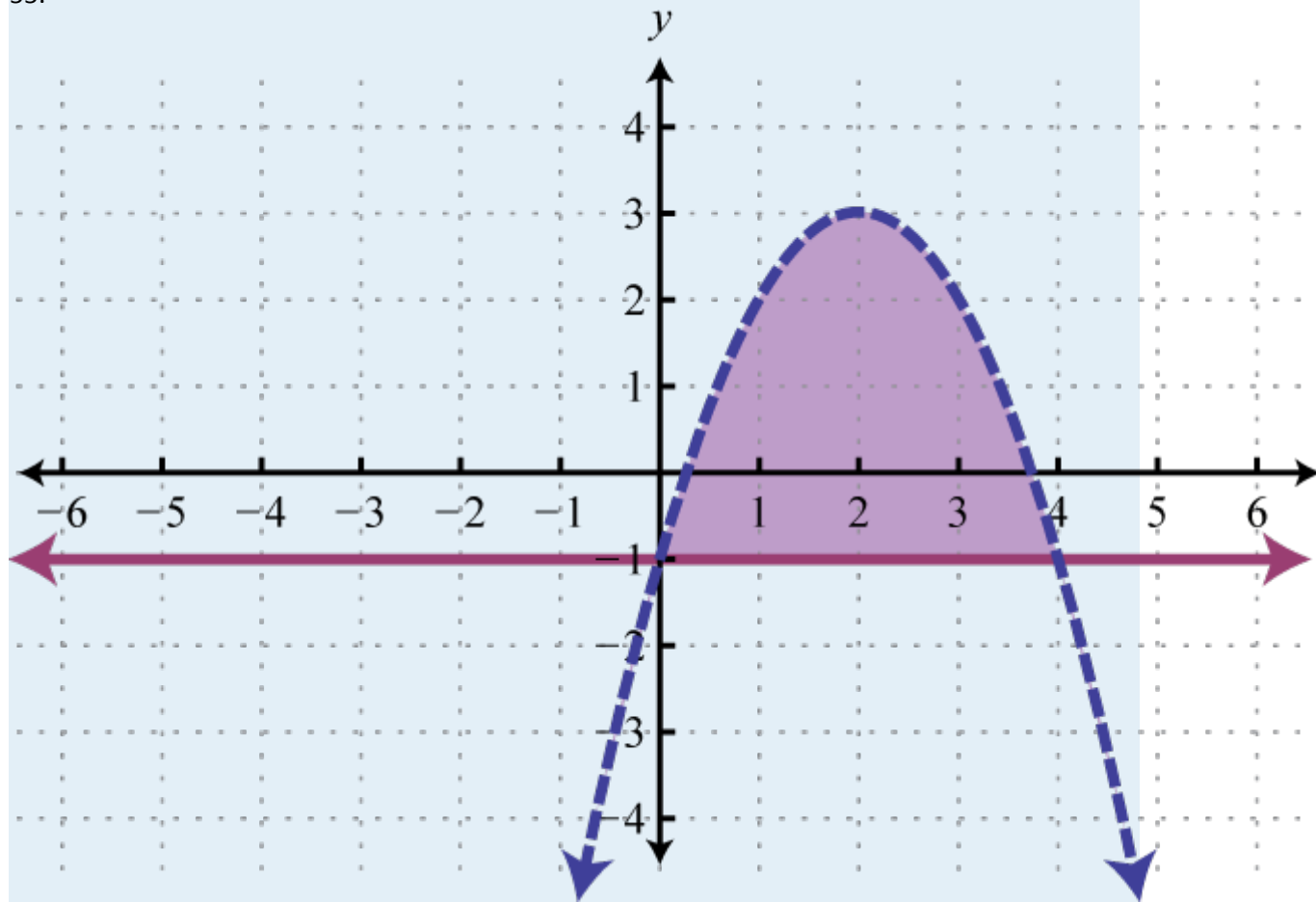


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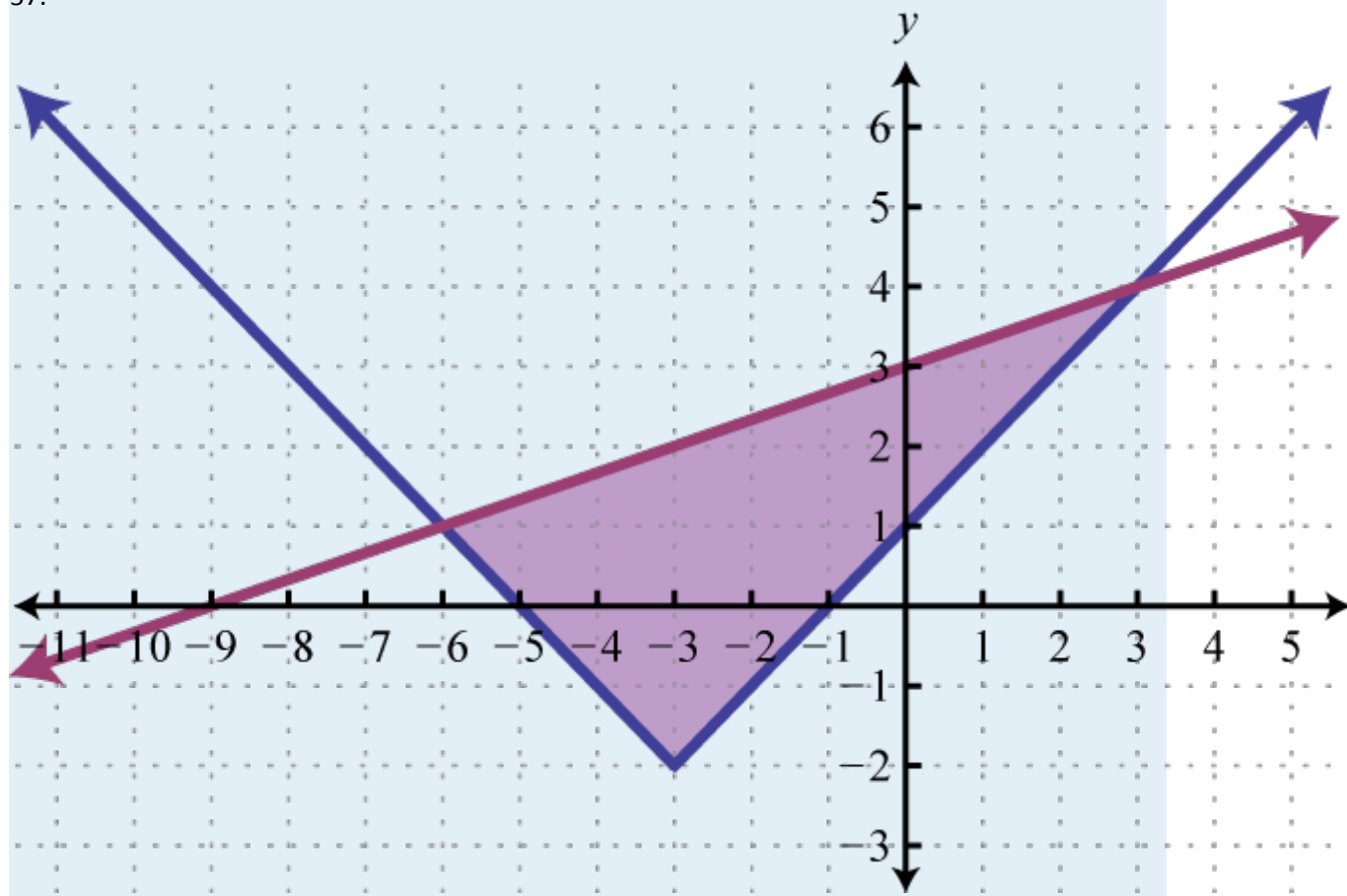


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57.



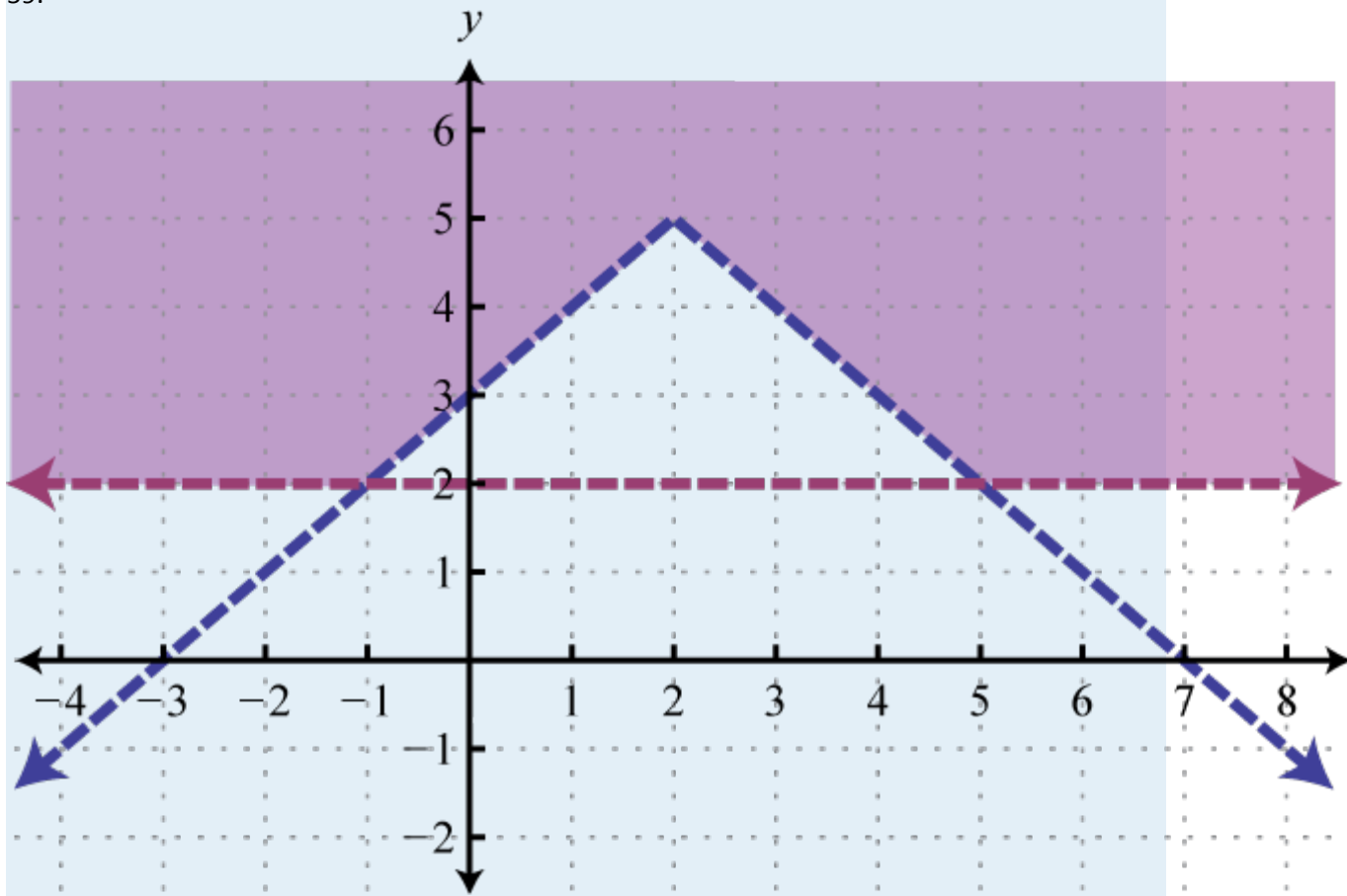
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59.



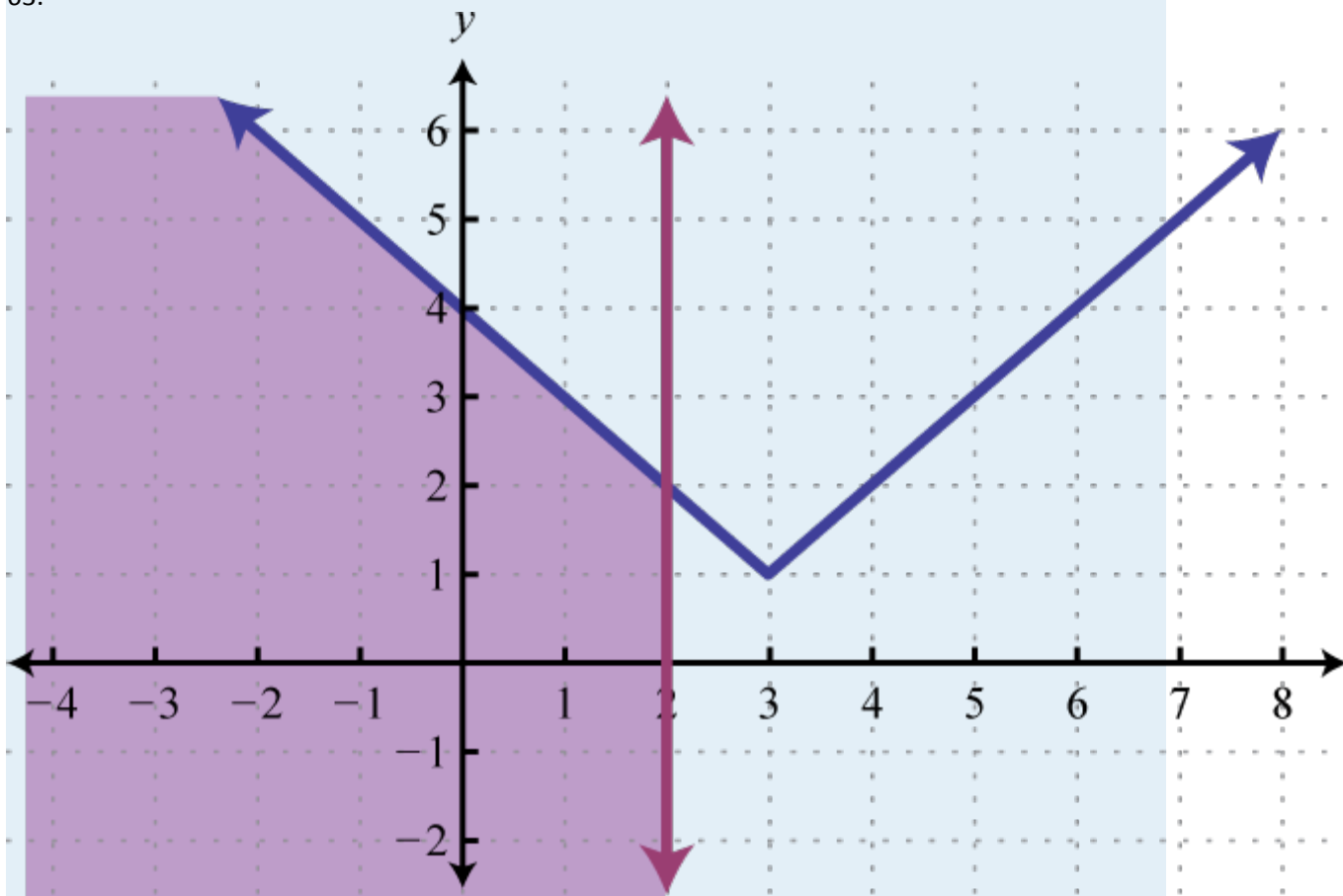
60.

61.  $\emptyset$

62.



63.

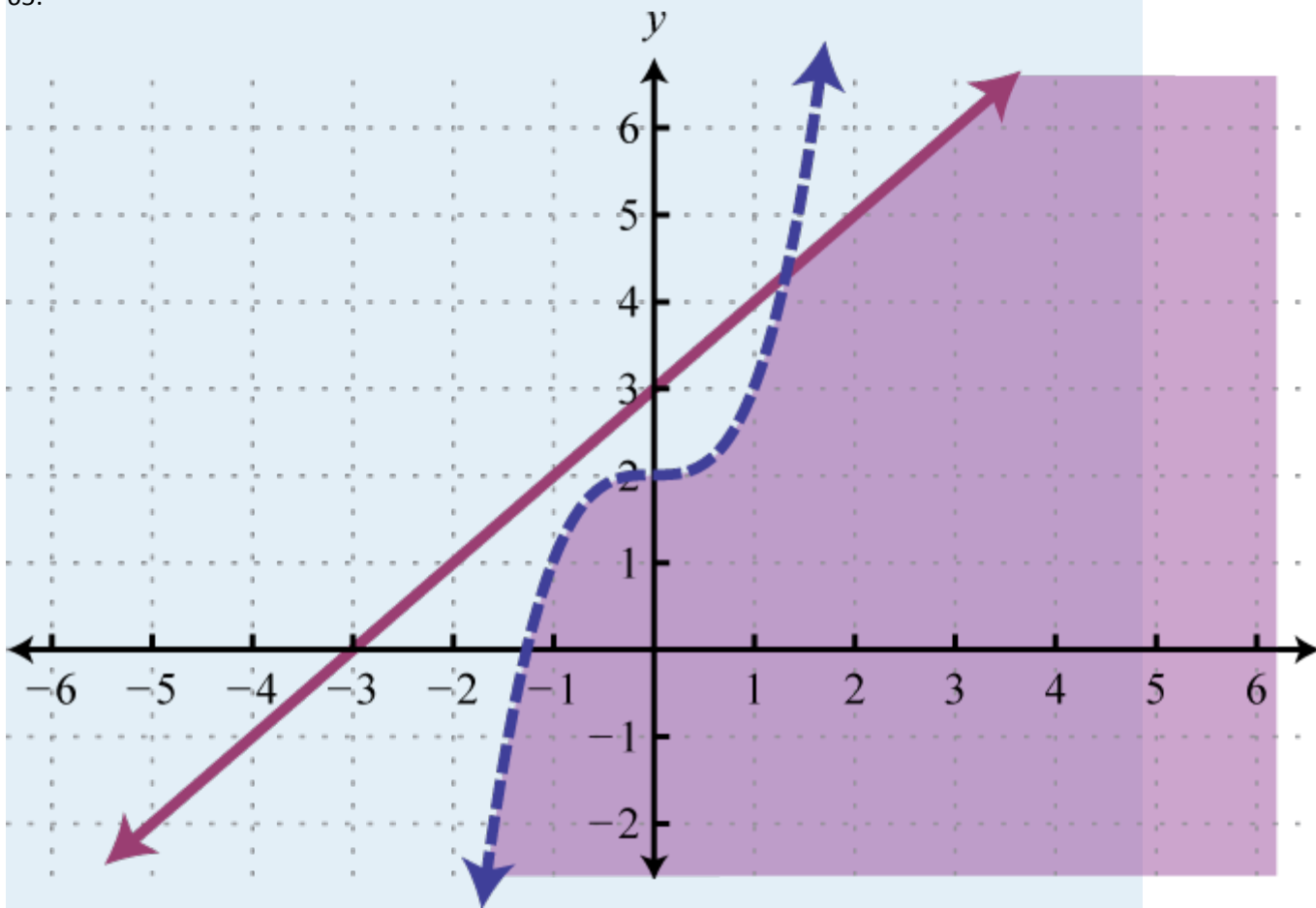


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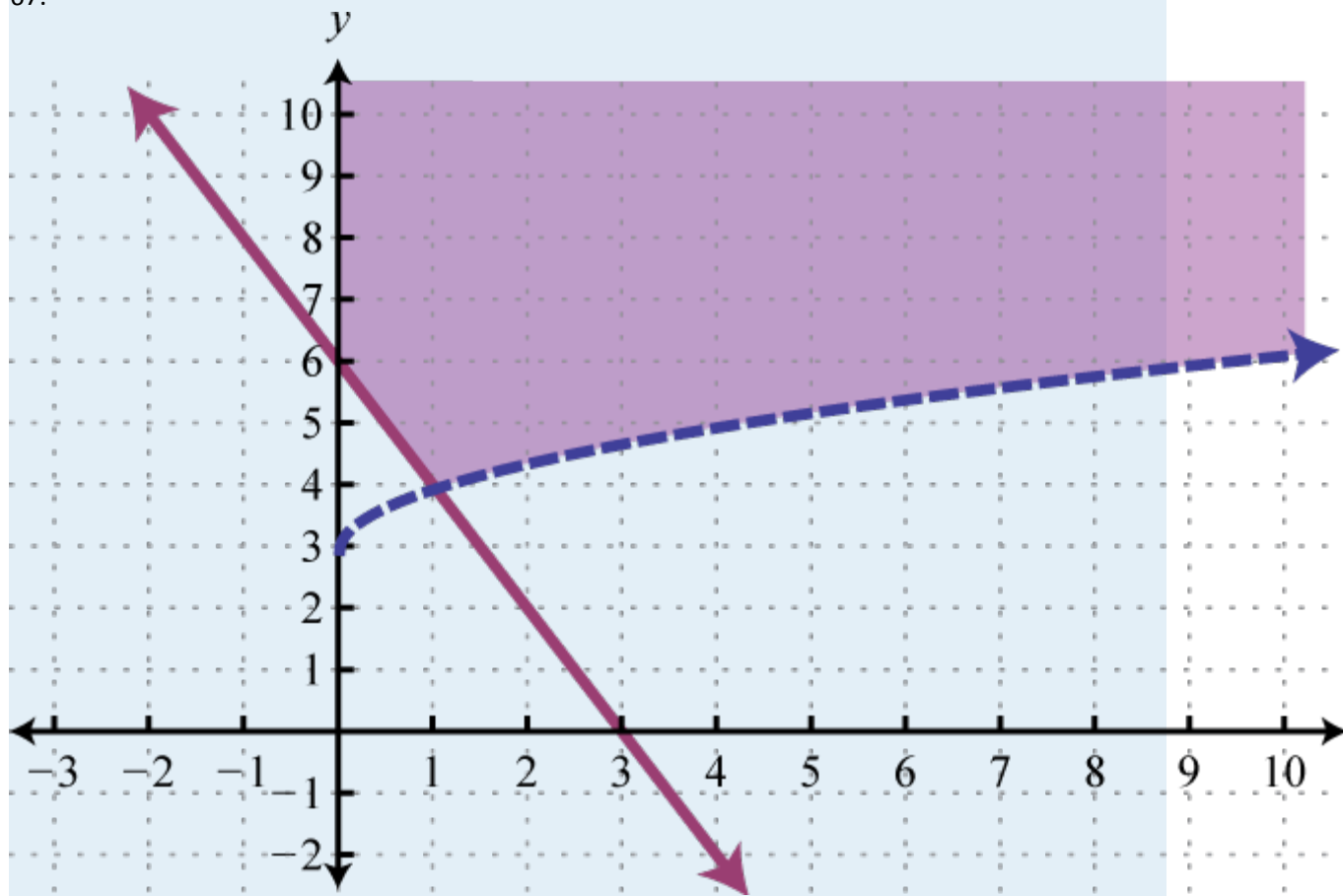


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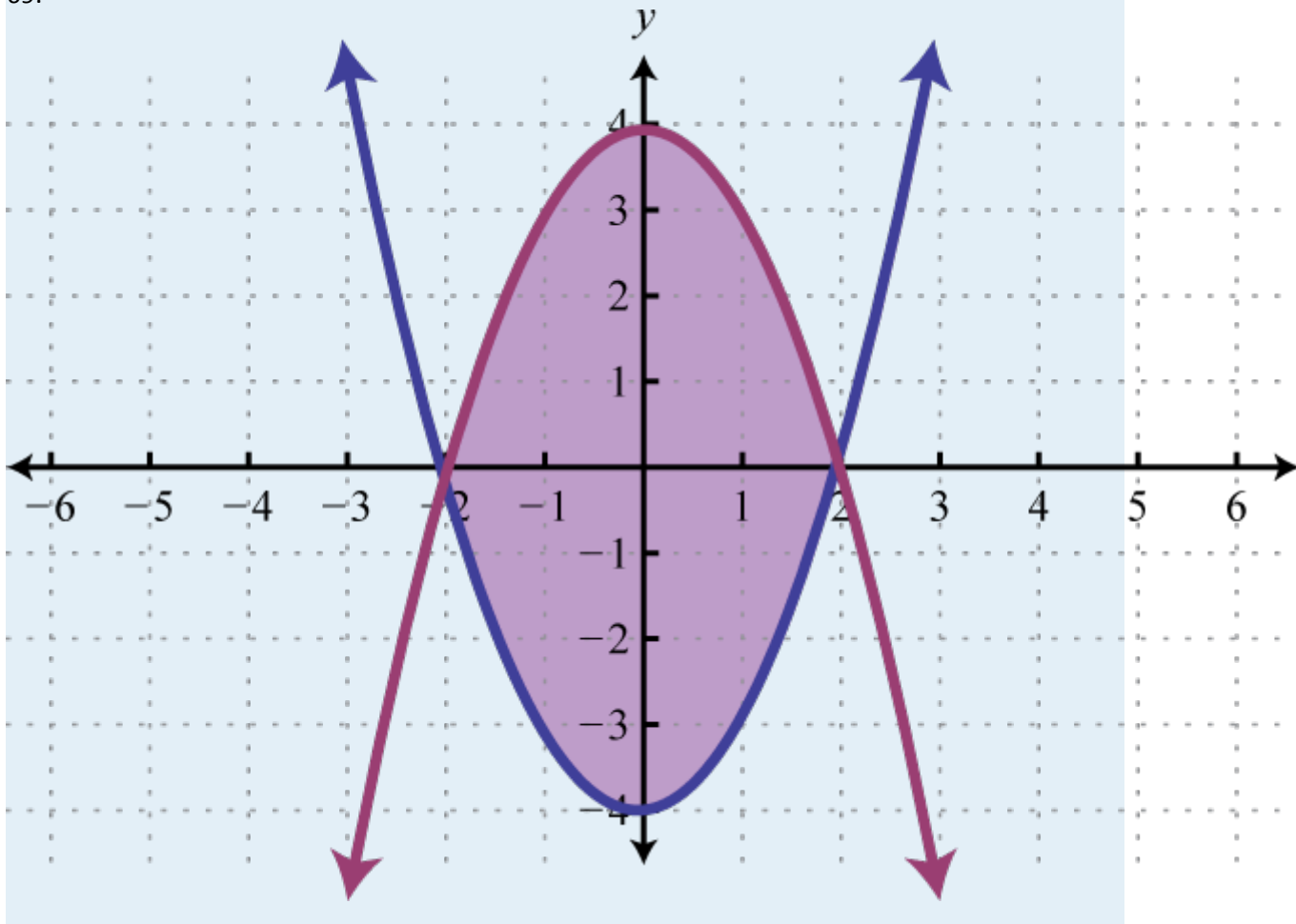


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## 3.8 Review Exercises and Sample Exam

### REVIEW EXERCISES LINEAR SYSTEMS AND THEIR SOLUTIONS

Determine whether or not the given ordered pair is a solution to the given system.

1.

;

2. ;

3. ;

4. ;

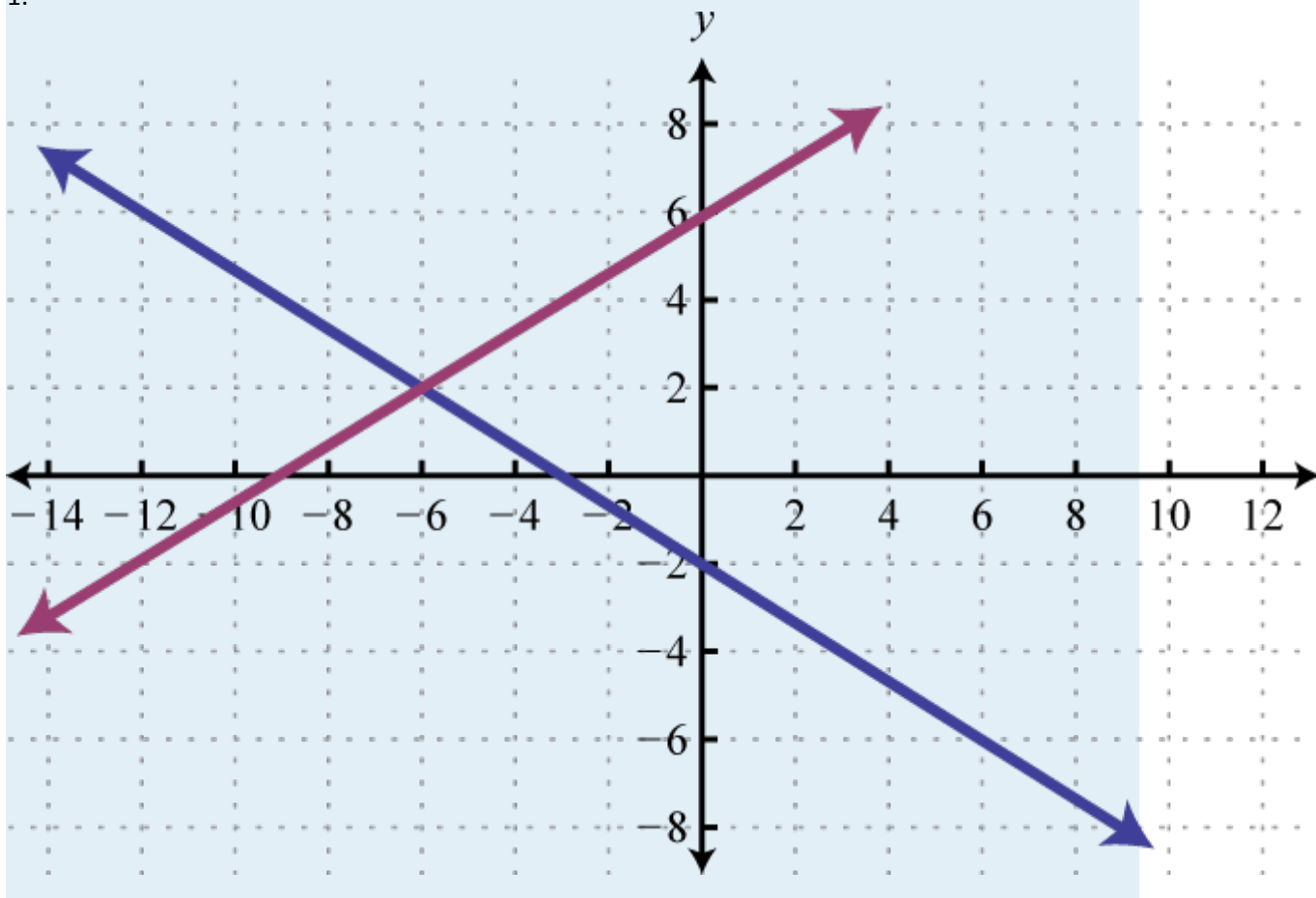
Given the graphs, determine the simultaneous solution.

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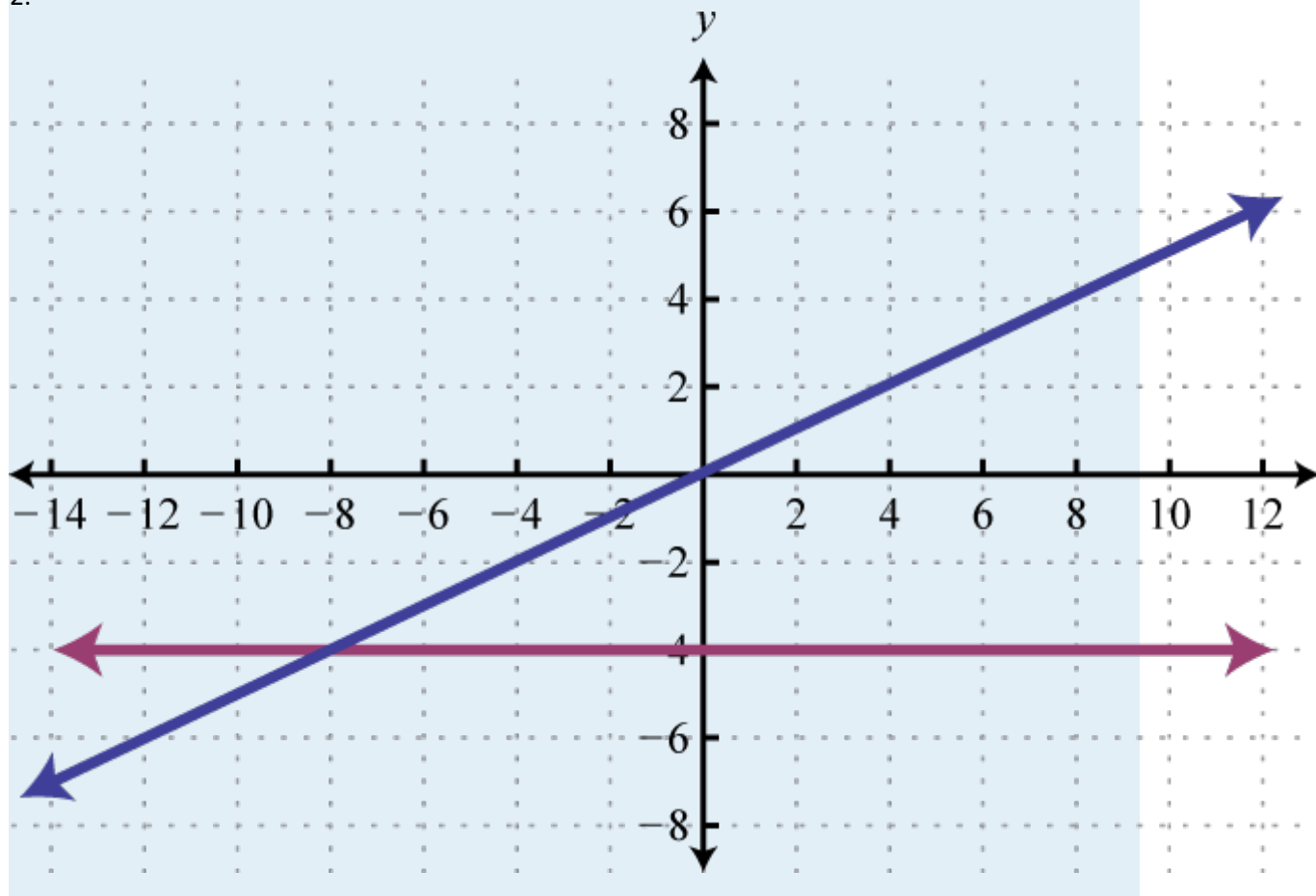
1.



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2.

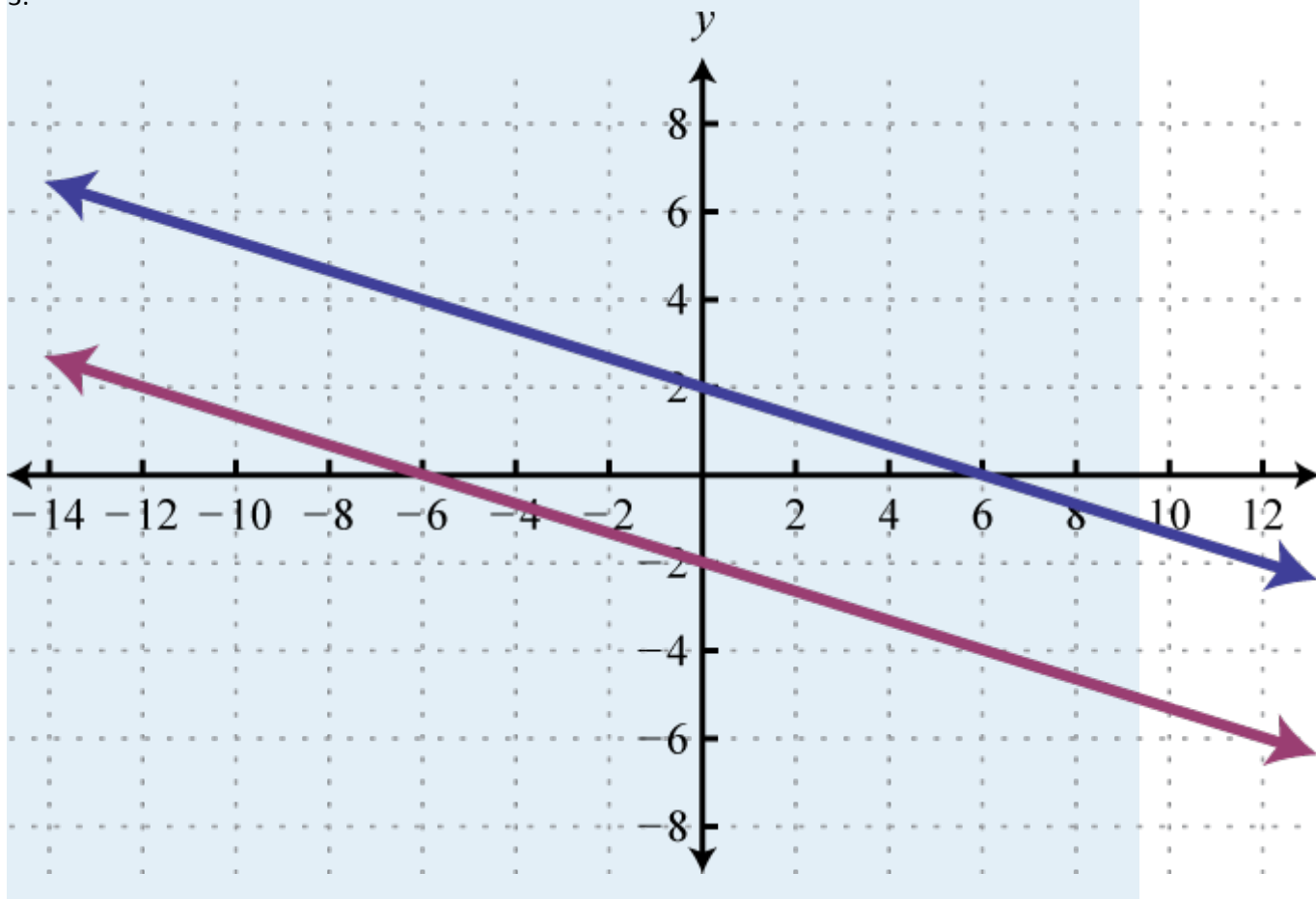


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3.

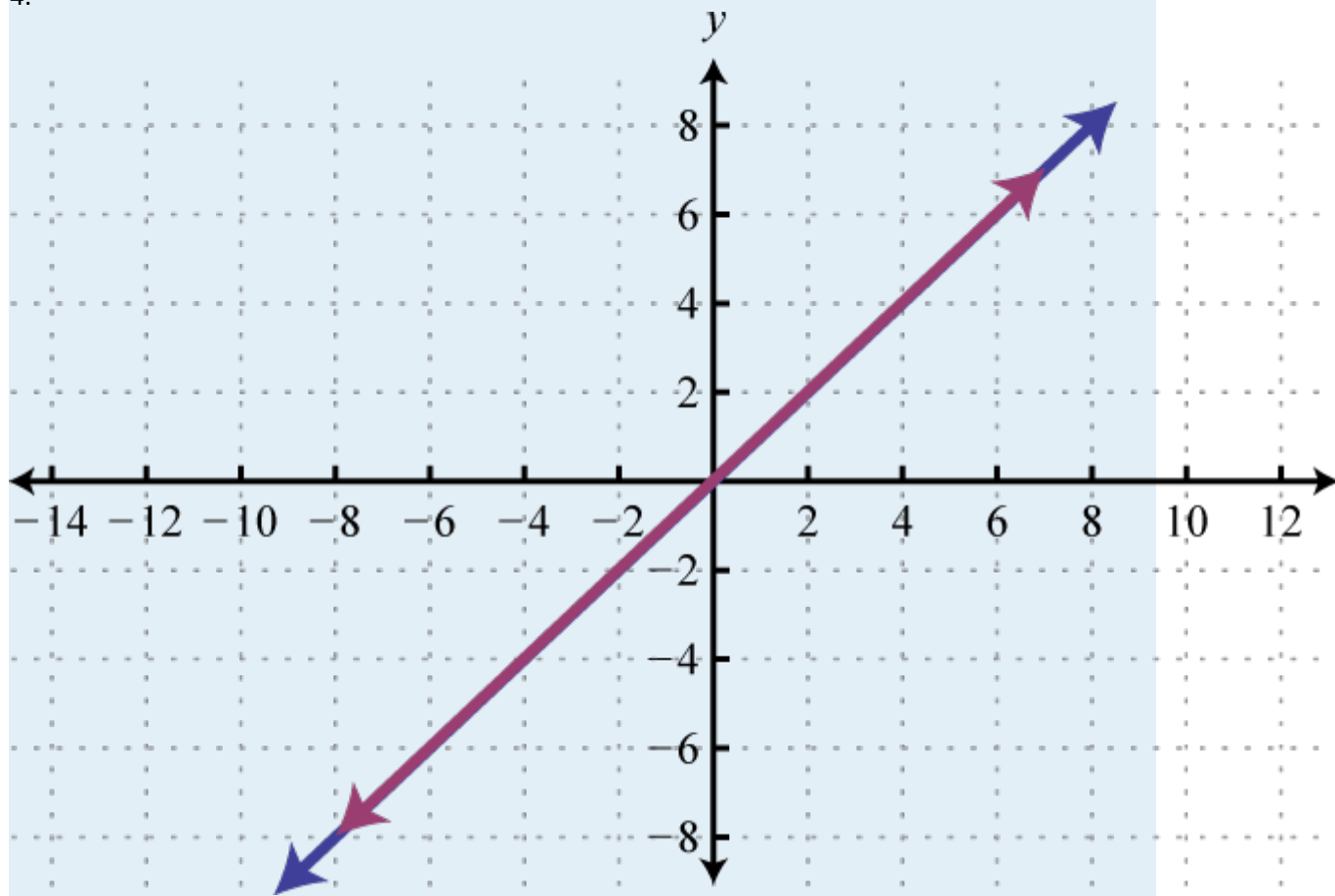


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4.



Solve by graphing.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.



## SOLVING LINEAR SYSTEMS WITH TWO VARIABLES

Solve by substitution.

- 17.
- 18.
- 19.
- 20.

Solve by elimination.

- 17.
- 18.
- 19.
- 20.

Solve using any method.

- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.

## APPLICATIONS OF LINEAR SYSTEMS WITH TWO VARIABLES

Set up a linear system and solve.

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33. The sum of two integers is 32. The larger is 4 less than twice the smaller. Find the integers.
34. The sum of 2 times a larger integer and 3 times a smaller integer is 54. When twice the smaller integer is subtracted from the larger, the result is  $-1$ . Find the integers.
35. The length of a rectangle is 2 centimeters less than three times its width and the perimeter measures 44 centimeters. Find the dimensions of the rectangle.
36. The width of a rectangle is one-third of its length. If the perimeter measures 5313 centimeters, then find the dimensions of the rectangle.
37. The sum of a larger integer and 3 times a smaller is 61. When twice the smaller integer is subtracted from the larger, the result is 1. Find the integers.
38. A total of \$8,600 was invested in two accounts. One account earned % annual interest and the other earned % annual interest. If the total interest for one year was \$431.25, how much was invested in each account?
39. A jar consisting of only nickels and dimes contains 76 coins. If the total value is \$6, how many of each coin are in the jar?
40. A nurse wishes to obtain 32 ounces of a 1.2% saline solution. How much of a 1% saline solution must she mix with a 2.6% saline solution to achieve the desired mixture?
41. A light aircraft flying with the wind can travel 330 miles in 2 hours. The aircraft can fly the same distance against the wind in 3 hours. Find the speed of the wind.
42. An executive was able to average 52 miles per hour to the airport in her car and then board an airplane that averaged 340 miles per hour. If the total 640-mile business trip took 4 hours, how long did she spend on the airplane?

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## SOLVING LINEAR SYSTEMS WITH THREE VARIABLES

Determine whether the given ordered triple is a solution to the given system.

43.  $(-2, -1, 3);$

44.  $(5, -3, -2);$

45.  $;$

46.  $;$

Solve.

43.

44.

45.

46.

47.

48.

49.

50.

Set up a linear system and solve.

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43. The sum of three integers is 24. The larger is equal to the sum of the two smaller integers. Three times the smaller is equal to the larger. Find the integers.

44. The sports center sold 120 tickets to the Friday night basketball game for a total of \$942. A general admission ticket cost \$12, a student ticket cost \$6, and a child ticket cost \$4. If the sum of the general admission and student tickets totaled 105, then how many of each ticket were sold?

45. A 16-ounce mixed nut product containing 13.5% peanuts is to be packaged. The packager has a three-mixed nut product containing 6%, 10%, and 50% peanut concentrations in stock. If the amount of 50% peanut product is to be one-quarter that of the 10% peanut product, then how much of each will be needed to produce the desired peanut concentration?

46. Water is to be mixed with two acid solutions to produce a 25-ounce solution containing 6% acid. The acid mixtures on hand contain 10% and 25% acid. If the amount of 25% acid is to be one-half the amount of the 10% acid solution, how much water will be needed?

## MATRICES AND GAUSSIAN ELIMINATION

Construct the corresponding augmented matrix.

59.

60.

61.

62.

Solve using matrices and Gaussian elimination.

59.



60.

61.

62.

63.

64.

65.

66.

## DETERMINANTS AND CRAMER'S RULE

Calculate the determinant.

71.

72.

73.

74.

75.

76.

77.

78.

Solve using Cramer's rule.

71.

72.

73.

74.

75.

76.

77.

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78.

79.

80.

## SYSTEMS OF INEQUALITIES WITH TWO VARIABLES

Determine whether or not the given point is a solution to the system of inequalities.

89.  $(-6, 1)$ ;

90. ;

91.  $(-4, -2)$ ;

92. ;

93.  $(-3, -2)$ ;

94. ;

Graph the solution set.

89.

90.

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91.

92.

93.

94.

95.

96.

## ANSWERS

1. Yes

2.

3. No

4.

5.  $(-6, 2)$

6.

7.  $\emptyset$

8.

9.  $(4, -2)$

10.

11.

12.

13.  $(4, 5)$

14.

15.  $(6, 0)$

16.

17.  $(2, -4)$

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- 18.
- 19.
- 20.
- 21.
- 22.
23. (4, 3)
- 24.
25. (5, 2)
- 26.
- 27.
- 28.
29.  $\emptyset$
- 30.
- 31.
- 32.
33. 12, 20
- 34.
35. Length: 16 centimeters; width: 6 centimeters
- 36.
37. 12, 25
- 38.
39. The jar contains 32 nickels and 44 dimes.
- 40.
41. 27.5 miles per hour
- 42.

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43. No
- 44.
45. Yes
- 46.
- 47.
- 48.
49.  $(-2, -1, 3)$
- 50.
51.  $\emptyset$
- 52.
- 53.
- 54.
55. 4, 8, 12
- 56.
57. 6 oz of the 6% peanut stock, 8 oz of the 10% peanut stock, and 2 oz of the 50% peanut stock should be mixed.
- 58.
- 59.
- 60.
- 61.
- 62.
63.  $(5, -4)$
- 64.
- 65.
- 66.

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- 67.
- 68.
- 69.  $(-2, 2, 2)$
- 70.
- 71.  $-22$
- 72.
- 73.  $-14$
- 74.
- 75.  $-1$
- 76.
- 77.  $0$
- 78.
- 79.
- 80.
- 81.
- 82.
- 83.  $(2, -5, 1)$
- 84.
- 85.
- 86.
- 87.
- 88.
- 89. Yes
- 90.
- 91. Yes

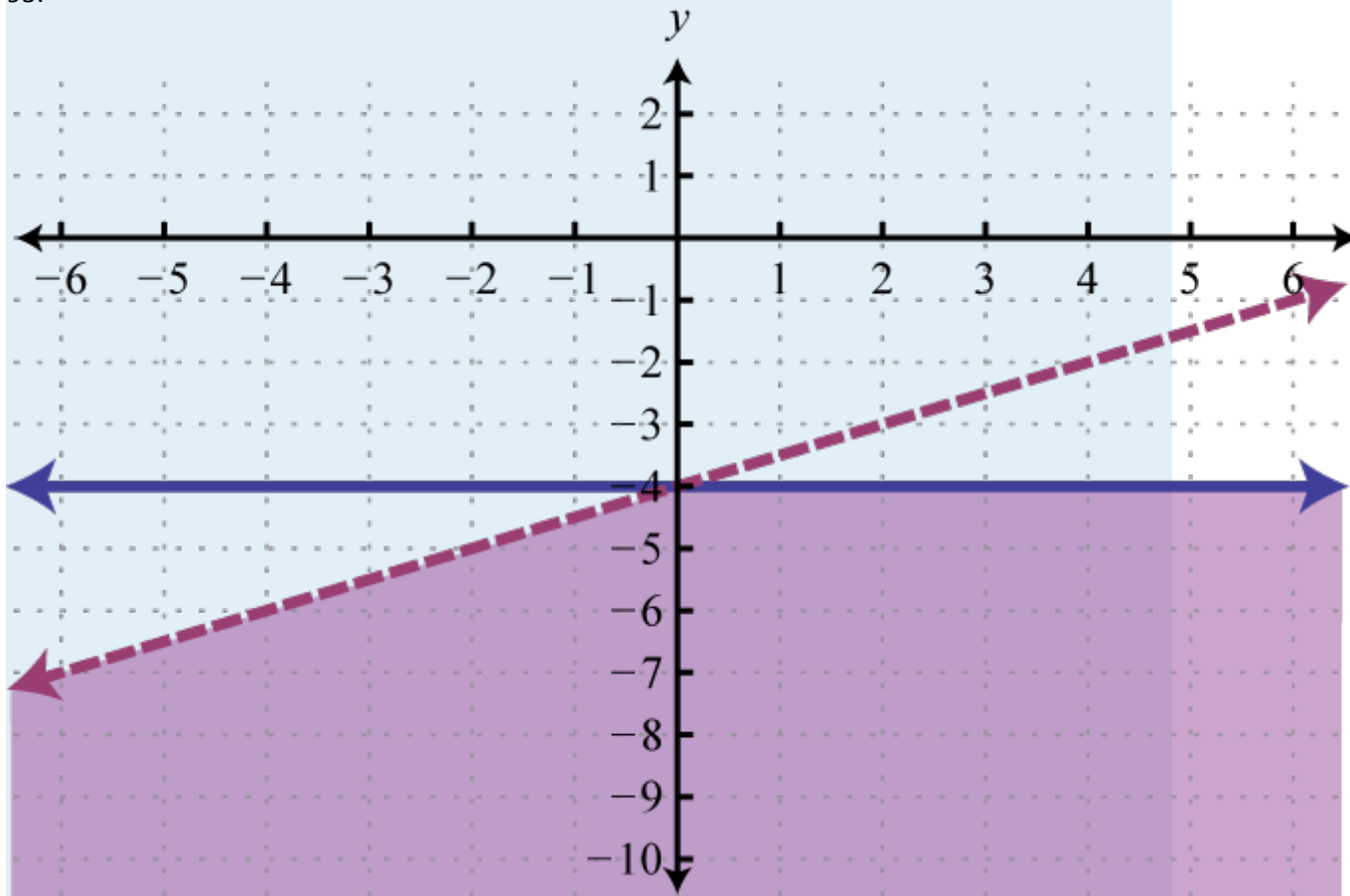


92.

93. Yes

94.

95.



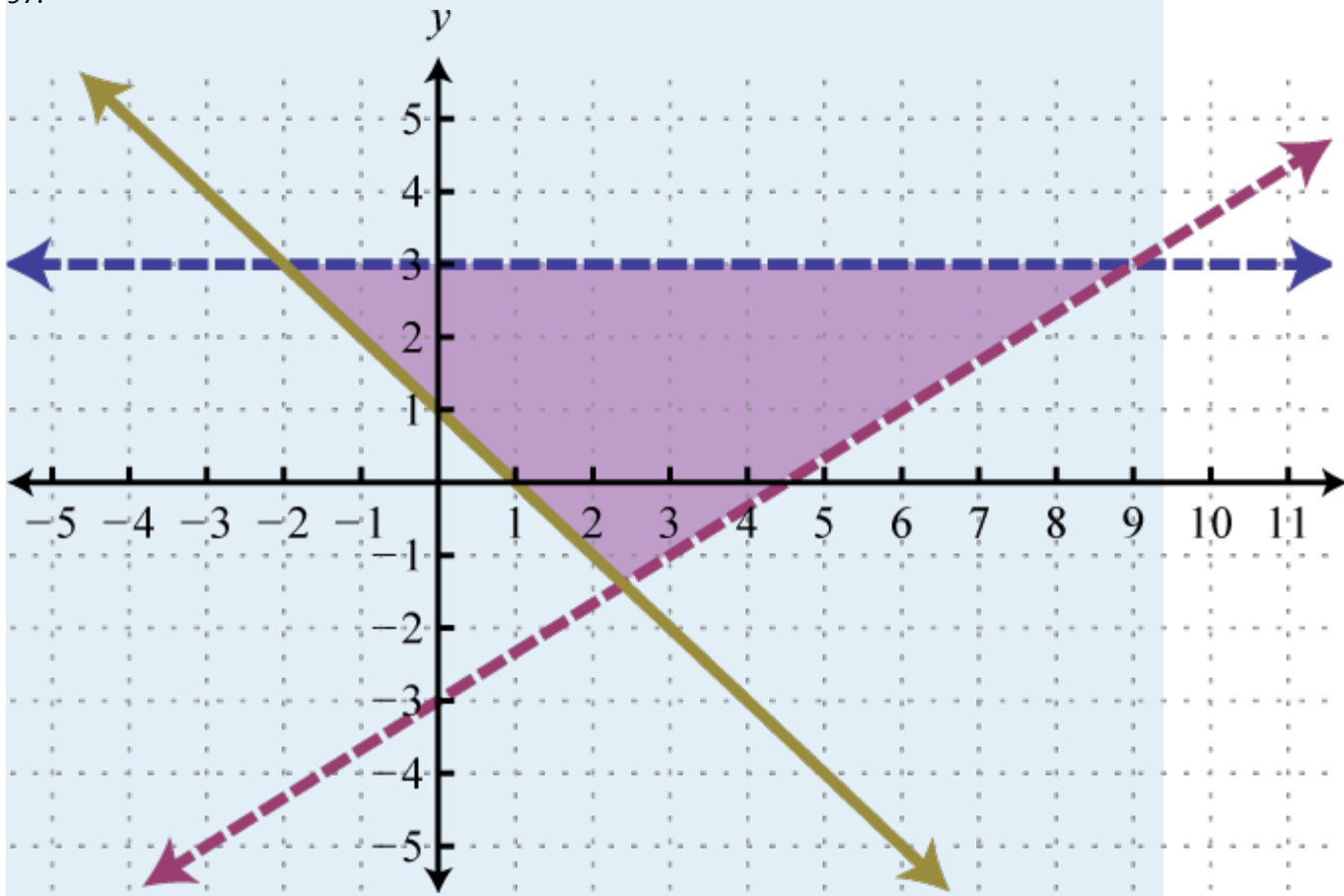
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97.

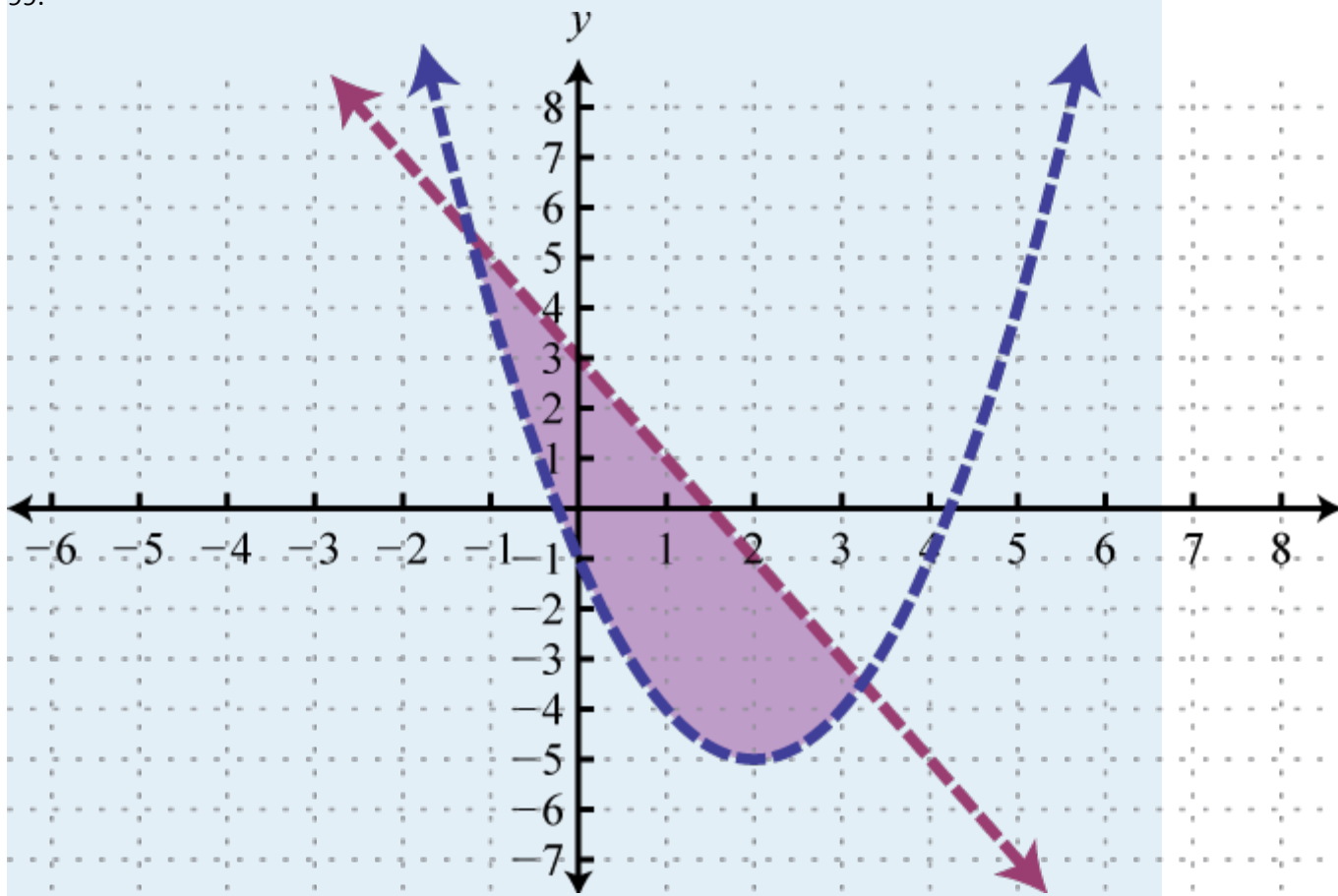


98.

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99.

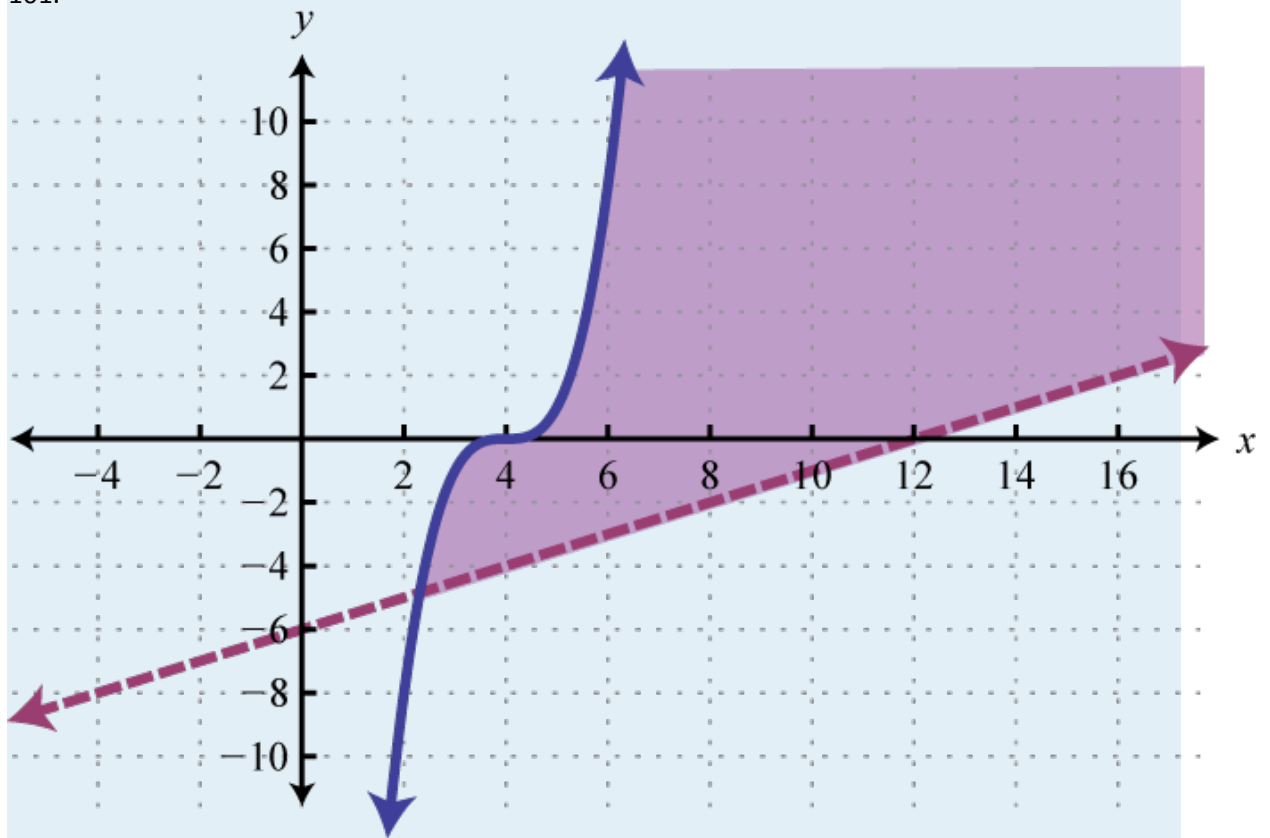


100.

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101.



102.

## SAMPLE EXAM

1. Determine whether or not is a solution to .
2. Determine whether or not  $(-3, 2, -5)$  is a solution to .

Solve by graphing.

- 1.
- 2.
- 3.

Solve by substitution.

- 1.

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2.

3.

Solve.

1.

2.

3.

4.

5.

Solve using any method.

1.

2.

3.

4.

Graph the solution set.

1.

2.

Use algebra to solve the following.

1. The length of a rectangle is 1 inch less than twice that of its width. If the perimeter measures 49 inches, then find the dimensions of the rectangle.
2. Joe's \$4,000 savings is in two accounts. One account earns 3.1% annual interest and the other earns 4.9% annual interest. His total interest for the year is \$174.40. How much does he have in each account?
3. One solution contains 40% alcohol and another contains 72% alcohol. How much of each should be mixed together to obtain 16 ounces of a 62% alcohol solution?

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4. Jerry took two buses on the 193-mile trip to visit his grandmother. The first bus averaged 46 miles per hour and the second bus was able to average 52 miles per hour. If the total trip took 4 hours, then how long was spent in each bus?

5. A total of \$8,500 was invested in three interest earning accounts. The interest rates were 2%, 3%, and 6%. If the total simple interest for one year was \$380 and the amount invested at 6% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account?

6. A mechanic wishes to mix 6 gallons of a 22% antifreeze solution. In stock he has a 60% and an 80% antifreeze concentrate. Water is to be added in the amount that is equal to twice the amount of both concentrates combined. How much water is needed?

## ANSWERS

1. Yes

2.

3.  $(-4, 1)$

4.

5.

6.

7.  $(-8, -3)$

8.

9.  $(4, -3)$

10.

11.  $\emptyset$

12.

13.

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14.

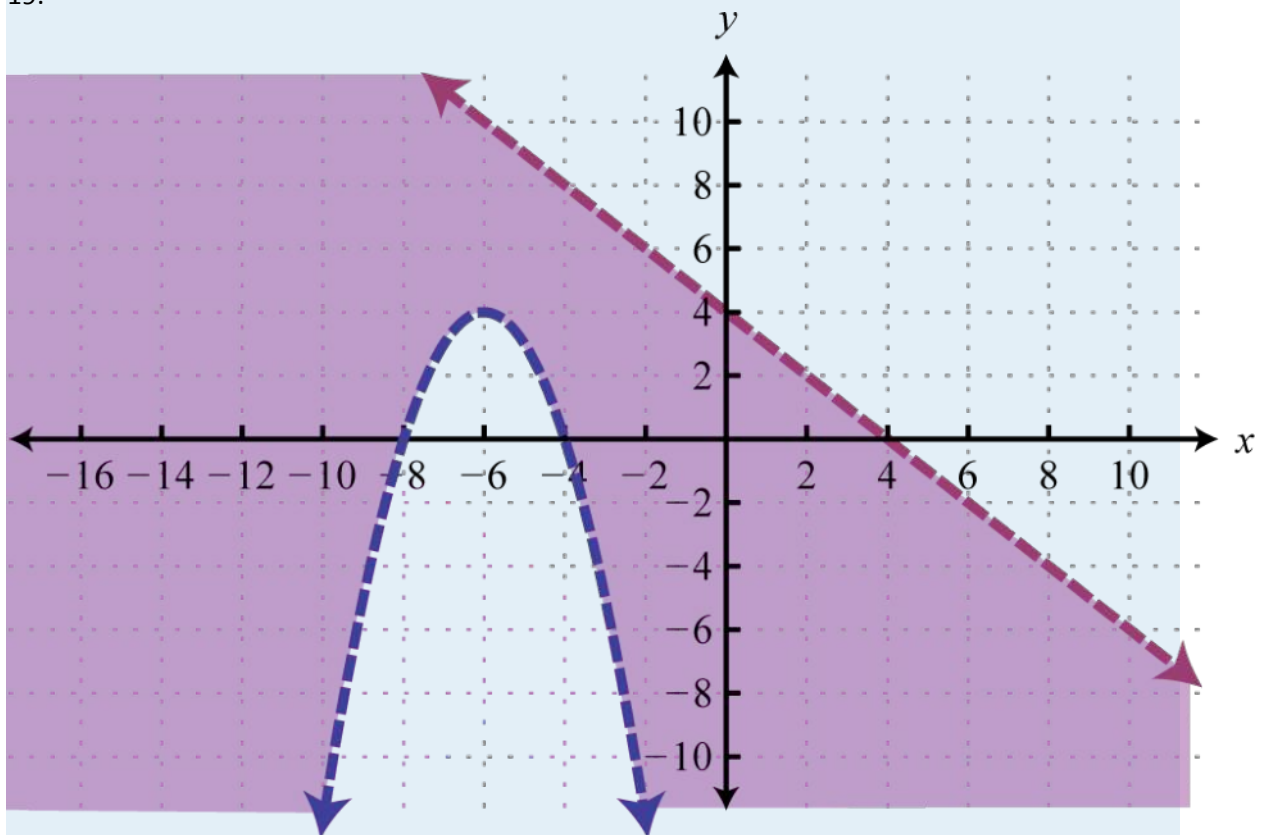
15.  $\emptyset$

16.

17.  $(2, 2, 2)$

18.

19.



20.

21. Joe has \$1,200 in the account earning 3.1% interest and \$2,800 in the account earning 4.9% interest.

22.

23. Jerry spent 2.5 hours in the first bus and 1.5 hours in the second.

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24.

25. 4 gallons of water is needed.

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# Chapter 4

## Polynomial and Rational Functions

### 4.1 Algebra of Functions

#### LEARNING OBJECTIVES

1. Identify and evaluate polynomial functions.
2. Add and subtract functions.
3. Multiply and divide functions.
4. Add functions graphically.

#### Polynomial Functions

Any polynomial with one variable is a function and can be written in the form

Here  $an$  represents any real number and  $n$  represents any whole number. The degree of a polynomial with one variable is the largest exponent of all the terms. Typically, we arrange terms of polynomials in descending order based on their degree and classify them as follows:

In this textbook, we call any polynomial with degree higher than 3 an  $n$ th-degree polynomial. For example, if the degree is 4, we call it a fourth-degree polynomial; if the degree is 5, we call it a fifth-degree polynomial, and so on.

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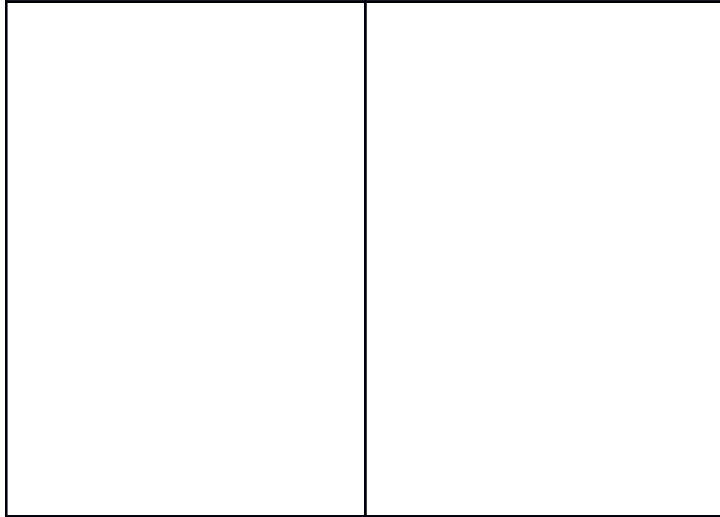


## Example 1

Given , find  $f(2)$  and  $f(4)$ .

Solution:

Replace each instance of  $x$  with the value given inside the parentheses.

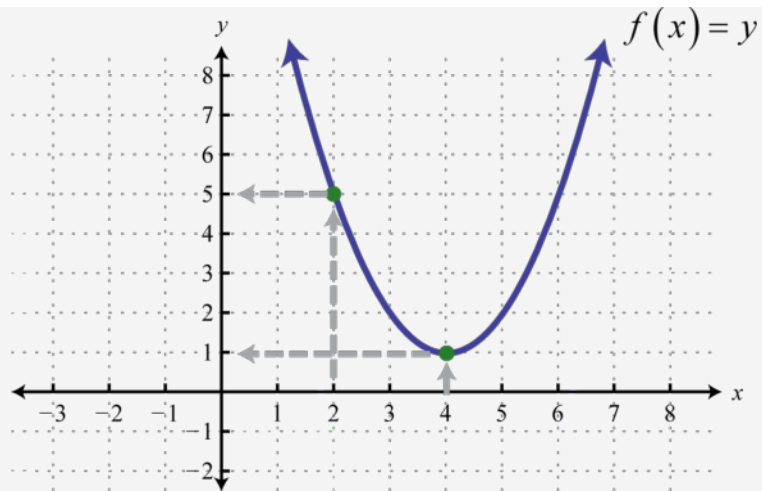


We can write  $f(2)=5$  and  $f(4)=1$ . Remember that  $f(x)=y$  and so we can interpret these results on the graph as follows:

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Answer:  $f(2)=5$ ;  $f(4)=1$

Often we will be asked to evaluate polynomials for algebraic expressions.

## Example 2

Given , find  $g(-2u)$  and  $g(x-2)$ .

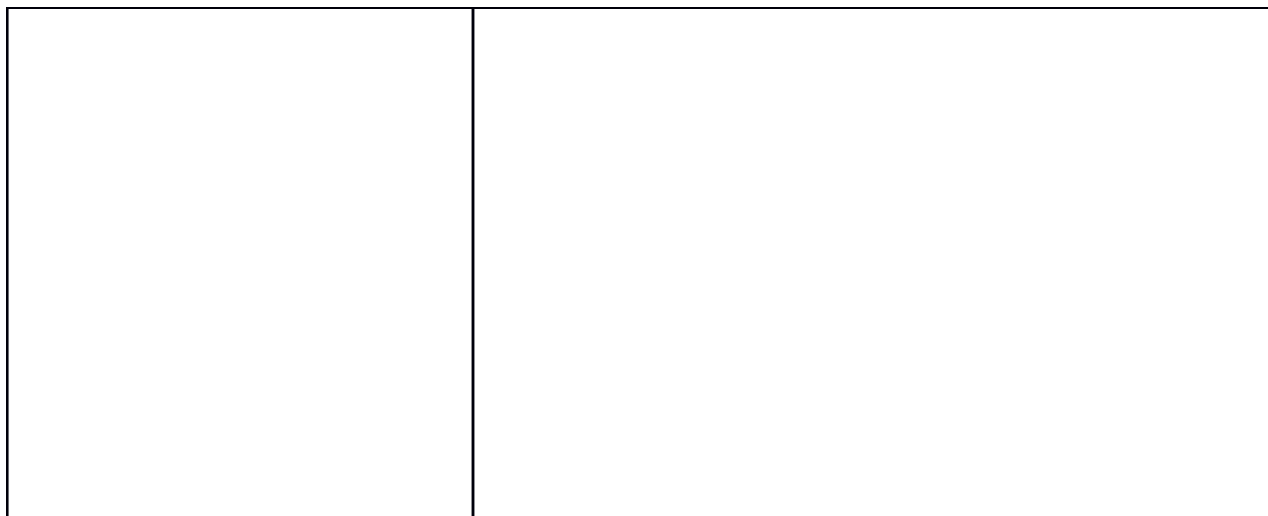
Solution:

Replace  $x$  with the expressions given inside the parentheses.

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Answer: and

The height of an object launched upward, ignoring the effects of air resistance, can be modeled with the following quadratic function:

With this formula, the height  $h(t)$  can be calculated at any given time  $t$  after the object is launched. The letter  $g$  represents acceleration due to gravity on the surface of the Earth, which is 32 feet per second squared (or, using metric units,  $g = 9.8$  meters per second squared). The variable  $v_0$ , pronounced “*v-naught*,” or sometimes “*v-zero*,” represents the initial velocity of the object, and  $s_0$  represents the initial height from which the object was launched.

### Example 3

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An object is launched from the ground at a speed of 64 feet per second. Write a function that models the height of the object and use it to calculate the objects height at 1 second and at 3.5 seconds.

Solution:

We know that the acceleration due to gravity is  $g=32$  feet per second squared and we are given the initial velocity  $v_0=64$  feet per second. Since the object is launched from the ground, the initial height is  $s_0=0$  feet. Create the mathematical model by substituting these coefficients into the following formula:

Use this model to calculate the height of the object at 1 second and 3.5 seconds.

Answer: ; At 1 second the object is at a height of 48 feet, and at 3.5 seconds it is at a height of 28 feet.

Try this! An object is dropped from a height of 6 meters. Write a function that models the height of the object and use it to calculate the object's height 1 second after it is dropped.

Answer: ; At 1 second the object is at a height of 1.1 meters.

## Adding and Subtracting Functions

The notation used to indicate **addition** and **subtraction** of functions follows:

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When using function notation, be careful to group the entire function and add or subtract accordingly.

## Example 4

Given  $f(x) = 2x + 3$  and  $g(x) = x - 1$ , find  $(f+g)(x)$  and  $(f-g)(x)$ .

Solution:

The notation  $f + g$  indicates that we should add the given expressions.

The notation  $f - g$  indicates that we should subtract the given expressions. When subtracting, the parentheses become very important. Recall that we can eliminate them after applying the distributive property.

Answer:  $(f+g)(x) = 3x + 2$  and  $(f-g)(x) = x + 4$

We may be asked to evaluate the sum or difference of two functions. We have the option to first find the sum or difference in general and then use the resulting function to

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evaluate for the given variable, or evaluate each first and then find the sum or difference.

## Example 5

Evaluate  $(f-g)(3)$  given and

Solution:

First, find  $(f-g)(x)$ .

Therefore,

Next, substitute 3 in for the variable  $x$ .

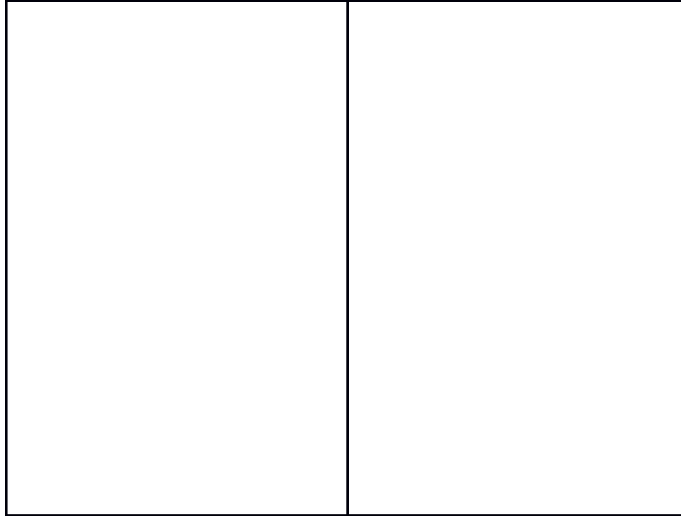
Hence  $(f-g)(3)=34$ .

Alternate Solution: Since  $(f-g)(3)=f(3)-g(3)$ , we can find  $f(3)$  and  $g(3)$  and then subtract the results.

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Therefore,

Notice that we obtain the same answer.

Answer:  $(f-g)(3)=34$

Note: If multiple values are to be evaluated, it is best to find the sum or difference in general first and then use it to evaluate.

Try this! Evaluate  $(f+g)(-1)$  given and

Answer: 2

## Multiplying and Dividing Functions

The notation used to indicate multiplication and division of functions follows:

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Multiplication of functions:	$(f \cdot g)(x) = f(x) \cdot g(x)$
Division of functions:	

## Example 6

Given  $f(x) = 2x^2 + 3x - 4$  and  $g(x) = x - 1$ , find  $(f \cdot g)(x)$  and  $(f/g)(x)$ .

Solution:

The notation  $f \cdot g$  indicates that we should multiply. Apply the distributive property and simplify.

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The notation  $f/g$  indicates that we should divide. For this quotient, assume  $x \neq 0$ .

Answer: and where  $x \neq 0$ .

## Example 7

Given  $f(x) = 6x - 5$  and  $g(x) = 2x + 4$ , evaluate  $(f \cdot g)(0)$  and  $(f \cdot g)(-1)$

Solution:

Begin by finding  $(f \cdot g)(x)$ .

Therefore, and we have,

--	--

Answer:  $(f \cdot g)(0) = 5$  and  $(f \cdot g)(-1) = -44$

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Try this! Evaluate  $(f \cdot g)(-1)$  given  $f(x) = x^3 + x - 8$  and

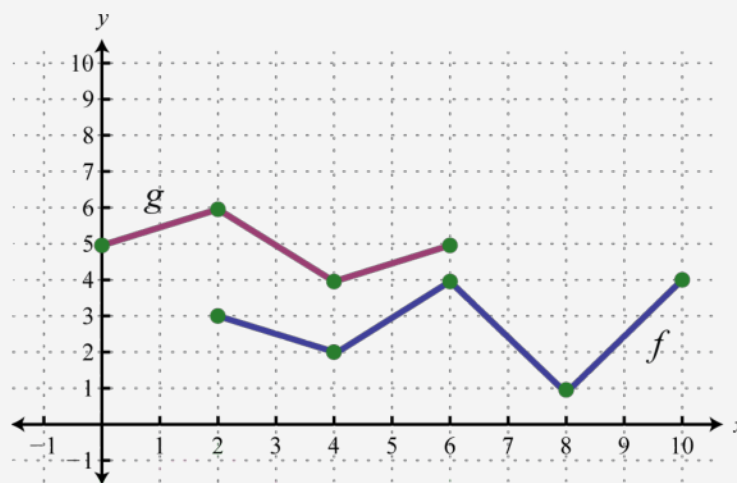
Answer:  $-120$

## Adding Functions Graphically

Here we explore the geometry of adding functions. One way to do this is to use the fact that  $(f + g)(x) = f(x) + g(x)$ . Add the functions together using  $x$ -values for which both  $f$  and  $g$  are defined.

### Example 8

Use the graphs of  $f$  and  $g$  to graph  $f + g$ . Also, give the domain of  $f + g$ .



Solution:

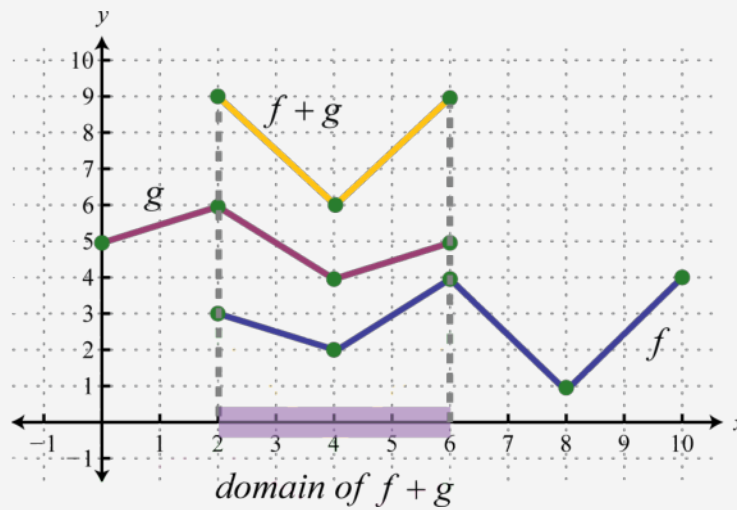
In this case, both functions are defined for  $x$ -values between 2 and 6. We will use 2, 4, and 6 as representative values in the domain of  $f + g$  to sketch its graph.

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Sketch the graph of  $f+g$  using the three ordered pair solutions (2,9), (4,6), and (6,9).



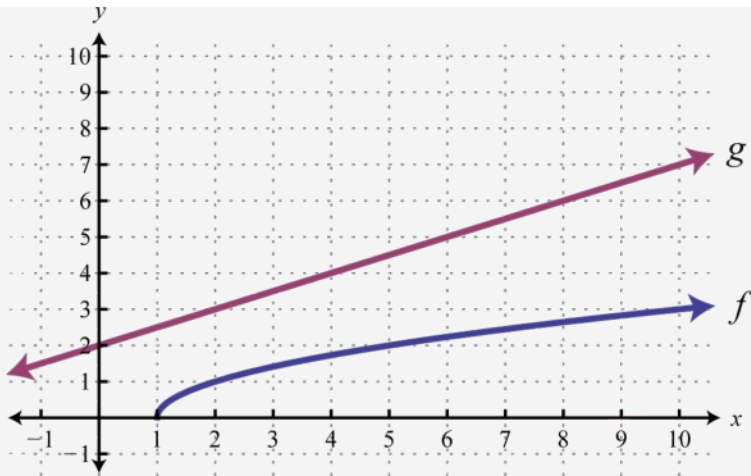
Answer:  $f+g$  graphed above has domain  $[2, 6]$ .

## Example 9

Use the graphs of  $f$  and  $g$  to graph  $f+g$ . Also, give the domain of  $f+g$ .

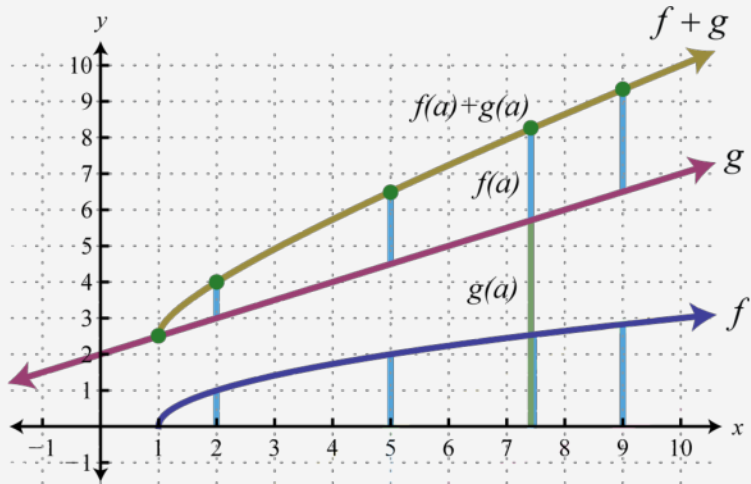






Solution:

Another way to add nonnegative functions graphically is to copy the line segment formed from the  $x$ -axis to one of the functions onto the other as illustrated below.



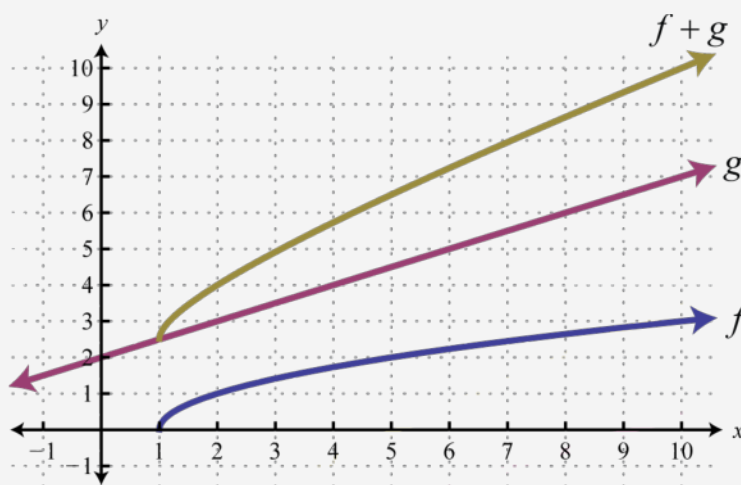
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The line segment from the  $x$ -axis to the function  $f$  represents  $f(a)$ . Copy this line segment onto the other function over the same point; the endpoint represents  $f(a) + g(a)$ . Doing this for a number of points allows us to obtain a quick sketch of the combined graph. In this example, the domain of  $f + g$  is limited to the  $x$ -values for which  $f$  is defined.

Answer: Domain:  $[1, \infty)$



In general, the domain of  $f + g$  is the intersection of the domain of  $f$  with the domain of  $g$ . In fact, this is the case for all of the arithmetic operations with an extra consideration for division. When dividing functions, we take extra care to remove any values that make the denominator zero. This will be discussed in more detail as we progress in algebra.

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## KEY TAKEAWAYS

Any polynomial with one variable is a function and can be written in the form  $P(x)$ . The degree of the polynomial is the largest exponent of all the terms.

Use function notation to streamline the evaluating process. Substitute the value or expression inside the parentheses for each instance of the variable.

The notation  $(f+g)(x)$  indicates that we should add  $f(x)+g(x)$ .

The notation  $(f-g)(x)$  indicates that we should subtract  $f(x)-g(x)$ .

The notation  $(f \cdot g)(x)$  indicates that we should multiply  $f(x)g(x)$ .

The notation  $(f/g)(x)$  indicates that we should divide

The domain of the function that results from these arithmetic operations is the intersection of the domain of each function. The domain of a quotient is further restricted to values that do not evaluate to zero in the denominator.

## TOPIC EXERCISES

### PART A: POLYNOMIAL FUNCTIONS

Evaluate.

1. Given  $f(x) = 2x^2 - 3x + 1$ , find  $f(-3)$ ,  $f(0)$ , and  $f(5)$ .
2. Given  $f(x) = x^2 + 4x - 5$ , find  $f(-1)$ ,  $f(0)$ , and  $f(3)$ .
3. Given  $g(x) = x^2 - 2x + 3$ , find  $g(-2)$ ,  $g(0)$ , and  $g(3)$ .
4. Given  $g(x) = x^2 - 5x + 6$ , find  $g(-5)$ ,  $g(0)$ , and  $g(3)$ .
5. Given  $s(x) = x^2 - x + 2$ , find  $s(-1)$ ,  $s(0)$ , and  $s(2)$ .
6. Given  $p(x) = x^2 - 3x + 2$ , find  $p(-3)$ ,  $p(-1)$ , and  $p(2)$ .
7. Given  $f(x) = x^2 - 2x + 1$ , find  $f(-2)$ ,  $f(-1)$ , and  $f(0)$ .
8. Given  $f(x) = x^2 - 2x + 1$ , find  $f(-2)$ ,  $f(-1)$ , and  $f(0)$ .

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9. Given , find  $f(2t)$  and  $f(2t-1)$ .
10. Given , find  $f(-3t)$  and  $f(2-3t)$ .
11. Given , find  $g(-5a)$  and  $g(5-2x)$ .
12. Given , find  $g(-4u)$  and  $g(3x-1)$ .
13. Given , find  $f(2a)$  and  $f(x-2)$ .
14. Given , find  $f(-3x)$  and  $f(2x+1)$ .
15. Given , find  $g(x^2)$  and  $g(x-4)$ .
16. Given , find  $g(-2x^3)$  and  $g(3x-1)$ .

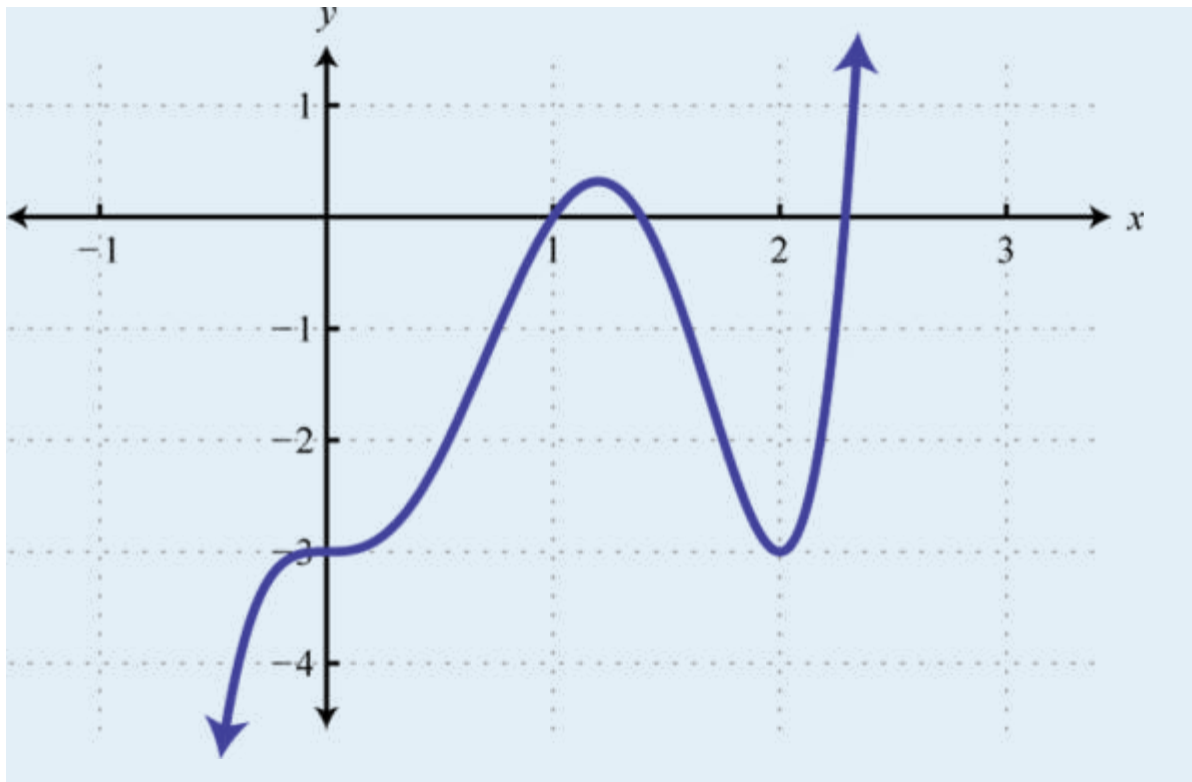
Given the function calculate  $f(x+h)$ .

1.  $f(x)=5x-3$
- 2.
- 3.
- 4.
- 5.

Given the graph of the polynomial function  $f$  find the function values.

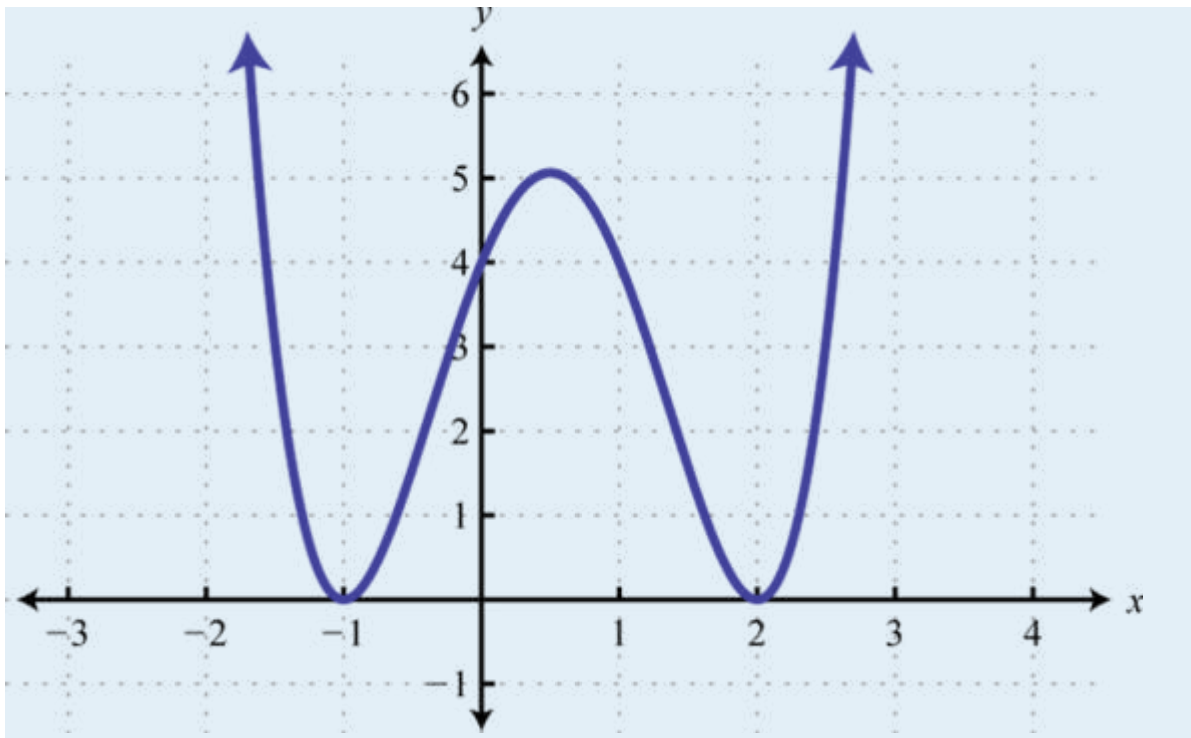
1. Find  $f(0)$ ,  $f(1)$ , and  $f(2)$ .





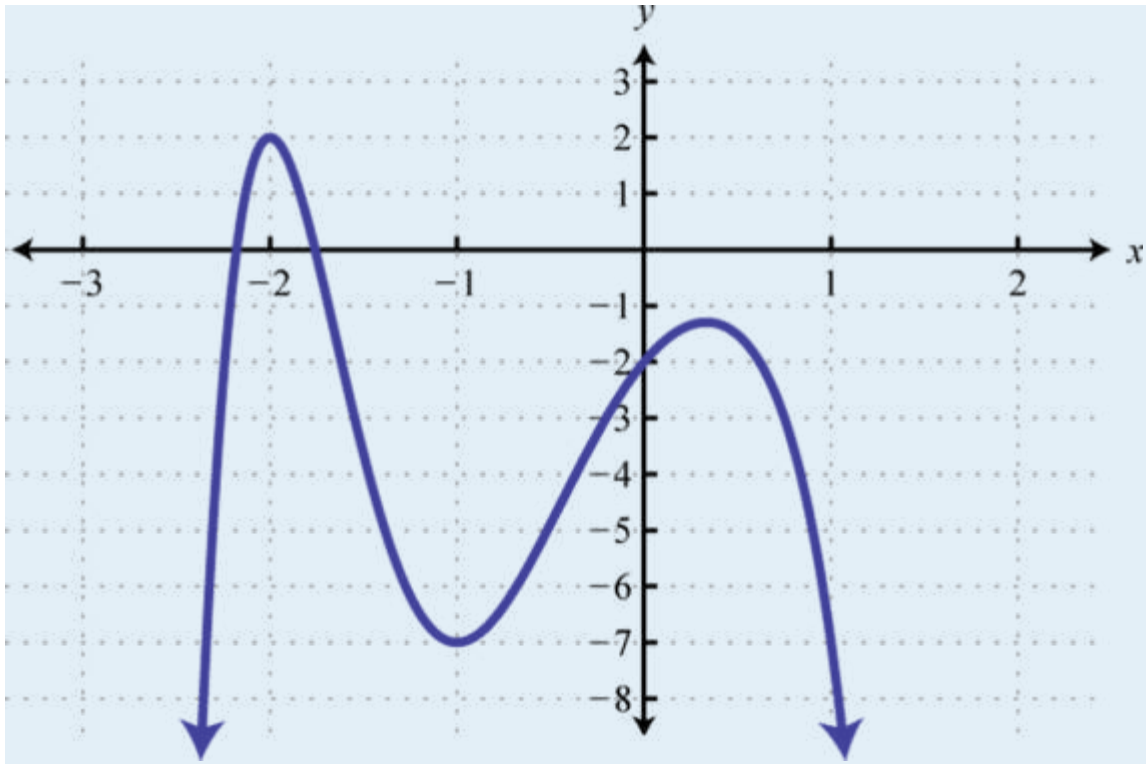
2. Find  $f(-1)$ ,  $f(0)$ , and  $f(1)$ .





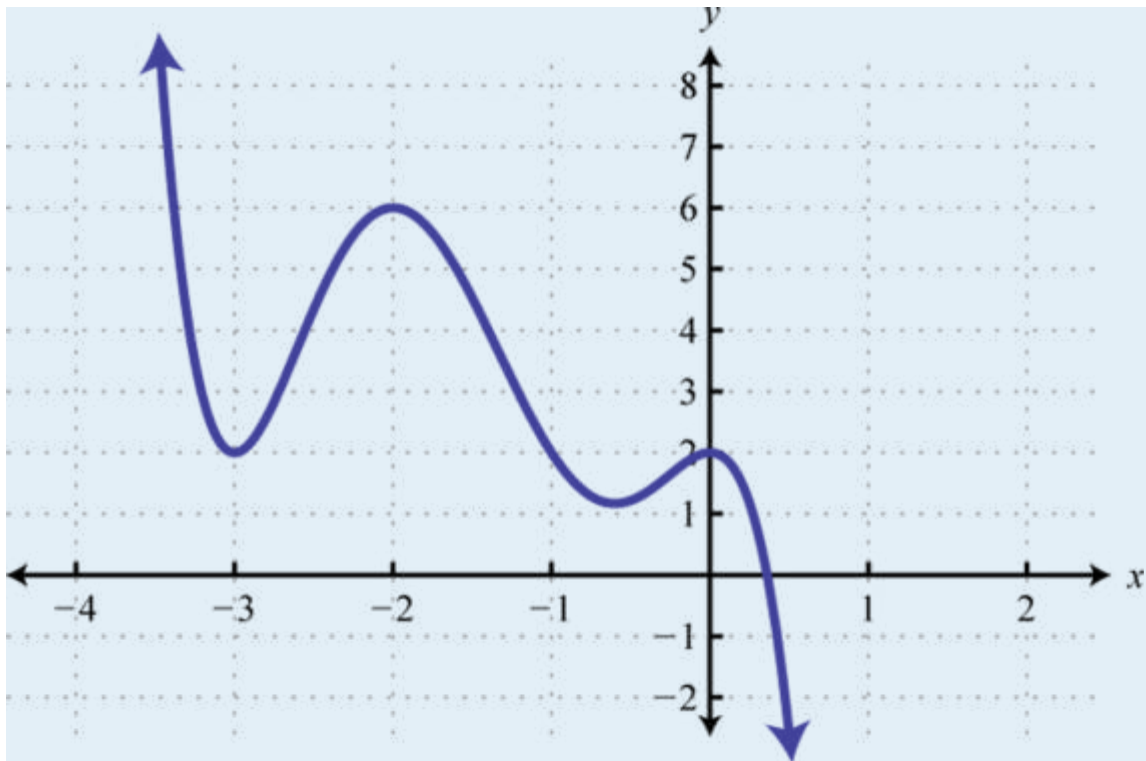
3. Find  $f(-2)$ ,  $f(-1)$ , and  $f(0)$ .





4. Find  $f(-3)$ ,  $f(-2)$ , and  $f(0)$ .





5. A projectile is launched upward from the ground at a speed of 48 feet per second. Write a function that models the height of the projectile and use it to calculate the height every  $\frac{1}{2}$  second after launch. Sketch a graph that shows the height of the projectile with respect to time.
6. An object is tossed upward from a 48-foot platform at a speed of 32 feet per second. Write a function that models the height of the object and use it to calculate the height every  $\frac{1}{2}$  second after the object is tossed. Sketch a graph that shows the height of the object with respect to time.
7. An object is dropped from a 128-foot bridge. Write a function that models the height of the object, and use it to calculate the height at 1 second and 2 seconds after it has been dropped.

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8. An object is dropped from a 500-foot building. Write a function that models the height of the object, and use it to calculate the distance the object falls in the 1<sup>st</sup> second, 2<sup>nd</sup> second, and the 3<sup>rd</sup> second.
9. A bullet is fired straight up into the air at 320 meters per second. Ignoring the effects of air friction, write a function that models the height of the bullet, and use it to calculate the bullet's height 1 minute after it was fired into the air.
10. A book is dropped from a height of 10 meters. Write a function that gives the height of the book, and use it to determine how far it will fall in  $1\frac{1}{4}$  seconds.

## PART B: ADDING AND SUBTRACTING FUNCTIONS

Given functions  $f$  and  $g$ , find  $(f+g)$  and  $(f-g)$ .

31.  $f(x)=5x-3, g(x)=4x-1$
32.  $f(x)=3x+2, g(x)=7x-5$
33.  $f(x)=2-3x, g(x)=1-x$
34.  $f(x)=8x-5, g(x)=-7x+4$
35. ;
36. ;
37. ;  $g(x)=6x+11$
38.  $f(x)=9x+5,$
39. ;
40. ;
41. ,
42. ,

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Given  $f$  and  $g$ , evaluate the following.

31.  $(f+g)(-2)$

32.  $(f+g)(3)$

33.  $(f-g)(-2)$

34.  $(f-g)(3)$

35.  $(g-f)(-2)$

36.  $(g-f)(3)$

37.  $(f+f)(1)$

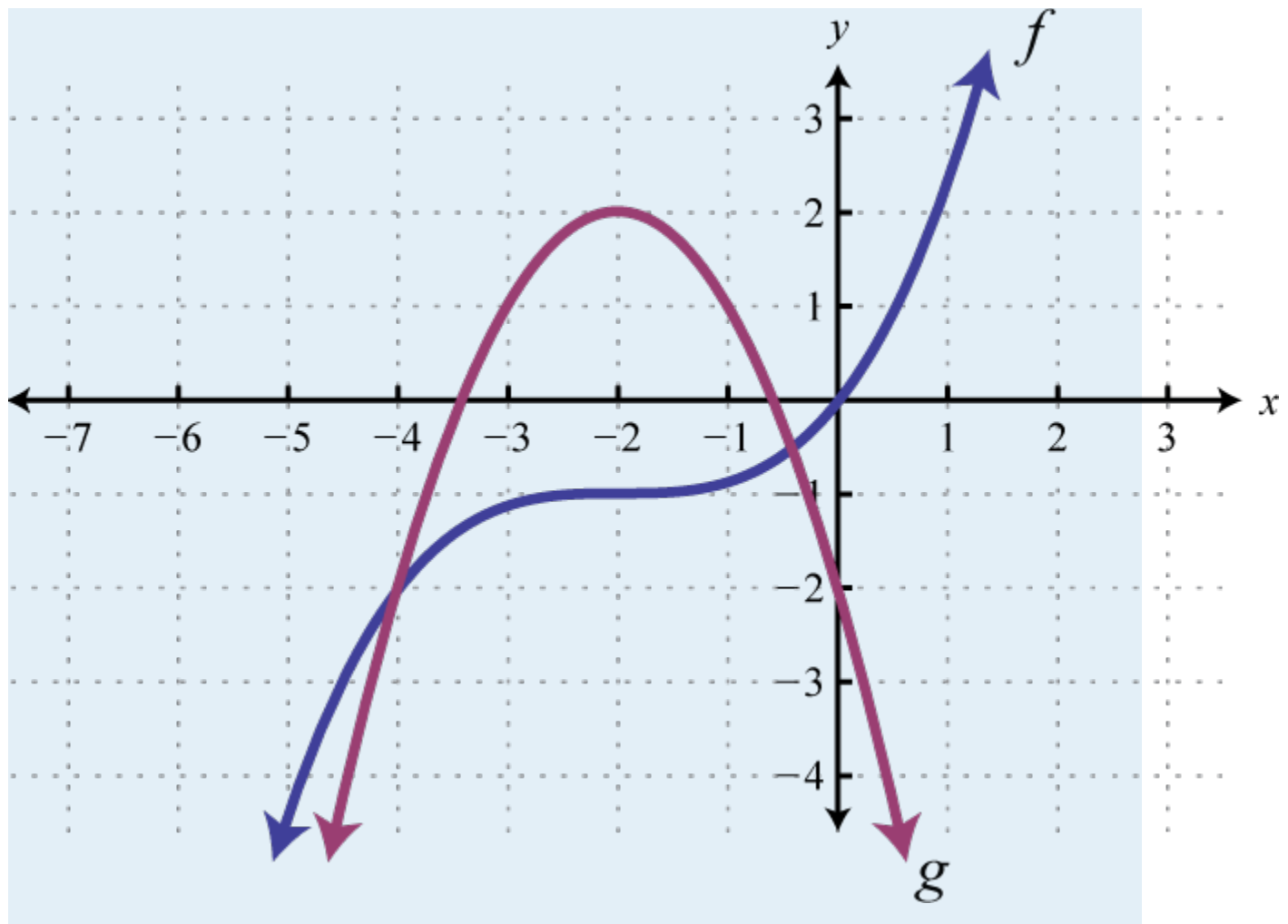
38.  $(g+g)(-1)$

Given the graphs of  $f$  and  $g$ , evaluate the following.

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31.  $(f+g)(-4)$
32.  $(f-g)(-4)$
33.  $(f+g)(-2)$
34.  $(f-g)(-2)$
35.  $(f+g)(0)$
36.  $(f-g)(0)$

### PART C: MULTIPLYING AND DIVIDING FUNCTIONS

Given  $f$  and  $g$ , find  $f \cdot g$ .

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57.  $f(x)=5x, g(x)=x-3$   
 58.  $f(x)=x-4, g(x)=6x$   
 59.  $f(x)=2x-3, g(x)=3x+4$   
 60.  $f(x)=5x-1, g(x)=2x+1$   
 61.  $f(x)=3x+4, g(x)=3x-4$   
 62.  $f(x)=x+5, g(x)=x-5$   
 63.  $f(x)=x-2,$   
 64.  $f(x)=2x-3,$   
 65. ,  
 66. ,  
 67. ,  
 68. ,

Given  $f$  and  $g$ , find  $f/g$ . (Assume all expressions in the denominator are nonzero.)

57. ,  $g(x)=4x$   
 58. ,  $g(x)=2x$   
 59. ,  
 60. ,  
 61. ,  $g(x)=x+2$   
 62. ,  $g(x)=x-3$   
 63. ,  $g(x)=2x-3$   
 64. ,  $g(x)=3x-1$   
 65. ,  $g(x)=3x+2$

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66.  $f(x) = 5x - 1$

Given  $f$  and  $g$  evaluate the following.

57.  $(f \cdot g)(-1)$

58.  $(f \cdot g)(1)$

59.  $(f/g)(-2)$

60.  $(f/g)(-3)$

61.  $(g \cdot f)(0)$

62.  $(g/f)(1)$

63.  $(g \cdot g)(-1)$

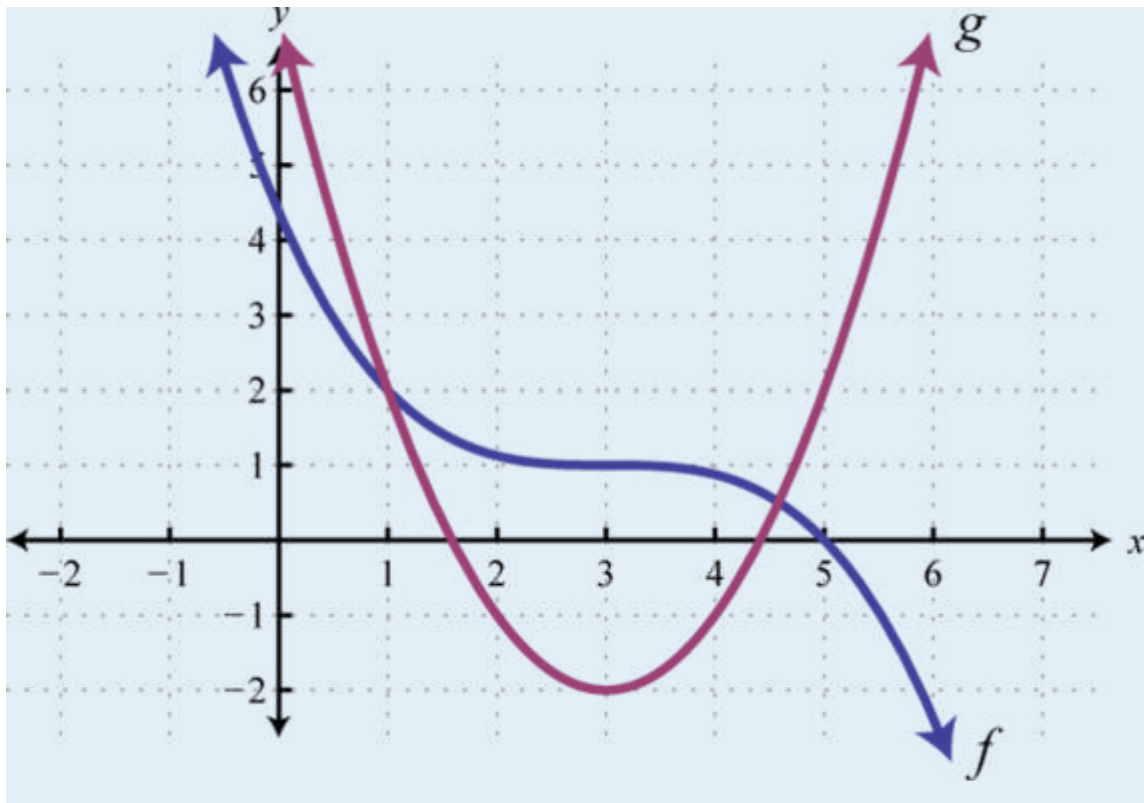
64.  $(f \cdot f)(-1)$

Given the graphs of  $f$  and  $g$  evaluate the following.

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- 57.  $(f \cdot g)(3)$
- 58.  $(f \cdot g)(5)$
- 59.  $(f/g)(5)$
- 60.  $(f/g)(3)$
- 61.  $(f \cdot g)(1)$
- 62.  $(f/g)(1)$

Given  $f(x) = 2x^2 - 3x + 1$ ,  $g(x) = x^2 - 4x + 3$ , and  $h(x) = -5x$ , find the following. (Assume all expressions in the denominator are nonzero.)

- 57.  $(f-g)(x)$
- 58.  $(g-f)(x)$
- 59.  $(g \cdot h)(x)$



60.  $(f/h)(x)$

61.  $(h+g)(x)$

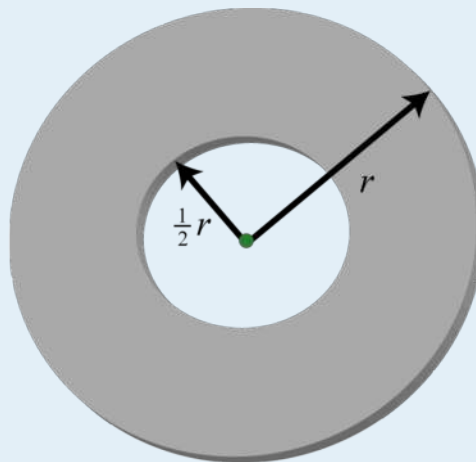
62.  $(h \cdot f)(x)$

63.  $(g/h)(2)$

64.  $(g-h)(-3)$

65. The revenue in dollars from selling MP3 players is given by the function , where  $n$  represents the number of units sold ( $0 \leq n < 833$ ). The cost in dollars of producing the MP3 players is given by the formula  $C(n) = 1200 + 42n$  where  $n$  represents the number of units produced. Write a function that models the profit of producing and selling  $n$  MP3 players. Use the function to determine the profit generated from producing and selling 225 MP3 players. Recall that profit equals revenues less costs.

66. The inner radius of a washer is 12 that of the outer radius.



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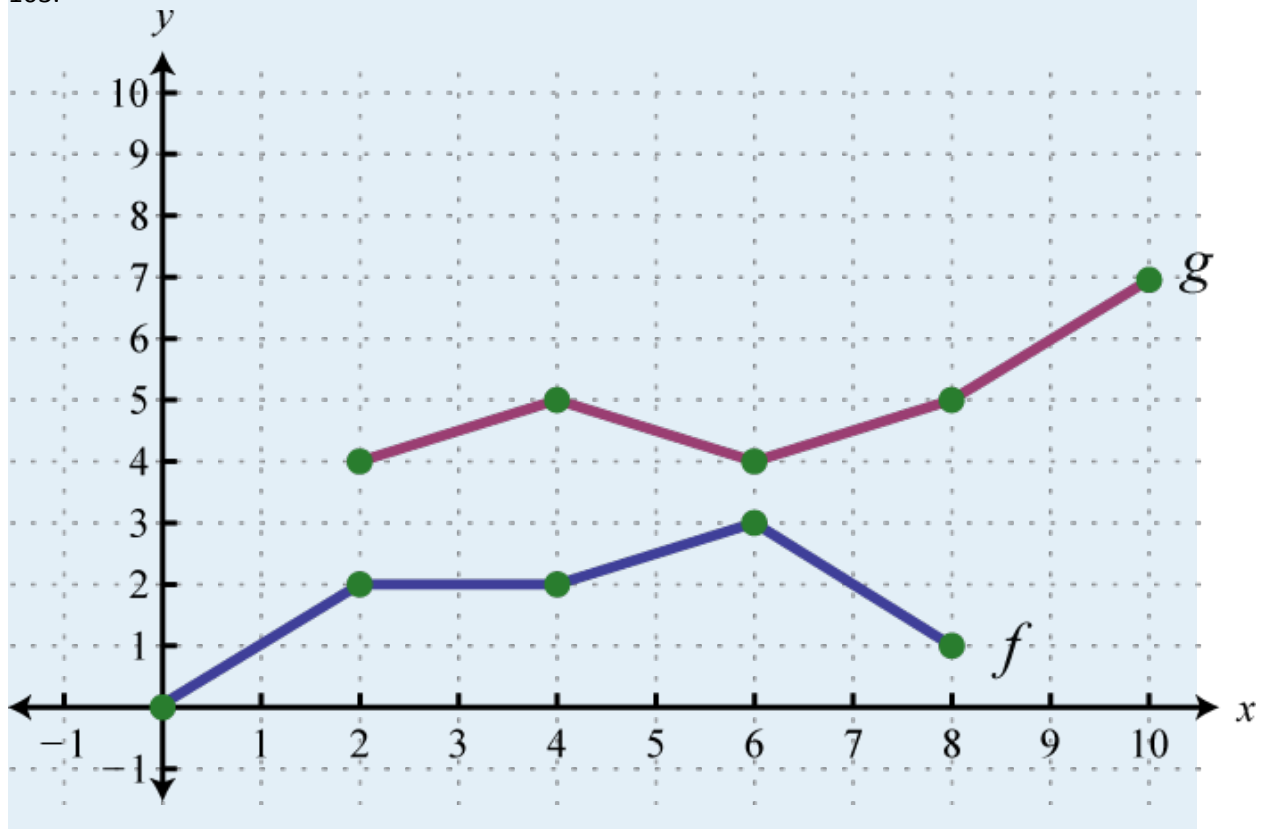
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## PART D: ADDING FUNCTIONS GEOMETRICALLY

Use the graphs of  $f$  and  $g$  to graph  $f+g$ . Also, give the domain of  $f+g$ .

103.



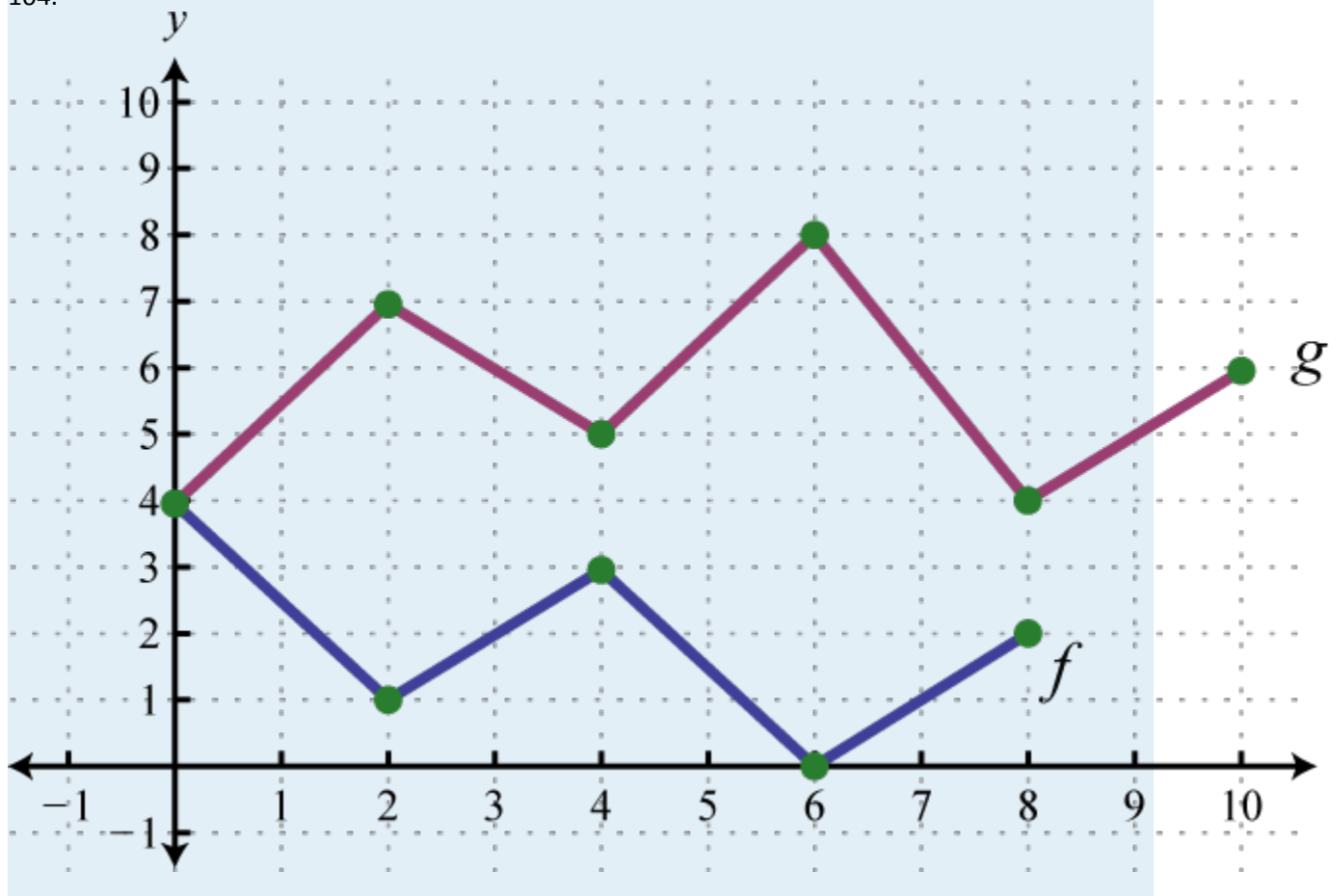
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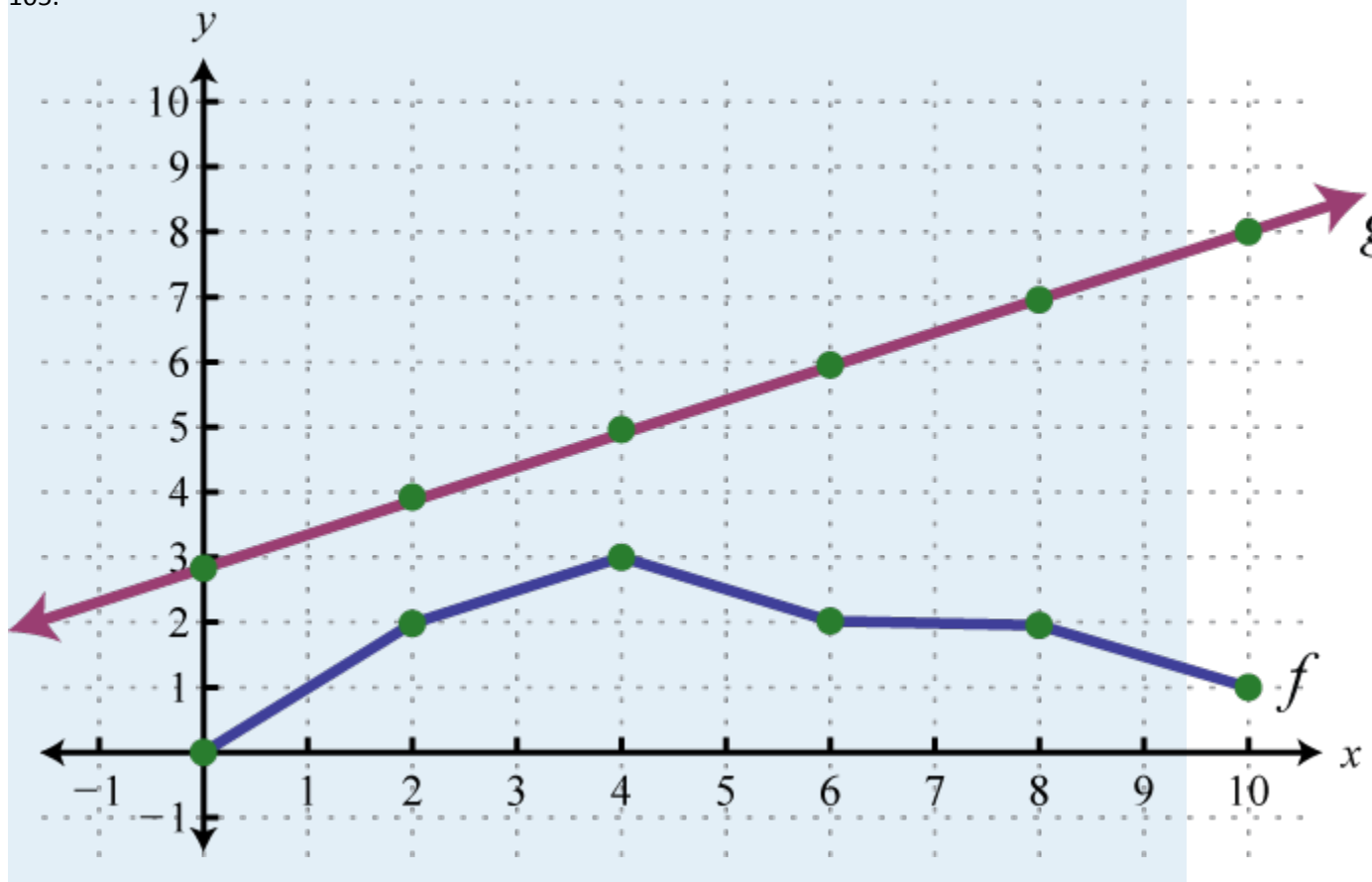


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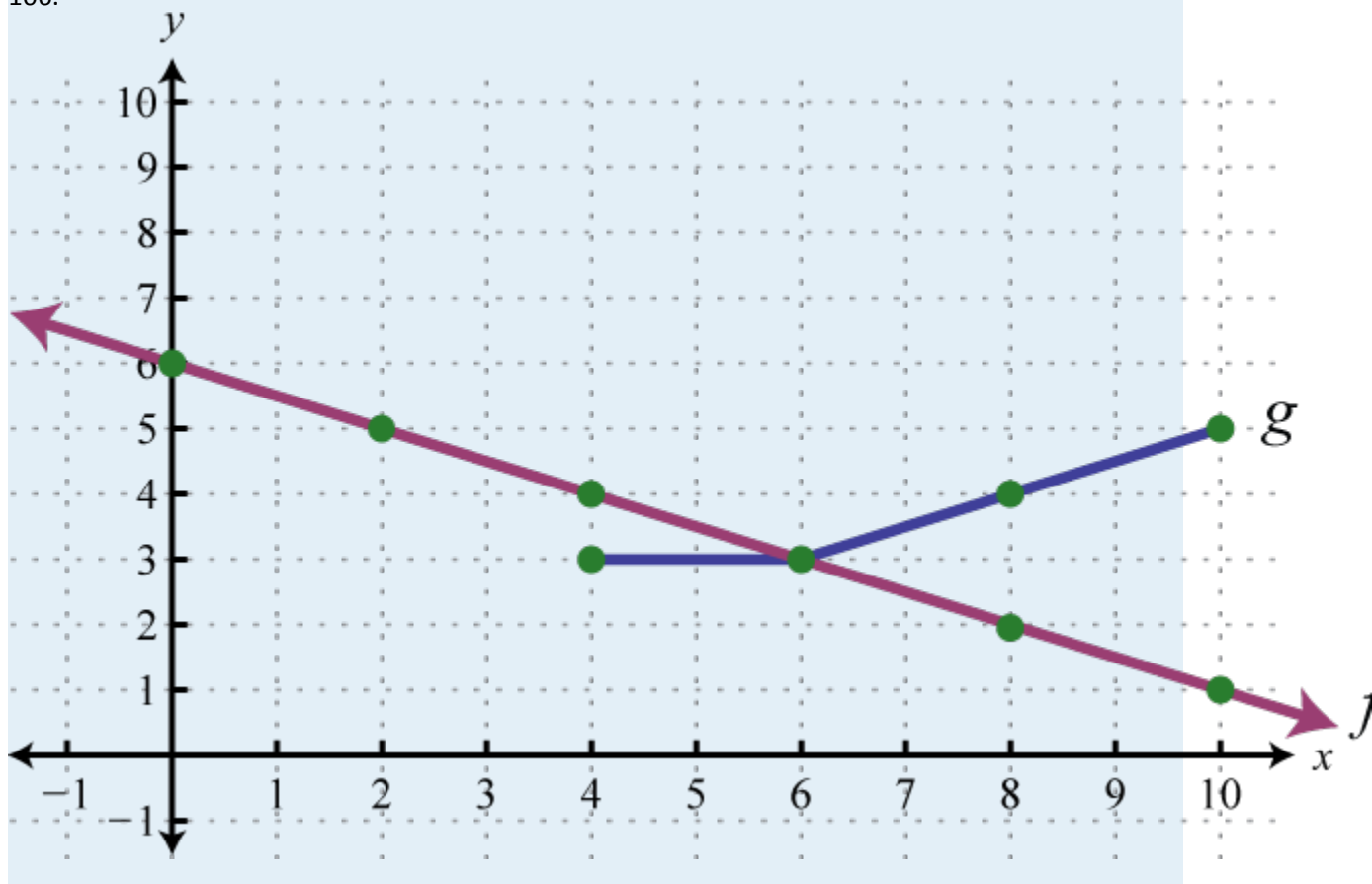


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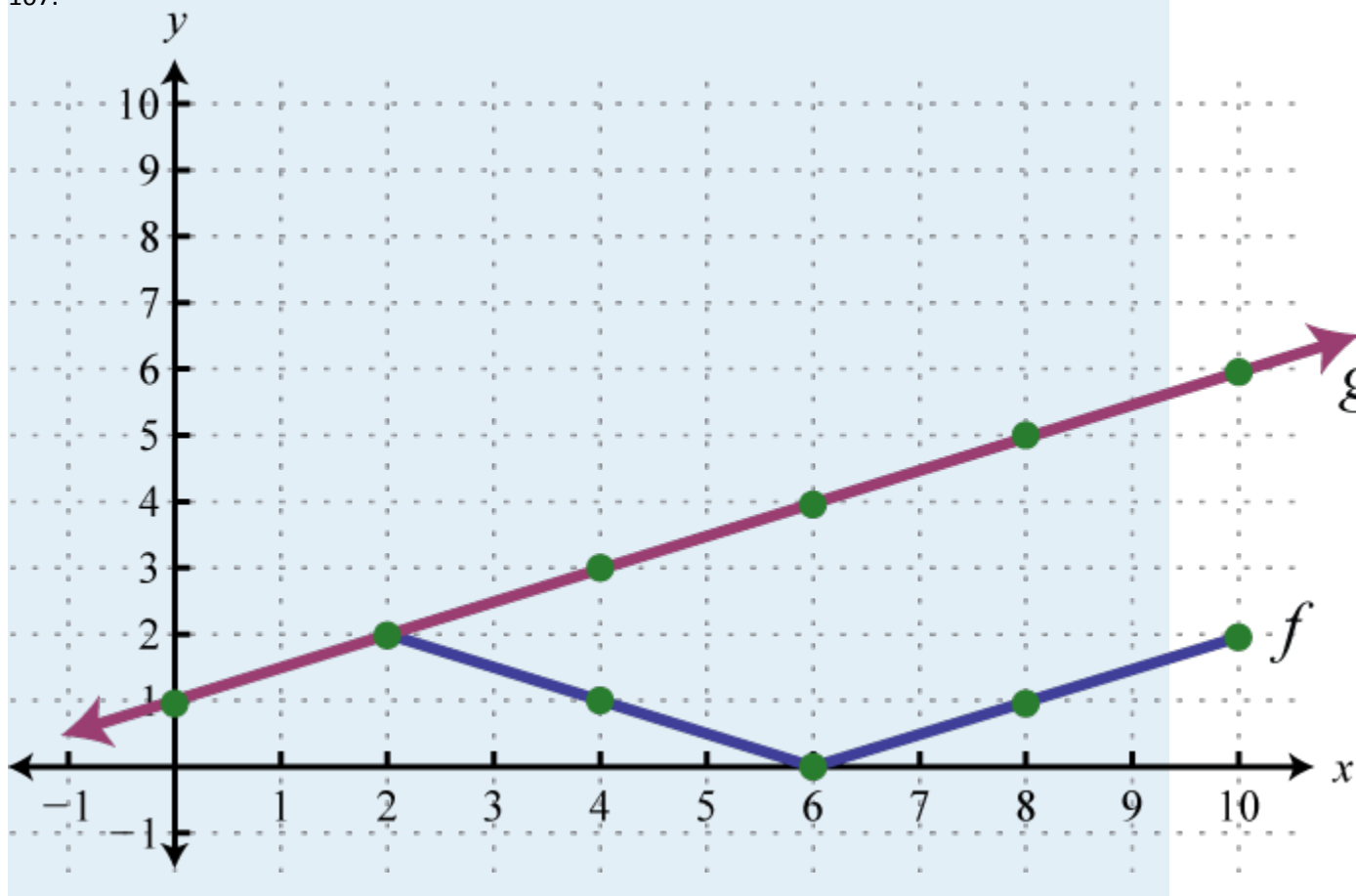


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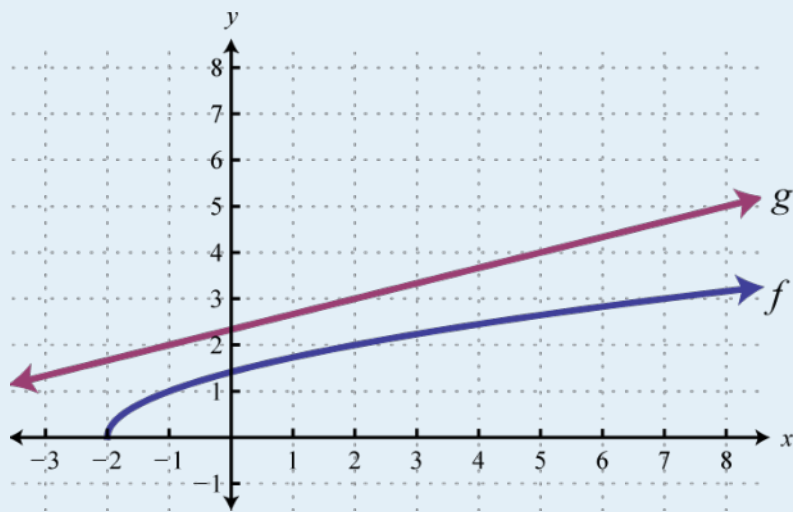
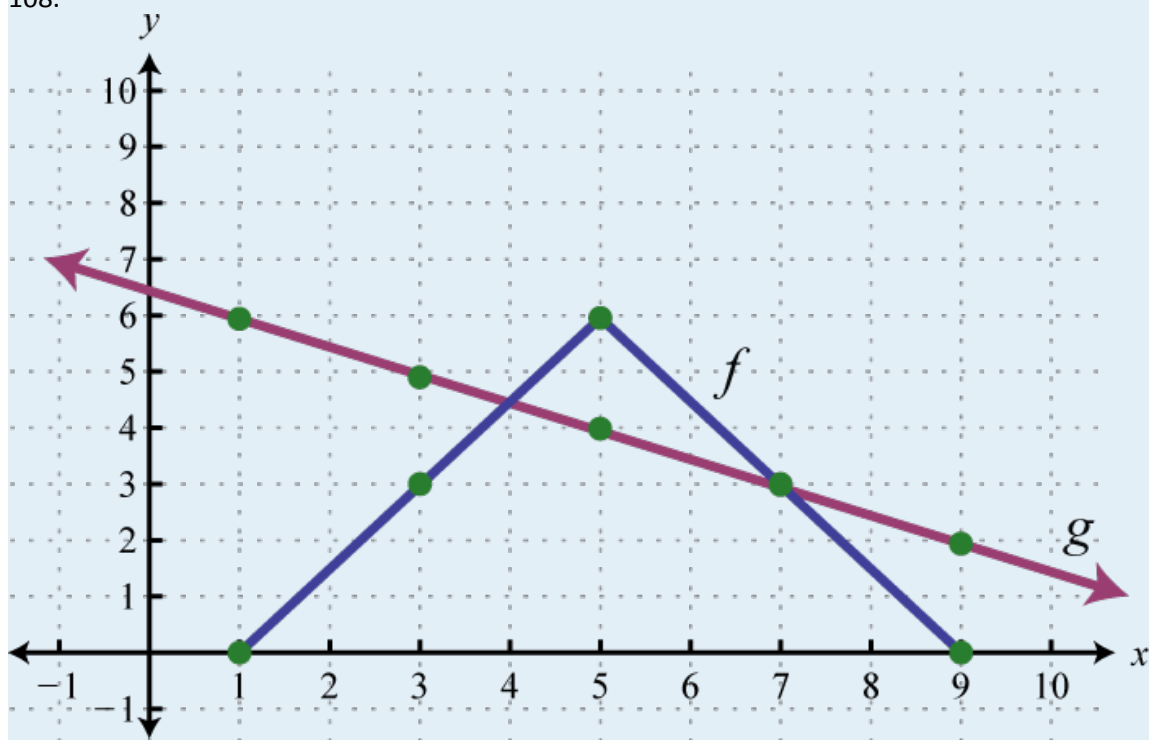


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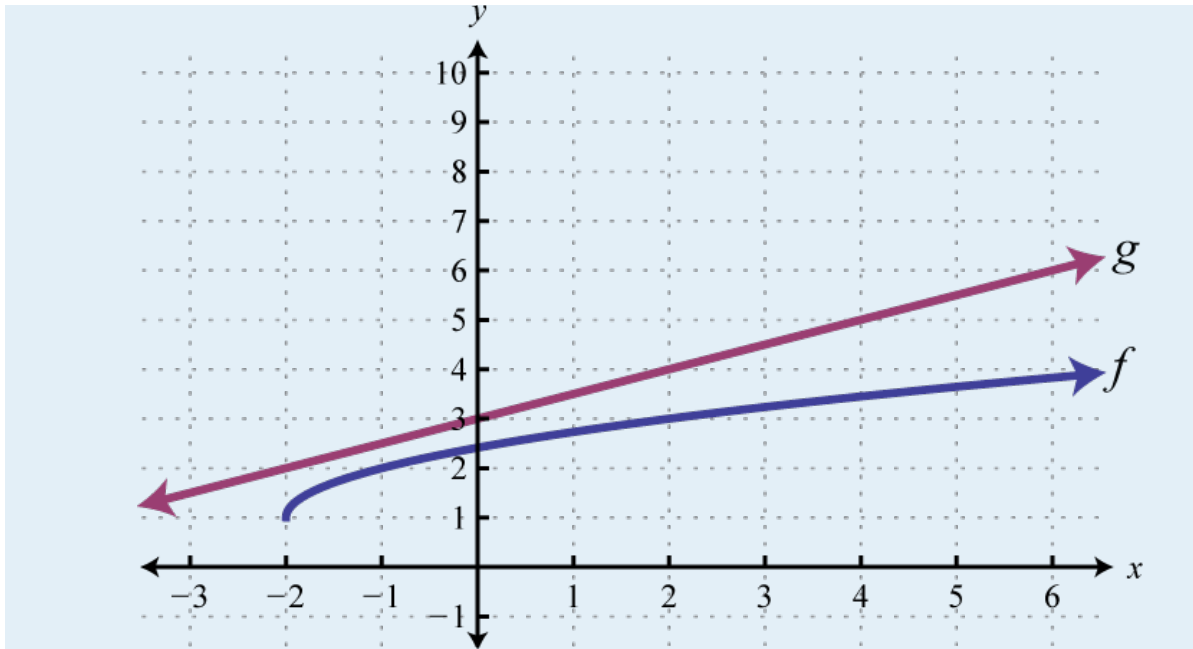
108.



109.

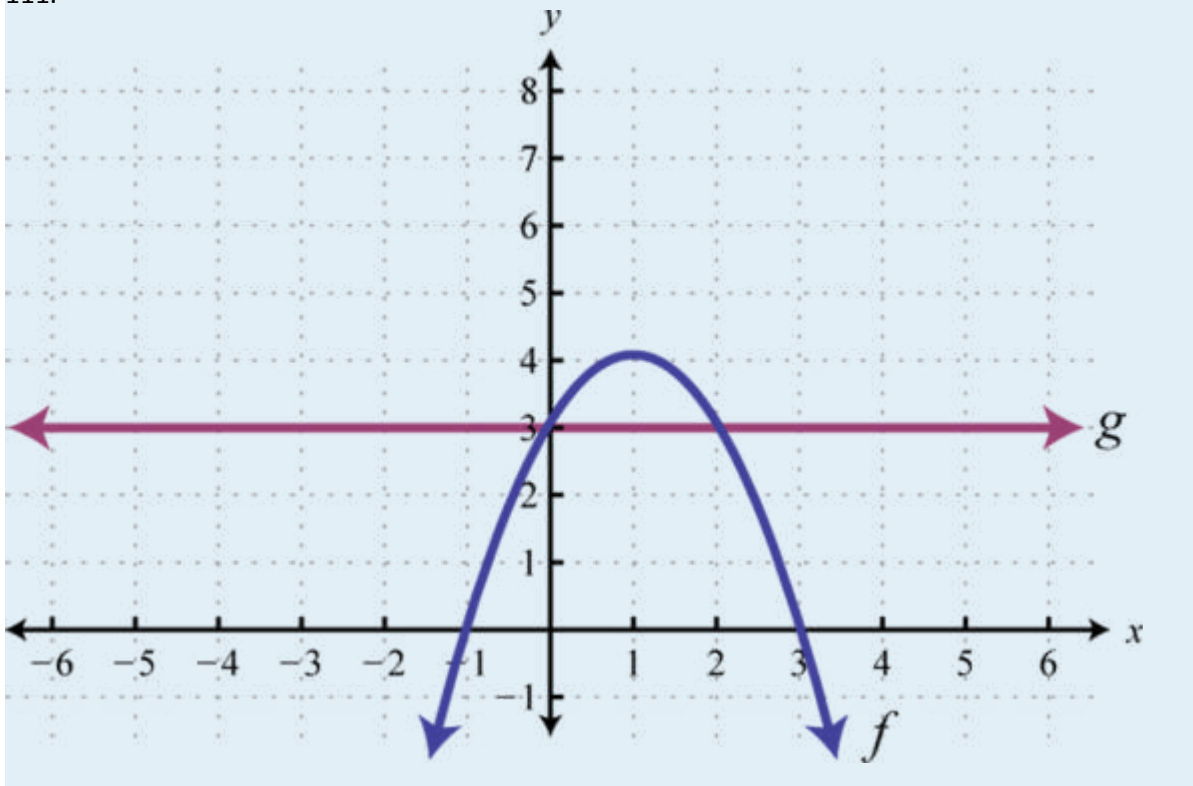
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110.

111.

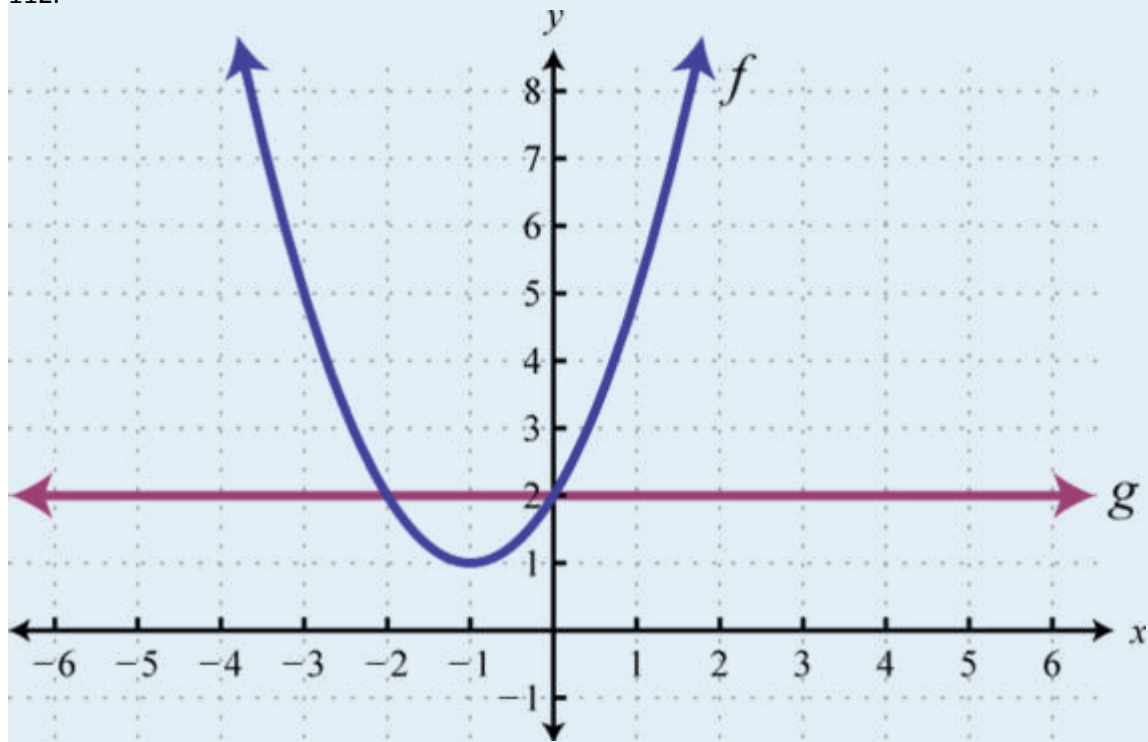


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112.



## PART E: DISCUSSION BOARD

113. Which arithmetic operations on functions are commutative? Explain.

114. Explore ways we can add functions graphically if they happen to be negative.

## ANSWERS

1.  $f(-3)=42; f(0)=3; f(5)=-22$

2.

3.  $g(-2)=-7; g(0)=7; g(3)=28$

4.

5.  $s(-1)=0; s(0)=-3; s(2)=75$

6.

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7.  $f(-2)=0; f(-1)=-63; f(0)=-64$

8.

9. ;

10.

11. ;

12.

13. ;

14.

15. ;

16.

17.  $f(x+h)=5x+5h-3$

18.

19.

20.

21.  $f(0)=-3; f(1)=0; f(2)=-3$

22.

23.  $f(-2)=2; f(-1)=-7; f(0)=-2$

24.

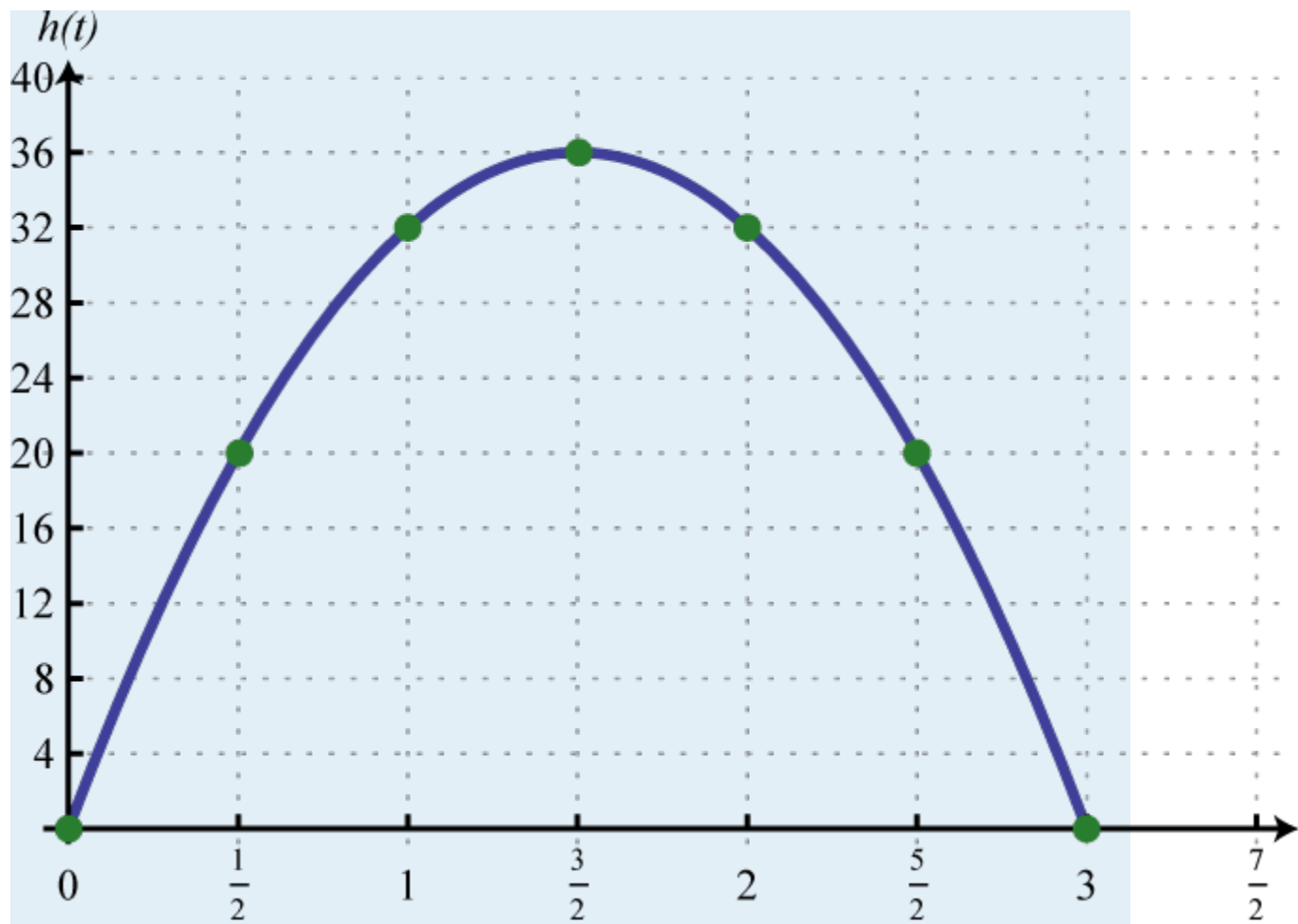
25. ;

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26.

27. ; At 1 second the object's height is 112 feet and at 2 seconds its height is 64 feet.

28.

29. ; 1,560 meters

30.

31.  $(f+g)(x)=9x-4$ ;  $(f-g)(x)=x-2$

32.

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33.  $(f+g)(x)=-4x+3$ ;  $(f-g)(x)=-2x+1$

34.

35. ;  $(f-g)(x)=-7x+9$

36.

37. ;

38.

39. ;

40.

41. ;

42.

43. 11

44.

45. -27

46.

47. 27

48.

49. -10

50.

51. -4

52.

53. 1

54.

55. -2

56.

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- 57.
- 58.
- 59.
- 60.
- 61.
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
- 69.
- 70.
- 71.
- 72.
- 73.
- 74.
- 75.
- 76.
- 77.
- 78.
- 79. 50
- 80.
- 81. 15



82.

83. 0

84.

85. 25

86.

87. -2

88.

89. 0

90.

91. 4

92.

93.

94.

95.

96.

97.

98.

99.

100.

101. ; \$9,881.25

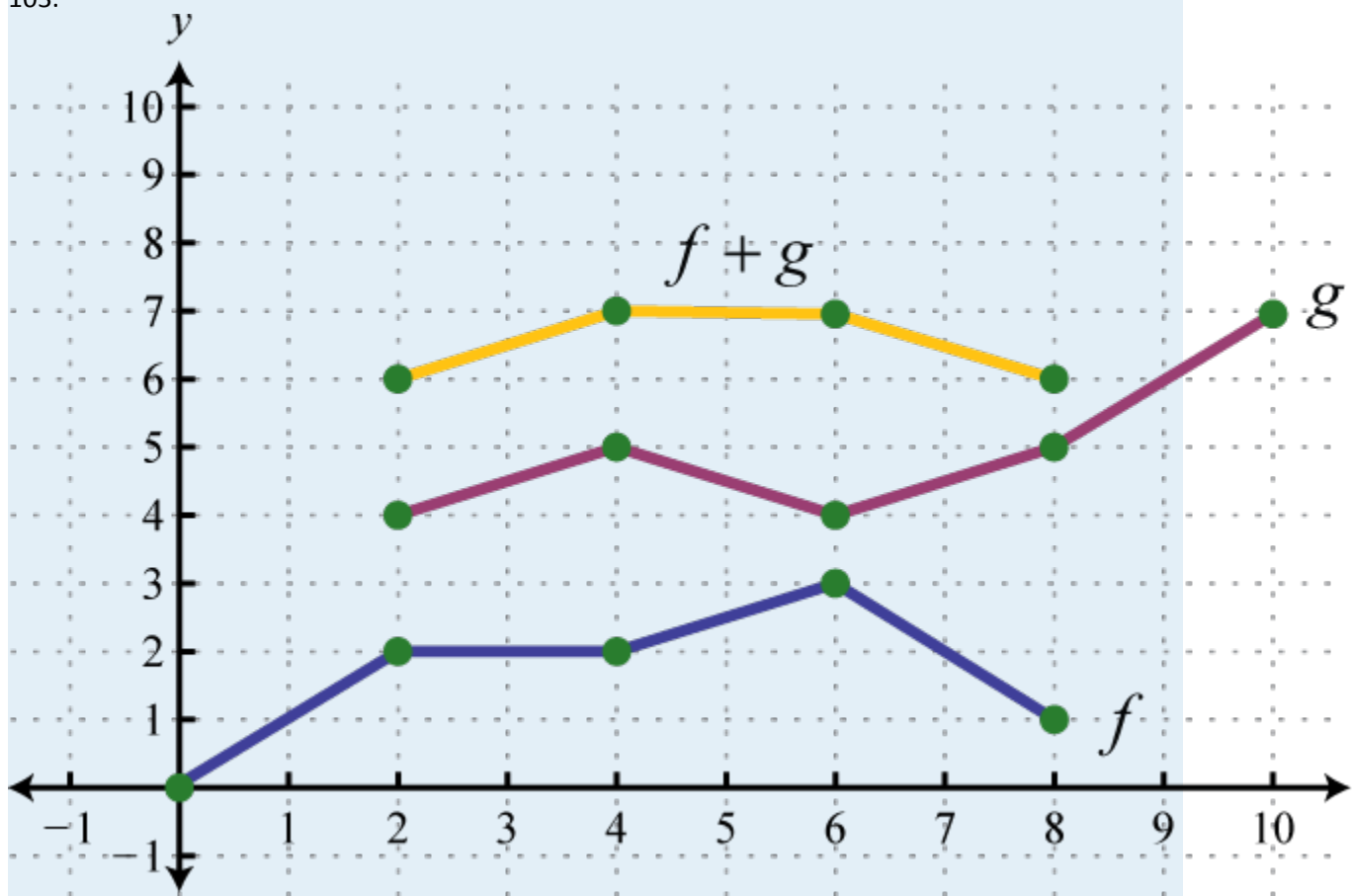
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103.

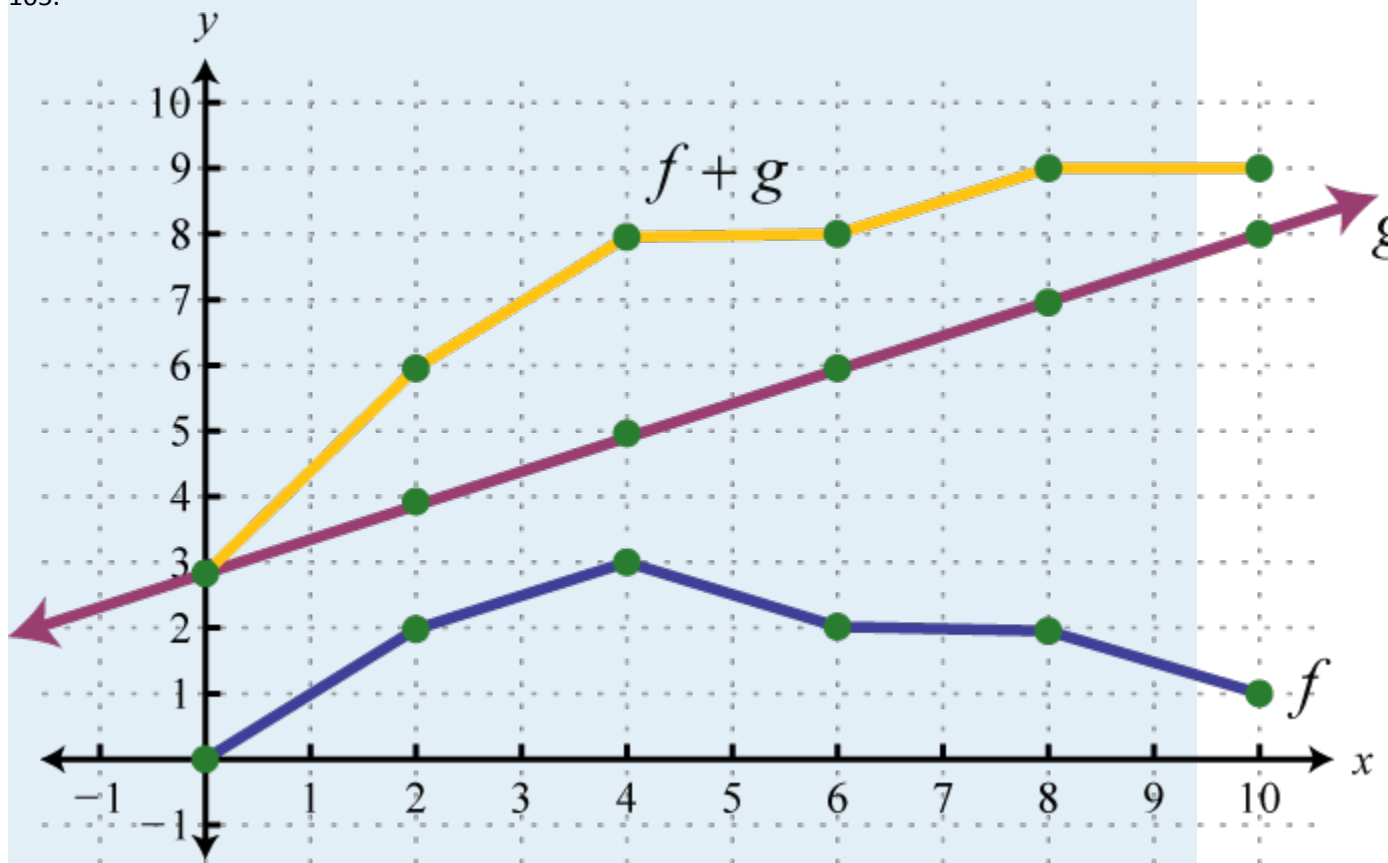


[2,8]

104.



105.



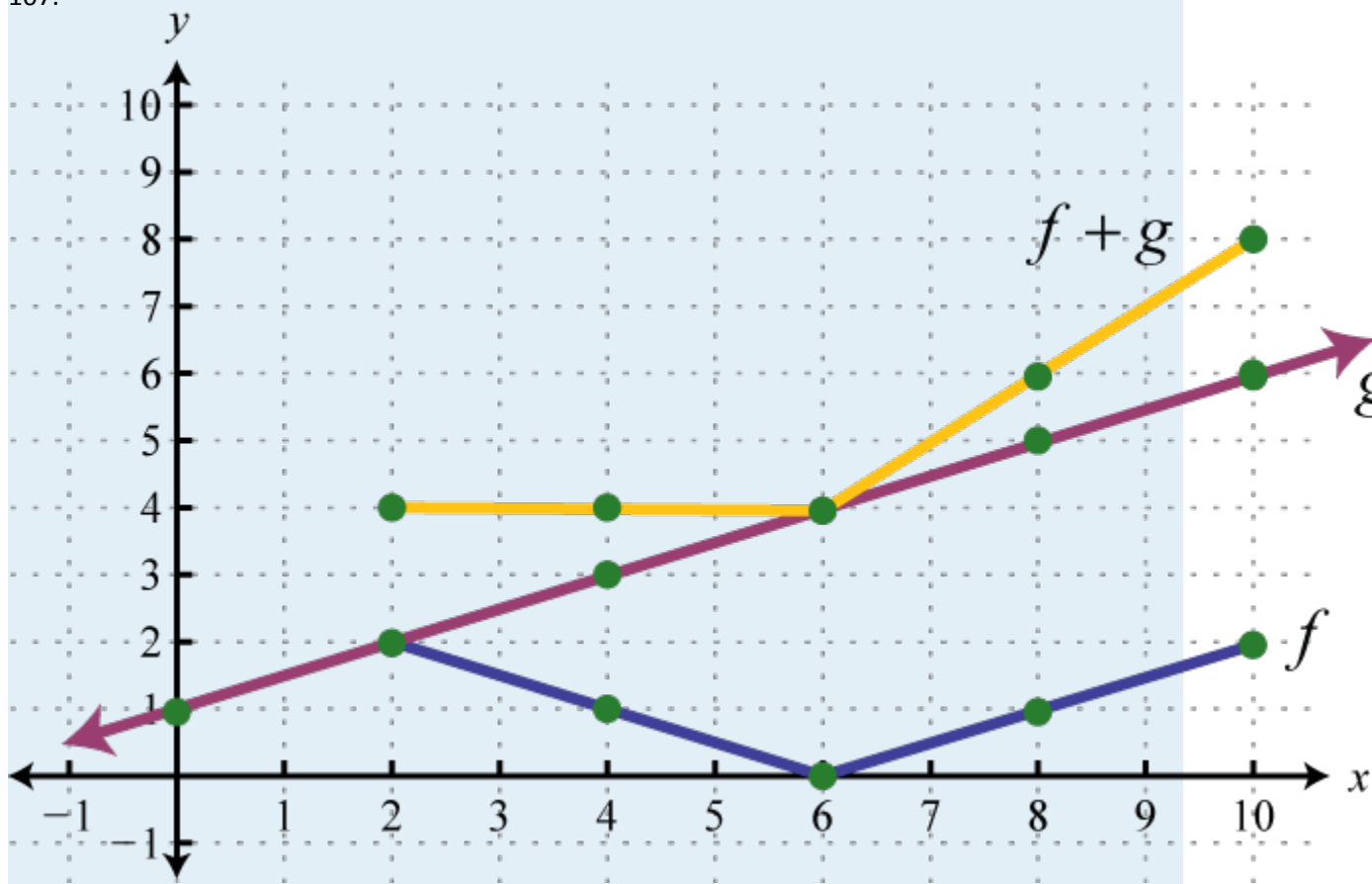
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107.



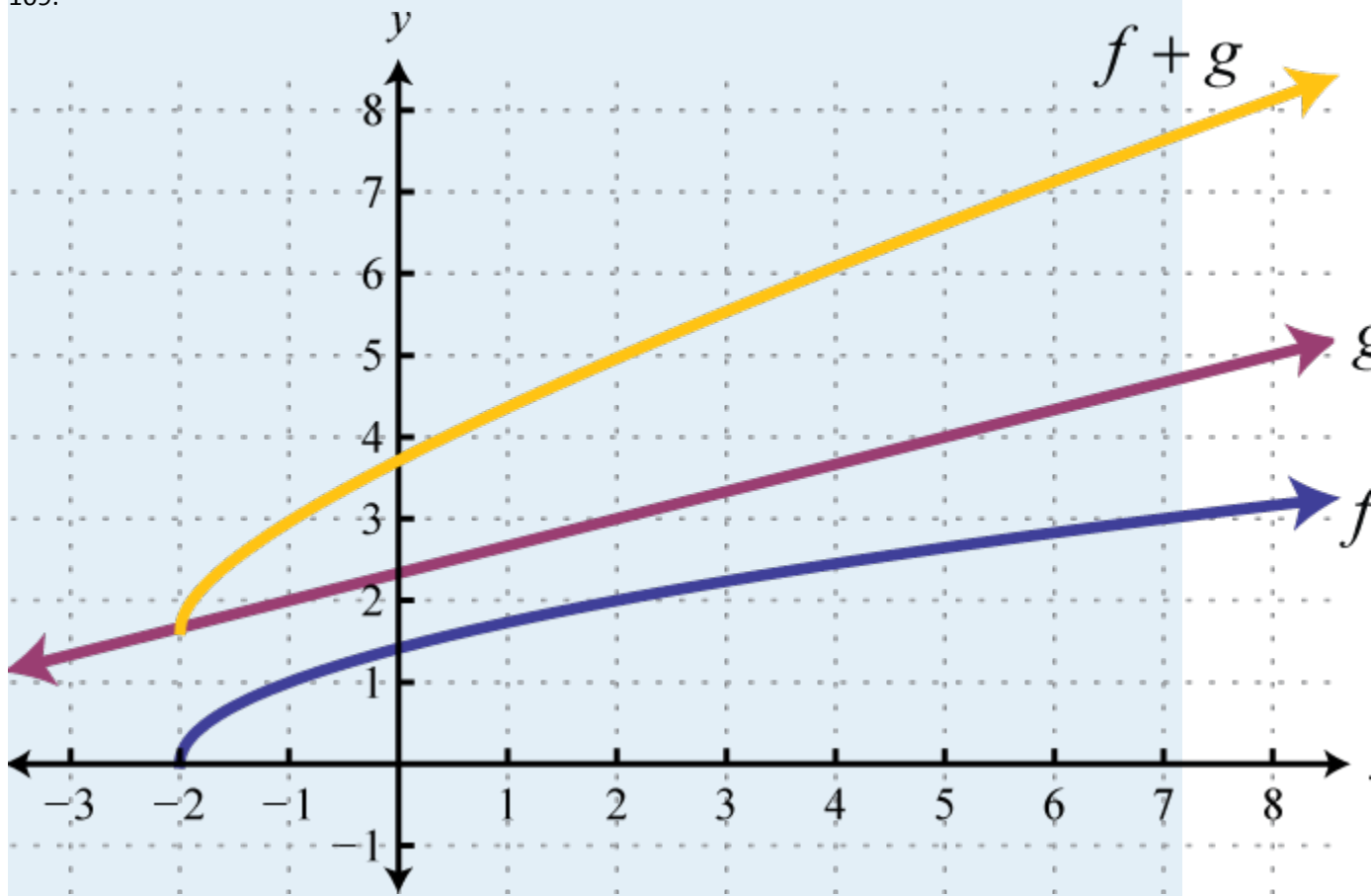
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109.



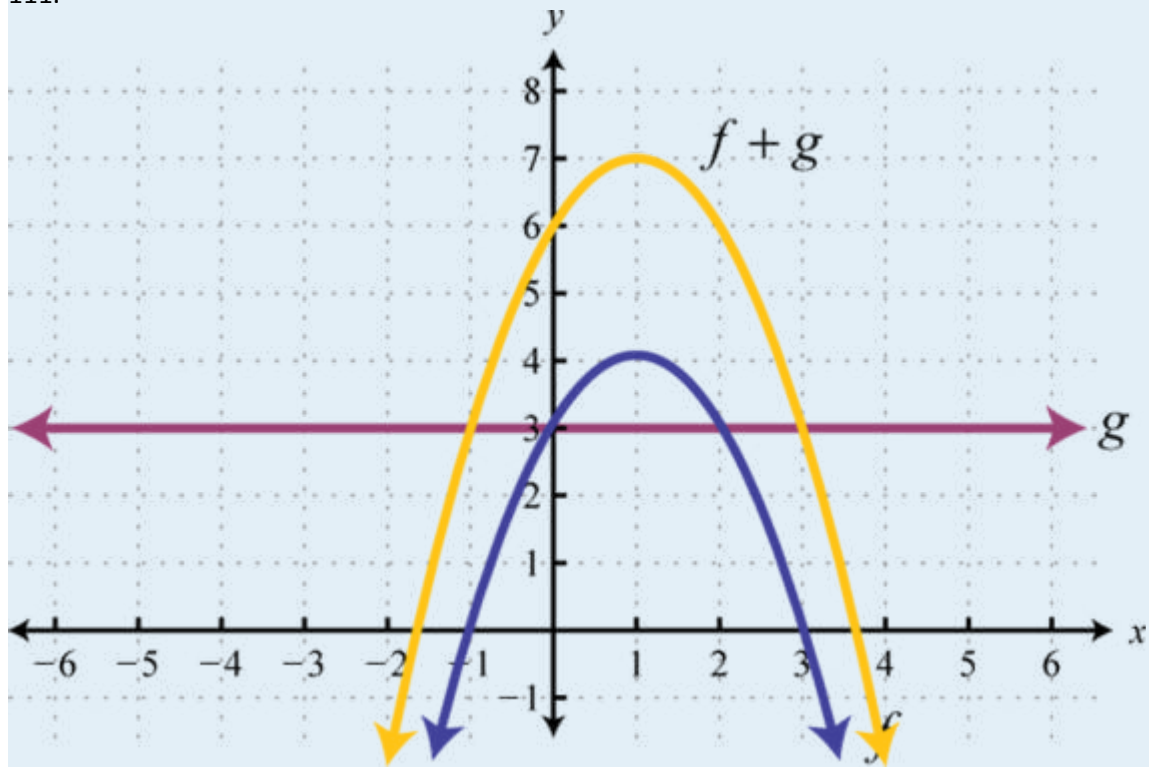
$[-2, \infty)$

110.





111.



$(-\infty, \infty)$

112.

113. Answer may vary



## 4.2 Factoring Polynomials

### LEARNING OBJECTIVES

1. Determine the greatest common factor (GCF) of monomials.
2. Factor out the GCF of a polynomial.
3. Factor a four-term polynomial by grouping.
4. Factor special binomials.

### Determining the GCF of Monomials

The process of writing a number or expression as a product is called **factoring**. If we write the monomial  $ax^2y$ , we say that the product is a **factorization** of  $ax^2y$  and  $a$ ,  $x^2$ , and  $y$  are **factors**. Typically, there are many ways to factor a monomial. Some factorizations of  $ax^2y$  follow:

Given two or more monomials, it will be useful to find the **greatest common monomial factor (GCMF)** of each. The GCMF of the monomials is the product of the common variable factors and the GCF of the coefficients.

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## Example 1

Find the GCF of and

Solution:

Begin by finding the GCF of the coefficients. In this case,  $25=5\cdot5$  and  $15=3\cdot5$ . It should be clear that

$$\text{GCF}(25, 15)=5$$

Next determine the common variable factors with the smallest exponents.

The common variable factors are  $x^3$ ,  $y^2$ , and  $z$ . Therefore, given the two monomials,

Answer:

It is worth pointing out that the GCF divides both expressions evenly.

Furthermore, we can write the following:

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The factors  $2x^2y$  and  $3xy^2$  share no common monomial factors other than 1; they are **relatively prime**.

## Example 2

Determine the GCF of the following three expressions:  $2x^2y$  and  $3xy^2$

Solution:

Begin by finding the GCF of the coefficients. To do this, determine the prime factorization of each and then multiply the common factors with the smallest exponents.

Therefore, the GCF of the coefficients of the three monomials is 1.

Next, determine the common factors of the variables.

The variable factors in common are  $x$ ,  $y$ , and  $1$ . Therefore,

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Note that the variable  $c$  is not common to all three expressions and thus is not included in the GCF.

Answer:

## Factoring out the GCF

The application of the distributive property is the key to multiplying polynomials. For example,

The process of factoring a polynomial involves applying the distributive property in reverse to write each polynomial as a product of polynomial factors.

Consider factoring the result of the opening example:

We see that the distributive property allows us to write the polynomial as a product of the two factors  $c$  and  $(2xy+1)$ . Note that in this case,  $c$  is the GCF of the terms of the polynomial.

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**Factoring out the greatest common factor (GCF)** of a polynomial involves rewriting it as a product where a factor is the GCF of all of its terms.

To factor out the GCF of a polynomial, we first determine the GCF of all of its terms. Then we can divide each term of the polynomial by this factor as a means to determine the remaining factor after applying the distributive property in reverse.

### Example 3

Factor out the GCF:

Solution:

In this case, the  $GCF(18, 30, 6) = 6$ , and the common variable factor with the smallest exponent is  $x^3$ . The GCF of the polynomial is .

The missing factor can be found by dividing each term of the original expression by the GCF.

Apply the distributive property (in reverse) using the terms found in the previous step.

If the GCF is the same as one of the terms, then, after the GCF is factored out, a constant term 1 will remain. The importance of remembering the constant term becomes clear when performing the check using the distributive property.

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Answer:

## Example 4

Factor out the GCF:

Solution:

The GCF of the terms is The last term does not have a variable factor of  $z$ , and thus  $z$  cannot be a part of the greatest common factor. If we divide each term by we obtain

and can write

Answer:

Try this! Factor out the GCF:

Answer:

Of course, not every polynomial with integer coefficients can be factored as a product of polynomials with integer coefficients other than 1 and itself. If this is the case, then we say that it is a **prime polynomial**. For example, a linear factor such as  $10x-9$  is prime. However, it can be factored as follows:

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If an  $x$  is factored out, the resulting factor is not a polynomial. If any constant is factored out, the resulting polynomial factor will not have integer coefficients. Furthermore, some linear factors are not prime. For example,

$$5x-10=5(x-2)$$

In general, any linear factor of the form  $ax+b$ , where  $a$  and  $b$  are relatively prime integers, is prime.

## Factoring by Grouping

In this section, we outline a technique for factoring polynomials with four terms. First, review a preliminary example where the terms have a common binomial factor.

### Example 5

Factor:  $7x(3x-2)-(3x-2)$ .

Solution:

Begin by rewriting the second term  $-(3x-2)$  as  $-1(3x-2)$ . Next, consider  $(3x-2)$  as a common binomial factor and factor it out as follows:

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Answer:  $(3x-2)(7x-1)$

**Factoring by grouping** is a technique that enables us to factor polynomials with four terms into a product of binomials. This involves an intermediate step where a common binomial factor will be factored out. For example, we wish to factor

Begin by grouping the first two terms and the last two terms. Then factor out the GCF of each grouping:

$$\begin{aligned} & \underbrace{3x^3 - 12x^2}_{\text{group}} + \underbrace{2x - 8}_{\text{group}} \\ &= 3x^2(x-4) + 2(x-4) \end{aligned}$$

In this form, the polynomial is a binomial with a common binomial factor,  $(x-4)$ .

Therefore,

We can check by multiplying.

✓

## Example 6

Factor by grouping:

Solution:

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The GCF for the first group is  $6a^3$ . We have to choose  $5$  or  $-5$  to factor out of the second group.

$$\begin{aligned} & \underbrace{24a^4 - 18a^3}_{\text{group}} - \underbrace{20a + 15}_{\text{group}} \\ &= 6a^3(4a - 3) + 5(-4a + 3) \quad \times \\ &= 6a^3(4a - 3) - 5(4a - 3) \quad \checkmark \end{aligned}$$

Factoring out  $+5$  does not result in a common binomial factor. If we choose to factor out  $-5$ , then we obtain a common binomial factor and can proceed. Note that when factoring out a negative number, we change the signs of the factored terms.

$$\begin{aligned} 24a^4 - 18a^3 - 20a + 15 &= \underbrace{24a^4 - 18a^3}_{\text{group}} - \underbrace{20a + 15}_{\text{group}} \\ &= 6a^3(4a - 3) - 5(4a - 3) \\ &= 6a^3(4a - 3) - 5(4a - 3) \\ &= (4a - 3)(6a^3 - 5) \\ &= (4a - 3)(6a^3 - 5) \end{aligned}$$

Answer: Check by multiplying; this is left to the reader as an exercise.

Sometimes we must first rearrange the terms in order to obtain a common factor.

## Example 7

Factor:

Solution:

Simply factoring the GCF out of the first group and last group does not yield a common binomial factor.

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$$\begin{aligned} & \underbrace{ab - 2a^2b}_{\text{group}} + \underbrace{a^3 - 2b^2}_{\text{group}} \\ &= ab(1 - 2a) + 1(a^3 - 2b^2) \end{aligned}$$

We must rearrange the terms, searching for a grouping that produces a common factor. In this example, we have a workable grouping if we switch the terms  $a^3$  and  $ab$ .

$$\begin{aligned} ab - 2a^2b + a^3 - 2b^2 &= \underbrace{a^3 - 2a^2b}_{\text{group}} + \underbrace{ab - 2b^2}_{\text{group}} \\ &= a^2(a - 2b) + b(a - 2b) \\ &= (a - 2b)(a^2 + b) \end{aligned}$$

Answer:

Try this! Factor:

Answer:

Not all factorable four-term polynomials can be factored with this technique. For example,

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This four-term polynomial cannot be grouped in any way to produce a common binomial factor. Despite this, the polynomial is not prime and can be written as a product of polynomials. It can be factored as follows:

Factoring such polynomials is something that we will learn to do as we move further along in our study of algebra. For now, we will limit our attempt to factor four-term polynomials to using the factor by grouping technique.

## Factoring Special Binomials

A binomial is a polynomial with two terms. We begin with the special binomial called **difference of squares**:

To verify the above formula, multiply.

We use this formula to factor certain special binomials.

### Example 8

Factor:

Solution:

Identify the binomial as difference of squares and determine the square factors of each term.

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*Difference of Squares*

$$x^2 - 9y^2 \qquad x^2 - 9y^2$$

↓                      ↓                      ↓

Here we can write

Substitute into the difference of squares formula where  $a=x$  and  $b=3y$ .

Multiply to check.



Answer:  $(x+3y)(x-3y)$

## Example 9

Factor:

Solution:

First, identify this expression as a difference of squares.

Use  $a=x$  and  $b=2x-1$  in the formula for a difference of squares and then simplify.

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Answer:  $(3x-1)(-x+1)$

Given any real number  $b$ , a polynomial of the form is prime. Furthermore, the **sum of squares** does not have a general factored equivalent. Care should be taken not to confuse this with a perfect square trinomial.

Therefore,

For example, the sum of squares binomial  $x^2+9$  is prime. Two other special binomials of interest are the **sum** and **difference of cubes**:

We can verify these formulas by multiplying.

✓

✓

The process for factoring sums and differences of cubes is very similar to that of differences of squares. We first identify  $a$  and  $b$  and then substitute into the appropriate formula. The separate formulas for the sum and difference of cubes allow us to always choose  $a$  and  $b$  to be positive.

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## Example 10

Factor:

Solution:

First, identify this binomial as a difference of cubes.

$$\begin{array}{ccc} \text{Difference} & \text{of} & \text{Cubes} \\ \downarrow & & \downarrow \quad \downarrow \\ x^3 - 8y^3 & & x^3 - 8y^3 \end{array}$$

Next, identify what is being cubed.

In this case,  $a=x$  and  $b=2y$ . Substitute into the difference of cubes formula.

We can check this factorization by multiplying.



Answer:

It may be the case that the terms of the binomial have a common factor. If so, it will be difficult to identify it as a special binomial until we first factor out the GCF.

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## Example 11

Factor:

Solution:

The terms are not perfect squares or perfect cubes. However, notice that they do have a common factor. First, factor out the GCF,  $3xy$ .

The resulting binomial factor is a sum of cubes with  $a=3x$  and  $b=y$ .

Answer:

When the degree of the special binomial is greater than two, we may need to apply the formulas multiple times to obtain a complete factorization. A polynomial is **completely factored** when it is prime or is written as a product of prime polynomials.

## Example 12

Factor completely:

Solution:

First, identify what is being squared.

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To do this, recall the power rule for exponents, When exponents are raised to a power, multiply them. With this in mind, we find

Therefore, and Substitute into the formula for difference of squares.

At this point, notice that the factor is itself a difference of two squares and thus can be further factored using and  $b=3y$ . The factor is prime and cannot be factored using real numbers.

Answer:

When factoring, always look for resulting factors to factor further.

## Example 13

Factor completely:

Solution:

This binomial is both a difference of squares and difference of cubes.

When confronted with a binomial that is a difference of both squares and cubes, as this is, make it a rule to factor using difference of squares first. Therefore, and Substitute into the difference of squares formula.

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The resulting two binomial factors are sum and difference of cubes. Each can be factored further. Therefore, we have

$$64x^6 - y^6 = (8x^3 + y^3) \cdot (8x^3 - y^3) \\ = \underbrace{(2x + y)(4x^2 - 2xy + y^2)} \cdot \underbrace{(2x - y)(4x^2 + 2xy + y^2)}$$

The trinomial factors are prime and the expression is completely factored.

Answer:

As an exercise, factor the previous example as a difference of cubes first and then compare the results. Why do you think we make it a rule to factor using difference of squares first?

Try this! Factor:

Answer:

## KEY TAKEAWAYS

The GCF of two or more monomials is the product of the GCF of the coefficients and the common variable factors with the smallest power.

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If the terms of a polynomial have a greatest common factor, then factor out that GCF using the distributive property. Divide each term of the polynomial by the GCF to determine the terms of the remaining factor.

Some four-term polynomials can be factored by grouping the first two terms and the last two terms. Factor out the GCF of each group and then factor out the common binomial factor.

When factoring by grouping, you sometimes have to rearrange the terms to find a common binomial factor. After factoring out the GCF, the remaining binomial factors must be the same for the technique to work.

When factoring special binomials, the first step is to identify it as a sum or difference. Once we identify the binomial, we then determine the values of  $a$  and  $b$  and then substitute into the appropriate formula.

If a binomial is both a difference of squares and cubes, then first factor it as a difference of squares.

## TOPIC EXERCISES

### PART A: FACTORING OUT THE GCF

Determine the GCF of the given expressions.

1.  $6x^2y, 9xy^2$
2.  $12xy, 16y^3$
3.  $50xy, 35xy, 10xy$
4.  $12xy, 36xy, 18xy$
5.  $15abc^5, 75ab^3c, 45abc$
6.  $12a^6bc, 48abc, 125abc$
7.  $60x(2x-1), 42x(2x-1), 6x(2x-1)$
8.  $14y(y-8), 28y(y-8), 35y(y-8)$

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9.  $10ab(a+b)$  ,  $48ab(a+b)^2$  ,  $26ab(a+b)$
10.  $45ab(a-b)$  ,  $36ab(a-b)$  ,  $63ab(a-b)$

Determine the missing factor.

1.  $18x^4 - 6x^3 + 2x^2 = 2x^2( \quad ? \quad )$
2.  $6x^5 - 9x^3 - 3x = 3x( \quad ? \quad )$
3.  $-10y^6 + 6y^4 - 4y^2 = -2y^2( \quad ? \quad )$
4.  $-27y^9 - 9y^6 + 3y^3 = -3y^3( \quad ? \quad )$
5.  $12x^3y^2 - 8x^2y^3 + 8xy = 4xy( \quad ? \quad )$
6.  $10x^4y^3 - 50x^3y^2 + 15x^2y^2 = 5xy( \quad ? \quad )$
7.  $14a^4b^5 - 21a^3b^4 - 7a^2b^3 = 7a^2b^3( \quad ? \quad )$
8.  $15a^5b^4 + 9a^4b^2 - 3a^2b = 3a^2b( \quad ? \quad )$
9.  $x^{3n} + x^{2n} + x^n = x^n( \quad ? \quad )$
10.  $y^{4n} + y^{3n} - y^{2n} = y^{2n}( \quad ? \quad )$

Factor out the GCF.

1.  $12x^4 - 16x^3 + 4x^2$
2.  $15x^5 - 10x^4 - 5x^3$
3.  $20y^8 + 28y^6 + 40y^3$
4.  $18y^7 - 24y^5 - 30y^3$
5.  $2a^4b^3 - 6a^3b^2 + 8a^2b$
6.  $28a^3b^3 - 21a^2b^4 - 14ab^5$
7.  $2x^3y^5 - 4x^4y^4 + x^2y^3$
8.  $3x^5y - 2x^4y^2 + x^3y^3$
9.  $5x^2(2x+3) - 3(2x+3)$
10.  $y^2(y-1) + 9(y-1)$

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11.  $9x^2(3x-1)+(3x-1)$

12.  $7y^2(5y+2)-(5y+2)$

13.  $x^{5n}-x^{3n}+x^n$

14.  $y^{6n}-y^{3n}-y^{2n}$

## PART B: FACTORING BY GROUPING

Factor by grouping.

35.  $2x^3+3x^2+2x+3$

36.  $5x^3+25x^2+x+5$

37.  $6x^3-3x^2+4x-2$

38.  $3x^3-2x^2-15x+10$

39.  $x^3-x^2-3x+3$

40.  $6x^3-15x^2-2x+5$

41.  $2x^3+7x^2-10x-35$

42.  $3x^3-x^2+24x-8$

43.  $14y^4+10y^3-7y-5$

44.  $5y^4+2y^3+20y+8$

45.  $x^{4n}+x^{3n}+2x^{2n}+2$

46.  $x^{5n}+x^{3n}+3x^{2n}+3$

47.  $x^3-x^2y+xy^2-y^3$

48.  $x^3+x^2y-2xy^2-2y^3$

49.  $3x^3y^2+9x^2y^3-x-3y$

50.  $2x^3y^3-x^2y^3+2x-y$

51.  $a^2b-4ab^2-3a+12b$

52.  $a^2b+3ab^2+5a+15b$

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53.  $a^4+a^2b^3+a^2b+b^4$   
 54.  $a^3b+2a^2+3ab^4+6b^3$   
 55.  $3ax+10by-5ay-6bx$   
 56.  $a^2x-5b^2y-5a^2y+b^2x$   
 57.  $x^4y^2-x^3y^3+x^2y^4-xy^5$   
 58.  $2x^5y^2+4x^4y^2+18x^3y+36x2y$   
 59.  $a^5b^2+a^4b^4+a^3b^3+a^2b^5$   
 60.  $3a^6b+3a^5b^2+9a^4b^2+9a^3b^3$

## PART C: FACTORING SPECIAL BINOMIALS

Factor.

61.  $x^2-64$   
 62.  $x^2-100$   
 63.  $9-4y^2$   
 64.  $25-y^2$   
 65.  $x^2-81y^2$   
 66.  $x^2-49y^2$   
 67.  $a^2b^2-4$   
 68.  $1-9a^2b^2$   
 69.  $a^2b^2-c^2$   
 70.  $4a^2-b^2c^2$   
 71.  $x^4-64$   
 72.  $36-y^4$   
 73.  $(2x+5)^2-x^2$   
 74.  $(3x-5)^2-x^2$

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75.  $y^2 - (y-3)^2$
76.  $y^2 - (2y+1)^2$
77.  $(2x+5)^2 - (x-3)^2$
78.  $(3x-1)^2 - (2x-3)^2$
79.  $x^4 - 16$
80.  $81x^4 - 1$
81.  $x^4y^4 - 1$
82.  $x^4 - y^4$
83.  $x^8 - y^8$
84.  $y^8 - 1$
85.  $x^{2n} - y^{2n}$
86.  $x^{2n}y^{2n} - 4$
87.  $x^{4n} - y^{4n}$
88.  $x^{4n}y^{4n} - 16$
89.  $x^3 - 27$
90.  $8x^3 - 125$
91.  $8y^3 + 27$
92.  $64x^3 + 343$
93.  $x^3 - y^3$
94.  $x^3 + y^3$
95.  $8a^3b^3 + 1$
96.  $27a^3 - 8b^3$
97.  $x^3y^3 - 125$
98.  $216x^3 + y^3$

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99.  $x^3+(x+3)^3$
100.  $y^3-(2y-1)^3$
101.  $(2x+1)^3-x^3$
102.  $(3y-5)^3-y^3$
103.  $x^{3n}-y^{3n}$
104.  $x^{3n}+y^{3n}$
105.  $a^6+64$
106.  $64a^6-1$
107.  $x^6-y^6$
108.  $x^6+y^6$
109.  $x^{6n}-y^{6n}$
110.  $x^{6n}+y^{6n}$
111. Given  $f(x)=2x-1$ , show that  $(f+f)(x)=2f(x)$ .
112. Given  $f(x)=x^2-3x+2$ , show that  $(f+f)(x)=2f(x)$ .
113. Given  $f(x)=mx+b$ , show that  $(f+f)(x)=2f(x)$ .
114. Given  $f(x)=ax^2+bx+c$ , show that  $(f+f)(x)=2f(x)$ .
115. Given  $f(x)=ax^2+bx+c$ , show that  $(f-f)(x)=0$ .
116. Given  $f(x)=mx+b$ , show that  $(f-f)(x)=0$ .

## PART D: DISCUSSION BOARD

117. What can be said about the degree of a factor of a polynomial? Give an example.
118. If a binomial falls into both categories, difference of squares and difference of cubes, which would be best to use for factoring, and why? Create an example that illustrates this situation and factor it using both formulas.





119. Write your own examples for each of the three special types of binomial. Factor them and share your results.

## ANSWERS

1.  $3x^2$

2.

3.  $5xy^2$

4.

5.  $15ab^2c$

6.

7.  $6x(2x-1)$

8.

9.  $2ab^2(a+b)^2$

10.

11.  $(9x^2-3x+1)$

12.

13.  $(5y^4-3y^2+2)$

14.

15.  $(3x^2y-2xy^2+2)$

16.

17.  $(2a^2b^2-3ab-1)$

18.

19.  $(x^{2n}+x^n+1)$

20.

21.  $4x^2(3x^2-4x+1)$

22.

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23.  $4y^3(5y^5+7y^3+10)$

24.

25.  $2a^2b(a^2b^2-3ab+4)$

26.

27.  $x^2y^3(2xy^2-4x^2y+1)$

28.

29.  $(2x+3)(5x^2-3)$

30.

31.  $(3x-1)(9x^2+1)$

32.

33.  $x^n(x^{4n}-x^{2n}+1)$

34.

35.  $(2x+3)(x^2+1)$

36.

37.  $(2x-1)(3x^2+2)$

38.

39.  $(x-1)(x^2-3)$

40.

41.  $(2x+7)(x^2-5)$

42.

43.  $(7y+5)(2y^3-1)$

44.

45.  $(x^n+1)(x^{3n}+2)$

46.

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47.  $(x-y)(x^2+y^2)$
- 48.
49.  $(x+3y)(3x^2y^2-1)$
- 50.
51.  $(a-4b)(ab-3)$
- 52.
53.  $(a^2+b)(a^2+b^3)$
- 54.
55.  $(a-2b)(3x-5y)$
- 56.
57.  $xy^2(x-y)(x^2+y^2)$
- 58.
59.  $a^2b^2(a^2+b)(a+b^2)$
- 60.
61.  $(x+8)(x-8)$
- 62.
63.  $(3+2y)(3-2y)$
- 64.
65.  $(x+9y)(x-9y)$
- 66.
67.  $(ab+2)(ab-2)$
- 68.
69.  $(ab+c)(ab-c)$
- 70.

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71.  $(x^2+8)(x^2-8)$
- 72.
73.  $(3x+5)(x+5)$
- 74.
75.  $3(2y-3)$
- 76.
77.  $(3x+2)(x+8)$
- 78.
79.  $(x^2+4)(x+2)(x-2)$
- 80.
81.  $(x^2y^2+1)(xy+1)(xy-1)$
- 82.
83.  $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$
- 84.
85.  $(x^n+y^n)(x^n-y^n)$
- 86.
87.  $(x^{2n}+y^{2n})(x^n+y^n)(x^n-y^n)$
- 88.
89.  $(x-3)(x^2+3x+9)$
- 90.
91.  $(2y+3)(4y^2-6y+9)$
- 92.
93.  $(x-y)(x^2+xy+y^2)$
- 94.

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95.  $(2ab+1)(4a^2b^2-2ab+1)$
- 96.
97.  $(xy-5)(x^2y^2+5xy+25)$
- 98.
99.  $(2x+3)(x^2+3x+9)$
- 100.
101.  $(x+1)(7x^2+5x+1)$
- 102.
103.  $(x^n-y^n)(x^{2n}+x^ny^n+y^{2n})$
- 104.
105.  $(a^2+4)(a^4-4a^2+16)$
- 106.
107.  $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$
- 108.
109.  $(x^n+y^n)(x^{2n}-x^ny^n+y^{2n})$   
 $\times(x^n-y^n)(x^{2n}+x^ny^n+y^{2n})$
- 110.
111. Answer may vary
- 112.
113. Answer may vary
- 114.
115. Answer may vary
- 116.
117. Answer may vary
- 118.

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119. Answer may vary

## 4.3 Factoring Trinomials

### LEARNING OBJECTIVES

1. Factor trinomials of the form  $x^2+bx+c$ .
2. Factor trinomials of higher degree.
3. Factor trinomials of the form  $ax^2+bx+c$ .
4. Factor trinomials using the AC method.

#### Factoring Trinomials of the Form $x^2+bx+c$

Some trinomials of the form  $x^2+bx+c$  can be factored as a product of binomials. If a trinomial of this type factors, then we have:

This gives us

$$b=n+m \quad \text{and} \quad c=mn$$

In short, if the leading coefficient of a factorable trinomial is 1, then the factors of the last term must add up to the coefficient of the middle term. This observation is the key to factoring trinomials using the technique known as the **trial and error (or guess and check) method.**

#### Example 1

Factor:  $x^2+12x+20$ .

Solution:

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We begin by writing two sets of blank parentheses. If a trinomial of this form factors, then it will factor into two linear binomial factors.

$$x^2+12x+20=(\quad)(\quad)$$

Write the factors of the first term in the first space of each set of parentheses. In this case, factor  $x^2=x \cdot x$ .

$$x^2+12x+20=(x \quad)(x \quad)$$

Determine the factors of the last term whose sum equals the coefficient of the middle term. To do this, list all of the factorizations of 20 and search for factors whose sum equals 12.

Choose  $20 = 2 \cdot 10$  because  $2 + 10 = 12$ . Write in the last term of each binomial using the factors determined in the previous step.

$$x^2+12x+20=(x + 2)(x + 10)$$

This can be visually interpreted as follows:

<i>First term</i>	<i>Middle term</i>	<i>Last term</i>
$(x+2)(x+10)$	$(x+2)(x+10)$	$(x+2)(x+10)$
$\xrightarrow{x^2}$	$\xrightarrow{10x}$ $\xrightarrow{+ 2x}$ <hr/> $12x$	$\xrightarrow{20}$



Check by multiplying the two binomials.



Answer:  $(x+2)(x+10)$

Since multiplication is commutative, the order of the factors does not matter.

If the last term of the trinomial is positive, then either both of the constant factors must be negative or both must be positive.

## Example 2

Factor:  $x^2y^2 - 7xy + 12$ .

Solution:

First, factor  $x^2y^2 = xy \cdot xy$ .

$$x^2y^2 - 7xy + 12 = (xy \quad ?)(xy \quad ?)$$

Next, search for factors of 12 whose sum is  $-7$ .

In this case, choose  $-3$  and  $-4$  because  $(-3)(-4) = +12$  and  $-3 + (-4) = -7$ .

Check.



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Answer:  $(xy-3)(xy-4)$

If the last term of the trinomial is negative, then one of its factors must be negative.

### Example 3

Factor:  $x^2-4xy-12y^2$ .

Solution:

Begin by factoring the first term  $x^2=x \cdot x$ .

$$x^2-4xy-12y^2=(x \quad ?)(x \quad ?)$$

The factors of 12 are listed below. In this example, we are looking for factors whose sum is  $-4$ .

Therefore, the coefficient of the last term can be factored as  $-12=2(-6)$ , where  $2+(-6)=-4$ . Because the last term has a variable factor of  $y^2$ , use  $-12y^2=2y(-6y)$  and factor the trinomial as follows:

Multiply to check.

✓

Answer:  $(x+2y)(x-6y)$

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Often our first guess will not produce a correct factorization. This process may require repeated trials. For this reason, the check is very important and is not optional.

## Example 4

Factor:  $a^2+10a-24$ .

Solution:

The first term of this trinomial,  $a^2$ , factors as  $a \cdot a$ .

$$a^2+10a-24=(a \quad ?)(a \quad ?)$$

Consider the factors of 24:

Suppose we choose the factors 4 and 6 because  $4 + 6 = 10$ , the coefficient of the middle term. Then we have the following incorrect factorization:

When we multiply to check, we find the error.

**x**

In this case, the middle term is correct but the last term is not. Since the last term in the original expression is negative, we need to choose factors that are opposite in sign.

Therefore, we must try again. This time we choose the factors  $-2$  and  $12$

because  $-2+12=10$ .

$$a^2+10a-24=(a-2)(a+12)$$

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Now the check shows that this factorization is correct.



Answer:  $(a-2)(a+12)$

If we choose the factors wisely, then we can reduce much of the guesswork in this process. However, if a guess is not correct, do not get discouraged; just try a different set of factors. Keep in mind that some polynomials are prime. For example, consider the trinomial  $x^2+3x+20$  and the factors of 20:

There are no factors of 20 whose sum is 3. Therefore, the original trinomial cannot be factored as a product of two binomials with integer coefficients. The trinomial is prime.

## Factoring Trinomials of Higher Degree

We can use the trial and error technique to factor trinomials of higher degree.

### Example 5

Factor:  $x^4+6x^2+5$ .

Solution:

Begin by factoring the first term  $x^4=x^2 \cdot x^2$ .

$$x^4+6x^2+5=(x^2 \quad ?)(x^2 \quad ?)$$

Since 5 is prime and the coefficient of the middle term is positive, choose +1 and +5 as the factors of the last term.

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Notice that the variable part of the middle term is  $x^2$  and the factorization checks out.



Answer:  $(x^2+1)(x^2+5)$

## Example 6

Factor:  $x^{2n}+4x^n-21$  where  $n$  is a positive integer.

Solution:

Begin by factoring the first term  $x^{2n}=x^n \cdot x^n$ .

$$x^{2n}+4x^n-21=(x^n \quad ?)(x^n \quad ?)$$

Factor  $-21=7(-3)$  because  $7+(-3)=+4$  and write

Answer:  $(x^n+7)(x^n-3)$  The check is left to the reader.

Try this! Factor:  $x^6-x^3-42$ .

Answer:  $(x^3+6)(x^3-7)$

## Factoring Trinomials of the Form $ax^2+bx+c$

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Factoring trinomials of the form  $ax^2+bx+c$  can be challenging because the middle term is affected by the factors of both  $a$  and  $c$ . In general,

This gives us,

$$a=pq \quad \text{and} \quad b=pn+qm, \quad \text{where} \quad c=mn$$

In short, when the leading coefficient of a trinomial is something other than 1, there will be more to consider when determining the factors using the trial and error method. The key lies in the understanding of how the middle term is obtained. Multiply  $(5x+3)$   $(2x+3)$  and carefully follow the formation of the middle term.

$$\begin{aligned} (5x+3)(2x+3) &= 5x \cdot 2x + \underbrace{5x \cdot 3 + 3 \cdot 2x}_{\text{middle term}} + 3 \cdot 3 \\ &= 10x^2 + 15x + 6x + 9 \\ &= 15x^2 + 21x + 9 \end{aligned}$$

As we have seen before, the product of the first terms of each binomial is equal to the first term of the trinomial. The middle term of the trinomial is the sum of the products of the outer and inner terms of the binomials. The product of the last terms of each binomial is equal to the last term of the trinomial. Visually, we have the following:

$$\begin{array}{c} \text{Outer product} \\ 15x \\ \text{---} \\ (5x+3)(2x+3) = 15x^2 + 21x + 9 \\ \text{---} \\ 6x \\ \text{Inner product} \end{array}$$

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For this reason, we need to look for products of the factors of the first and last terms whose sum is equal to the coefficient of the middle term. For example, to factor  $6x^2+29x+35$ , look at the factors of 6 and 35.

The combination that produces the coefficient of the middle term is  $2 \cdot 7 + 3 \cdot 5 = 14 + 15 = 29$ . Make sure that the outer terms have coefficients 2 and 7, and that the inner terms have coefficients 5 and 3. Use this information to factor the trinomial.

We can always check by multiplying; this is left to the reader.

## Example 7

Factor:  $5x^2+16xy+3y^2$ .

Solution:

Since the leading coefficient and the last term are both prime, there is only one way to factor each.

$$5=1 \cdot 5 \quad \text{and} \quad 3=1 \cdot 3$$

Begin by writing the factors of the first term,  $5x^2$ , as follows:

$$5x^2+16xy+3y^2=(x \quad ?)(5x \quad ?)$$

The middle and last term are both positive; therefore, the factors of 3 are chosen as positive numbers. In this case, the only choice is in which grouping to place these factors.

$$(x+y)(5x+3y) \quad \text{or} \quad (x+3y)(5x+y)$$

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Determine which grouping is correct by multiplying each expression.

x

✓

Answer:  $(x+3y)(5x+y)$

## Example 8

Factor:  $18a^2b^2 - ab - 4$ .

Solution:

First, consider the factors of the coefficients of the first and last terms.

We are searching for products of factors whose sum equals the coefficient of the middle term,  $-1$ . After some thought, we can see that the sum of 8 and  $-9$  is  $-1$  and the combination that gives this follows:

$$2(4) + 9(-1) = 8 - 9 = -1$$

Factoring begins at this point with two sets of blank parentheses.

$$18a^2b^2 - ab - 4 = ( \quad ) ( \quad )$$

Use  $2ab$  and  $9ab$  as factors of  $18a^2b^2$ .

$$18a^2b^2 - ab - 4 = (2ab \quad ?)(9ab \quad ?)$$

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Next use the factors 1 and 4 in the correct order so that the inner and outer products are  $-9ab$  and  $8ab$  respectively.

$$18a^2b^2 - ab - 4 = (2ab - 1)(9ab + 4)$$

Answer:  $(2ab - 1)(9ab + 4)$ . The complete check is left to the reader.

It is a good practice to first factor out the GCF, if there is one. Doing this produces a trinomial factor with smaller coefficients. As we have seen, trinomials with smaller coefficients require much less effort to factor. This commonly overlooked step is worth identifying early.

## Example 9

Factor:  $12y^3 - 26y^2 - 10y$ .

Solution:

Begin by factoring out the GCF.

$$12y^3 - 26y^2 - 10y = 2y(6y^2 - 13y - 5)$$

After factoring out  $2y$ , the coefficients of the resulting trinomial are smaller and have fewer factors. We can factor the resulting trinomial using  $6 = 2(3)$  and  $5 = (5)(1)$ . Notice that these factors can produce  $-13$  in two ways:

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Because the last term is  $-5$ , the correct combination requires the factors 1 and 5 to be opposite signs. Here we use  $2(1) = 2$  and  $3(-5) = -15$  because the sum is  $-13$  and the product of  $(1)(-5) = -5$ .

Check.



The factor  $2y$  is part of the factored form of the original expression; be sure to include it in the answer.

Answer:  $2y(2y-5)(3y+1)$

It is a good practice to consistently work with trinomials where the leading coefficient is positive. If the leading coefficient is negative, factor it out along with any GCF. Note that sometimes the factor will be  $-1$ .

## Example 10

Factor:  $-18x^6 - 69x^4 + 12x^2$ .

Solution:

In this example, the GCF is  $3x^2$ . Because the leading coefficient is negative we begin by factoring out  $-3x^2$ .

$$-18x^6 - 69x^4 + 12x^2 = -3x^2(6x^4 + 23x^2 - 4)$$

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At this point, factor the remaining trinomial as usual, remembering to write the  $-3x^2$  as a factor in the final answer. Use  $6 = 1(6)$  and  $-4 = 4(-1)$  because  $1(-1)+6(4)=23$ . Therefore,

Answer:  $-3x^2(x^2+4)(6x^2-1)$ . The check is left to the reader.

Try this! Factor:  $-12a^5b+a^3b^3+ab^5$ .

Answer:  $-ab(3a^2-b^2)(4a^2+b^2)$

### Factoring Using the AC Method

An alternate technique for factoring trinomials, called the **AC method**, makes use of the grouping method for factoring four-term polynomials. If a trinomial in the form  $ax^2+bx+c$  can be factored, then the middle term,  $bx$ , can be replaced with two terms with coefficients whose sum is  $b$  and product is  $ac$ . This substitution results in an equivalent expression with four terms that can be factored by grouping.

### Example 11

Factor using the AC method:  $18x^2-31x+6$ .

Solution:

Here  $a = 18$ ,  $b = -31$ , and  $c = 6$ .

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Factor 108, and search for factors whose sum is  $-31$ .



In this case, the sum of the factors  $-27$  and  $-4$  equals the middle coefficient,  $-31$ .  
Therefore,  $-31x = -27x - 4x$ , and we can write

Factor the equivalent expression by grouping.

Answer:  $(2x-3)(9x-2)$

## Example 12

Factor using the AC method:  $4x^2y^2 - 7xy - 15$ .

Solution:

Here  $a = 4$ ,  $b = -7$ , and  $c = -15$ .

Factor  $-60$  and search for factors whose sum is  $-7$ .



The sum of factors  $5$  and  $-12$  equals the middle coefficient,  $-7$ .

Replace  $-7xy$  with  $5xy - 12xy$ .

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Answer:  $(4xy+5)(xy-3)$ . The check is left to the reader.

If factors of  $ac$  cannot be found to add up to  $b$  then the trinomial is prime.

## KEY TAKEAWAYS

If a trinomial of the form  $x^2+bx+c$  factors into the product of two binomials, then the coefficient of the middle term is the sum of factors of the last term.

If a trinomial of the form  $ax^2+bx+c$  factors into the product of two binomials, then the coefficient of the middle term will be the sum of certain products of factors of the first and last terms.

If the trinomial has a greatest common factor, then it is a best practice to first factor out the GCF before attempting to factor it into a product of binomials.

If the leading coefficient of a trinomial is negative, then it is a best practice to first factor that negative factor out before attempting to factor the trinomial.

Factoring is one of the more important skills required in algebra. For this reason, you should practice working as many problems as it takes to become proficient.

## TOPIC EXERCISES

### PART A: FACTORING TRINOMIALS OF THE FORM $x^2+bx+c$

Factor.

1.  $x^2+5x-6$
2.  $x^2+5x+6$
3.  $x^2+4x-12$

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4.  $x^2+3x-18$
5.  $x^2-14x+48$
6.  $x^2-15x+54$
7.  $x^2+11x-30$
8.  $x^2-2x+24$
9.  $x^2-18x+81$
10.  $x^2-22x+121$
11.  $x^2-xy-20y^2$
12.  $x^2+10xy+9y^2$
13.  $x^2y^2+5xy-50$
14.  $x^2y^2-16xy+48$
15.  $a^2-6ab-72b^2$
16.  $a^2-21ab+80b^2$
17.  $u^2+14uv-32v^2$
18.  $m^2+7mn-98n^2$
19.  $(x+y)^2-2(x+y)-8$
20.  $(x-y)^2-2(x-y)-15$
21.  $x^4-7x^2-8$
22.  $x^4+13x^2+30$
23.  $x^4-8x^2-48$
24.  $x^4+25x^2+24$
25.  $y^4-20y^2+100$
26.  $y^4+14y^2+49$
27.  $x^4+3x^2y^2+2y^4$

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28.  $x^4 - 8x^2y^2 + 15y^4$
29.  $a^4b^4 - 4a^2b^2 + 4$
30.  $a^4 + 6a^2b^2 + 9b^4$
31.  $x^6 - 18x^3 - 40$
32.  $x^6 + 18x^3 + 45$
33.  $x^6 - x^3y^3 - 6y^6$
34.  $x^6 + x^3y^3 - 20y^6$
35.  $x^6y^6 + 2x^3y^3 - 15$
36.  $x^6y^6 + 16x^3y^3 + 48$
37.  $x^{2n} + 12x^n + 32$
38.  $x^{2n} + 41x^n + 40$
39.  $x^{2n} + 2ax^n + a^2$
40.  $x^{2n} - 2ax^n + a^2$

## PART B: FACTORING TRINOMIALS OF THE FORM $ax^2 + bx + c$

Factor.

41.  $3x^2 + 20x - 7$
42.  $2x^2 - 9x - 5$
43.  $6a^2 + 13a + 6$
44.  $4a^2 + 11a + 6$
45.  $6x^2 + 7x - 10$
46.  $4x^2 - 25x + 6$
47.  $24y^2 - 35y + 4$

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48.  $10y^2 - 23y + 12$
49.  $14x^2 - 11x + 9$
50.  $9x^2 + 6x + 8$
51.  $4x^2 - 28x + 49$
52.  $36x^2 - 60x + 25$
53.  $27x^2 - 6x - 8$
54.  $24x^2 + 17x - 20$
55.  $6x^2 + 23xy - 4y^2$
56.  $10x^2 - 21xy - 27y^2$
57.  $8a^2b^2 - 18ab + 9$
58.  $12a^2b^2 - ab - 20$
59.  $8u^2 - 26uv + 15v^2$
60.  $24m^2 - 26mn + 5n^2$
61.  $4a^2 - 12ab + 9b^2$
62.  $16a^2 + 40ab + 25b^2$
63.  $5(x+y)^2 - 9(x+y) + 4$
64.  $7(x-y)^2 + 15(x-y) - 18$
65.  $7x^4 - 22x^2 + 3$
66.  $5x^4 - 41x^2 + 8$
67.  $4y^6 - 3y^3 - 10$
68.  $12y^6 + 4y^3 - 5$
69.  $5a^4b^4 - a^2b^2 - 18$
70.  $21a^4b^4 + 5a^2b^2 - 4$
71.  $6x^6y^6 + 17x^3y^3 + 10$

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72.  $16x^6y^6+46x^3y^3+15$
73.  $8x^{2n}-10x^n-25$
74.  $30x^{2n}-11x^n-6$
75.  $36x^{2n}+12ax^n+a^2$
76.  $9x^{2n}-12ax^n+4a^2$
77.  $-3x^2+14x+5$
78.  $-2x^2+13x-20$
79.  $-x^2-10x+24$
80.  $-x^2+8x+48$
81.  $54-12x-2x^2$
82.  $60+5x-5x^2$
83.  $4x^3+16x^2+20x$
84.  $2x^4-12x^3+14x^2$
85.  $2x^3-8x^2y-24xy^2$
86.  $6x^3-9x^2y-6xy^2$
87.  $4a^3b-4a^2b^2-24ab^3$
88.  $15a^4b-33a^3b^2+6a^2b^3$
89.  $3^x5^y+30x^3y^3+75xy^5$
90.  $45x^5y^2-60x^3y^4+20xy^6$

Factor.

41.  $4-25x^2$
42.  $8x^3-y^3$
43.  $9x^2-12xy+4y^2$
44.  $30a^2-57ab-6b^2$

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45.  $10a^2 - 5a - 6ab + 3b$
46.  $3x^3 - 4x^2 + 9x - 12$
47.  $x^2 + 4y^2$
48.  $x^2 - x + 2$
49.  $15a^3b^2 + 6a^2b^3 - 3ab^4$
50.  $54x^2 - 63x$

### PART D: DISCUSSION BOARD

101. Create your own trinomial of the form  $ax^2 + bx + c$  that factors. Share it, along with the solution, on the discussion board.
102. Create a trinomial of the form  $ax^2 + bx + c$  that does not factor and share it along with the reason why it does not factor.

### ANSWERS

1.  $(x-1)(x+6)$
- 2.
3.  $(x-2)(x+6)$
- 4.
5.  $(x-6)(x-8)$
- 6.
7. Prime
- 8.
9.  $(x-9)^2$
- 10.
11.  $(x-5y)(x+4y)$
- 12.

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13.  $(xy-5)(xy+10)$
- 14.
15.  $(a+6b)(a-12b)$
- 16.
17.  $(u-2v)(u+16v)$
- 18.
19.  $(x+y-4)(x+y+2)$
- 20.
21.  $(x^2-8)(x^2+1)$
- 22.
23.  $(x^2+4)(x^2-12)$
- 24.
25.  $(y^2-10)^2$
- 26.
27.  $(x^2+y^2)(x^2+2y^2)$
- 28.
29.  $(a^2b^2-2)^2$
- 30.
31.  $(x^3-20)(x^3+2)$
- 32.
33.  $(x^3+2y^3)(x^3-3y^3)$
- 34.
35.  $(x^3y^3-3)(x^3y^3+5)$
- 36.

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37.  $(x^n+4)(x^n+8)$

38.

39.  $(x^n+a)^2$

40.

41.  $(3x-1)(x+7)$

42.

43.  $(2a+3)(3a+2)$

44.

45.  $(6x-5)(x+2)$

46.

47.  $(8y-1)(3y-4)$

48.

49. Prime

50.

51.  $(2x-7)^2$

52.

53.  $(9x+4)(3x-2)$

54.

55.  $(6x-y)(x+4y)$

56.

57.  $(4ab-3)(2ab-3)$

58.

59.  $(2u-5v)(4u-3v)$

60.

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61.  $(2a-3b)^2$
- 62.
63.  $(x+y-1)(5x+5y-4)$
- 64.
65.  $(x^2-3)(7x^2-1)$
- 66.
67.  $(y^3-2)(4y^3+5)$
- 68.
69.  $(a^2b^2-2)(5a^2b^2+9)$
- 70.
71.  $(6x^3y^3+5)(x^3y^3+2)$
- 72.
73.  $(2x^n-5)(4x^n+5)$
- 74.
75.  $(6x^n+a)^2$
- 76.
77.  $-(x-5)(3x+1)$
- 78.
79.  $-(x-2)(x+12)$
- 80.
81.  $-2(x-3)(x+9)$
- 82.
83.  $4x(x^2+4x+5)$
- 84.

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85.  $2x(x+2y)(x-6y)$
- 86.
87.  $4ab(a-3b)(a+2b)$
- 88.
89.  $3xy(x^2+5y^2)^2$
- 90.
91.  $(2-5x)(2+5x)$
- 92.
93.  $(3x-2y)^2$
- 94.
95.  $(2a-1)(5a-3b)$
- 96.
97. Prime
- 98.
99.  $3ab^2(5a^2+2ab-b^2)$
- 100.
101. Answer may vary

## 4.4 Solve Polynomial Equations by Factoring

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## LEARNING OBJECTIVES

1. Review general strategies for factoring.
2. Solve polynomial equations by factoring.
3. Find roots of a polynomial function.
4. Find polynomial equations given the solutions.

### Reviewing General Factoring Strategies

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a general guideline for factoring polynomials:

1. Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF).
2. Determine the number of terms in the polynomial.
  1. Factor four-term polynomials by grouping.
- b. Factor trinomials (3 terms) using “trial and error” or the AC method.
- c. Factor binomials (2 terms) using the following special products:
  3. Look for factors that can be factored further.
  4. Check by multiplying.

Note: If a binomial is both a difference of squares and a difference cubes, then first factor it as difference of squares. This will result in a more complete factorization. In

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addition, not all polynomials with integer coefficients factor. When this is the case, we say that the polynomial is prime.

If an expression has a GCF, then factor this out first. Doing so is often overlooked and typically results in factors that are easier to work with. Furthermore, look for the resulting factors to factor further; many factoring problems require more than one step. A polynomial is completely factored when none of the factors can be factored further.

## Example 1

Factor:  $54x^4 - 36x^3 - 24x^2 + 16x$ .

Solution:

This four-term polynomial has a GCF of  $2x$ . Factor this out first.

Now factor the resulting four-term polynomial by grouping and look for resulting factors to factor further.

$$\begin{aligned} 54x^4 - 36x^3 - 24x^2 + 16x &= 2x \left( \underbrace{27x^3 - 18x^2}_{\text{group}} - \underbrace{12x + 8}_{\text{group}} \right) \\ &= 2x(9x^2(3x - 2) - 4(3x - 2)) \\ &= 2x(3x - 2) \underbrace{(9x^2 - 4)}_{\substack{\text{difference} \\ \text{of squares}}} \\ &= 2x(3x - 2)(3x - 2)(3x + 2) \end{aligned}$$

Answer:  $2x(3x - 2)^2(3x + 2)$ . The check is left to the reader.

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## Example 2

Factor:  $x^4 - 3x^2 - 4$ .

Solution:

This trinomial does not have a GCF.

The factor  $(x^2 + 1)$  is prime and the trinomial is completely factored.

Answer:  $(x^2 + 1)(x + 2)(x - 2)$

## Example 3

Factor:  $x^6 + 6x^3 - 16$ .

Solution:

Begin by factoring  $x^6 = x^3 \cdot x^3$  and look for the factors of 16 that add to 6.

The factor  $(x^3 - 2)$  cannot be factored any further using integers and the factorization is complete.

Answer:  $(x^3 - 2)(x + 2)(x^2 + 2x + 4)$

Try this! Factor:  $9x^4 + 17x^2 - 2$

Answer:  $(3x + 1)(3x - 1)(x^2 + 2)$

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## Solving Polynomial Equations by Factoring

In this section, we will review a technique that can be used to solve certain polynomial equations. We begin with the **zero-product** property:

$$a \cdot b = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

The zero-product property is true for any number of factors that make up an equation. In other words, if any product is equal to zero, then at least one of the variable factors must be equal to zero. If an expression is equal to zero and can be factored into linear factors, then we will be able to set each factor equal to zero and solve for each equation.

### Example 4

Solve:  $2x(x-4)(5x+3)=0$ .

Solution:

Set each variable factor equal to zero and solve.

To check that these are solutions we can substitute back into the original equation to see if we obtain a true statement. Note that each solution produces a zero factor. This is left to the reader.

Answer: The solutions are 0, 4, and

Of course, most equations will not be given in factored form.

### Example 5

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Solve:

Solution:

Begin by factoring the left side completely.

Set each factor equal to zero and solve.

Answer: The solutions are 14, -5, and 5.

Using the zero-product property after factoring an equation that is equal to zero is the key to this technique. However, the equation may not be given equal to zero, and so there may be some preliminary steps before factoring. The steps required to **solve by factoring** are outlined in the following example.

## Example 6

Solve:  $15x^2 + 3x - 8 = 5x - 7$ .

Solution:

Step 1: Express the equation in standard form, equal to zero. In this example, subtract  $5x$  from and add 7 to both sides.

Step 2: Factor the expression.

$$(3x-1)(5x+1)=0$$

Step 3: Apply the zero-product property and set each variable factor equal to zero.

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$$3x-1=0 \quad \text{or} \quad 5x+1=0$$

Step 4: Solve the resulting linear equations.

Answer: The solutions are and . The check is optional.

### Example 7

Solve:  $(3x+2)(x+1)=4$ .

Solution:

This quadratic equation appears to be factored; hence it might be tempting to set each factor equal to 4. However, this would lead to incorrect results. We must rewrite the equation equal to zero, so that we can apply the zero-product property.

Once it is in standard form, we can factor and then set each factor equal to zero.

Answer: The solutions are and  $-2$ .

## Finding Roots of Functions

Recall that any polynomial with one variable is a function and can be written in the form,

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A **root** of a function is a value in the domain that results in zero. In other words, the roots occur when the function is equal to zero,  $f(x)=0$ .

## Example 8

Find the roots:  $f(x)=(x+2)^2-4$ .

Solution:

To find roots we set the function equal to zero and solve.

Next, set each factor equal to zero and solve.

We can show that these  $x$ -values are roots by evaluating.

✓

✓

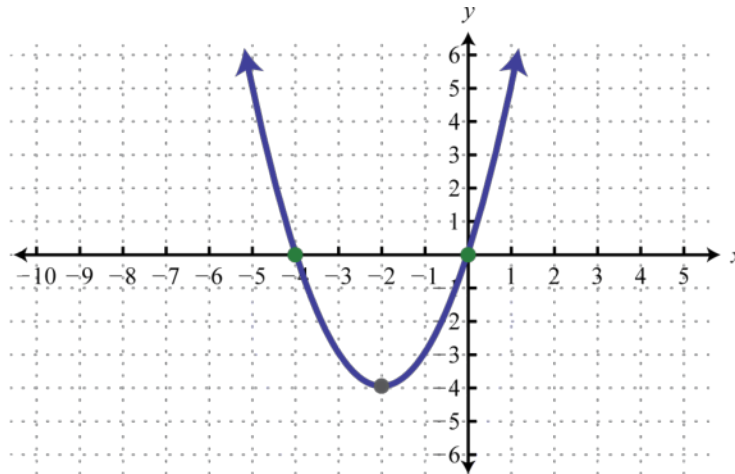
Answer: The roots are 0 and  $-4$ .

If we graph the function in the previous example we will see that the roots correspond to the  $x$ -intercepts of the function. Here the function  $f$  is a basic parabola shifted 2 units to the left and 4 units down.

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## Example 9

Find the roots:  $f(x)=x^4-5x^2+4$ .

Solution:

To find roots we set the function equal to zero and solve.

Next, set each factor equal to zero and solve.

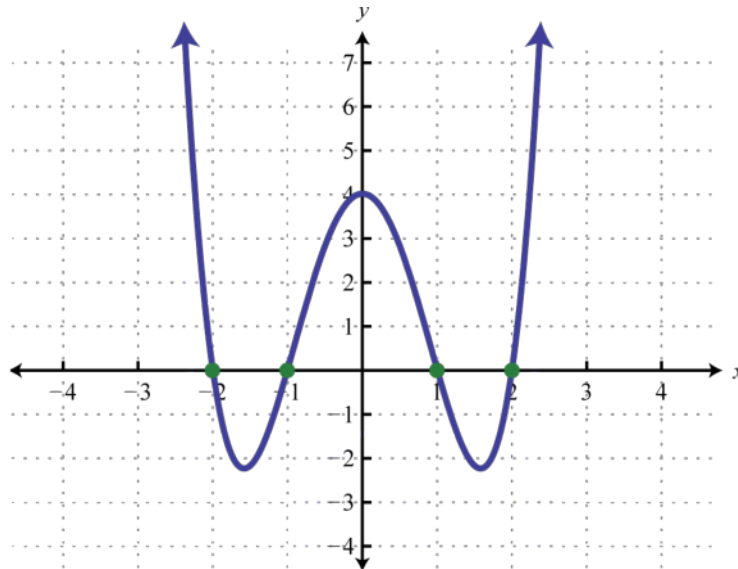
Answer: The roots are  $-1$ ,  $1$ ,  $-2$ , and  $2$ .

Graphing the previous function is not within the scope of this course. However, the graph is provided below:

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Notice that the degree of the polynomial is 4 and we obtained four roots. In general, for any polynomial function with one variable of degree  $n$ , the **fundamental theorem of algebra** guarantees  $n$  real roots or fewer. We have seen that many polynomials do not factor. This does not imply that functions involving these unfactorable polynomials do not have real roots. In fact, many polynomial functions that do not factor do have real solutions. We will learn how to find these types of roots as we continue in our study of algebra.

## Example 10

Find the roots:  $f(x) = -x^2 + 10x - 25$ .

Solution:

To find roots we set the function equal to zero and solve.

Next, set each variable factor equal to zero and solve.

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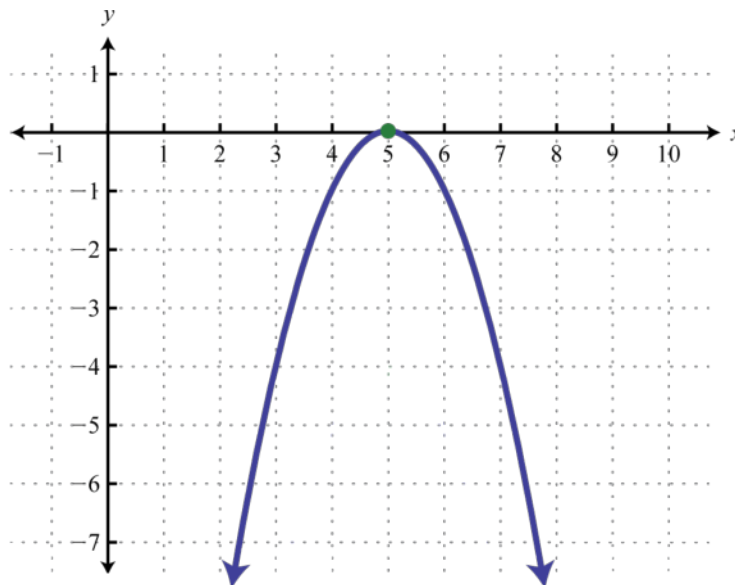
A solution that is repeated twice is called a double root. In this case, there is only one solution.

Answer: The root is 5.

The previous example shows that a function of degree 2 can have one root. From the factoring step, we see that the function can be written

$$f(x) = -(x-5)^2$$

In this form, we can see a reflection about the  $x$ -axis and a shift to the right 5 units. The vertex is the  $x$ -intercept, illustrating the fact that there is only one root.



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Try this! Find the roots of  $f(x)=x^3+3x^2-x-3$ .

Answer:  $\pm 1, -3$

## Example 11

Assuming dry road conditions and average reaction times, the safe stopping distance in feet is given by  $d(x) = \frac{x^2}{20} + x$ , where  $x$  represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 40 feet.

Solution:

We are asked to find the speed  $x$  where the safe stopping distance  $d(x)=40$  feet.

To solve for  $x$ , rewrite the resulting equation in standard form. In this case, we will first multiply both sides by 20 to clear the fraction.

Next factor and then set each factor equal to zero.

The negative answer does not make sense in the context of this problem.

Consider  $x=20$  miles per hour to be the only solution.

Answer: 20 miles per hour

## Finding Equations with Given Solutions

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We can use the zero-product property to find equations, given the solutions. To do this, the steps for solving by factoring are performed in reverse.

## Example 12

Find a quadratic equation with integer coefficients, given solutions and

Solution:

Given the solutions, we can determine two linear factors. To avoid fractional coefficients, we first clear the fractions by multiplying both sides by the denominator.

The product of these linear factors is equal to zero when or

$$(2x+3)(3x-1)=0$$

Multiply the binomials and present the equation in standard form.

We may check our equation by substituting the given answers to see if we obtain a true statement. Also, the equation found above is not unique and so the check becomes essential when our equation looks different from someone else's. This is left as an exercise.

Answer:  $6x^2+7x-3=0$

## Example 13

Find a polynomial function with real roots 1, -2, and 2.

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Solution:

Given solutions to  $f(x)=0$  we can find linear factors.

Apply the zero-product property and multiply.

Answer:  $f(x)=x^3-x^2-4x+4$

Try this! Find a polynomial equation with integer coefficients, given solutions and

Answer:  $8x^2+2x-3=0$

## KEY TAKEAWAYS

Factoring and the zero-product property allow us to solve equations.

To solve a polynomial equation, first write it in standard form. Once it is equal to zero, factor it and then set each variable factor equal to zero. The solutions to the resulting equations are the solutions to the original.

Not all polynomial equations can be solved by factoring. We will learn how to solve polynomial equations that do not factor later in the course.

A polynomial function can have at most a number of real roots equal to its degree. To find roots of a function, set it equal to zero and solve.

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To find a polynomial equation with given solutions, perform the process of solving by factoring in reverse.

## TOPIC EXERCISES

### PART A: GENERAL FACTORING

Factor completely.

1.  $50x^2 - 18$
2.  $12x^3 - 3x$
3.  $10x^3 + 65x^2 - 35x$
4.  $15x^4 + 7x^3 - 4x^2$
5.  $6a^4b - 15a^3b^2 - 9a^2b^3$
6.  $8a^3b - 44a^2b^2 + 20ab^3$
7.  $36x^4 - 72x^3 - 4x^2 + 8x$
8.  $20x^4 + 60x^3 - 5x^2 - 15x$
9.  $3x^5 + 2x^4 - 12x^3 - 8x^2$
10.  $10x^5 - 4x^4 - 90x^3 + 36x^2$
11.  $x^4 - 23x^2 - 50$
12.  $2x^4 - 31x^2 - 16$
13.  $-2x^5 - 6x^3 + 8x$
14.  $-36x^5 + 69x^3 + 27x$
15.  $54x^5 - 78x^3 + 24x$
16.  $4x^6 - 65x^4 + 16x^2$
17.  $x^6 - 7x^3 - 8$
18.  $x^6 - 25x^3 - 54$

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19.  $3x^6+4x^3+1$

20.  $27x^6-28x^3+1$

**PART B: SOLVING POLYNOMIAL EQUATIONS BY FACTORING**

Solve.

21.  $(6x-5)(x+7)=0$

22.  $(x+9)(3x-8)=0$

23.  $5x(2x-5)(3x+1)=0$

24.  $4x(5x-1)(2x+3)=0$

25.  $(x-1)(2x+1)(3x-5)=0$

26.  $(x+6)(5x-2)(2x+9)=0$

27.  $(x+4)(x-2)=16$

28.  $(x+1)(x-7)=9$

29.  $(6x+1)(x+1)=6$

30.  $(2x-1)(x-4)=39$

31.  $x^2-15x+50=0$

32.  $x^2+10x-24=0$

33.  $3x^2+2x-5=0$

34.  $2x^2+9x+7=0$

35.

36.

37.  $6x^2-5x-2=30x+4$

38.  $6x^2-9x+15=20x-13$

39.  $5x^2-23x+12=4(5x-3)$

40.  $4x^2+5x-5=15(3-2x)$

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41.  $(x+6)(x-10)=4(x-18)$
42.  $(x+4)(x-6)=2(x+4)$
43.  $4x^3-14x^2-30x=0$
44.  $9x^3+48x^2-36x=0$
- 45.
- 46.
47.  $-10x^3-28x^2+48x=0$
48.  $-2x^3+15x^2+50x=0$
49.  $2x^3-x^2-72x+36=0$
50.  $4x^3-32x^2-9x+72=0$
51.  $45x^3-9x^2-5x+1=0$
52.  $x^3-3x^2-x+3=0$
53.  $x^4-5x^2+4=0$
54.  $4x^4-37x^2+9=0$

Find the roots of the given functions.

21.  $f(x)=x^2+10x-24$
22.  $f(x)=x^2-14x+48$
23.  $f(x)=-2x^2+7x+4$
24.  $f(x)=-3x^2+14x+5$
25.  $f(x)=16x^2-40x+25$
26.  $f(x)=9x^2-12x+4$
27.  $g(x)=8x^2+3x$

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28.  $g(x)=5x^2-30x$

29.  $p(x)=64x^2-1$

30.  $q(x)=4x^2-121$

31.

32.

33.  $g(x)=x^4-13x^2+36$

34.  $g(x)=4x^4-13x^2+9$

35.  $f(x)=(x+5)^2-1$

36.  $g(x)=-(x+5)^2+9$

37.  $f(x)=-(3x-5)^2$

38.  $g(x)=-(x+2)^2+4$

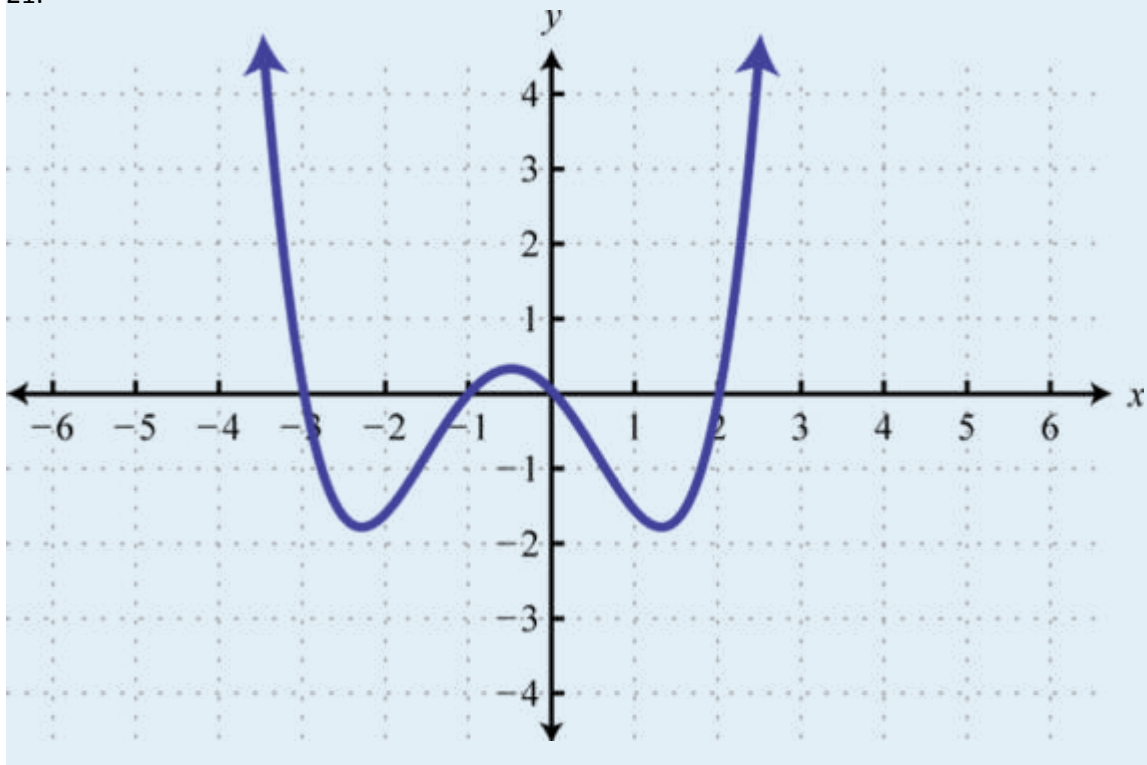
Given the graph of a function, determine the real roots.

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21.

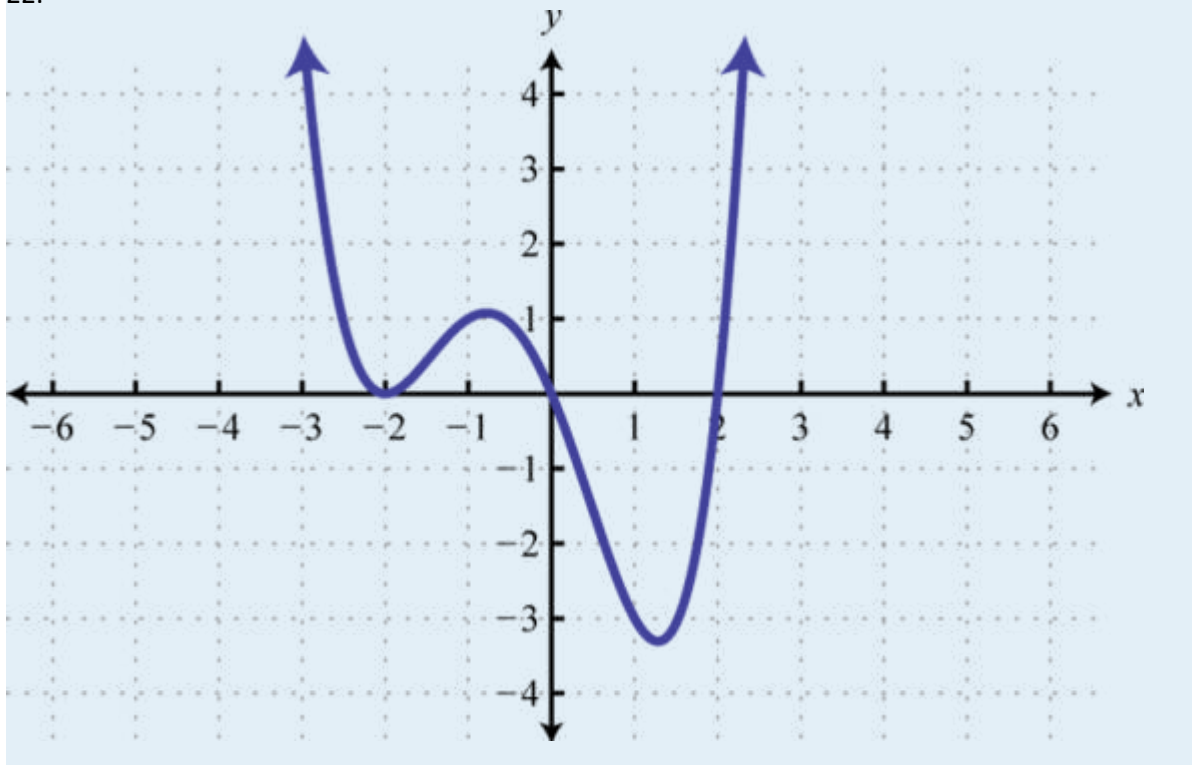


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22.



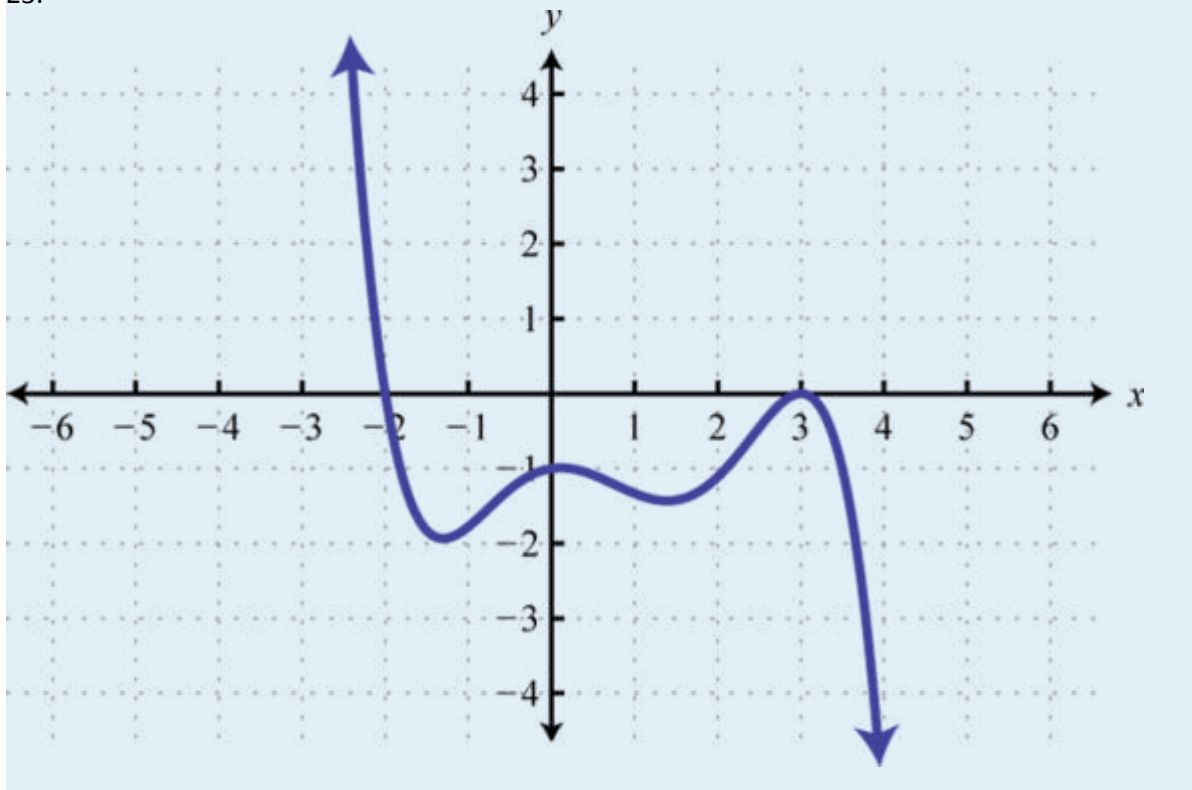
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23.

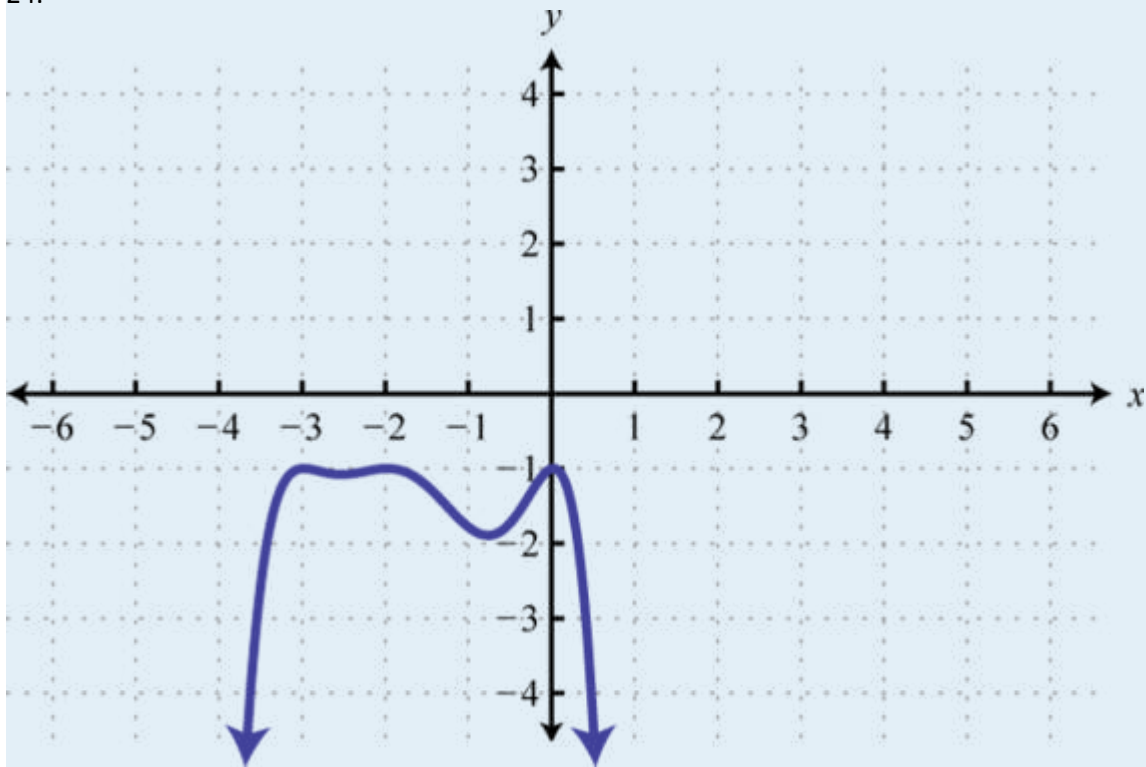


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24.



25. The sides of a square measure  $x-2$  units. If the area is 36 square units, then find  $x$ .

26. The sides of a right triangle have lengths that are consecutive even integers. Find the lengths of each side. (Hint: Apply the Pythagorean theorem)

27. The profit in dollars generated by producing and selling  $n$  bicycles per week is given by the formula  $P(n)=-5n^2+400n-6000$ . How many bicycles must be produced and sold to break even?

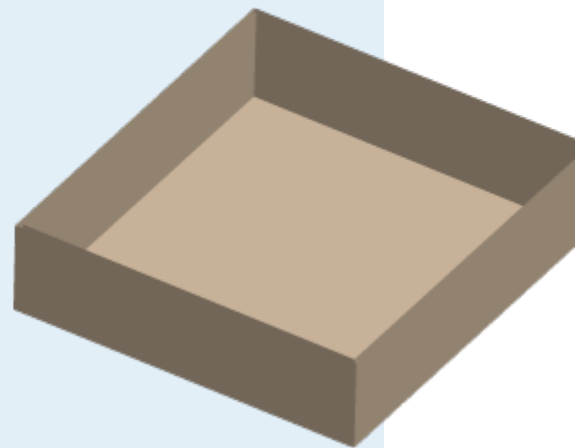
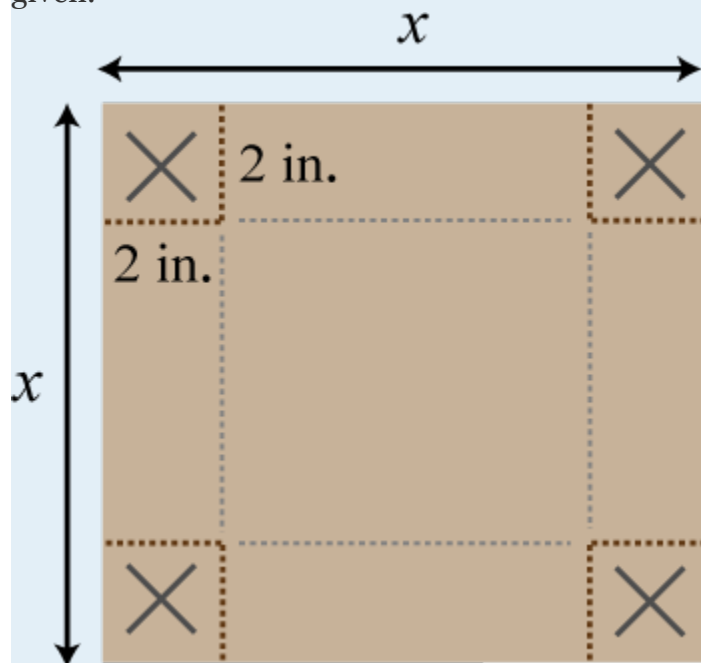
28. The height in feet of an object dropped from the top of a 64-foot building is given by  $h(t)=-16t^2+64$  where  $t$  represents the time in seconds after it is dropped. How long will it take to hit the ground?

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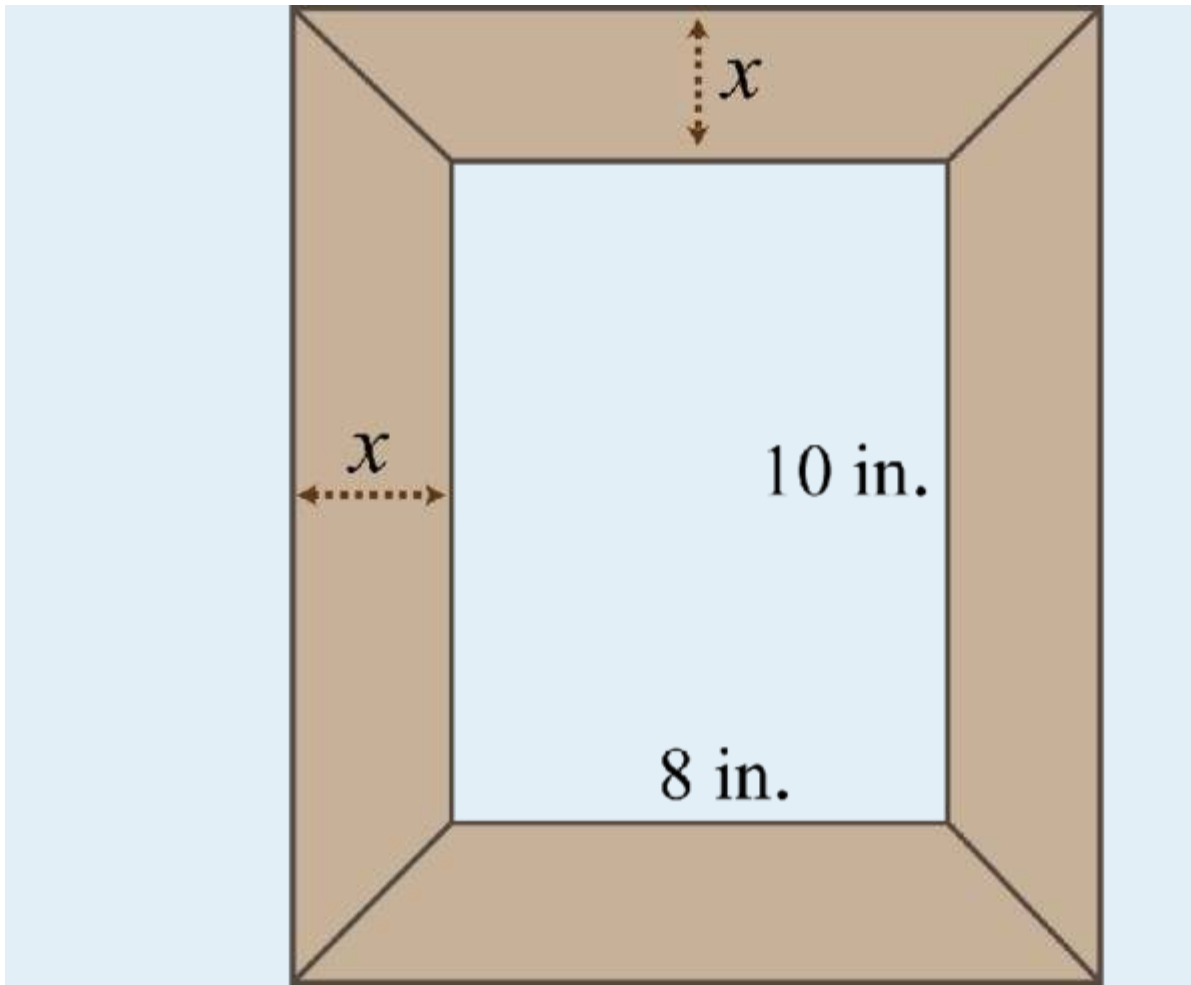
29. A box can be made by cutting out the corners and folding up the edges of a square sheet of cardboard. A template for a cardboard box of height 2 inches is given.



What is the length of each side of the cardboard sheet if the volume of the box is to be 98 cubic inches?

30. The height of a triangle is 4 centimeters less than twice the length of its base. If the total area of the triangle is 48 square centimeters, then find the lengths of the base and height.

31. A uniform border is to be placed around an  $8 \times 10$  inch picture.



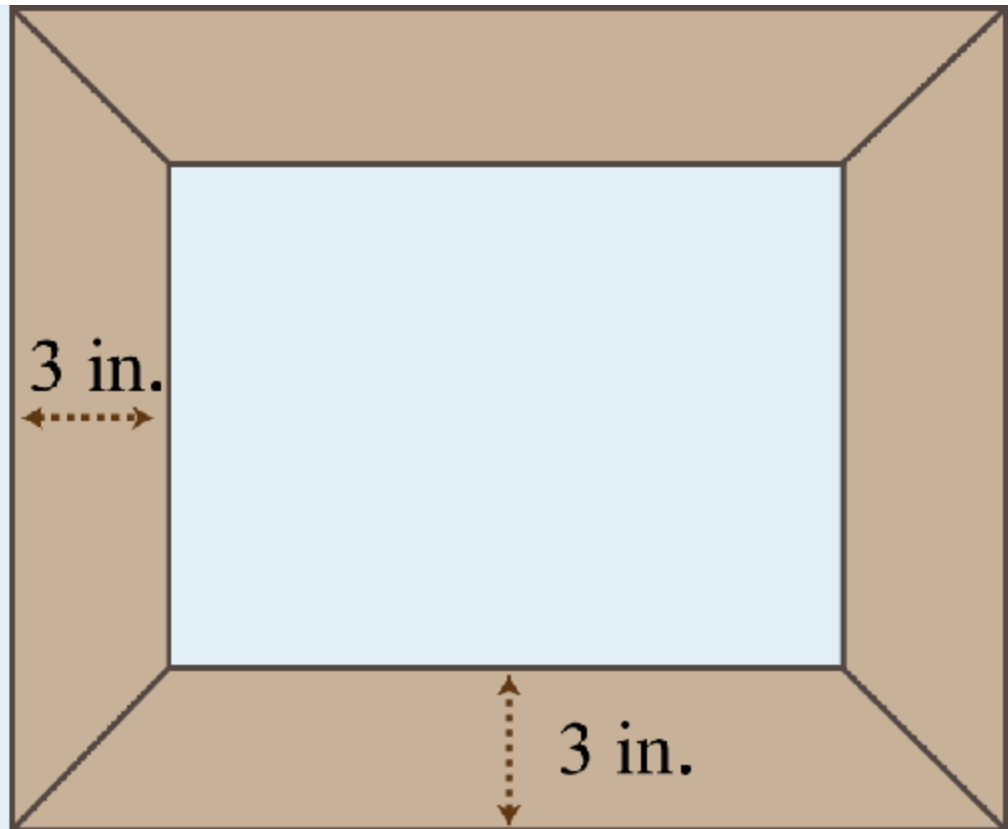
If the total area including the border must be 168 square inches, then how wide should the border be?

32. The area of a picture frame including a 3-inch wide border is 120 square inches.

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If the width of the inner area is 2 inches less than its length, then find the dimensions of the inner area.

33. Assuming dry road conditions and average reaction times, the safe stopping distance in feet is given by  $d = \frac{x^2}{20}$  where  $x$  represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 75 feet.

34. A manufacturing company has determined that the daily revenue in thousands of dollars is given by the formula  $R(n) = 12n - 0.6n^2$  where  $n$  represents the number of palettes of product sold ( $0 \leq n < 20$ ). Determine the number of palettes sold in a day if the revenue was 45 thousand dollars.

## PART C: FINDING EQUATIONS WITH GIVEN SOLUTIONS

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Find a polynomial equation with the given solutions.

- 87.  $-3, 5$
- 88.  $-1, 8$
- 89.  $2,$
- 90.  $, 5$
- 91.  $0, -4$
- 92.  $0, 7$
- 93.  $\pm 7$
- 94.  $\pm 2$
- 95.  $-3, 1, 3$
- 96.  $-5, -1, 1$

Find a function with the given roots.

- 87.  $,$
- 88.  $,$
- 89.
- 90.
- 91.  $5$  double root
- 92.  $-3$  double root
- 93.  $-1, 0, 3$
- 94.  $-5, 0, 2$

Recall that if  $|X|=p$ , then  $X=-p$  or  $X=p$ . Use this to solve the following absolute value equations.

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- 87.
- 88.
- 89.
- 90.
- 91.
- 92.

## PART D: DISCUSSION BOARD

111. Explain to a beginning algebra student the difference between an equation and an expression.
112. What is the difference between a root and an  $x$ -intercept? Explain.
113. Create a function with three real roots of your choosing. Graph it with a graphing utility and verify your results. Share your function on the discussion board.
114. Research and discuss the fundamental theorem of algebra.

## ANSWERS

1.  $2(5x+3)(5x-3)$
- 2.
3.  $5x(x+7)(2x-1)$
- 4.
5.  $3a^2b(2a+b)(a-3b)$
- 6.
7.  $4x(x-2)(3x+1)(3x-1)$
- 8.
9.  $x^2(3x+2)(x+2)(x-2)$

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10.

11.  $(x^2+2)(x+5)(x-5)$

12.

13.  $-2x(x^2+4)(x-1)(x+1)$

14.

15.  $6x(x+1)(x-1)(3x+2)(3x-2)$

16.

17.  $(x+1)(x^2-x+1)(x-2)(x^2+2x+4)$

18.

19.  $(3x^3+1)(x+1)(x^2-x+1)$

20.

21.  $-7,$

22.

23.  $0, , -$

24.

25.  $-, 1,$

26.

27.  $-6, 4$

28.

29.  $-,$

30.

31.  $5, 10$

32.

33.  $-, 1$

34.

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- 35.  $-5$
- 36.
- 37.  $-6$
- 38.
- 39.  $8$
- 40.
- 41.  $2, 6$
- 42.
- 43.  $0, -5$
- 44.
- 45.  $0, \pm 32$
- 46.
- 47.  $-4, 0,$
- 48.
- 49.  $\pm 6,$
- 50.
- 51.  $\pm,$
- 52.
- 53.  $\pm 1, \pm 2$
- 54.
- 55.  $2, -12$
- 56.
- 57.  $-4$
- 58.

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- 59.
60.  $- , 0$
- 61.
62.  $\pm$
- 63.
64.  $\pm, 5$
- 65.
66.  $\pm 2, \pm 3$
- 67.
68.  $-6, -4$
- 69.
- 70.
71.  $-3, -1, 0, 2$
- 72.
73.  $-2, 3$
- 74.
75. 8 units
- 76.
77. 20 or 60 bicycles
- 78.
79. 11 in
- 80.
81. 2 inches
- 82.

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83. 30 miles per hour

84.

87.  $x^2 - 2x - 15 = 0$

88.

89.  $3x^2 - 7x + 2 = 0$

90.

91.  $x^2 + 4x = 0$

92.

93.  $x^2 - 49 = 0$

94.

95.  $x^3 - x^2 - 9x + 9 = 0$

96.

97.  $f(x) = 6x^2 - 7x + 2$

98.

99.  $f(x) = 16x^2 - 9$

100.

101.  $f(x) = x^2 - 10x + 25$

102.

103.  $f(x) = x^3 - 2x^2 - 3x$

104.

105.  $\pm 4, 0$

106.

107.  $\pm 1, 3$

108.

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109.  $-2, 1, 4$

110.

111. Answer may vary

112.

113. Answer may vary

## 4.5 Rational Functions: Multiplication and Division

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## LEARNING OBJECTIVES

1. Identify restrictions to the domain of a rational function.
2. Simplify rational functions.
3. Multiply and divide rational functions.

### Identifying Restrictions and Simplifying Rational Functions

Rational functions have the form

where  $p(x)$  and  $q(x)$  are polynomials and  $q(x) \neq 0$ . The

domain of a rational function consists of all real numbers  $x$  except those where the denominator  $q(x) = 0$ . **Restrictions** are the real numbers for which the expression is not defined. We often express the domain of a rational function in terms of its restrictions. For example, consider the function

which can be written in factored form

Because rational expressions are undefined when the denominator is 0, we wish to find the values for  $x$  that make it 0. To do this, apply the zero-product property. Set each factor in the denominator equal to 0 and solve.

Therefore, the original function is defined for any real number except 2 and 3. We can express its domain using notation as follows:

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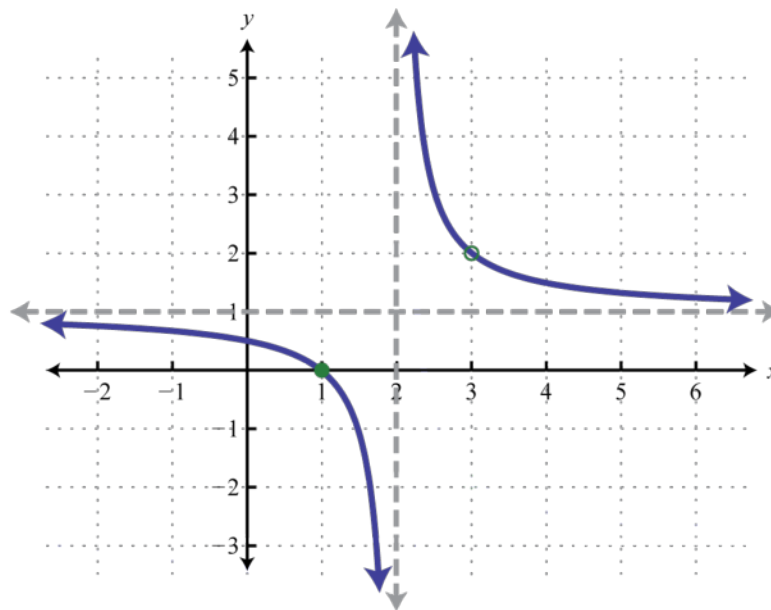
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The restrictions to the domain of a rational function are determined by the denominator. Once the restrictions are determined we can cancel factors and obtain an equivalent function as follows:

$$f(x) = \frac{(x-1)\cancel{(x-3)}}{(x-2)\cancel{(x-3)}} = \frac{x-1}{x-2}$$

It is important to note that 1 is *not* a restriction to the domain because the expression is defined as 0 when the numerator is 0. In fact,  $x=1$  is a root. This function is graphed below:



Notice that there is a vertical asymptote at the restriction  $x=2$  and the graph is left undefined at the restriction  $x=3$  as indicated by the open dot, or hole, in the graph. Graphing rational functions in general is beyond the scope of this textbook. However, it

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is useful at this point to know that the restrictions are an important part of the graph of rational functions.

## Example 1

State the restrictions and simplify:

Solution:

In this example, the function is undefined where  $x$  is 0.

Therefore, the domain consists of all real numbers  $x$ , where  $x \neq 0$ . With this understanding, we can simplify by reducing the rational expression to lowest terms. Cancel common factors.

Answer:  $g(x) = 4x^2$ , where  $x \neq 0$

## Example 2

State the restrictions and simplify:

Solution:

First, factor the numerator and denominator.

Any  $x$ -value that makes the denominator zero is a restriction. To find the restrictions, first set the denominator equal to zero and then solve

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Therefore, With this understanding, we can cancel any common factors.

Answer: , where

We define the opposite of a polynomial  $P$  to be  $-P$ . Finding the opposite of a polynomial requires the application of the distributive property. For example, the opposite of the polynomial  $(x-3)$  is written as

This leads us to the **opposite binomial property**,  $-(a-b)=(b-a)$ . Care should be taken not to confuse this with the fact that  $(a+b)=(b+a)$ . This is the case because addition is commutative. In general,

Also, it is important to recall that

In other words, a negative fraction is shown by placing the negative sign in either the numerator, in front of the fraction bar, or in the denominator. Generally, negative denominators are avoided.

### Example 3

State the restrictions and simplify:

Solution:

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Begin by factoring the numerator and denominator.

Answer: , where  $x \neq 5$

It is important to remember that we can only cancel factors of a product. A common mistake is to cancel terms. For example,

Try this! State the restrictions and simplify:

Answer: , where

In some examples, we will make a broad assumption that the denominator is nonzero. When we make that assumption, we do not need to determine the restrictions.

## Example 4

Simplify: (*Assume all denominators are nonzero.*)

Solution:

Factor the numerator by grouping. Factor the denominator using the formula for a difference of squares.

Next, cancel common factors.

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Note: When the entire numerator or denominator cancels out a factor of 1 always remains.

Answer:

## Example 5

Given  $f(x)=x^2-2x+5$ , simplify

Solution:

Begin by calculating  $f(3)$ .

Next, substitute into the quotient that is to be simplified.

Answer:  $x+1$ , where  $x \neq 3$

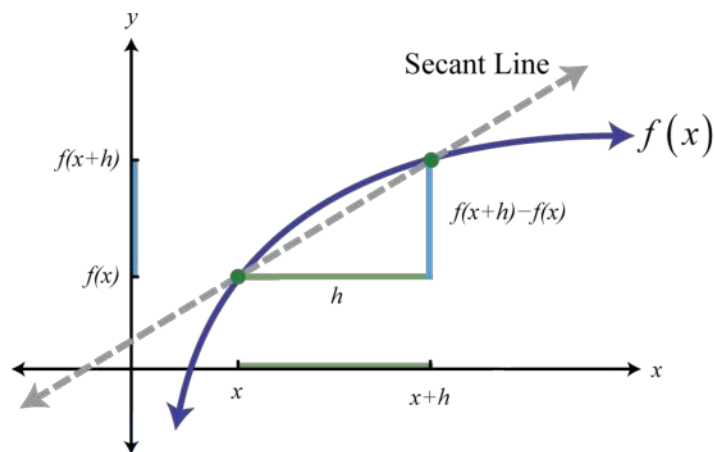
An important quantity in higher level mathematics is the **difference quotient**:

This quantity represents the slope of the line connecting two points on the graph of a function. The line passing through the two points is called a **secant line**.

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Calculating the difference quotient for many different functions is an important skill to learn in intermediate algebra. We will encounter this quantity often as we proceed in this textbook. When calculating the difference quotient we assume the denominator is nonzero.

### Example 6

Given  $g(x) = -2x^2 + 1$ , simplify

Solution:

Answer:  $-4x - 2h$

Try this! Given  $f(x) = x^2 - x - 1$ , simplify

Answer:  $2x - 1 + h$

## Multiplying and Dividing Rational Functions

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When multiplying fractions, we can multiply the numerators and denominators together and then reduce. Multiplying rational expressions is performed in a similar manner. In general, given polynomials  $P$ ,  $Q$ ,  $R$ , and  $S$ , where  $Q \neq 0$  and  $S \neq 0$ , we have

The restrictions to the domain of a product consist of the restrictions of each function.

## Example 7

Given  $f(x) = \frac{3x-5}{x+3}$  and  $g(x) = \frac{x+3}{x-5}$ , find  $(f \cdot g)(x)$  and determine the restrictions to the domain.

Solution:

In this case, the domain of  $f$  consists of all real numbers except 5, and the domain of  $g$  consists of all real numbers except  $-3$ . Therefore, the domain of the product consists of all real numbers except 5 and  $-3$ . Multiply the functions and then simplify the result.

Answer:  $(f \cdot g)(x) = (3x-5)(x+3)$ , where

To divide two fractions, we multiply by the reciprocal of the divisor. Dividing rational expressions is performed in a similar manner. In general, given polynomials  $P$ ,  $Q$ ,  $R$ , and  $S$ , where  $Q \neq 0$ ,  $R \neq 0$ , and  $S \neq 0$ , we have

The restrictions to the domain of a quotient will consist of the restrictions of each function as well as the restrictions on the reciprocal of the divisor.

## Example 8

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Given  $f(x) = \frac{1}{x+3}$  and  $g(x) = \frac{1}{x-7}$ , find  $(f/g)(x)$  and determine the restrictions.

Solution:

In this case, the domain of  $f$  consists of all real numbers except  $-3$  and  $7$ , and the domain of  $g$  consists of all real numbers except  $7$  and  $-7$ . In addition, the reciprocal of  $g(x)$  has a restriction of  $-3$  and  $7$ . Therefore, the domain of this quotient consists of all real numbers except  $-3$ ,  $7$ , and  $\pm 7$ .

Answer:  $\frac{x-7}{x+3}$ , where  $x \neq -3, 7, \pm 7$

Recall that multiplication and division operations are to be performed from left to right.

## Example 9

Simplify:  $\frac{2x^2 - 5x + 2}{x^2 - 4}$  (Assume all denominators are nonzero.)

Solution:

Begin by replacing the factor that is to be divided by multiplication of its reciprocal.

Answer:

Try this! Given  $f(x) = \frac{1}{x+2}$  and  $g(x) = \frac{1}{x-3}$ , calculate  $(f/g)(x)$  and determine the restrictions.

Answer:  $\frac{x-3}{x+2}$ , where  $x \neq -2, 3, \pm 3$

If a cost function  $C$  represents the cost of producing  $x$  units, then the **average cost** is the cost divided by the number of units produced.

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## Example 10

A manufacturer has determined that the cost in dollars of producing sweaters is given by  $C(x)=0.01x^2-3x+1200$ , where  $x$  represents the number of sweaters produced daily. Determine the average cost of producing 100, 200, and 300 sweaters per day.

Solution:

Set up a function representing the average cost.

Next, calculate  $(100)$ ,  $(200)$ , and  $(300)$ .

Answer: The average cost of producing 100 sweaters per day is \$10.00 per sweater. If 200 sweaters are produced, the average cost per sweater is \$5.00. If 300 are produced, the average cost per sweater is \$4.00.

## KEY TAKEAWAYS

Simplifying rational expressions is similar to simplifying fractions. First, factor the numerator and denominator and then cancel the common factors. Rational expressions are simplified if there are no common factors other than 1 in the numerator and the denominator.

Simplified rational functions are equivalent for values in the domain of the original function. Be sure to state the restrictions unless the problem states that the denominators are assumed to be nonzero.

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After multiplying rational expressions, factor both the numerator and denominator and then cancel common factors. Make note of the restrictions to the domain. The values that give a value of 0 in the denominator for all expressions are the restrictions.

To divide rational expressions, multiply the numerator by the reciprocal of the divisor.

The restrictions to the domain of a product consist of the restrictions to the domain of each factor.

## TOPIC EXERCISES

### PART A: SIMPLIFYING RATIONAL FUNCTIONS

Simplify the function and state its domain using interval notation.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.

State the restrictions and simplify the given rational expressions.

- 1.
- 2.

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- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.

Simplify the given rational expressions. Assume all variable expressions in the denominator are nonzero.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.

Given the function, simplify the rational expression.

1. Given  $f(x)=x^2-8$ , simplify

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2. Given  $f(x)=x^2+4x-1$ , simplify
3. Given  $g(x)=x^2-3x+1$ , simplify
4. Given  $g(x)=x^2-2x$ , simplify
5. Given  $f(x)=4x^2+6x+1$ , simplify
6. Given  $f(x)=9x^2+1$ , simplify

For the given function, simplify the difference quotient

1.  $f(x)=5x-3$
2.  $f(x)=3-2x$
3.  $f(x)=x^2-3$
4.  $f(x)=x^2+8x$
5.  $f(x)=x^2-x+5$
6.  $f(x)=4x^2+3x-2$
7.  $f(x)=ax^2+bx+c$
8.  $f(x)=ax^2+bx$
9.  $f(x)=x^3+1$
10.  $f(x)=x^3-x+2$

## PART B: MULTIPLYING AND DIVIDING RATIONAL FUNCTIONS

Simplify the product  $f \cdot g$  and state its domain using interval notation.

- 51.
- 52.
- 53.
- 54.
- 55.
- 56.

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Simplify the quotient  $f/g$  and state its domain using interval notation.

51.

52.

53.

54.

55.

56.

Multiply or divide as indicated, state the restrictions, and simplify.

51.

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61.

62.

Perform the operations and simplify. Assume all variable expressions in the denominator are nonzero.

51.

52.

53.

54.

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55.

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60.

61. A manufacturer has determined that the cost in dollars of producing bicycles is given by  $C(x)=0.5x^2-x+6200$ , where  $x$  represents the number of bicycles produced weekly. Determine the average cost of producing 50, 100, and 150 bicycles per week.

62. The cost in dollars of producing custom lighting fixtures is given by the function  $C(x)=x^2-20x+1200$ , where  $x$  represents the number of fixtures produced in a week. Determine the average cost per unit if 20, 40, and 50 units are produced in a week.

63. A manufacturer has determined that the cost in dollars of producing electric scooters is given by the function  $C(x)=3x(x-100)+32,000$ , where  $x$  represents the number of scooters produced in a month. Determine the average cost per scooter if 50 are produced in a month.

64. The cost in dollars of producing a custom injected molded part is given by  $C(n)=1,900+0.01n$ , where  $n$  represents the number of parts produced. Calculate the average cost of each part if 2,500 custom parts are ordered.

65. The cost in dollars of an environmental cleanup is given by the function  $C(p)=1000p^2$ , where  $p$  represents the percentage of the area to be cleaned up ( $0 \leq p < 1$ ). Use the function to determine the cost of cleaning up 50% of an affected area and the cost of cleaning up 80% of the area.



66. The value of a new car is given by the function  $V(t)=16,500(t+1)^{-1}$  where  $t$  represents the age of the car in years. Determine the value of the car when it is 6 years old.

## PART D: DISCUSSION BOARD

91. Describe the restrictions to the rational expression . Explain.
92. Describe the restrictions to the rational expression . Explain.
93. Explain why  $x=5$  is a restriction to
94. Explain to a beginning algebra student why we cannot cancel  $x$  in the rational expression
95. Research and discuss the importance of the difference quotient. What does it represent and in what subject does it appear?

## ANSWERS

1.  $f(x)=5x^4$ ;  
Domain:  $(-\infty,0)\cup(0,\infty)$
- 2.
3. ; Domain:  $(-\infty,-8)\cup(-8,\infty)$
- 4.
5. ; Domain:
- 6.
7. ; Domain:
- 8.
9.  $h(x)=x-1$ ; Domain:  $(-\infty,-1)\cup(-1,\infty)$
- 10.
11. ;

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12.

13. ;  $x \neq -2, 7$

14.

15. ;

16.

17. ;  $x \neq \pm 3$

18.

19. ;  $x \neq 1, \pm 4$

20.

21.

22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.  $x+5$ , where  $x \neq 5$

35.

36.  $x-4$ , where  $x \neq -1$

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37.

38.  $2(x+2)$ , where

39.

40. 5

41.

42.  $2x+h$

43.

44.  $2x-1+h$

45.

46.  $2ax+b+ah$

47.

48.  $3x^2+3xh+h^2$

49.

51. ; Domain:  $(-\infty,0)\cup(0,2)\cup(2,\infty)$

52.

53. ; Domain:  $(-\infty,-2)\cup(-2,0)\cup(0,\infty)$

54.

55. ; Domain:

56.

57.  $(f/g)(x)=10x(5x-1)$ ; Domain:

58.

59. ; Domain:

60.

61. ; Domain:

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62.

63. ;  $x \neq -12, 0$

64.

65. ;  $x \neq -8, 0, 4$

66.

67.  $5(x+1)$ ;

68.

69. ;  $x \neq -7, -2, 0, 3$

70.

71. ;

72.

73. ;

74.

75.

76.

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83.

84.

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85. If 50 bicycles are produced, the average cost per bicycle is \$148. If 100 are produced, the average cost is \$111. If 150 bicycles are produced, the average cost is \$115.33.

86.

87. If 50 scooters are produced, the average cost of each is \$490.

88.

89. A 50% cleanup will cost \$25,000. An 80% cleanup will cost \$100,000.

90.

91. Answer may vary

92.

93. Answer may vary

94.

95. Answer may vary





## 4.6 Rational Functions: Addition and Subtraction

### LEARNING OBJECTIVES

1. Add and subtract rational functions.
2. Simplify complex rational expressions.

### Adding and Subtracting Rational Functions

Adding and subtracting rational expressions is similar to adding and subtracting fractions. Recall that if the denominators are the same, we can add or subtract the numerators and write the result over the common denominator. When working with rational expressions, the common denominator will be a polynomial. In general, given polynomials  $P$ ,  $Q$ , and  $R$ , where  $Q \neq 0$ , we have the following:

The set of restrictions to the domain of a sum or difference of rational expressions consists of the restrictions to the domains of each expression.

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## Example 1

Subtract:

Solution:

The denominators are the same. Hence we can subtract the numerators and write the result over the common denominator. Take care to distribute the negative 1.

Answer: , where  $x \neq \pm 8$

To add rational expressions with unlike denominators, first find equivalent expressions with common denominators. Do this just as you have with fractions. If the denominators of fractions are relatively prime, then the least common denominator (LCD) is their product. For example,

Multiply each fraction by the appropriate form of 1 to obtain equivalent fractions with a common denominator.

In general, given polynomials  $P$ ,  $Q$ ,  $R$ , and  $S$ , where  $Q \neq 0$  and  $S \neq 0$ , we have the following:

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## Example 2

Given  $f(x) = \frac{2x+1}{x+1}$  and  $g(x) = \frac{3x+1}{x+1}$ , find  $f+g$  and state the restrictions.

Solution:

Here the LCD is the product of the denominators  $(3x+1)(x+1)$ . Multiply by the appropriate factors to obtain rational expressions with a common denominator before adding.

The domain of  $f$  consists all real numbers except  $-1$ , and the domain of  $g$  consists of all real numbers except  $-1$ . Therefore, the domain of  $f + g$  consists of all real numbers except  $-1$  and  $-13$ .

Answer:  $(\frac{5x+2}{x+1})$ , where

It is not always the case that the LCD is the product of the given denominators. Typically, the denominators are not relatively prime; thus determining the LCD requires some thought. Begin by factoring all denominators. The LCD is the product of all factors with the highest power.

## Example 3

Given  $f(x) = \frac{2x+1}{x+1}$  and  $g(x) = \frac{3x+1}{x+1}$ , find  $f-g$  and state the restrictions to the domain.

Solution:

To determine the LCD, factor the denominator of  $g$ .

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In this case the  $LCD=(3x-1)(x-1)$ . Multiply  $f$  by 1 in the form of to obtain equivalent algebraic fractions with a common denominator and then subtract.

The domain of  $f$  consists of all real numbers except , and the domain of  $g$  consists of all real numbers except 1 and . Therefore, the domain of  $f - g$  consists of all real numbers except 1 and .

Answer: , where  $x \neq 1$

## Example 4

Simplify and state the restrictions:

Solution:

Begin by applying the opposite binomial property  $6-x=-(x-6)$ .

Next, find equivalent fractions with the  $LCD=(x+6)(x-6)$  and then simplify.

Answer: , where  $x \neq \pm 6$

Try this! Simplify and state the restrictions:

Answer: , where  $x \neq \pm 1$

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Rational expressions are sometimes expressed using negative exponents. In this case, apply the rules for negative exponents before simplifying the expression.

## Example 5

Simplify and state the restrictions:  $5a^{-2} + (2a+5)^{-1}$ .

Solution:

Recall that . Begin by rewriting the rational expressions with negative exponents as fractions.

Then find the LCD and add.

Answer: , where

## Simplifying Complex Rational Expressions

A **complex rational expression** is defined as a rational expression that contains one or more rational expressions in the numerator or denominator or both. For example,

is a complex rational expression. We simplify a complex rational expression by finding an equivalent fraction where the numerator and denominator are polynomials. There are two methods for simplifying complex rational expressions, and we will outline the steps for both methods. For the sake of clarity, assume that variable expressions used as denominators are nonzero.

### Method 1: Simplify Using Division

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We begin our discussion on simplifying complex rational expressions using division. Before we can multiply by the reciprocal of the denominator, we must simplify the numerator and denominator separately. The goal is to first obtain single algebraic fractions in the numerator and the denominator. The steps for simplifying a complex algebraic fraction are illustrated in the following example.

## Example 6

Simplify: .

Solution:

Step 1: Simplify the numerator and denominator to obtain a single algebraic fraction divided by another single algebraic fraction. In this example, find equivalent terms with a common denominator in both the numerator and denominator before adding and subtracting.

At this point we have a single algebraic fraction divided by another single algebraic fraction.

Step 2: Multiply the numerator by the reciprocal of the denominator.

Step 3: Factor all numerators and denominators completely.

Step 4: Cancel all common factors.

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Answer:

## Example 7

Simplify: .

Solution:

Obtain a single algebraic fraction in the numerator and in the denominator.

Next, multiply the numerator by the reciprocal of the denominator, factor, and then cancel.

Answer:

Try this! Simplify using division: .

Answer:

Sometimes complex rational expressions are expressed using negative exponents.

## Example 8

Simplify: .

Solution:

We begin by rewriting the expression without negative exponents.

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Obtain single algebraic fractions in the numerator and denominator and then multiply by the reciprocal of the denominator.

Apply the opposite binomial property  $(y-2x)=-(-2x-y)$  and then cancel.

Answer:

### **Method 2: Simplify Using the LCD**

An alternative method for simplifying complex rational expressions involves clearing the fractions by multiplying the expression by a special form of 1. In this method, multiply the numerator and denominator by the least common denominator (LCD) of all given fractions.

## **Example 9**

Simplify: .

Solution:

Step 1: Determine the LCD of all the fractions in the numerator and denominator. In this case, the denominators of the given fractions are 1,  $x$ , and  $x^2$ . Therefore, the LCD is  $x^2$ .

Step 2: Multiply the numerator and denominator by the LCD. This step should clear the fractions in both the numerator and denominator.

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This leaves us with a single algebraic fraction with a polynomial in the numerator and in the denominator.

Step 3: Factor the numerator and denominator completely.

Step 4: Cancel all common factors.

Note: This was the same problem presented in Example 6 and the results here are the same. It is worth taking the time to compare the steps involved using both methods on the same problem.

Answer:

It is important to point out that multiplying the numerator and denominator by the same nonzero factor is equivalent to multiplying by 1 and does not change the problem.

Try this! Simplify using the LCD: .

Answer:

## KEY TAKEAWAYS

Adding and subtracting rational expressions is similar to adding and subtracting fractions. A common denominator is required. If the denominators are the same, then we can add or subtract the numerators and write the result over the common denominator.

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The set of restrictions to the domain of a sum or difference of rational functions consists of the restrictions to the domains of each function.

Complex rational expressions can be simplified into equivalent expressions with a polynomial numerator and polynomial denominator. They are reduced to lowest terms if the numerator and denominator are polynomials that share no common factors other than 1.

One method of simplifying a complex rational expression requires us to first write the numerator and denominator as a single algebraic fraction. Then multiply the numerator by the reciprocal of the denominator and simplify the result.

Another method for simplifying a complex rational expression requires that we multiply it by a special form of 1. Multiply the numerator and denominator by the LCD of all the denominators as a means to clear the fractions. After doing this, simplify the remaining rational expression.

## TOPIC EXERCISES

### PART A: ADDING AND SUBTRACTING RATIONAL FUNCTIONS

State the restrictions and simplify.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.

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- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.
- 15.
- 16.
- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.
- 25.
- 26.

Given  $f$  and  $g$ , simplify the sum  $f+g$  and difference  $f-g$ . Also, state the domain using interval notation.

- 1.
- 2.
- 3.
- 4.



- 5.
- 6.
- 7.
- 8.
- 9.
- 10.

State the restrictions and simplify.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.

Simplify the given algebraic expressions. Assume all variable expressions in the denominator are nonzero.

1.  $x^{-2}+y^{-2}$
2.  $x^{-2}+(2y)^{-2}$
3.  $2x^{-1}+y^{-2}$
4.  $x^{-2}-4y^{-1}$
5.  $16x^{-2}+y^2$
6.  $xy^{-1}-yx^{-1}$

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7.  $3(x+y)^{-1}+x^{-2}$
8.  $2(x-y)^{-2}-(x-y)^{-1}$
9.  $a^{-2}-(a+b)^{-1}$
10.  $(a-b)^{-1}-(a+b)^{-1}$
11.  $x^{-n}+y^{-n}$
12.  $xy^{-n}+yx^{-n}$

## PART B: SIMPLIFYING COMPLEX RATIONAL EXPRESSIONS

Simplify. Assume all variable expressions in the denominators are nonzero.

- 59.
- 60.
- 61.
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
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- 70.
- 71.
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- 75.
- 76.

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- 77.
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- 89.
- 90.
- 91.
- 92.
- 93.
- 94.
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- 96.
- 97.
- 98.
- 99.
- 100.
- 101.
- 102.

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103.

104.

105. Given , simplify

106. Given , simplify .

107. Given , simplify the difference quotient

108. Given , simplify the difference quotient

## PART C: DISCUSSION BOARD

109. Explain why the domain of a sum of rational functions is the same as the domain of the difference of those functions.

110. Two methods for simplifying complex rational expressions have been presented in this section. Which of the two methods do you feel is more efficient, and why?

## ANSWERS

1. ;

2.

3. ;

4.

5. ;  $x \neq 0$

6.

7. ;  $x \neq 1$

8.

9. ;

10.

11. ;  $x \neq 0, 2$

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- 12.
13. ;  $x \neq 0, 7$
- 14.
15. ;  $x \neq \pm 5$
- 16.
17. ;  $x \neq 0, -4$
- 18.
19. ;  $x \neq -7, 0, 7$
- 20.
21. ;
- 22.
23. ;  $x \neq 0$
- 24.
25. ;  $x \neq \pm 2$
- 26.
27. ; ; Domain:  $(-\infty, 0) \cup (0, \infty)$
- 28.
29. ; ; Domain:  $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
- 30.
31. ; ; Domain:
- 32.
33. ; ; Domain:
- 34.
35. ; ; Domain:





36.

37. ;  $x \neq 0$

38.

39. ;

40.

41. ;  $x \neq \pm 1$

42.

43. ;

44.

45. 0;

46.

47.

48.

49.

50.

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59.

60.

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62.

64.

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88.

87.

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92.

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93.

96.

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98.

97.

100.

99.

102.

101.

104.

103.

106.

105.

108.

107.

109. Answer may vary



## 4.7 Solving Rational Equations

### LEARNING OBJECTIVES

1. Solve rational equations.
2. Solve literal equations, or formulas, involving rational expressions.
3. Solve applications involving the reciprocal of unknowns.

## Solving Rational Equations

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A **rational equation** is an equation containing at least one rational expression.

Rational expressions typically contain a variable in the denominator. For this reason, we will take care to ensure that the denominator is not 0 by making note of restrictions and checking our solutions. Solving rational equations involves clearing fractions by multiplying both sides of the equation by the least common denominator (LCD).

## Example 1

Solve:

Solution:

We first make a note of the restriction on  $x$ ,  $x \neq 0$ . We then multiply both sides by the LCD, which in this case equals  $2x^2$ .

Check your answer. Substitute  $x=5$  into the original equation and see if you obtain a true statement.



Answer: The solution is 5.

After multiplying both sides of the previous example by the LCD, we were left with a linear equation to solve. This is not always the case; sometimes we will be left with quadratic equation.

## Example 2

Solve:

Solution:

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In this example, there are two restrictions,  $x \neq 4$  and  $x \neq 2$ . Begin by multiplying both sides by the LCD,  $(x-2)(x-4)$ .

After distributing and simplifying both sides of the equation, a quadratic equation remains. To solve, rewrite the quadratic equation in standard form, factor, and then set each factor equal to 0.

Check to see if these values solve the original equation.

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<b>Check</b> $x=0$	<b>Check</b> $x=-4$
✓	✓

Answer: The solutions are 0 and  $-4$ .

Up to this point, all of the possible solutions have solved the original equation. However, this may not always be the case. Multiplying both sides of an equation by variable factors may lead to **extraneous solutions**, which are solutions that do not solve the original equation. A complete list of steps for solving a rational equation is outlined in the following example.

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## Example 3

Solve:

Solution:

Step 1: Factor all denominators and determine the LCD.

The LCD is  $(3x+1)(x-5)$ .

Step 2: Identify the restrictions. In this case,  $x \neq -\frac{1}{3}$  and  $x \neq 5$ .

Step 3: Multiply both sides of the equation by the LCD. Distribute carefully and then simplify.

Step 4: Solve the resulting equation. Here the result is a quadratic equation. Rewrite it in standard form, factor, and then set each factor equal to 0.

Step 5: Check for extraneous solutions. Always substitute into the original equation, or the factored equivalent. In this case, choose the factored equivalent to check:

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*Check*

*Check*  $x=5$

✓

✗

*Undefined terms!*

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Here 5 is an extraneous solution and is not included in the solution set. It is important to note that 5 is a restriction.

Answer: The solution is

If this process produces a solution that happens to be a restriction, then disregard it as a solution.

Try this! Solve:

Answer:

Sometimes all potential solutions are extraneous, in which case we say that there is no solution to the original equation. In the next two examples, we demonstrate two ways in which rational equation can have no solutions.

## Example 4

Solve:

Solution:

To identify the LCD, first factor the denominators.

Multiply both sides by the LCD,  $(x+4)(x-1)$ , distributing carefully.

The equation is a contradiction and thus has no solution.

Answer: No solution,  $\emptyset$

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## Example 5

Solve:

Solution:

First, factor the denominators.

Take note that the restrictions on the domain are To clear the fractions, multiply by the LCD,  $(2x+3)(2x-3)$ .

Both of these values are restrictions of the original equation; hence both are extraneous.

Answer: No solution,  $\emptyset$

It is important to point out that this technique for clearing algebraic fractions only works for equations. Do not try to clear algebraic fractions when simplifying expressions. As a reminder, an example of each is provided below.

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Expression	Equation

Expressions are to be simplified and equations are to be solved. If we multiply the expression by the LCD,  $x(2x+1)$ , we obtain another expression that is not equivalent.



<b>Incorrect</b>	<b>Correct</b>
	

Rational equations are sometimes expressed using negative exponents.

### Example 6

Solve:  $6+x^{-1}=x^{-2}$ .

Solution:

Begin by removing the negative exponents.

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Here we can see the restriction,  $x \neq 0$ . Next, multiply both sides by the LCD,  $x^2$ .

Answer: ,

A **proportion** is a statement of equality of two ratios.

This proportion is often read “ $a$  is to  $b$  as  $c$  is to  $d$ .” Given any nonzero real numbers  $a$ ,  $b$ ,  $c$ , and  $d$  that satisfy a proportion, multiply both sides by the product of the denominators to obtain the following:

This shows that cross products are equal, and is commonly referred to as **cross multiplication**.

Cross multiply to solve proportions where terms are unknown.

## Example 7

Solve:

Solution:

When cross multiplying, be sure to group  $5n-1$ .

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$$\frac{5n-1}{5} = \frac{3n}{2}$$

$$(5n-1) \cdot 2 = 5 \cdot 3n$$

Apply the distributive property in the next step.

Answer:

Cross multiplication can be used as an alternate method for solving rational equations. The idea is to simplify each side of the equation to a single algebraic fraction and then cross multiply.

## Example 8

Solve:

Solution:

Obtain a single algebraic fraction on the left side by subtracting the equivalent fractions with a common denominator.

Note that  $x \neq 0$ , cross multiply, and then solve for  $x$ .

Next, set each variable factor equal to zero.

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The check is left to the reader.

Answer:  $-8, 4$

Try this! Solve:

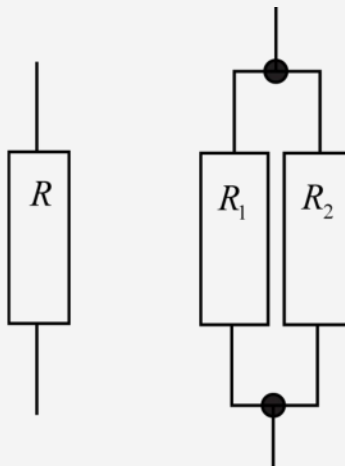
Answer:  $2, 3$

## Solving Literal Equations and Applications Involving Reciprocals

Literal equations, or formulas, are often rational equations. Hence the techniques described in this section can be used to solve for particular variables. Assume that all variable expressions in the denominator are nonzero.

### Example 9

The reciprocal of the combined resistance  $R$  of two resistors  $R_1$  and  $R_2$  in parallel is given by the formula  $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ . Solve for  $R$  in terms of  $R_1$  and  $R_2$ .



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Solution:

The goal is to isolate  $R$  on one side of the equation. Begin by multiplying both sides of the equation by the LCD,  $RR_1R_2$ .

Answer:

Try this! Solve for

Answer:

Recall that the reciprocal of a nonzero number  $n$  is  $\frac{1}{n}$ . For example, the reciprocal of 5 is  $\frac{1}{5}$  and  $5 \cdot \frac{1}{5} = 1$ . In this section, the applications will often involve the key word “reciprocal.” When this is the case, we will see that the algebraic setup results in a rational equation.

## Example 10

A positive integer is 3 less than another. If the reciprocal of the smaller integer is subtracted from twice the reciprocal of the larger, then the result is  $\frac{1}{6}$ . Find the two integers.

Solution:

Let  $n$  represent the larger positive integer.

Let  $n - 3$  represent the smaller positive integer.

Set up an algebraic equation.

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$$\overbrace{2\left(\frac{1}{n}\right)}^{\text{twice the reciprocal of the larger}} - \overbrace{\frac{1}{n-3}}^{\text{the reciprocal of the smaller is subtracted}} \underbrace{=}_{\text{the result is}} \frac{1}{20}$$

Solve this rational expression by multiplying both sides by the LCD. The LCD is  $20n(n-3)$ .

Here we have two viable possibilities for the larger integer  $n$ . For this reason, we will we have two solutions to this problem.

If  $n=8$ , then  $n-3=8-3=5$ .

If  $n=15$ , then  $n-3=15-3=12$ .

As a check, perform the operations indicated in the problem.



<b>Check 8 and 5.</b>	<b>Check 15 and 12.</b>
✓	✓

Answer: Two sets of positive integers solve this problem:  $\{5, 8\}$  and  $\{12, 15\}$ .

Try this! When the reciprocal of the larger of two consecutive even integers is subtracted from 4 times the reciprocal of the smaller, the result is  $\frac{1}{6}$ . Find the integers.

Answer: 4, 6

## KEY TAKEAWAYS

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Begin solving rational equations by multiplying both sides by the LCD. The resulting equivalent equation can be solved using the techniques learned up to this point.

Multiplying both sides of a rational equation by a variable expression introduces the possibility of extraneous solutions. Therefore, we must check the solutions against the set of restrictions. If a solution is a restriction, then it is not part of the domain and is extraneous.

When multiplying both sides of an equation by an expression, distribute carefully and multiply each term by that expression.

If all of the resulting solutions are extraneous, then the original equation has no solutions.

## TOPIC EXERCISES

### PART A: SOLVING RATIONAL EQUATIONS

Solve.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.

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- 12.
- 13.
- 14.
- 15.
- 16.
- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.
- 25.
- 26.
- 27.
- 28.
- 29.
- 30.
- 31.
- 32.

Solve the following equations involving negative exponents.

1.  $2x^{-1} = 2x^{-2} - x^{-1}$
2.  $3 + x(x+1)^{-1} = 2(x+1)^{-1}$
3.  $x^{-2} - 64 = 0$
4.  $1 - 4x^{-2} = 0$



5.  $x - (x+2)^{-1} = -2$   
6.  $2x - 9(2x-1)^{-1} = 1$   
7.  $2x^{-2} + (x-12)^{-1} = 0$   
8.  $-2x^{-2} + 3(x+4)^{-1} = 0$

Solve by cross multiplying.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.

Simplify or solve, whichever is appropriate.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.



7.

8.

Find the roots of the given function.

1.

2.

3.

4.

5.

6.

7.

Given , find  $x$  when  $f(x)=2$ .

8.

Given , find  $x$  when

9.

Given , find  $x$  when  $f(x)=1$ .

10.

Given , find  $x$  when  $f(x)=3$ .

Find the  $x$ - and  $y$ -intercepts.

1.

2.

3.

4.

5.

6.

Find the points where the given functions coincide. (Hint: Find the points where  $f(x)=g(x)$ .)

1.

,

2.

,  $g(x)=-x$

3.

,  $g(x)=x+1$

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4.

$$, g(x)=x+2$$

Recall that if  $|X|=p$ , then  $X=-p$  or  $X=p$ . Use this to solve the following absolute value equations.

1.

2.

3.

4.

5.

6.

## PART B: SOLVING LITERAL EQUATIONS

Solve for the given variable.

87. Solve for  $P$ :

88. Solve for  $A$ :

89. Solve for  $t$ :

90. Solve for  $n$ :

91. Solve for  $y$ :

92. Solve for  $m_1$ :

93. Solve for  $y$ :

94. Solve for  $y$ :

95. Solve for  $y$ :

96. Solve for  $y$ :

97. Solve for  $x$ :  $ax+cb=ac$

98. Solve for  $y$ :

Use algebra to solve the following applications.

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87. The value in dollars of a tablet computer is given by the function  $V(t)=460(t+1)^{-1}$ , where  $t$  represents the age of the tablet. Determine the age of the tablet if it is now worth \$100.

88. The value in dollars of a car is given by the function  $V(t)=24,000(0.5t+1)^{-1}$ , where  $t$  represents the age of the car. Determine the age of the car if it is now worth \$6,000.

*Solve for the unknowns.*

87. When 2 is added to 5 times the reciprocal of a number, the result is 12. Find the number.

88. When 1 is subtracted from 4 times the reciprocal of a number, the result is 11. Find the number.

89. The sum of the reciprocals of two consecutive odd integers is Find the integers.

90. The sum of the reciprocals of two consecutive even integers is Find the integers.

91. An integer is 4 more than another. If 2 times the reciprocal of the larger is subtracted from 3 times the reciprocal of the smaller, then the result is Find the integers.

92. An integer is 2 more than twice another. If 2 times the reciprocal of the larger is subtracted from 3 times the reciprocal of the smaller, then the result is Find the integers.

93. If 3 times the reciprocal of the larger of two consecutive integers is subtracted from 2 times the reciprocal of the smaller, then the result is Find the two integers.



94. If 3 times the reciprocal of the smaller of two consecutive integers is subtracted from 7 times the reciprocal of the larger, then the result is Find the two integers.

95. A positive integer is 5 less than another. If the reciprocal of the smaller integer is subtracted from 3 times the reciprocal of the larger, then the result is Find the two integers.

96. A positive integer is 6 less than another. If the reciprocal of the smaller integer is subtracted from 10 times the reciprocal of the larger, then the result is Find the two integers.

## PART C: DISCUSSION BOARD

111. Explain how we can tell the difference between a rational expression and a rational equation. How do we treat them differently? Give an example of each.

112. Research and discuss reasons why multiplying both sides of a rational equation by the LCD sometimes produces extraneous solutions.

## ANSWERS

1.

2.

3. -4

4.

5. -7, 3

6.

7. , 2

8.

9. -2,



10.

11.

12.

13.

14.

15.  $\emptyset$

16.

17.  $-2,$

18.

19.

20.

21.  $6$

22.

23.  $\emptyset$

24.

25.  $-8, 2$

26.

27.  $5$

28.

29.  $-6, 4$

30.

31.  $10$

32.

33.

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35.  $\pm$
- 36.
37.  $-3, -1$
- 38.
39.  $-6, 4$
- 40.
- 41.
- 42.
- 43.
- 44.
45.  $-16$
- 46.
- 47.
- 48.
49.  $-2, 0$
- 50.
51.  $-3, 2$
- 52.
53. Solve;  $-3,$
- 54.
55. Simplify;
- 56.
57. Simplify;
- 58.
59. Solve;

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- 60.
- 61.
- 62.
63.  $\pm 9$
- 64.
65.  $-$
- 66.
67.  $x = -$
- 68.
69.  $x = -4$
- 70.
71.  $x$ -intercept:  $(-, 0)$ ;  $y$ -intercept:  $(0, 5)$
- 72.
73.  $x$ -intercept:  $(, 0)$ ;  $y$ -intercept:  $(0, )$
- 74.
75.  $x$ -intercept:  $(, 0)$ ;  $y$ -intercept: none
- 76.
77.  $(-1, -1)$  and  $(1, 1)$
- 78.
79.  $(1, 2)$  and  $(3, 4)$
- 80.
81.  $- , -$
- 82.
83.  $2, 10$

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84.

85.  $-3, -2, -1, 6$

86.

87.  $P=2l+2w$

88.

89.

90.

91.

92.

93.

94.

95.

96.

97.

98.

99. 3.6 years old

100.

101.

102.

103. 5, 7

104.

105.  $\{-8, -4\}$  and  $\{12, 16\}$

106.

107.  $\{1, 2\}$  or  $\{-4, -3\}$

108.

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109. {4, 9} or {15, 20}

110.

111. Answer may vary

## 4.8 Applications and Variation

### LEARNING OBJECTIVES

1. Solve applications involving uniform motion (distance problems).
2. Solve work-rate applications.
3. Set up and solve applications involving direct, inverse, and joint variation.

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# Solving Uniform Motion Problems

**Uniform motion (or distance)** problems involve the formula  $D=rt$ , where the distance  $D$  is given as the product of the average rate  $r$  and the time  $t$  traveled at that rate. If we divide both sides by the average rate  $r$ , then we obtain the formula

For this reason, when the unknown quantity is time, the algebraic setup for distance problems often results in a rational equation. We begin any uniform motion problem by first organizing our data with a chart. Use this information to set up an algebraic equation that models the application.

## Example 1

Sally traveled 15 miles on the bus and then another 72 miles on a train. The train was 18 miles per hour faster than the bus, and the total trip took 2 hours. What was the average speed of the train?

Solution:

First, identify the unknown quantity and organize the data.

Let  $x$  represent the average speed (in miles per hour) of the bus.

Let  $x+18$  represent the average speed of the train.

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$$\text{Distance} = \text{Rate} \times \text{Time}$$

<i>Bus trip</i>	15 mi	$x$	
<i>Train trip</i>	72 mi	$x + 18$	
<b>Total</b>			2 hours

To avoid introducing two more variables for the time column, use the formula . The time for each leg of the trip is calculated as follows:

Use these expressions to complete the chart.

$\text{Distance} = \text{Rate} \times \text{Time}$			
<i>Bus trip</i>	15 mi	$x$	$\frac{15}{x}$
<i>Train trip</i>	72 mi	$x + 18$	$\frac{72}{x + 18}$
<b>Total</b>			2 hours

$$\frac{15}{x} + \frac{72}{x + 18} = 2$$

The algebraic setup is defined by the time column. Add the time spent on each leg of the trip to obtain a total of 2 hours:

$$\overbrace{\frac{15}{x}}^{\text{time spent on the bus}} + \overbrace{\frac{72}{x + 18}}^{\text{time spent on the train}} = \overbrace{2}^{\text{total time of trip}}$$

We begin solving this equation by first multiplying both sides by the LCD,  $x(x+18)$ .

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Solve the resulting quadratic equation by factoring.

Since we are looking for an average speed we will disregard the negative answer and conclude the bus averaged 30 mph. Substitute  $x=30$  in the expression identified as the speed of the train.

$$x+18=30+18=48$$

Answer: The speed of the train was 48 mph.

## Example 2

A boat can average 12 miles per hour in still water. On a trip downriver the boat was able to travel 29 miles with the current. On the return trip the boat was only able to travel 19 miles in the same amount of time against the current. What was the speed of the current?

Solution:

First, identify the unknown quantities and organize the data.

Let  $c$  represent the speed of the river current.

Next, organize the given data in a chart. Traveling downstream, the current will increase the speed of the boat, so it adds to the average speed of the boat. Traveling upstream, the current slows the boat, so it will subtract from the average speed of the boat.

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<i>Distance = Rate × Time</i>			
<i>trip downriver</i>	29 mi	$12 + c$	
<i>trip upriver</i>	19 mi	$12 - c$	
<b>Total</b>			

Use the formula to fill in the time column.

<i>Distance = Rate × Time</i>			
<i>trip downriver</i>	29 mi	$12 + c$	$\frac{29}{12 + c}$
<i>trip upriver</i>	19 mi	$12 - c$	$\frac{19}{12 - c}$
<b>Total</b>			



Because the boat traveled the same amount of time downriver as it did upriver, finish the algebraic setup by setting the expressions that represent the times equal to each other.

Since there is a single algebraic fraction on each side, we can solve this equation using cross multiplication.

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Answer: The speed of the current was miles per hour.

Try this! A jet aircraft can average 160 miles per hour in calm air. On a trip, the aircraft traveled 600 miles with a tailwind and returned the 600 miles against a headwind of the same speed. If the total round trip took 8 hours, then what was the speed of the wind?

Answer: 40 miles per hour

## Solving Work-Rate Problems

The rate at which a task can be performed is called a **work rate**. For example, if a painter can paint a room in 6 hours, then the task is to paint the room, and we can write

In other words, the painter can complete of the task per hour. If he works for less than 6 hours, then he will perform a fraction of the task. If he works for more than 6 hours, then he can complete more than one task. For example,

Obtain the amount of the task completed by multiplying the work rate by the amount of time the painter works. Typically, work-rate problems involve people or machines working together to complete tasks. In general, if  $t$  represents the time two people work together, then we have the following **work-rate formula**:

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Here and are the individual work rates.

### Example 3

Joe can paint a typical room in 2 hours less time than Mark. If Joe and Mark can paint 5 rooms working together in a 12 hour shift, how long does it take each to paint a single room?

Solution:

Let  $x$  represent the time it takes Mark to paint a typical room.

Let  $x - 2$  represent the time it takes Joe to paint a typical room.

Therefore, Mark's individual work-rate is rooms per hour and Joe's is rooms per hour.

Both men worked for 12 hours. We can organize the data in a chart, just as we did with distance problems.

	<i>Amount of Task Completed</i>	<i>Work - Rate</i>	<i>Time</i>
<i>Joe</i>	$\frac{1}{x-2} \cdot 12$	$\frac{1}{x-2}$	12
<i>Mark</i>	$\frac{1}{x} \cdot 12$	$\frac{1}{x}$	12
<i>Total</i>	5		

$$\frac{1}{x-2} \cdot 12 + \frac{1}{x} \cdot 12 = 5$$

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Working together, they can paint 5 total rooms in 12 hours. This leads us to the following algebraic setup:

Multiply both sides by the LCD,  $x(x-2)$ .

Solve the resulting quadratic equation by factoring.

We can disregard because back substituting into  $x - 2$  would yield a negative time to paint a room. Take  $x = 6$  to be the only solution and use it to find the time it takes Joe to paint a typical room.

$$x-2=6-2=4$$

Answer: Joe can paint a typical room in 4 hours and Mark can paint a typical room in 6 hours. As a check we can multiply both work rates by 12 hours to see that together they can paint 5 rooms.



## Example 4

It takes Bill twice as long to lay a tile floor by himself as it does Manny. After working together with Bill for 4 hours, Manny was able to complete the job in 2 additional hours. How long would it have taken Manny working alone?

Solution:

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Let  $x$  represent the time it takes Manny to lay the floor alone.

Let  $2x$  represent the time it takes Bill to lay the floor alone.

Manny's work rate is  $\frac{1}{x}$  of the floor per hour and Bill's work rate is  $\frac{1}{2x}$ . Bill worked on the job for 4 hours and Manny worked on the job for 6 hours.

	<i>Amount of Task Completed</i>	=	<i>Work - Rate</i>	$\times$	<i>Time</i>
<i>Manny</i>	$\frac{1}{x} \cdot 6$		$\frac{1}{x}$		6 hours
<i>Bill</i>	$\frac{1}{2x} \cdot 4$		$\frac{1}{2x}$		4 hours
<i>Total</i>	1				

This leads us to the following algebraic setup:

Solve.

Answer: It would have taken Manny 8 hours to complete the floor by him self.

Consider the work-rate formula where one task is to be completed.

Factor out the time  $t$  and then divide both sides by  $t$ . This will result in equivalent specialized work-rate formulas:

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In summary, we have the following equivalent work-rate formulas:

Try this! Matt can tile a countertop in 2 hours, and his assistant can do the same job in 3 hours. If Matt starts the job and his assistant joins him 1 hour later, then how long will it take to tile the countertop?

Answer: hours

## Solving Problems involving Direct, Inverse, and Joint variation

Many real-world problems encountered in the sciences involve two types of functional relationships. The first type can be explored using the fact that the distance  $s$  in feet an object falls from rest, without regard to air resistance, can be approximated using the following formula:

$$s=16t^2$$

Here  $t$  represents the time in seconds the object has been falling. For example, after 2 seconds the object will have fallen  $s=16(2)^2=16\cdot 4=64$  feet.

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Time $t$ in seconds	Distance $s=16t^2$ in feet
0	0
1	16

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Time $t$ in seconds	Distance $s=16t^2$ in feet
2	64
3	144

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Time $t$ in seconds	Distance $s=16t^2$ in feet
4	256

In this example, we can see that the distance varies over time as the product of a constant 16 and the square of the time  $t$ . This relationship is described as **direct variation** and 16 is called the **constant of variation**. Furthermore, if we divide both sides of  $s=16t^2$  by  $t^2$  we have

In this form, it is reasonable to say that  $s$  is proportional to  $t^2$ , and 16 is called the **constant of proportionality**. In general, we have

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<i>Key words</i>	<i>Translation</i>
“ $y$ varies directly as $x$ ”	$y=kx$

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<i>Key words</i>	<i>Translation</i>
“ $y$ is directly proportional to $x$ ”	
“ $y$ is proportional to $x$ ”	

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Here  $k$  is nonzero and is called the constant of variation or the constant of proportionality. Typically, we will be given information from which we can determine this constant.

## Example 5

An object's weight on Earth varies directly to its weight on the Moon. If a man weighs 180 pounds on Earth, then he will weigh 30 pounds on the Moon. Set up an algebraic equation that expresses the weight on Earth in terms of the weight on the Moon and use it to determine the weight of a woman on the Moon if she weighs 120 pounds on Earth.

Solution:

Let  $y$  represent weight on Earth.

Let  $x$  represent weight on the Moon.

We are given that the “weight on Earth varies directly to the weight on the Moon.”

$$y=kx$$

To find the constant of variation  $k$ , use the given information. A 180-lb man on Earth weighs 30 pounds on the Moon, or  $y = 180$  when  $x = 30$ .

$$180=k \cdot 30$$

Solve for  $k$ .

Next, set up a formula that models the given information.

$$y=6x$$

This implies that a person's weight on Earth is 6 times his weight on the Moon. To answer the question, use the woman's weight on Earth,  $y = 120$  lbs, and solve for  $x$ .

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Answer: The woman weighs 20 pounds on the Moon.

The second functional relationship can be explored using the formula that relates the intensity of light  $I$  to the distance from its source  $d$ .

Here  $k$  represents some constant. A foot-candle is a measurement of the intensity of light. One foot-candle is defined to be equal to the amount of illumination produced by a standard candle measured one foot away. For example, a 125-Watt fluorescent growing light is advertised to produce 525 foot-candles of illumination. This means that at a distance  $d=1$  foot,  $I=525$  foot-candles and we have:

Using  $k=525$  we can construct a formula which gives the light intensity produced by the bulb:

Here  $d$  represents the distance the growing light is from the plants. In the following chart, we can see that the amount of illumination fades quickly as the distance from the plants increases.

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distance $t$ in feet	Light Intensity
1	525
2	131.25

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distance $t$ in feet	Light Intensity
3	58.33
4	32.81

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	<b>Light Intensity</b>
<b>distance <math>t</math> in feet</b>	
5	21

This type of relationship is described as an **inverse variation**. We say that  $I$  is **inversely proportional** to the square of the distance  $d$ , where 525 is the constant of proportionality. In general, we have

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<i>Key words</i>	<i>Translation</i>
“ $y$ varies inversely as $x$ ”	
“ $y$ is inversely proportional to $x$ ”	

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Again,  $k$  is nonzero and is called the constant of variation or the constant of proportionality.

## Example 6

The weight of an object varies inversely as the square of its distance from the center of Earth. If an object weighs 100 pounds on the surface of Earth (approximately 4,000 miles from the center), how much will it weigh at 1,000 miles above Earth's surface?

Solution:

Let  $w$  represent the weight of the object.

Let  $d$  represent the object's distance from the center of Earth.

Since " $w$  varies inversely as the square of  $d$ ," we can write

Use the given information to find  $k$ . An object weighs 100 pounds on the surface of Earth, approximately 4,000 miles from the center. In other words,  $w = 100$  when  $d = 4,000$ :

Solve for  $k$ .

Therefore, we can model the problem with the following formula:

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To use the formula to find the weight, we need the distance from the center of Earth. Since the object is 1,000 miles above the surface, find the distance from the center of Earth by adding 4,000 miles:

$$d=4,000+1,000=5,000 \text{ miles}$$

To answer the question, use the formula with  $d = 5,000$ .

Answer: The object will weigh 64 pounds at a distance 1,000 miles above the surface of Earth.

Lastly, we define relationships between multiple variables, described as **joint variation**. In general, we have



<i>Key Words</i>	<i>Translation</i>
“ $y$ varies jointly as $x$ and $z$ ”	
“ $y$ is jointly proportional to $x$ and $z$ ”	

$$y=kxz$$

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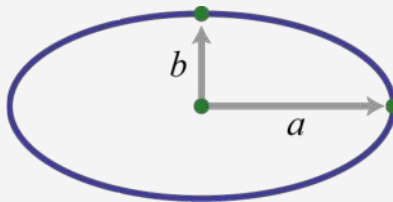
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Here  $k$  is nonzero and is called the constant of variation or the constant of proportionality.

### Example 7

The area of an ellipse varies jointly as  $a$ , half of the ellipse's major axis, and  $b$ , half of the ellipse's minor axis as pictured. If the area of an ellipse is  $300\pi \text{ cm}^2$ , where  $a=10$  cm and  $b=30$  cm, what is the constant of proportionality? Give a formula for the area of an ellipse.



Solution:

If we let  $A$  represent the area of an ellipse, then we can use the statement “area varies jointly as  $a$  and  $b$ ” to write

$$A=kab$$

To find the constant of variation  $k$ , use the fact that the area is  $300\pi$  when  $a=10$  and  $b=30$ .

Therefore, the formula for the area of an ellipse is

$$A=\pi ab$$

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Answer: The constant of proportionality is  $\pi$  and the formula for the area of an ellipse is  $A=ab\pi$ .

Try this! Given that  $y$  varies directly as the square of  $x$  and inversely with  $z$ , where  $y = 2$  when  $x = 3$  and  $z = 27$ , find  $y$  when  $x = 2$  and  $z = 16$ .

Answer:

## KEY TAKEAWAYS

When solving distance problems where the time element is unknown, use the equivalent form of the uniform motion formula, , to avoid introducing more variables.

When solving work-rate problems, multiply the individual work rate by the time to obtain the portion of the task completed. The sum of the portions of the task results in the total amount of work completed.

The setup of variation problems usually requires multiple steps. First, identify the key words to set up an equation and then use the given information to find the constant of variation  $k$ . After determining the constant of variation, write a formula that models the problem. Once a formula is found, use it to answer the question.

## TOPIC EXERCISES

### PART A: SOLVING UNIFORM MOTION PROBLEMS

Use algebra to solve the following applications.

1. Every morning Jim spends 1 hour exercising. He runs 2 miles and then he bikes 16 miles. If Jim can bike twice as fast as he can run, at what speed does he average on his bike?

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2. Sally runs 3 times as fast as she walks. She ran for  $\frac{1}{3}$  of a mile and then walked another 3 miles. The total workout took 1 hour. What was Sally's average walking speed?
3. On a business trip, an executive traveled 720 miles by jet and then another 80 miles by helicopter. If the jet averaged 3 times the speed of the helicopter, and the total trip took 4 hours, what was the average speed of the jet?
4. A triathlete can run 3 times as fast as she can swim and bike 6 times as fast as she can swim. The race consists of a  $\frac{1}{3}$  mile swim, 3 mile run, and a 12 mile bike race. If she can complete all of these events in 1 hour, then how fast can she swim, run and bike?
5. On a road trip, Marty was able to drive an average 4 miles per hour faster than George. If Marty was able to drive 39 miles in the same amount of time George drove 36 miles, what was Marty's average speed?
6. The bus is 8 miles per hour faster than the trolley. If the bus travels 9 miles in the same amount of time the trolley can travel 7 miles, what is the average speed of each?
7. Terry decided to jog the 5 miles to town. On the return trip, she walked the 5 miles home at half of the speed that she was able to jog. If the total trip took 3 hours, what was her average jogging speed?
8. James drove the 24 miles to town and back in 1 hour. On the return trip, he was able to average 20 miles per hour faster than he averaged on the trip to town. What was his average speed on the trip to town?
9. A light aircraft was able to travel 189 miles with a 14 mile per hour tailwind in the same time it was able to travel 147 miles against it. What was the speed of the aircraft in calm air?

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10. A jet flew 875 miles with a 30 mile per hour tailwind. On the return trip, against a 30 mile per hour headwind, it was able to cover only 725 miles in the same amount of time. How fast was the jet in calm air?
11. A helicopter averaged 90 miles per hour in calm air. Flying with the wind it was able to travel 250 miles in the same amount of time it took to travel 200 miles against it. What is the speed of the wind?
12. Mary and Joe took a road-trip on separate motorcycles. Mary's average speed was 12 miles per hour less than Joe's average speed. If Mary drove 115 miles in the same time it took Joe to drive 145 miles, what was Mary's average speed?
13. A boat averaged 12 miles per hour in still water. On a trip downstream, with the current, the boat was able to travel 26 miles. The boat then turned around and returned upstream 33 miles. How fast was the current if the total trip took 5 hours?
14. If the river current flows at an average 3 miles per hour, a tour boat can make an 18-mile tour downstream with the current and back the 18 miles against the current in 4 hours. What is the average speed of the boat in still water?
15. Jose drove 10 miles to his grandmother's house for dinner and back that same evening. Because of traffic, he averaged 20 miles per hour less on the return trip. If it took hour longer to get home, what was his average speed driving to his grandmother's house?
16. Jerry paddled his kayak, upstream against a 1 mph current, for 12 miles. The return trip, downstream with the 1 mph current, took one hour less time. How fast did Jerry paddle the kayak in still water?
17. James and Mildred left the same location in separate cars and met in Los Angeles 300 miles away. James was able to average 10 miles an hour faster than

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Mildred on the trip. If James arrived 1 hour earlier than Mildred, what was Mildred's average speed?

18. A bus is 20 miles per hour faster than a bicycle. If Bill boards a bus at the same time and place that Mary departs on her bicycle, Bill will arrive downtown 5 miles away hour earlier than Mary. What is the average speed of the bus?

## **PART B: SOLVING WORK-RATE PROBLEMS**

Use algebra to solve the following applications.

19. Mike can paint the office by himself in 4 hours. Jordan can paint the office in 6 hours. How long will it take them to paint the office working together?

20. Barry can lay a brick driveway by himself in 3 days. Robert does the same job in 5 days. How long will it take them to lay the brick driveway working together?

21. A larger pipe fills a water tank twice as fast as a smaller pipe. When both pipes are used, they fill the tank in 10 hours. If the larger pipe is left off, how long would it take the smaller pipe to fill the tank?

22. A newer printer can print twice as fast as an older printer. If both printers working together can print a batch of flyers in 45 minutes, then how long would it take the older printer to print the batch working alone?

23. Mary can assemble a bicycle for display in 2 hours. It takes Jane 3 hours to assemble a bicycle. How long will it take Mary and Jane, working together, to assemble 5 bicycles?

24. Working alone, James takes twice as long to assemble a computer as it takes Bill. In one 8-hour shift, working together, James and Bill can assemble 6 computers. How long would it take James to assemble a computer if he were working alone?

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25. Working alone, it takes Harry one hour longer than Mike to install a fountain. Together they can install 10 fountains in 12 hours. How long would it take Mike to install 10 fountains by himself?
26. Working alone, it takes Henry 2 hours longer than Bill to paint a room. Working together they painted 2 rooms in 6 hours. How long would it have taken Henry to paint the same amount if he were working alone?
27. Manny, working alone, can install a custom cabinet in 3 hours less time than his assistant. Working together they can install the cabinet in 2 hours. How long would it take Manny to install the cabinet working alone?
28. Working alone, Garret can assemble a garden shed in 5 hours less time than his brother. Working together, they need 6 hours to build the garden shed. How long would it take Garret to build the shed working alone?
29. Working alone, the assistant-manager takes 2 more hours than the manager to record the inventory of the entire shop. After working together for 2 hours, it took the assistant-manager 1 additional hour to complete the inventory. How long would it have taken the manager to complete the inventory working alone?
30. An older printer can print a batch of sales brochures in 16 minutes. A newer printer can print the same batch in 10 minutes. After working together for some time, the newer printer was shut down and it took the older printer 3 more minutes to complete the job. How long was the newer printer operating?

## PART C: SOLVING VARIATION PROBLEMS

Translate each of the following sentences into a mathematical formula.

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31. The distance  $D$  an automobile can travel is directly proportional to the time  $t$  that it travels at a constant speed.
32. The extension of a hanging spring  $d$  is directly proportional to the weight  $w$  attached to it.
33. An automobile's braking distance  $d$  is directly proportional to the square of the automobile's speed  $v$ .
34. The volume  $V$  of a sphere varies directly as the cube of its radius  $r$ .
35. The volume  $V$  of a given mass of gas is inversely proportional to the pressure  $p$  exerted on it.
36. Every particle of matter in the universe attracts every other particle with a force  $F$  that is directly proportional to the product of the masses  $m_1$  and  $m_2$  of the particles, and it is inversely proportional to the square of the distance  $d$  between them.
37. Simple interest  $I$  is jointly proportional to the annual interest rate  $r$  and the time  $t$  in years a fixed amount of money is invested.
38. The time  $t$  it takes an object to fall is directly proportional to the square root of the distance  $d$  it falls.

Construct a mathematical model given the following:

31.  $y$  varies directly as  $x$ , and  $y = 30$  when  $x = 6$ .
32.  $y$  varies directly as  $x$ , and  $y = 52$  when  $x = 4$ .
33.  $y$  is directly proportional to  $x$ , and  $y = 12$  when  $x = 3$ .
34.  $y$  is directly proportional to  $x$ , and  $y = 120$  when  $x = 20$ .
35.  $y$  is inversely proportional to  $x$ , and  $y = 3$  when  $x = 9$ .
36.  $y$  is inversely proportional to  $x$ , and  $y = 21$  when  $x = 3$ .

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37.  $y$  varies inversely as  $x$ , and  $y = 2$  when  $x =$ .
38.  $y$  varies inversely as  $x$ , and  $y =$  when  $x =$ .
39.  $y$  is jointly proportional to  $x$  and  $z$ , where  $y = 2$  when  $x = 1$  and  $z = 3$ .
40.  $y$  is jointly proportional to  $x$  and  $z$ , where  $y = 15$  when  $x = 3$  and  $z = 7$ .
41.  $y$  varies jointly as  $x$  and  $z$ , where  $y =$  when  $x =$  and  $z = 12$ .
42.  $y$  varies jointly as  $x$  and  $z$ , where  $y = 5$  when  $x =$  and  $z =$ .
43.  $y$  varies directly as the square of  $x$ , where  $y = 45$  when  $x = 3$ .
44.  $y$  varies directly as the square of  $x$ , where  $y = 3$  when  $x =$ .
45.  $y$  is inversely proportional to the square of  $x$ , where  $y = 27$  when  $x =$ .
46.  $y$  is inversely proportional to the square of  $x$ , where  $y = 9$  when  $x =$ .
47.  $y$  varies jointly as  $x$  and the square of  $z$ , where  $y = 6$  when  $x =$  and  $z =$ .
48.  $y$  varies jointly as  $x$  and  $z$  and inversely as the square of  $w$ , where  $y = 5$  when  $x = 1$ ,  $z = 3$ , and  $w =$ .
49.  $y$  varies directly as the square root of  $x$  and inversely as the square of  $z$ , where  $y = 15$  when  $x = 25$  and  $z = 2$ .
50.  $y$  varies directly as the square of  $x$  and inversely as  $z$  and the square of  $w$ , where  $y = 14$  when  $x = 4$ ,  $w = 2$ , and  $z = 2$ .

Solve applications involving variation.

31. Revenue in dollars is directly proportional to the number of branded sweatshirts sold. The revenue earned from selling 25 sweatshirts is \$318.75. Determine the revenue if 30 sweatshirts are sold.
32. The sales tax on the purchase of a new car varies directly as the price of the car. If an \$18,000 new car is purchased, then the sales tax is \$1,350. How much sales tax is charged if the new car is priced at \$22,000?

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33. The price of a share of common stock in a company is directly proportional to the earnings per share (EPS) of the previous 12 months. If the price of a share of common stock in a company is \$22.55, and the EPS is published to be \$1.10, determine the value of the stock if the EPS increases by \$0.20.
34. The distance traveled on a road trip varies directly with the time spent on the road. If a 126-mile trip can be made in 3 hours, then what distance can be traveled in 4 hours?
35. The circumference of a circle is directly proportional to its radius. The circumference of a circle with radius 7 centimeters is measured as  $14\pi$  centimeters. What is the constant of proportionality?
36. The area of circle varies directly as the square of its radius. The area of a circle with radius 7 centimeters is determined to be  $49\pi$  square centimeters. What is the constant of proportionality?
37. The surface area of a sphere varies directly as the square of its radius. When the radius of a sphere measures 2 meters, the surface area measures  $16\pi$  square meters. Find the surface area of a sphere with radius 3 meters.
38. The volume of a sphere varies directly as the cube of its radius. When the radius of a sphere measures 3 meters, the volume is  $36\pi$  cubic meters. Find the volume of a sphere with radius 1 meter.
39. With a fixed height, the volume of a cone is directly proportional to the square of the radius at the base. When the radius at the base measures 10 centimeters, the volume is 200 cubic centimeters. Determine the volume of the cone if the radius of the base is halved.

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40. The distance  $d$  an object in free fall drops varies directly with the square of the time  $t$  that it has been falling. If an object in free fall drops 36 feet in 1.5 seconds, then how far will it have fallen in 3 seconds?

Hooke's law suggests that the extension of a hanging spring is directly proportional to the weight attached to it. The constant of variation is called the spring constant.

*Figure 4.1*



*Robert Hooke (1635–1703)*

31. A hanging spring is stretched 5 inches when a 20-pound weight is attached to it. Determine its spring constant.
32. A hanging spring is stretched 3 centimeters when a 2-kilogram weight is attached to it. Determine the spring constant.
33. If a hanging spring is stretched 3 inches when a 2-pound weight is attached, how far will it stretch with a 5-pound weight attached?
34. If a hanging spring is stretched 6 centimeters when a 4-kilogram weight is attached to it, how far will it stretch with a 2-kilogram weight attached?

The braking distance of an automobile is directly proportional to the square of its speed.

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31. It takes 36 feet to stop a particular automobile moving at a speed of 30 miles per hour. How much breaking distance is required if the speed is 35 miles per hour?

32. After an accident, it was determined that it took a driver 80 feet to stop his car. In an experiment under similar conditions, it takes 45 feet to stop the car moving at a speed of 30 miles per hour. Estimate how fast the driver was moving before the accident.

*Figure 4.2*



*Robert Boyle (1627–1691)*

Boyle's law states that if the temperature remains constant, the volume  $V$  of a given mass of gas is inversely proportional to the pressure  $p$  exerted on it.

31. A balloon is filled to a volume of 216 cubic inches on a diving boat under 1 atmosphere of pressure. If the balloon is taken underwater approximately 33 feet, where the pressure measures 2 atmospheres, then what is the volume of the balloon?



32. A balloon is filled to 216 cubic inches under a pressure of 3 atmospheres at a depth of 66 feet. What would the volume be at the surface, where the pressure is 1 atmosphere?
33. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 72-pound boy is sitting 3 feet from the fulcrum, how far from the fulcrum must a 54-pound boy sit to balance the seesaw?
34. The current  $I$  in an electrical conductor is inversely proportional to its resistance  $R$ . If the current is  $\frac{1}{2}$  ampere when the resistance is 100 ohms, what is the current when the resistance is 150 ohms?
35. The amount of illumination  $I$  is inversely proportional to the square of the distance  $d$  from a light source. If 70 foot-candles of illumination is measured 2 feet away from a lamp, what level of illumination might we expect 3 feet away from the lamp?
36. The amount of illumination  $I$  is inversely proportional to the square of the distance  $d$  from a light source. If 40 foot-candles of illumination is measured 3 feet away from a lamp, at what distance can we expect 10 foot-candles of illumination?
37. The number of men, represented by  $y$ , needed to lay a cobblestone driveway is directly proportional to the area  $A$  of the driveway and inversely proportional to the amount of time  $t$  allowed to complete the job. Typically, 3 men can lay 1,200 square feet of cobblestone in 4 hours. How many men will be required to lay 2,400 square feet of cobblestone in 6 hours?
38. The volume of a right circular cylinder varies jointly as the square of its radius and its height. A right circular cylinder with a 3-centimeter radius and a height of 4 centimeters has a volume of  $36\pi$  cubic centimeters. Find a formula for the volume of a right circular cylinder in terms of its radius and height.

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39. The period  $T$  of a pendulum is directly proportional to the square root of its length  $L$ . If the length of a pendulum is 1 meter, then the period is approximately 2 seconds. Approximate the period of a pendulum that is 0.5 meter in length.

40. The time  $t$  it takes an object to fall is directly proportional to the square root of the distance  $d$  it falls. An object dropped from 4 feet will take 12 second to hit the ground. How long will it take an object dropped from 16 feet to hit the ground?

Newton's universal law of gravitation states that every particle of matter in the universe attracts every other particle with a force  $F$  that is directly proportional to the product of the masses  $m_1$  and  $m_2$  of the particles and inversely proportional to the square of the distance  $d$  between them. The constant of proportionality is called the gravitational constant.

*Figure 4.3*



*Sir Isaac Newton (1643–1727)*

*Source: Portrait of Isaac Newton by Sir Godfrey Kneller, from*

*<http://commons.wikimedia.org/wiki/File:GodfreyKneller-IsaacNewton-1689>.*

*[http://commons.wikimedia.org/wiki/File:Frans\\_Hals\\_-\\_Portret\\_](http://commons.wikimedia.org/wiki/File:Frans_Hals_-_Portret_)*

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van\_Ren%C3%A9\_Descartes.jpg.

31. If two objects with masses 50 kilograms and 100 kilograms are meter apart, then they produce approximately  $1.34 \times 10^{-6}$  newtons (N) of force. Calculate the gravitational constant.
32. Use the gravitational constant from the previous exercise to write a formula that approximates the force  $F$  in newtons between two masses  $m_1$  and  $m_2$ , expressed in kilograms, given the distance  $d$  between them in meters.
33. Calculate the force in newtons between Earth and the Moon, given that the mass of the Moon is approximately  $7.3 \times 10^{22}$  kilograms, the mass of Earth is approximately  $6.0 \times 10^{24}$  kilograms, and the distance between them is on average  $1.5 \times 10^{11}$  meters.
34. Calculate the force in newtons between Earth and the Sun, given that the mass of the Sun is approximately  $2.0 \times 10^{30}$  kilograms, the mass of Earth is approximately  $6.0 \times 10^{24}$  kilograms, and the distance between them is on average  $3.85 \times 10^8$  meters.
35. If  $y$  varies directly as the square of  $x$ , then how does  $y$  change if  $x$  is doubled?
36. If  $y$  varies inversely as square of  $t$ , then how does  $y$  change if  $t$  is doubled?
37. If  $y$  varies directly as the square of  $x$  and inversely as the square of  $t$ , then how does  $y$  change if both  $x$  and  $t$  are doubled?

## ANSWERS

1. 20 miles per hour
- 2.
3. 240 miles per hour
- 4.

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5. 52 miles per hour

6.

7. 5 miles per hour

8.

9. 112 miles per hour

10.

11. 10 miles per hour

12.

13. 1 mile per hour

14.

15. 40 miles per hour

16.

17. 50 miles per hour

18.

19. 2 hours

20.

21. 30 hours

22.

23. 6 hours

24.

25. 20 hours

26.

27. 3 hours

28.

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29. 4 hours

30.

31.  $D=kt$

32.

33.  $d=kv^2$

34.

35.  $V=$

36.

37.  $I=krt$

38.

39.  $y=5x$

40.

41.  $y=4x$

42.

43.  $y=$

44.

45.  $y=$

46.

47.  $y=xz$

48.

49.  $y=xz$

50.

51.  $y=5x^2$

52.

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- 53.
- 54.
55.  $y=54xz^2$
- 56.
- 57.
- 58.
59. \$382.50
- 60.
61. \$26.65
- 62.
63.  $2\pi$
- 64.
65.  $36\pi$  square meters
- 66.
67. 50 cubic centimeters
- 68.
- 69.
70. 7.5 inches
- 71.
72. 49 feet
- 73.
74. 108 cubic inches
- 75.
76. 4 feet
- 77.



78. 1,120 foot-candles
- 79.
80. 4 men
- 81.
82. 1.4 seconds
- 83.
84.  $6.7 \times 10^{-11} \text{ N m}^2/\text{kg}^2$
- 85.
86.  $1.98 \times 10^{20} \text{ N}$
- 87.
88.  $y$  changes by a factor of 4
- 89.
90.  $y$  remains unchanged

## 4.9 Review Exercises and Sample Exam

### REVIEW EXERCISES ALGEBRA OF FUNCTIONS

#### Evaluate

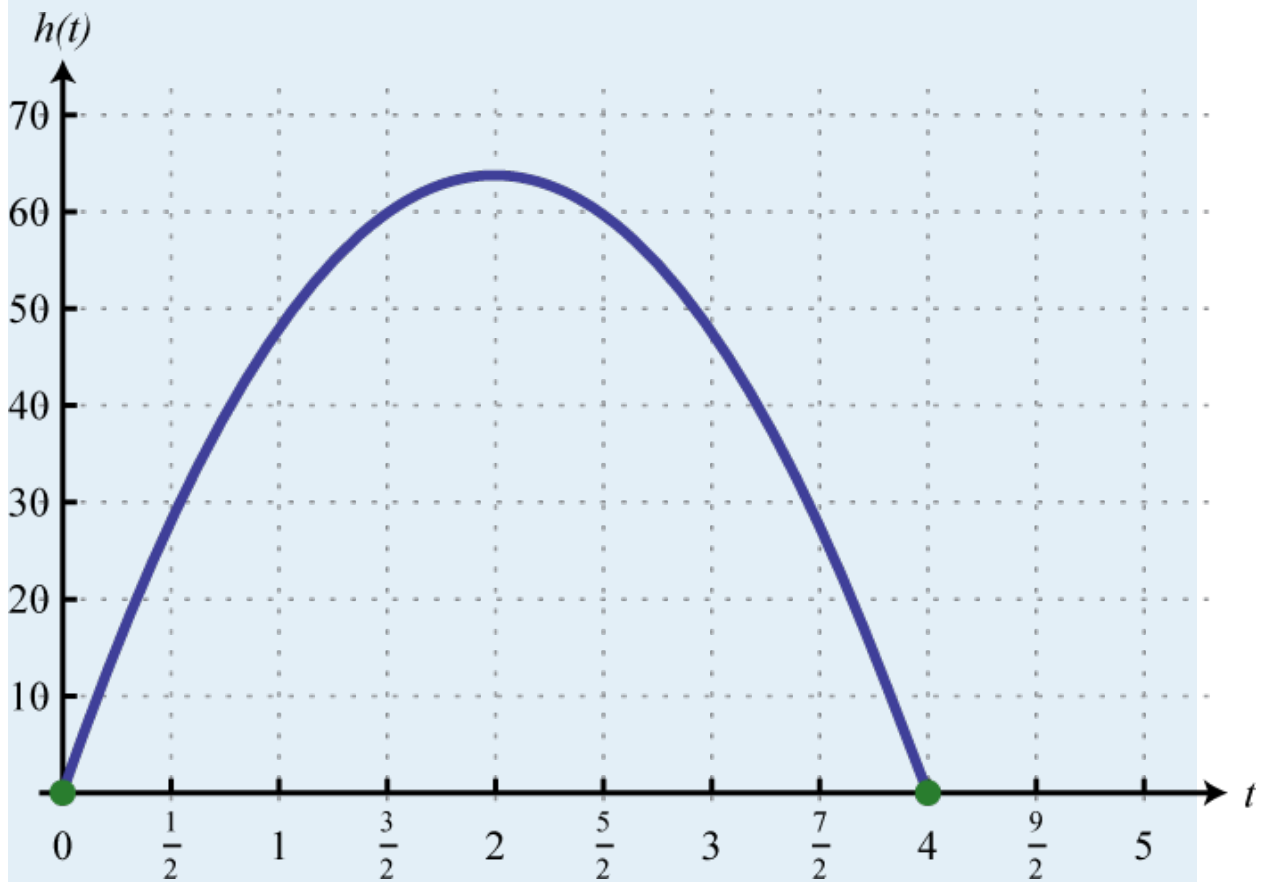
- Given  $f(x) = 2x^2 - x + 6$ , find  $f(-3)$ ,  $f(0)$ , and  $f(10)$ .
- Given  $g(x) = -x^2 + 4x - 1$ , find  $g(-1)$ ,  $g(0)$ , and  $g(3)$ .
- Given  $h(t) = -t^3 - 2t^2 + 3$ , find  $h(-3)$ ,  $h(0)$ , and  $h(2)$ .
- Given  $p(x) = x^4 - 2x^2 + x$ , find  $p(-1)$ ,  $p(0)$ , and  $p(2)$ .
- The following graph gives the height  $h(t)$  in feet of a projectile over time  $t$  in seconds.

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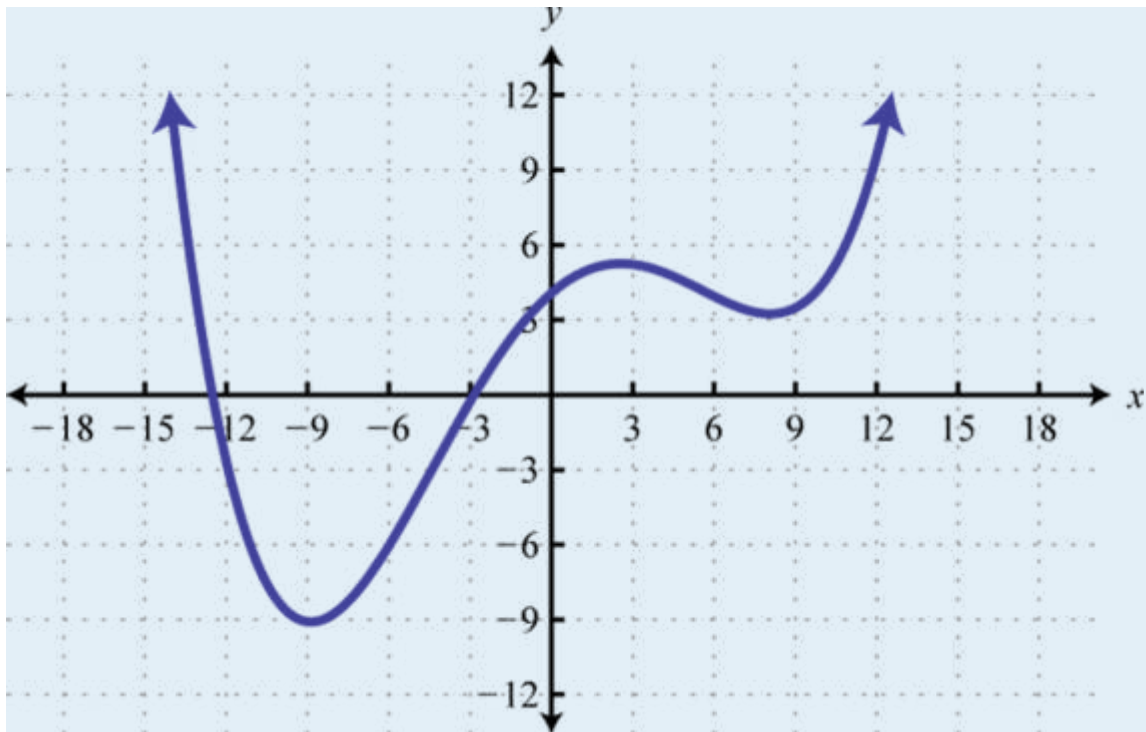






- a. Use the graph to determine the height of the projectile at 2.5 seconds.
  - b. At what time does the projectile reach its maximum height?
  - c. How long does it take the projectile to return to the ground?
6. Given the graph of the function  $f$ , find  $f(-9)$ ,  $f(-3)$ , and  $f(12)$ .





7. From the ground, a bullet is fired straight up into the air at 340 meters per second. Ignoring the effects of air friction, write a function that models the height of the bullet and use it to calculate the bullet's height after one-quarter of a second. (Round off to the nearest meter.)
8. An object is tossed into the air at an initial speed of 30 feet per second from a rooftop 10 feet high. Write a function that models the height of the object and use it to calculate the height of the object after 1 second.

Perform the operations.

1. Given  $f(x)=5x^2-3x+1$  and  $g(x)=2x^2-x-1$ , find  $(f+g)(x)$ .
2. Given  $f(x)=x^2+3x-8$  and  $g(x)=x^2-5x-7$ , find  $(f-g)(x)$ .
3. Given  $f(x)=3x^2-x+2$  and  $g(x)=2x-3$ , find  $(f \cdot g)(x)$ .



4. Given  $f(x)=27x^5-15x^3-3x^2$  and  $g(x)=3x^2$ , find  $(f/g)(x)$ .
5. Given  $g(x)=x^2-x+1$ , find  $g(-3u)$ .
6. Given  $g(x)=x^3-1$ , find  $g(x-1)$ .

Given  $f(x)=16x^3-12x^2+4x$ ,  $g(x)=x^2-x+1$ , and  $h(x)=4x$ , find the following:

1.  $(g \cdot h)(x)$
2.  $(f-g)(x)$
3.  $(g+f)(x)$
4.  $(f/h)(x)$
5.  $(f \cdot h)(-1)$
6.  $(g+h)(-3)$
7.  $(g-f)(2)$
- 8.

## FACTORING POLYNOMIALS

Factor out the greatest common factor (GCF).

23.  $2x^4-12x^3-2x^2$
24.  $18a^3b-3a^2b^2+3ab^3$
25.  $x^4y^3-3x^3y+x^2y$
26.  $x^{3n}-x^{2n}-x^n$

Factor by grouping.

23.  $2x^3-x^2+2x-1$
24.  $3x^3-x^2-6x+2$
25.  $x^3-5x^2y+xy^2-5y^3$
26.  $a^2b-a+ab^3-b^2$

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27.  $2x^4 - 4xy^3 + 2x^2y^2 - 4x^3y$

28.  $x^4y^2 - xy^5 + x^3y^4 - x^2y^3$

Factor the special binomials.

23.  $64x^2 - 1$

24.  $9 - 100y^2$

25.  $x^2 - 36y^2$

26.  $4 - (2x - 1)^2$

27.  $a^3b^3 + 125$

28.  $64x^3 - y^3$

29.  $81x^4 - y^4$

30.  $x^8 - 1$

31.  $x^6 - 64y^6$

32.  $1 - a^6b^6$

## FACTORIZING TRINOMIALS

Factor.

43.  $x^2 - 8x - 48$

44.  $x^2 - 15x + 54$

45.  $x^2 - 4x - 6$

46.  $x^2 - 12xy + 36y^2$

47.  $x^2 + 20xy + 75y^2$

48.  $-x^2 + 5x + 150$

49.  $-2y^2 + 20y + 48$

50.  $28x^2 + 20x + 3$

51.  $150x^2 - 100x + 6$

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52.  $24a^2 - 38ab + 3b^2$   
 53.  $27u^2 - 3uv - 4v^2$   
 54.  $16x^2y^2 - 78xy + 27$   
 55.  $16m^2 + 72mn + 81n^2$   
 56.  $4x^2 - 5x + 20$   
 57.  $25x^4 - 35x^2 + 6$   
 58.  $2x^4 + 7x^2 + 3$   
 59.  $x^6 + 3x^3y^3 - 10y^6$   
 60.  $a^6 - 8a^3b^3 + 15b^6$   
 61.  $x^{2n} - 2x^n + 1$   
 62.  $6x^{2n} - x^n - 2$

## SOLVE POLYNOMIAL EQUATIONS BY FACTORING

Factor completely.

63.  $45x^3 - 20x$   
 64.  $12x^4 - 70x^3 + 50x^2$   
 65.  $-20x^2 + 32x - 3$   
 66.  $-x^3y + 9xy^3$   
 67.  $24a^4b^2 + 3ab^5$   
 68.  $64a^6b^6 - 1$   
 69.  $64x^2 + 1$   
 70.  $x^3 + x^2y - xy^2 - y^3$

Solve by factoring.

63.  $9x^2 + 8x = 0$

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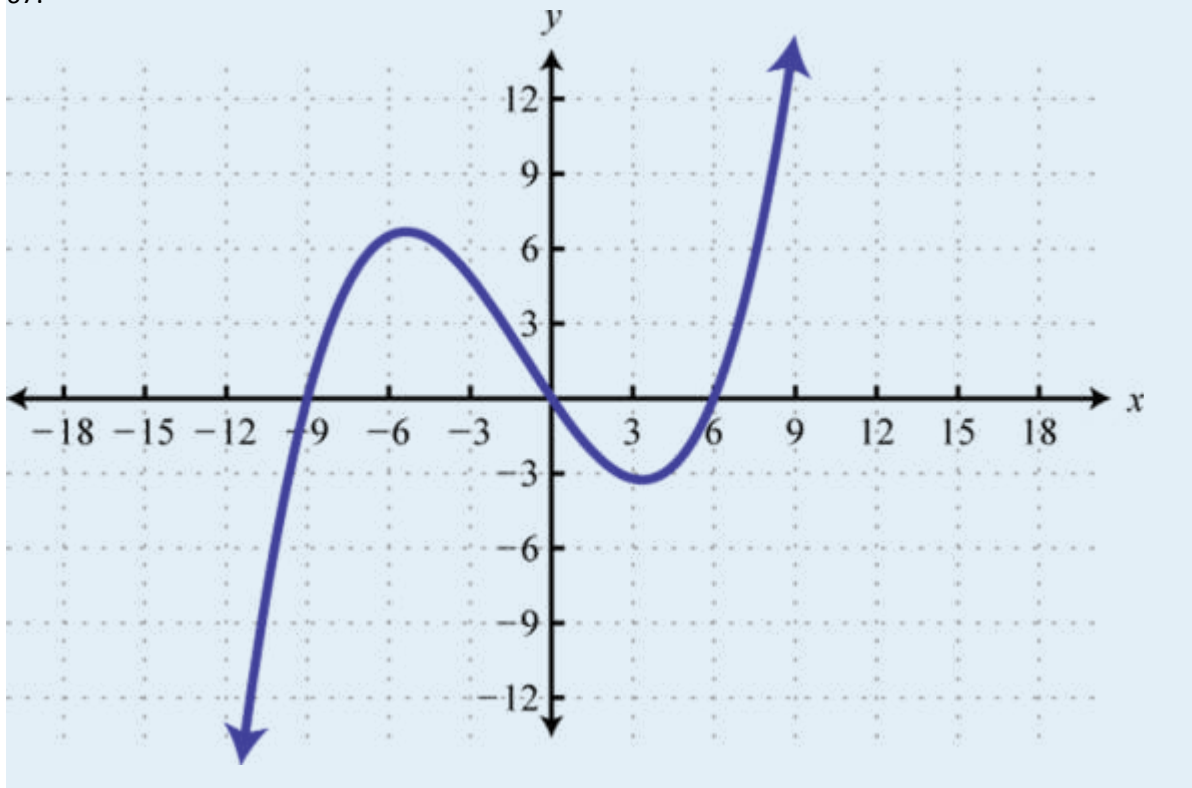
64.  $x^2 - 1 = 0$   
65.  $x^2 - 12x + 20 = 0$   
66.  $x^2 - 2x - 48 = 0$   
67.  $(2x + 1)(x - 2) = 3$   
68.  $2 - (x - 4)^2 = -7$   
69.  $(x - 6)(x + 3) = -18$   
70.  $(x + 5)(2x - 1) = 3(2x - 1)$   
71.  
72.  
73.  
74.

Find the roots of the given functions.

63.  $f(x) = 12x^2 - 8x$   
64.  $g(x) = 2x^3 - 18x$   
65.  $h(t) = -16t^2 + 64$   
66.  $p(x) = 5x^2 - 21x + 4$



67.

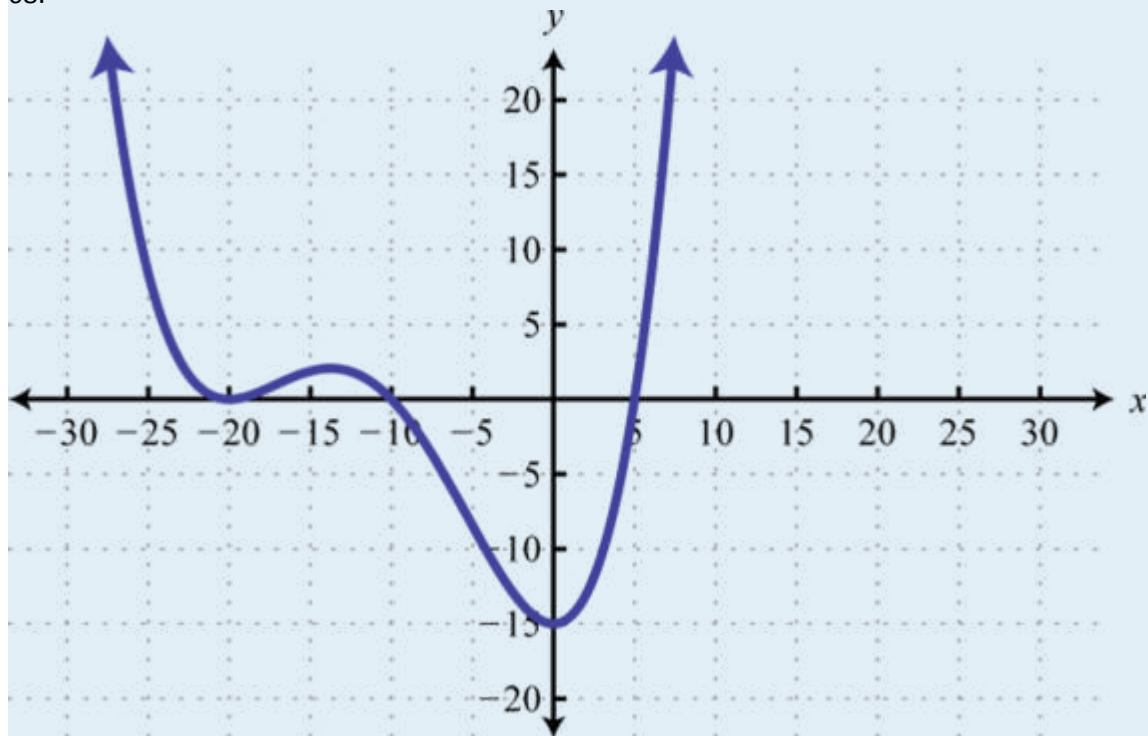


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68.



69. The height in feet of an object dropped from the top of a 16-foot ladder is given by  $h(t) = -16t^2 + 16$ , where  $t$  represents the time in seconds after the object has been dropped. How long will it take to hit the ground?

70. The length of a rectangle is 2 centimeters less than twice its width. If the area of the rectangle is 112 square centimeters, find its dimensions.

71. A triangle whose base is equal in measure to its height has an area of 72 square inches. Find the length of the base.

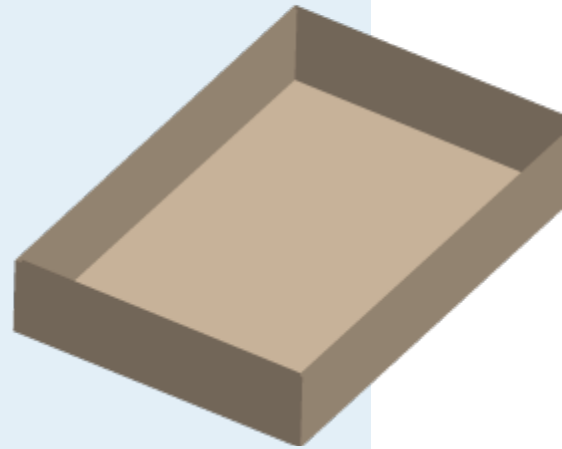
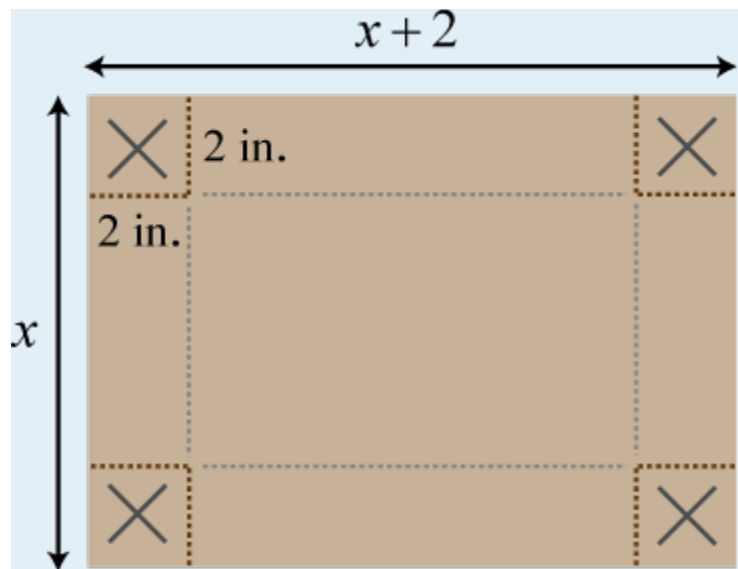
72. A box can be made by cutting out the corners and folding up the edges of a sheet of cardboard. A template for a rectangular cardboard box of height 2 inches is given.

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What are the dimensions of a cardboard sheet that will make a rectangular box with volume 240 cubic inches?

Solve or factor.

63.  $x^2 - 25$

64.  $x^2 - 121 = 0$

65.  $16x^2 - 22x - 3 = 0$

66.  $3x^2 - 14x - 5$

67.  $x^3 - x^2 - 2x - 2$

68.  $3x^2 = -15x$

Find a polynomial equation with integer coefficients, given the solutions.

63.  $5, -2$



64.

65.

66.  $\pm 10$

67.  $-4, 0, 3$

68.  $-8$  double root

## RATIONAL FUNCTIONS: MULTIPLICATION AND DIVISION

State the restrictions and simplify.

105.

106.

107.

108.

109.

110.

111.

112.

Perform the operations and simplify. Assume all variable expressions in the denominator are nonzero.

105.

106.

107.

108.

Perform the operations and state the restrictions.

105. Given  $f(x)$  and  $g(x)$ , find  $(f \cdot g)(x)$ .

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106. Given  $f$  and  $g$ , find  $(f \cdot g)(x)$ .
107. Given  $f$  and  $g$ , find  $(f/g)(x)$ .
108. Given  $f$  and  $g$ , find  $(f/g)(x)$ .
109. The daily cost in dollars of running a small business is given by  $C(x)=150+45x$  where  $x$  represents the number of hours the business is in operation. Determine the average cost per hour if the business is in operation for 8 hours in a day.
110. An electric bicycle manufacturer has determined that the cost of producing its product in dollars is given by the function  $C(n)=2n^2+100n+2,500$  where  $n$  represents the number of electric bicycles produced in a day. Determine the average cost per bicycle if 10 and 20 are produced in a day.
111. Given  $f(x)=3x-5$ , simplify
112. Given  $g(x)=2x^2-x+1$ , simplify

## RATIONAL FUNCTIONS: ADDITION AND SUBTRACTION

State the restrictions and simplify.

125.

126.

127.

128.

129.

130.

131.

132.

133.

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134.

135.

136.

Simplify. Assume that all variable expressions used as denominators are nonzero.

125.

126.

127.

128.

129.

130.

Perform the operations and state the restrictions.

125. Given  $f(x)$  and  $g(x)$ , find  $(f+g)(x)$ .

126. Given  $f(x)$  and  $g(x)$ , find  $(f+g)(x)$ .

127. Given  $f(x)$  and  $g(x)$ , find  $(f-g)(x)$ .

128. Given  $f(x)$  and  $g(x)$ , find  $(f-g)(x)$ .

## SOLVING RATIONAL EQUATIONS

Solve.

147.

148.

149.

150.

151.

152.

153.

154.

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155.

156.

157.

158.

159.

160.

161. Solve for  $a$ :

162. Solve for  $y$ :

163. A positive integer is 4 less than another. If the reciprocal of the larger integer is subtracted from twice the reciprocal of the smaller, the result is Find the two integers.

164. If 3 times the reciprocal of the larger of two consecutive odd integers is added to 7 times the reciprocal of the smaller, the result is Find the integers.

165. If the reciprocal of the smaller of two consecutive integers is subtracted from three times the reciprocal of the larger, the result is Find the integers.

166. A positive integer is twice that of another. The sum of the reciprocals of the two positive integers is Find the two integers.

## APPLICATIONS AND VARIATION

Use algebra to solve the following applications.

167. Manuel traveled 8 miles on the bus and another 84 miles on a train. If the train was 16 miles per hour faster than the bus, and the total trip took 2 hours, what was the average speed of the train?

168. A boat can average 10 miles per hour in still water. On a trip downriver, the boat was able to travel 7.5 miles with the current. On the return trip, the boat

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was only able to travel 4.5 miles in the same amount of time against the current. What was the speed of the current?

169. Susan can jog, on average, 1 miles per hour faster than her husband Bill. Bill can jog 10 miles in the same amount of time it takes Susan to jog 13 miles. How fast, on average, can Susan jog?

170. In the morning, Raul drove 8 miles to visit his grandmother and then returned later that evening. Because of traffic, his average speed on the return trip was that of his average speed that morning. If the total driving time was of an hour, what was his average speed on the return trip?

171. One pipe can completely fill a water tank in 6 hours while another smaller pipe takes 8 hours to fill the same tank. How long will it take to fill the tank to capacity if both pipes are turned on?

172. It takes Bill 3 minutes longer than Jerry to fill an order. Working together they can fill 15 orders in 30 minutes. How long does it take Bill to fill an order by himself?

173. Manny takes twice as long as John to assemble a skateboard. If they work together, they can assemble a skateboard in 6 minutes. How long would it take Manny to assemble the skateboard without John's help?

174. Working alone, Joe can complete the yard work in 30 minutes. It takes Mike 45 minutes to complete work on the same yard. How long would it take them working together?

Construct a mathematical model given the following:

167.  $y$  varies directly as  $x$ , where  $y = 30$  when  $x = 5$ .

168.  $y$  varies inversely as  $x$ , where  $y = 3$  when  $x = -2$ .

169.  $y$  is jointly proportional to  $x$  and  $z$ , where  $y = -50$  when  $x = -2$  and  $z = 5$ .

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170.  $y$  is directly proportional to the square of  $x$  and inversely proportional to  $z$ , where  $y = -6$  when  $x = 2$  and  $z = -8$ .

171. The distance an object in free fall varies directly with the square of the time that it has been falling. It is observed that an object falls 36 feet in 1 seconds. Find an equation that models the distance an object will fall, and use it to determine how far it will fall in 2 seconds.

172. After the brakes are applied, the stopping distance  $d$  of an automobile varies directly with the square of the speed  $s$  of the car. If a car traveling 55 miles per hour takes 181.5 feet to stop, how many feet will it take to stop if it is moving 65 miles per hour?

173. The weight of an object varies inversely as the square of its distance from the center of the Earth. If an object weighs 180 lbs on the surface of the Earth (approximately 4,000 miles from the center), then how much will it weigh at 2,000 miles above the Earth's surface?

174. The cost per person of renting a limousine varies inversely with the number of people renting it. If 5 people go in on the rental, the limousine will cost \$112 per person. How much will the rental cost per person if 8 people go in on the rental?

175. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 52-pound boy is sitting 3 feet away from the fulcrum, then how far from the fulcrum must a 44-pound boy sit? Round to the nearest tenth of a foot.

## ANSWERS

1.  $f(-3)=27; f(0)=6; f(10)=196$

2.

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3.  $h(-3)=12; h(0)=3; h(2)=-13$

4.

5. a. 60 feet; b. 2 seconds; c. 4 seconds

6.

7.  $h(t)=-4.9t^2+340t$ ; at 0.25 second, the bullet's height is about 85 meters.

8.

9.  $(f+g)(x)=7x^2-4x$

10.

11.  $(f \cdot g)(x)=6x^3-11x^2+7x-6$

12.

13.  $g(-3u)=9u^2+3u+1$

14.

15.  $(g \cdot h)(x)=4x^3-4x^2+4x$

16.

17.  $(g+f)(x)=16x^3-11x^2+3x+1$

18.

19.  $(f \cdot h)(-1)=128$

20.

21.  $(g-f)(2)=-85$

22.

23.  $2x^2(x^2-6x-1)$

24.

25.  $x^2y(x^2y^2-3x+1)$

26.

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27.  $(x^2+1)(2x-1)$
- 28.
29.  $(x^2+y^2)(x-5y)$
- 30.
31.  $2x(x-2y)(x^2+y^2)$
- 32.
33.  $(8x+1)(8x-1)$
- 34.
35.  $(x+6y)(x-6y)$
- 36.
37.  $(ab+5)(a^2b^2-5ab+25)$
- 38.
39.  $(9x^2+y^2)(3x+y)(3x-y)$
- 40.
41.  $(x+2y)(x^2-2xy+4y^2)(x-2y)(x^2+2xy+4y^2)$
- 42.
43.  $(x-12)(x+4)$
- 44.
45. Prime
- 46.
47.  $(x+5y)(x+15y)$
- 48.
49.  $-2(y-12)(y+2)$
- 50.

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51.  $2(15x-1)(5x-3)$
- 52.
53.  $(3u+v)(9u-4v)$
- 54.
55.  $(4m+9n)^2$
- 56.
57.  $(5x^2-6)(5x^2-1)$
- 58.
59.  $(x^3+5y^3)(x^3-2y^3)$
- 60.
61.  $(xn-1)^2$
- 62.
63.  $5x(3x+2)(3x-2)$
- 64.
65.  $-(10x-1)(2x-3)$
- 66.
67.  $3ab^2(2a+b)(4a^2-2ab+b^2)$
- 68.
69. Prime
- 70.
71. , 0
- 72.
73. 2, 10
- 74.

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75. -1,
- 76.
77. 0, 3
- 78.
- 79.
- 80.
81. -4, 0, 6
- 82.
83. 0,
- 84.
85.  $\pm 2$
- 86.
87. -9, 0, 6
- 88.
89. 1 second
- 90.
91. 12 inches
- 92.
93. Factor;  $(x+5)(x-5)$
- 94.
95. Solve;
- 96.
97. Factor;  $(x-1)(x^2-2)$
- 98.

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99.  $x^2 - 3x - 10 = 0$

100.

101.  $25x^2 - 16 = 0$

102.

103.  $x^3 + x^2 - 12x = 0$

104.

105.  $9x; x \neq 0$

106.

107. ;

108.

109. ;

110.

111. ;

112.

113.

114.

115.

116.

117. ;  $x \neq -10, -5, -3, 2$

118.

119. ;

120.

121. \$63.75 per hour

122.

123. 3

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- 124.
- 125. ;  $x \neq \pm 6$
- 126.
- 127. ;  $x \neq 0,5$
- 128.
- 129. ;
- 130.
- 131. ;  $x \neq 3,5$
- 132.
- 133. ;  $t \neq \pm 1$
- 134.
- 135. ;  $x \neq 0$
- 136.
- 137.
- 138.
- 139.
- 140.
- 141.
- 142.
- 143. ;  $x \neq -2,3$
- 144.
- 145. ;  $x \neq \pm 5$
- 146.
- 147.  $-9$
- 148.



149.  $-1, 4$
- 150.
151.  $-11, 0$
- 152.
153.  $\emptyset$
- 154.
155.  $-4$
- 156.
- 157.
- 158.
- 159.
- 160.
- 161.
- 162.
163.  $\{8, 12\}$
- 164.
165.  $\{5, 6\}$
- 166.
167. 48 miles per hour
- 168.
169. 6.5 miles per hour
- 170.
171. Approximately 2.6 hours
- 172.
173. 18 minutes

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174.

175.  $y=6x$

176.

177.  $y=5xz$

178.

179.  $d=16t^2$ ; 100 feet

180.

181. 80 lbs

182.

183. Approximately 3.5 feet

## SAMPLE EXAM

Given  $f(x)=x^2-x+4$ ,  $g(x)=5x-1$ , and  $h(x)=2x^2+x-3$ , find the following:

1.  $(g \cdot h)(x)$

2.  $(h-f)(x)$

3.  $(f+g)(-1)$

Factor.

1.  $x^3+16x-2x^2-32$

2.  $x^3-8y^3$

3.  $x^4-81$

4.  $25x^2y^2-40xy+16$

5.  $16x^3y+12x^2y^2-18xy^3$

Solve.

1.  $6x^2+24x=0$

2.  $(2x+1)(3x+2)=12$

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3.  $(2x+1)^2=23x+6$
4. Find a quadratic equation with integer coefficients given the solutions
5. Given  $f(x)=5x^2-x+4$ , simplify  $f(h)$ , where  $h \neq 0$ .

Simplify and state the restrictions.

- 1.
- 2.

Assume all variable expressions in the denominator are nonzero and simplify.

- 1.

Solve.

- 1.
- 2.

3. Find the root of the function defined by
4. Solve for  $y$ :

Use algebra to solve.

1. The height of an object dropped from a 64-foot building is given by the function  $h(t)=-16t^2+64$ , where  $t$  represents time in seconds after it was dropped.
  - 1.a. Determine the height of the object at  $t=2$  of a second.
  - 1.b. How long will it take the object to hit the ground?
2. One positive integer is 3 units more than another. When the reciprocal of the larger is subtracted from twice the reciprocal of the smaller, the result is  $\frac{1}{6}$ . Find the two positive integers.
3. A light airplane can average 126 miles per hour in still air. On a trip, the airplane traveled 222 miles with a tailwind. On the return trip, against a headwind of the same speed, the plane was only able to travel 156 miles in the same amount of time. What was the speed of the wind?





4. On the production line, it takes John 2 minutes less time than Mark to assemble a watch. Working together they can assemble 5 watches in 12 minutes. How long does it take John to assemble a watch working alone?

5. Write an equation that relates  $x$  and  $y$ , given that  $y$  varies inversely with the square of  $x$ , where  $y=-$  when  $x=3$ . Use it to find  $y$  when  $x=$ .

## ANSWERS

1.  $(g \cdot h)(x) = 10x^3 + 3x^2 - 16x + 3$

2.

3.  $(f+g)(-1) = 0$

4.

5.  $(x-2y)(x^2+2xy+4y^2)$

6.

7.  $(5xy-4)^2$

8.

9.  $-4, 0$

10.

11.  $- , 5$

12.

13.  $10x+5h-1$

14.

15.  $; x \neq 1, 7$

16.

17.  $-$

18.

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19. –
- 20.
21. a. 55 feet; b. 2 seconds
- 22.
23. 22 miles per hour
- 24.
25. ;  $y = -12$

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# Chapter 5

## Radical Functions and Equations

### 5.1 Roots and Radicals

#### LEARNING OBJECTIVES

1. Identify and evaluate square and cube roots.
2. Determine the domain of functions involving square and cube roots.
3. Evaluate  $n$ th roots.
4. Simplify radicals using the product and quotient rules for radicals.

#### Square and Cube Roots

Recall that a square root of a number is a number that when multiplied by itself yields the original number. For example, 5 is a square root of 25, because  $5 \cdot 5 = 25$ . Since  $5 \cdot 5 = 25$ , we can say that  $-5$  is a square root of 25 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the radical sign to denote the principal (nonnegative) square root and a negative sign in front of the radical  $-\sqrt{\quad}$  to denote the negative square root.

Zero is the only real number with one square root.

#### Example 1

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Evaluate.

1.

2.

Solution:

1.

2.

If the radicand, the number inside the radical sign, can be factored as the square of another number, then the square root of the number is apparent. In this case, we have the following property:

Or more generally,

The absolute value is important because  $a$  may be a negative number and the radical sign denotes the principal square root. For example,

Make use of the absolute value to ensure a positive result.

## Example 2

Simplify:

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Solution:

Here the variable expression  $x-2$  could be negative, zero, or positive. Since the sign depends on the unknown quantity  $x$ , we must ensure that we obtain the principal square root by making use of the absolute value.

Answer:

The importance of the use of the absolute value in the previous example is apparent when we evaluate using values that make the radicand negative. For example, when ,

Next, consider the square root of a negative number. To determine the square root of  $-25$ , you must find a number that when squared results in  $-25$ :

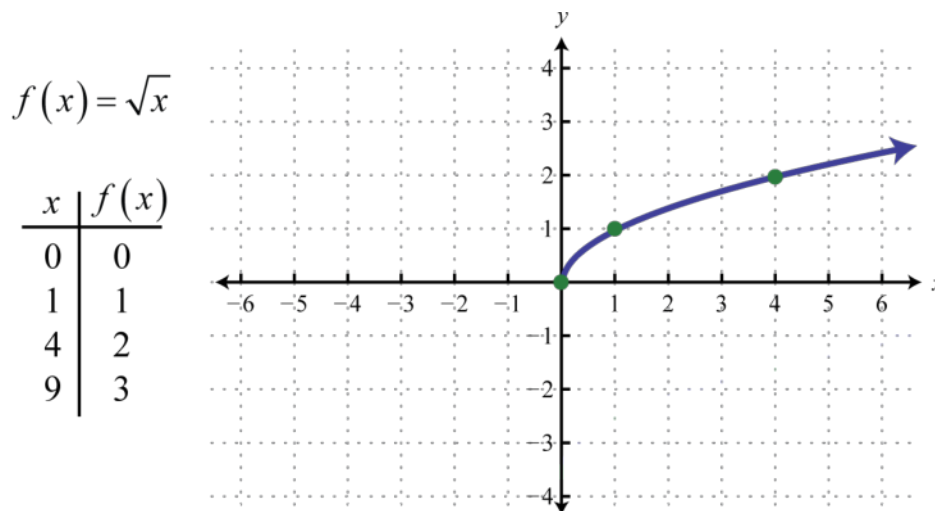
However, any real number squared always results in a positive number. The square root of a negative number is currently left undefined. For now, we will state that is not a real number. Therefore, the square root function given by is not defined to be a real number

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if the  $x$ -values are negative. The smallest value in the domain is zero. For example, and Recall the graph of the square root function.



The domain and range both consist of real numbers greater than or equal to zero:  $[0, \infty)$ . To determine the domain of a function involving a square root we look at the radicand and find the values that produce nonnegative results.

### Example 3

Determine the domain of the function defined by

Solution:

Here the radicand is This expression must be zero or positive. In other words,



Solve for  $x$ .

Answer: Domain:

A cube root of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol  $\sqrt[3]{\phantom{x}}$ , where 3 is called the index. For example,

The product of three equal factors will be positive if the factor is positive and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

In general, given any real number  $a$ , we have the following property:

When simplifying cube roots, look for factors that are perfect cubes.

## Example 4

Evaluate.

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- a.
- b.
- c.
- d.
- e.

Solution:

- a.
- b.
- c.
- d.
- e.

It may be the case that the radicand is not a perfect square or cube. If an integer is not a perfect power of the index, then its root will be irrational. For example,  $\sqrt[3]{2}$  is an irrational number that can be approximated on most calculators using the root button  $\sqrt{x}$ . Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$\sqrt[3]{2} =$$

Therefore, we have





Since cube roots can be negative, zero, or positive we do not make use of any absolute values.

### **Example 5**

Simplify:

Solution:

The cube root of a quantity cubed is that quantity.

Answer:

Try this! Evaluate:

Answer:  $-10$

Next, consider the cube root function:

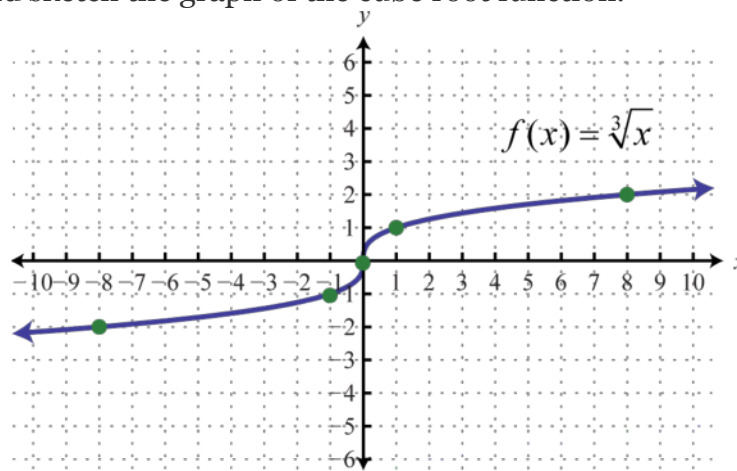
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Since the cube root could be either negative or positive, we conclude that the domain consists of all real numbers. Sketch the graph by plotting points. Choose some positive and negative values for  $x$ , as well as zero, and then calculate the corresponding  $y$ -values.

Plot the points and sketch the graph of the cube root function.



The graph passes the vertical line test and is indeed a function. In addition, the range consists of all real numbers.

### Example 6

Given , find , , and Sketch the graph of  $g$ .

Solution:

Replace  $x$  with the given values.

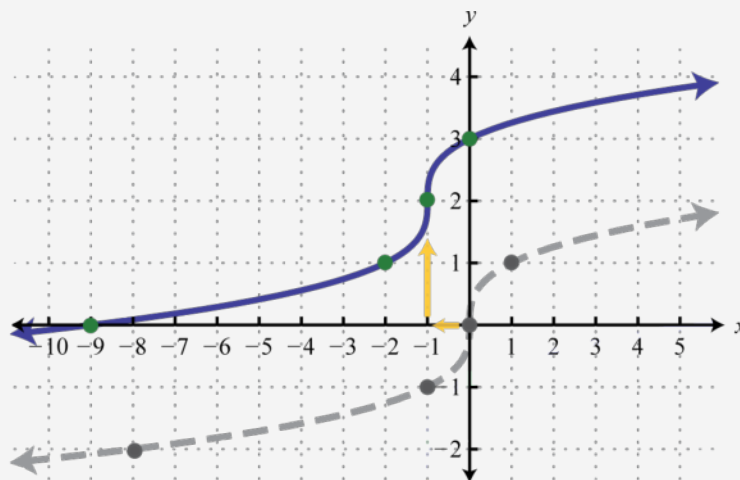
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We can also sketch the graph using the following translations:

Answer:



## nth Roots

For any integer  $n$ , we define

an **nth root** of a positive real number as a number that when raised to the  $n$ th power yields the original number. Given any nonnegative real number  $a$ , we have the following property:

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Here  $n$  is called the index and is called the radicand. Furthermore, we can refer to the entire expression as a radical. When the index is an integer greater than or equal to 4, we say “fourth root,” “fifth root,” and so on. The  $n$ th root of any number is apparent if we can write the radicand with an exponent equal to the index.

### Example 7

Simplify

- a.
- b.
- c.
- d.

Solution:

- a.
- b.
- c.
- d.

*Note:* If the index is 2, then the radical indicates a square root and it is customary to write the radical without the index;

We have already taken care to define the principal square root of a real number. At this point, we extend this idea to  $n$ th roots when  $n$  is even. For example, 3 is a fourth root of 81, because  $3^4 = 81$ . And since  $(-3)^4 = 81$ , we can say that  $-3$  is a fourth root of 81 as well. Hence we use the radical sign to denote

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the principal (nonnegative)  **$n$ th root** when  $n$  is even. In this case, for any real number  $a$ , we use the following property:

For example,

The negative  $n$ th root, when  $n$  is even, will be denoted using a negative sign in front of the radical

We have seen that the square root of a negative number is not real because any real number that is squared will result in a positive number. In fact, a similar problem arises for any even index:

We can see that a fourth root of  $-81$  is not a real number because the fourth power of any real number is always positive.

You are encouraged to try all of these on a calculator. What does it say?

### **Example 8**

Simplify.

1.

2.

3.

Solution:

Since the indices are even, use absolute values to ensure nonnegative results.

1.

2. is not a real number.

3.

When the index  $n$  is odd, the same problems do not occur. The product of an odd number of positive factors is positive and the product of an odd number of negative factors is negative. Hence when the index  $n$  is odd, there is only one real  $n$ th root for any real number  $a$ . And we have the following property:

### **Example 9**

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Simplify.

1.

2.

3.

Solution:

Since the indices are odd, the absolute value is not used.

1.

2.

3.

In summary, for any real number  $a$  we have,

When  $n$  is *odd*, the  $n$ th root is *positive or negative* depending on the sign of the radicand.

When  $n$  is *even*, the  $n$ th root is *positive or not real* depending on the sign of the radicand.

Try this! Simplify:

Answer: 16

## Simplifying Radicals

It will not always be the case that the radicand is a perfect power of the given index. If it is not, then we use the product rule for radicals and the quotient rule for radicals to simplify them. Given real numbers and ,

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Product Rule for Radicals:	
Quotient Rule for Radicals:	

A radical is simplified if it does not contain any factors that can be written as perfect powers of the index.

**Example 10**

Simplify:

Solution:

Here 150 can be written as

We can verify our answer on a calculator:

Also, it is worth noting that

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Answer:

Note: is the exact answer and 12.25 is an approximate answer. We present exact answers unless told otherwise.

### **Example 11**

Simplify:

Solution:

Use the prime factorization of 160 to find the largest perfect cube factor:

Replace the radicand with this factorization and then apply the product rule for radicals.

We can verify our answer on a calculator.

Answer:

### **Example 12**

Simplify:

Solution:

Here we note that the index is odd and the radicand is negative; hence the result will be negative. We can factor the radicand as follows:

Then simplify:

Answer:

### **Example 13**

Simplify:

Solution:

In this case, consider the equivalent fraction with in the numerator and in the denominator and then simplify.

Answer:

Try this! Simplify:

Answer:

## **KEY TAKEAWAYS**

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To simplify a square root, look for the largest perfect square factor of the radicand and then apply the product or quotient rule for radicals.

To simplify a cube root, look for the largest perfect cube factor of the radicand and then apply the product or quotient rule for radicals.

When working with  $n$ th roots,  $n$  determines the definition that applies. We use when  $n$  is odd and when  $n$  is even.

To simplify  $n$ th roots, look for the factors that have a power that is equal to the index  $n$  and then apply the product or quotient rule for radicals. Typically, the process is streamlined if you work with the prime factorization of the radicand.

## TOPIC EXERCISES

### PART A: SQUARE AND CUBE ROOTS

Simplify.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.

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- 15.
- 16.
- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.
- 25.
- 26.
- 27.
- 28.
- 29.
- 30.

Determine the domain of the given function.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.

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10.

Evaluate given the function definition.

1. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

2. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

3. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

4. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

5. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

6. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

7. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

8. Given  $f(x)$ , find  $f(2)$ ,  $f(3)$ , and  $f(4)$ .

Sketch the graph of the given function and give its domain and range.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

## PART B: NTH ROOTS

Simplify.

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- 61.
- 62.
- 63.
- 64.
- 65.
- 66.
- 67.
- 68.
- 69.
- 70.
- 71.
- 72.
- 73.
- 74.
- 75.
- 76.
- 77.
- 78.
- 79.
- 80.
- 81.
- 82.
- 83.
- 84.
- 85.
- 86.

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87.

88.

89.

90.

## PART C: SIMPLIFYING RADICALS

Simplify.

91.

92.

93.

94.

95.

96.

97.

98.

99.

100.

101.

102.

103.

104.

105.

106.

107.

108.

109.

110.

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111.

112.

113.

114.

115.

116.

117.

118.

119.

120.

121.

122.

123.

124.

Simplify. Give the exact answer and the approximate answer rounded to the nearest hundredth.

91.

92.

93.

94.

95.

96.

97.

98.

99.

100.

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Rewrite the following as a radical expression with coefficient 1.

91.

92.

93.

94.

95.

96.

97.

98.

99. Each side of a square has a length that is equal to the square root of the square's area. If the area of a square is 72 square units, find the length of each of its sides.

100. Each edge of a cube has a length that is equal to the cube root of the cube's volume. If the volume of a cube is 375 cubic units, find the length of each of its edges.

101. The current  $I$  measured in amperes is given by the formula where  $P$  is the power usage measured in watts and  $R$  is the resistance measured in ohms. If a 100 watt light bulb has 160 ohms of resistance, find the current needed. (Round to the nearest hundredth of an ampere.)

102. The time in seconds an object is in free fall is given by the formula where  $s$  represents the distance in feet the object has fallen. How long will it take an object to fall to the ground from the top of an 8-foot stepladder? (Round to the nearest tenth of a second.)

#### PART D: DISCUSSION BOARD

147. Explain why there are two real square roots for any positive real number and one real cube root for any real number.

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148. What is the square root of 1 and what is the cube root of 1? Explain why.
149. Explain why is not a real number and why is a real number.
150. Research and discuss the methods used for calculating square roots before the common use of electronic calculators.

## ANSWERS

1. 6
- 2.
- 3.
- 4.
5. -4
- 6.
7. 5
- 8.
9. Not a real number
- 10.
11. -3
- 12.
- 13.
- 14.
- 15.
- 16.
17. 4
- 18.
19. -6
- 20.

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21. -2

22.

23. 2

24.

25.

26.

27.

28.

29.

30.

31.

32.

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34.

35.

36.

37.

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40.

41. ; ;

42.

43. ; ;

44.

45. ; ;

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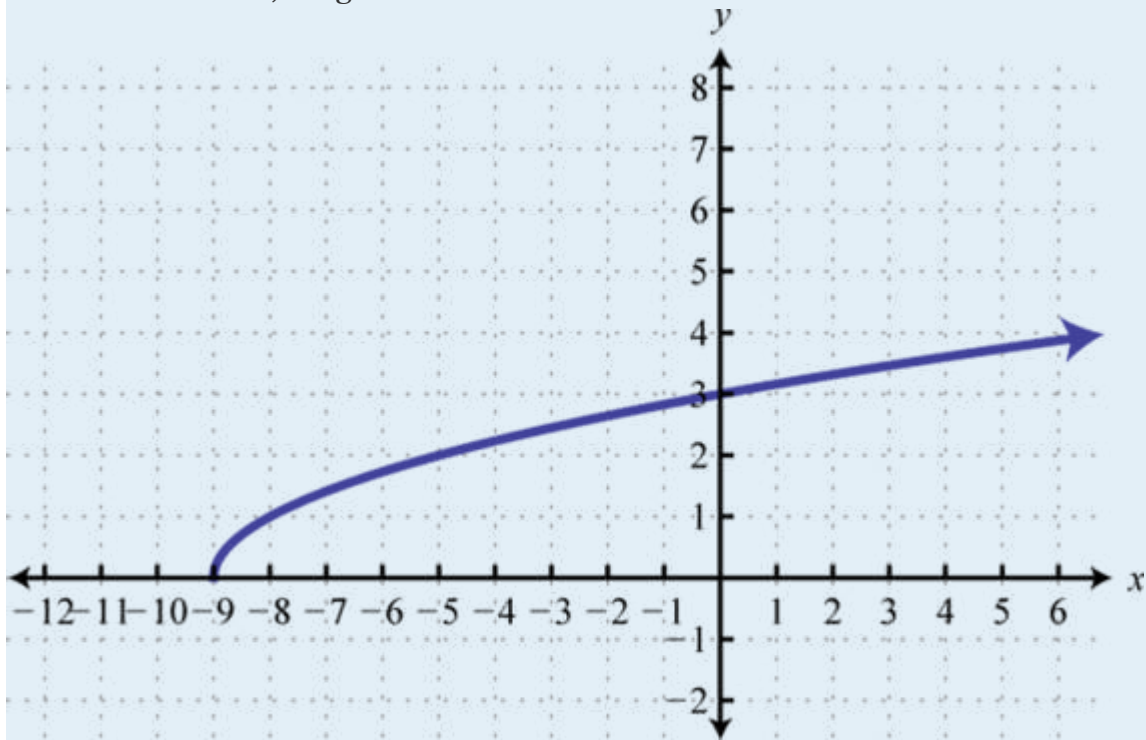


46.

47. ; ;

48.

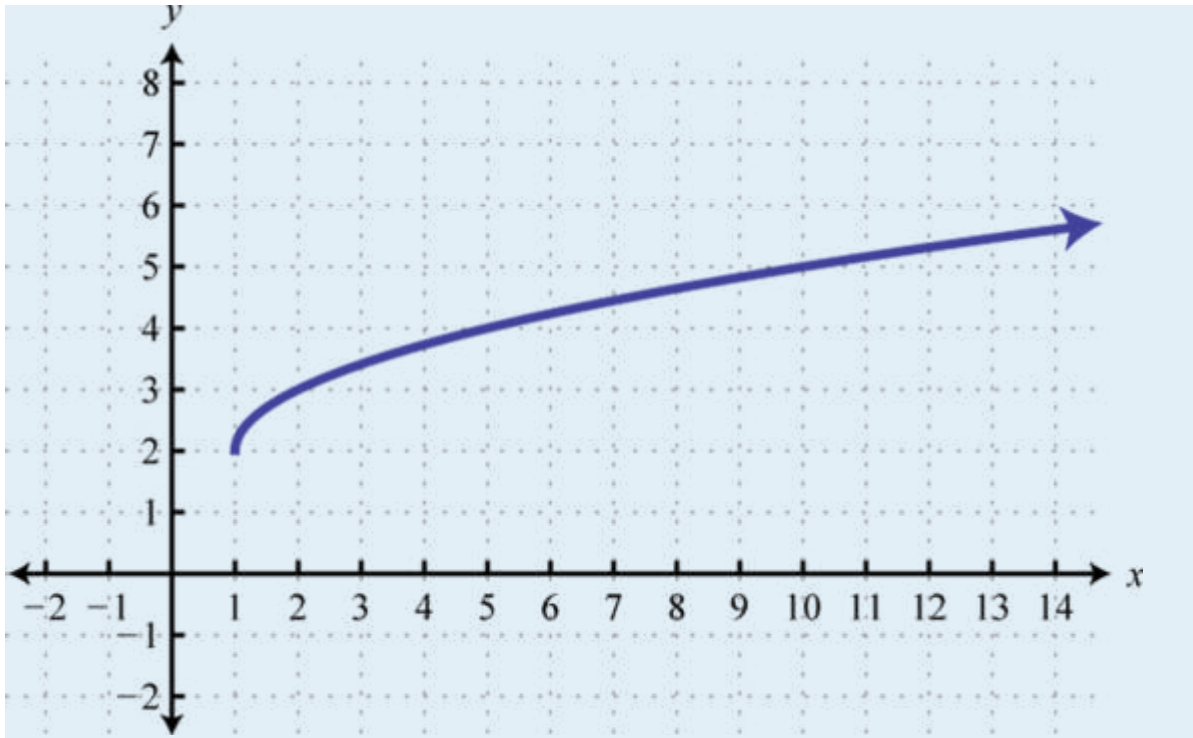
49. Domain: ; range:



50.

51. Domain: ; range:

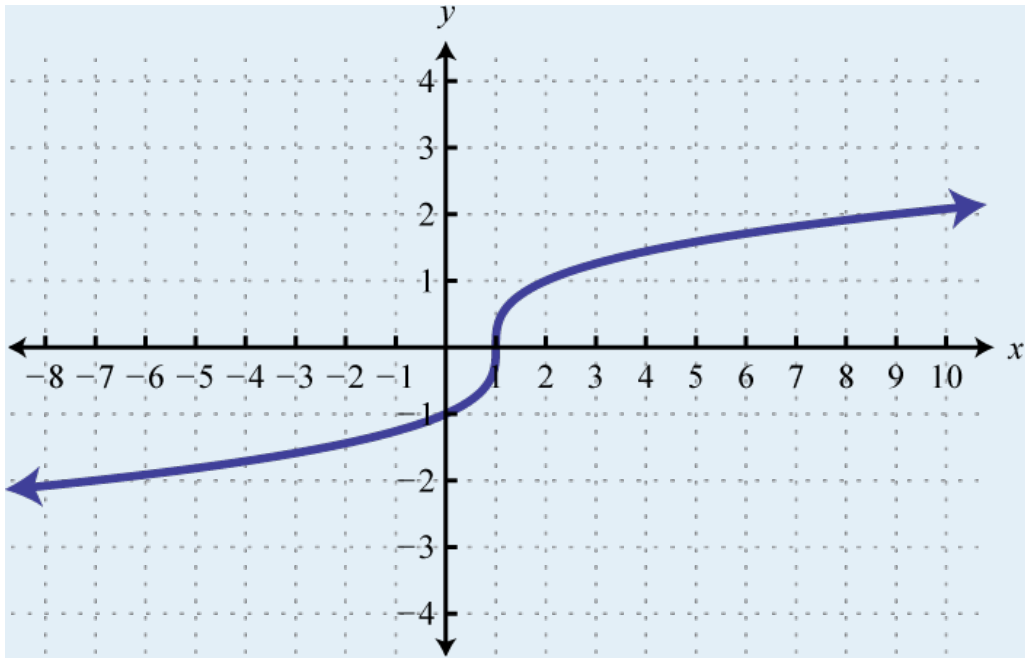




52.

53. Domain: ; range:





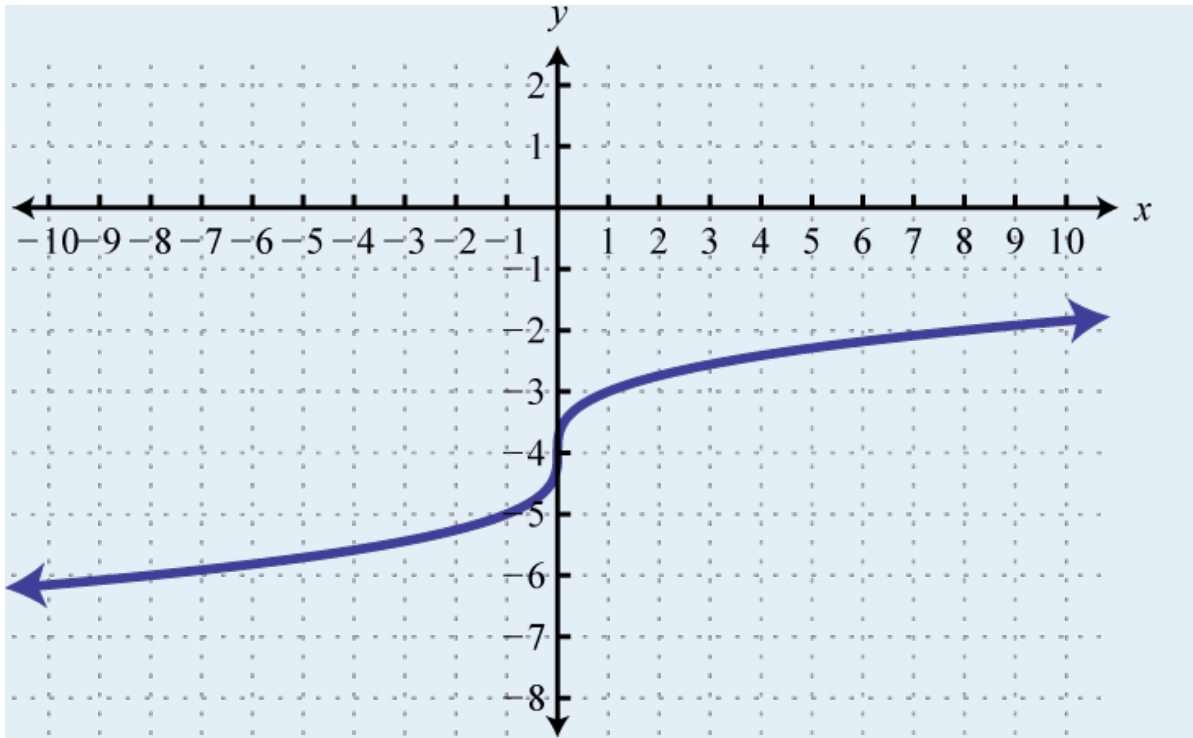
54.

55. Domain: ; range:

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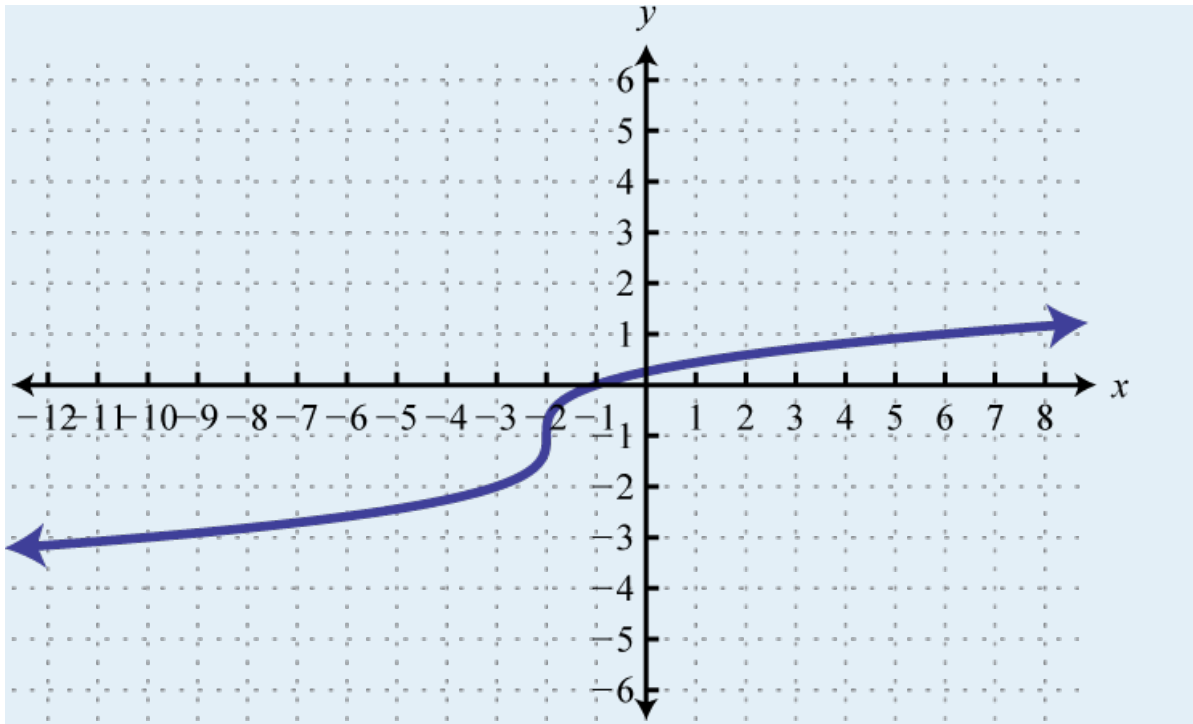




56.

57. Domain: ; range:

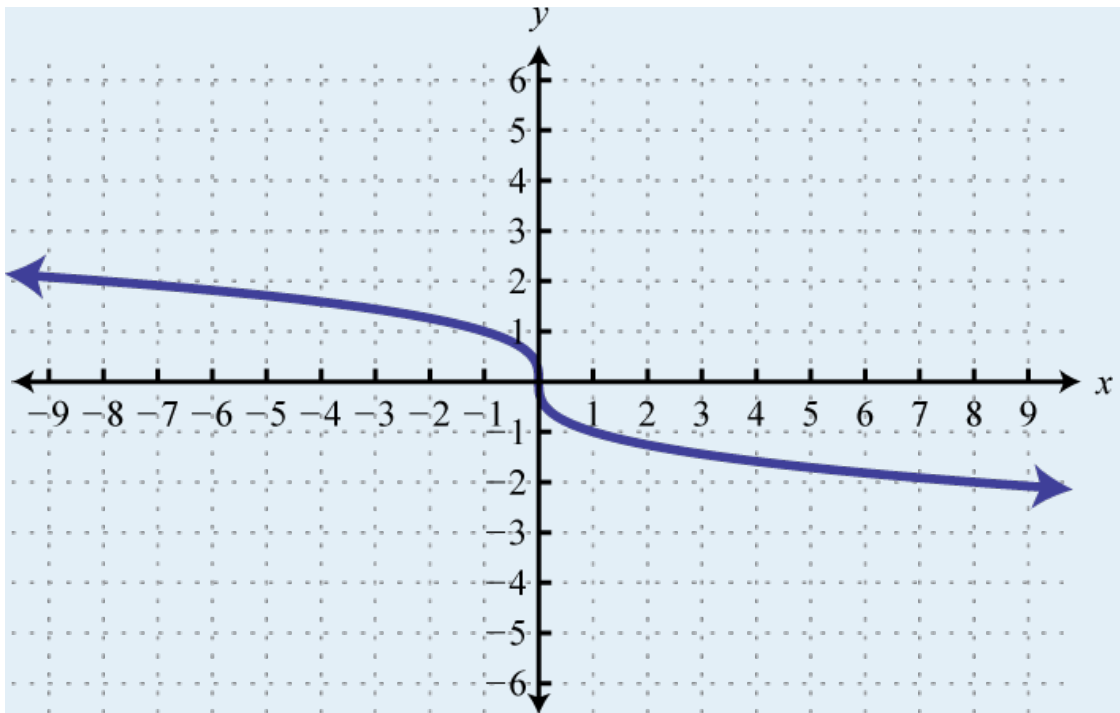




58.

59. Domain: ; range:





- 60.
- 61. 4
- 62.
- 63. 5
- 64.
- 65. 4
- 66.
- 67. 3
- 68.
- 69.
- 70.
- 71. -2



- 72.
- 73. -2
- 74.
- 75. Not a real number
- 76.
- 77. 18
- 78.
- 79. -20
- 80.
- 81. Not a real number
- 82.
- 83.
- 84.
- 85. 3
- 86.
- 87.
- 88.
- 89. 20
- 90.
- 91.
- 92.
- 93.
- 94.
- 95.
- 96.





- 97.
- 98.
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- 102.
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- 110.
- 111.
- 112.
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- 114.
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- 116.
- 117.
- 118.
- 119.
- 120.
- 121.
- 122.

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123.

124.

125. ; 7.75

126.

127. ; 1.40

128.

129. ; 6.21

130.

131. ; 1.32

132.

133. ; 4.70

134.

135.

136.

137.

138.

139.

140.

141.

142.

143. units

144.

145. Answer: 0.79 ampere

146.

147. Answer may vary

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148.

149. Answer may vary

150.

## 5.2 Simplifying Radical Expressions

### LEARNING OBJECTIVES

1. Simplify radical expressions using the product and quotient rule for radicals.
2. Use formulas involving radicals.

### Simplifying Radical Expressions

An algebraic expression that contains radicals is called a radical expression. We use the product and quotient rules to simplify them.

#### Example 1

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Simplify:

Solution:

Use the fact that when  $n$  is odd.

Answer:  $3x$

## Example 2

Simplify:

Solution:

Use the fact that when  $n$  is even.

Since  $y$  is a variable, it may represent a negative number. Thus we need to ensure that the result is positive by including the absolute value.

Answer:  $2|y|$

## Important Note

Typically, at this point in algebra we note that all variables are assumed to be positive. If this is the case, then  $y$  in the previous example is positive and the absolute value operator is not needed. The example can be simplified as follows.

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In this section, we will assume that all variables are positive. This allows us to focus on calculating  $n$ th roots without the technicalities associated with the principal  $n$ th root problem. For this reason, we will use the following property for the rest of the section,

When simplifying radical expressions, look for factors with powers that match the index.

### **Example 3**

Simplify:

Solution:

Begin by determining the square factors of 12,  $x^6$ , and  $y^3$ .

Make these substitutions, and then apply the product rule for radicals and simplify.

Answer:

### **Example 4**

Simplify:

Solution:

Begin by determining the square factors of 18,  $a^5$ , and  $b^8$ .

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Make these substitutions, apply the product and quotient rules for radicals, and then simplify.

Answer:

### **Example 5**

Simplify:

Solution:

Begin by determining the cubic factors of  $80$ ,  $x^5$ , and  $y^7$ .

Make these substitutions, and then apply the product rule for radicals and simplify.

Answer:

### **Example 6**

Simplify

Solution:

The coefficient  $9=3^2$ , and thus does not have any perfect cube factors. It will be left as the only remaining radicand because all of the other factors are cubes, as illustrated below:

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Replace the variables with these equivalents, apply the product and quotient rules for radicals, and then simplify.

Answer:

### **Example 7**

Simplify:

Solution:

Determine all factors that can be written as perfect powers of 4. Here, it is important to see that  $b^5 = b^4 \cdot b$ . Hence the factor  $b$  will be left inside the radical.

Answer:

### **Example 8**

Simplify:

Solution:

Notice that the variable factor  $x$  cannot be written as a power of 5 and thus will be left inside the radical. In addition,  $y^6 = y^5 \cdot y$ ; the factor  $y$  will be left inside the radical as well.

Answer:

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Tip: To simplify finding an  $n$ th root, divide the powers by the index.

If the index does not divide into the power evenly, then we can use the quotient and remainder to simplify. For example,

The quotient is the exponent of the factor outside of the radical, and the remainder is the exponent of the factor left inside the radical.

Try this! Simplify:

Answer:

## Formulas Involving Radicals

Formulas often consist of radical expressions. For example, the period of a pendulum, or the time it takes a pendulum to swing from one side to the other and back, depends on its length according to the following formula.

Here  $T$  represents the period in seconds and  $L$  represents the length in feet of the pendulum.

### Example 9

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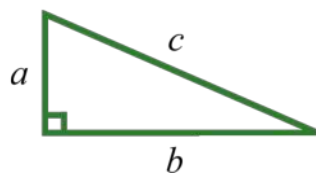
If the length of a pendulum measures 112 feet, then calculate the period rounded to the nearest tenth of a second.

Solution:

Substitute for  $L$  and then simplify.

Answer: The period is approximately 1.36 seconds.

Frequently you need to calculate the distance between two points in a plane. To do this, form a right triangle using the two points as vertices of the triangle and then apply the Pythagorean theorem. Recall that the Pythagorean theorem states that if given any right triangle with legs measuring  $a$  and  $b$  units, then the square of the measure of the hypotenuse  $c$  is equal to the sum of the squares of the legs:  $a^2 + b^2 = c^2$ . In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.



$$a^2 + b^2 = c^2$$

or

$$c = \sqrt{a^2 + b^2}$$

## Example 10

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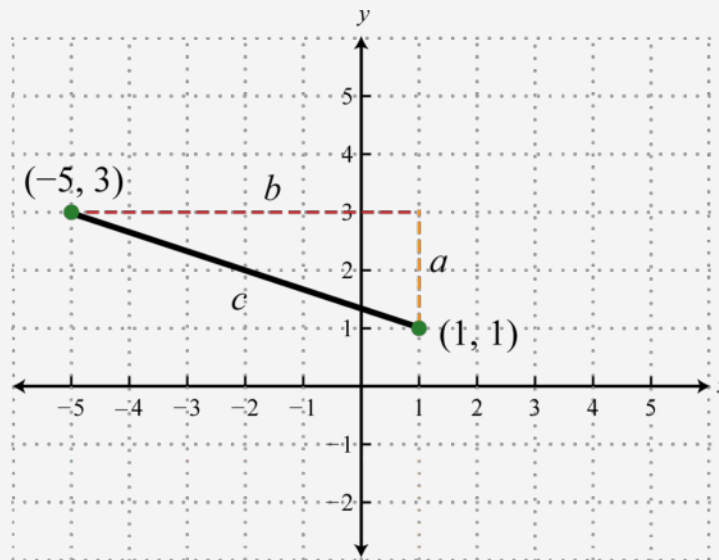
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Find the distance between  $(-5, 3)$  and  $(1, 1)$ .

Solution:

Form a right triangle by drawing horizontal and vertical lines through the two points. This creates a right triangle as shown below:



The length of leg  $b$  is calculated by finding the distance between the  $x$ -values of the given points, and the length of leg  $a$  is calculated by finding the distance between the given  $y$ -values.

Next, use the Pythagorean theorem to find the length of the hypotenuse.

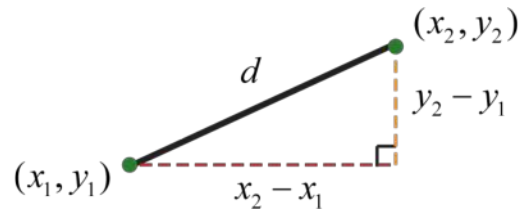
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Answer: The distance between the two points is units.

Generalize this process to produce a formula that can be used to algebraically calculate the distance between any two given points.



Given two points,  $(x_1, y_1)$  and  $(x_2, y_2)$ , the distance,  $d$ , between them is given by the distance formula,

### Example 11

Calculate the distance between  $(-4, 7)$  and  $(2, 1)$ .

Solution:

Use the distance formula with the following points.

It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

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Answer: The distance between the two points is units.

### **Example 12**

Do the three points  $(2, -1)$ ,  $(3, 2)$ , and  $(8, -3)$  form a right triangle?

Solution:

The Pythagorean theorem states that having side lengths that satisfy the property  $a^2+b^2=c^2$  is a necessary and sufficient condition of right triangles. In other words, if you can show that the sum of the squares of the leg lengths of the triangle is equal to the square of the length of the hypotenuse, then the triangle must be a right triangle. First, calculate the length of each side using the distance formula.

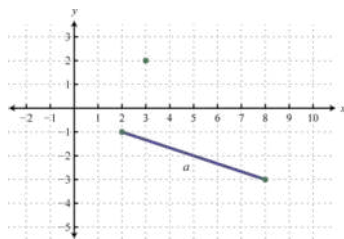
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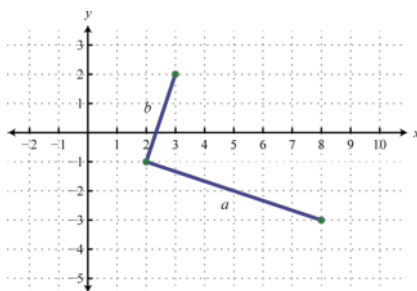


**Geometry**

**Calculation**



Points: (2, -1) and (8, -3)

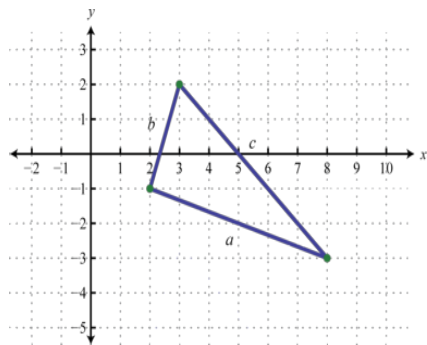


Points: (2, -1) and (3, 2)



**Geometry**

**Calculation**



Points: (3, 2) and (8, -3)

Now we check to see if  $a^2+b^2=c^2$ .



Answer: Yes, the three points form a right triangle.



Try this! The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On wet concrete, the speed  $v$  in miles per hour can be estimated by the formula  $v = \sqrt{2d}$ , where  $d$  represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes if the skid marks left behind measure 27 feet. Round to the nearest mile per hour.

Answer: 18 miles per hour

## KEY TAKEAWAYS

To simplify a radical expression, look for factors of the radicand with powers that match the index. If found, they can be simplified by applying the product and quotient rules for radicals, as well as the property  $\sqrt[n]{a^m} = a^{m/n}$ , where  $a$  is nonnegative.

A radical expression is simplified if its radicand does not contain any factors that can be written as perfect powers of the index.

We typically assume that all variable expressions within the radical are nonnegative. This allows us to focus on simplifying radicals without the technical issues associated with the principal  $n$ th root. If this assumption is not made, we will ensure a positive result by using absolute values when simplifying radicals with even indices.

## TOPIC EXERCISES

### PART A: SIMPLIFYING RADICAL EXPRESSIONS

Assume that the variable could represent any real number and then simplify.

- 1.
- 2.

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3.

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6.

7.

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10.

11.

12.

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14.

15.

16.

17.

18.

19.

20.

Simplify. (Assume all variable expressions represent positive numbers.)

1.

2.

3.

4.

5.

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6.

7.

8.

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31.

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55.

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56.

Rewrite the following as a radical expression with coefficient 1.

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

## PART B: FORMULAS INVOLVING RADICALS

The period  $T$  in seconds of a pendulum is given by the formula

where  $L$  represents the length in feet of the pendulum. Calculate the period, given each of the following lengths. Give the exact value and the approximate value rounded to the nearest tenth of a second.

87. 8 feet

88. 32 feet

89. foot

90. foot

The time  $t$  in seconds an object is in free fall is given by the formula

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where  $s$  represents the distance in feet the object has fallen. Calculate the time it takes an object to fall, given each of the following distances. Give the exact value and the approximate value rounded to the nearest tenth of a second.

87. 48 feet

88. 80 feet

89. 192 feet

90. 288 feet

91. The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed  $v$  in miles per hour can be estimated by the formula  $v = \sqrt{2d}$ , where  $d$  represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes on dry pavement if the skid marks left behind measure 27 feet. Round to the nearest mile per hour.

92. The radius  $r$  of a sphere can be calculated using the formula  $V = \frac{4}{3}\pi r^3$ , where  $V$  represents the sphere's volume. What is the radius of a sphere if the volume is  $36\pi$  cubic centimeters?

Given the function find the  $y$ -intercept

87.

88.

89.

90.

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91.

92.

Use the distance formula to calculate the distance between the given two points.

87.  $(5, -7)$  and  $(3, -8)$

88.  $(-9, 7)$  and  $(-8, 4)$

89.  $(-3, -4)$  and  $(3, -6)$

90.  $(-5, -2)$  and  $(1, -6)$

91.  $(-1, 1)$  and  $(-4, 10)$

92.  $(8, -3)$  and  $(2, -12)$

93.  $(0, -6)$  and  $(-3, 0)$

94.  $(0, 0)$  and  $(8, -4)$

95.  $(-, -)$  and  $(-1, -)$

96.  $(-, 2)$  and  $(-, -)$

Determine whether or not the three points form a right triangle. Use the Pythagorean theorem to justify your answer.

87.  $(2, -1)$ ,  $(-1, 2)$ , and  $(6, 3)$

88.  $(-5, 2)$ ,  $(-1, -2)$ , and  $(-2, 5)$

89.  $(-5, 0)$ ,  $(0, 3)$ , and  $(6, -1)$

90.  $(-4, -1)$ ,  $(-2, 5)$ , and  $(7, 2)$

91.  $(1, -2)$ ,  $(2, 3)$ , and  $(-3, 4)$

92.  $(-2, 1)$ ,  $(-1, -1)$ , and  $(1, 3)$

93.  $(-4, 0)$ ,  $(-2, -10)$ , and  $(3, -9)$

94.  $(0, 0)$ ,  $(2, 4)$ , and  $(-2, 6)$

## PART D: DISCUSSION BOARD

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121. Give a value for  $x$  such that Explain why it is important to assume that the variables represent nonnegative numbers.

122. Research and discuss the accomplishments of Christoph Rudolff. What is he credited for?

123. What is a surd, and where does the word come from?

124. Research ways in which police investigators can determine the speed of a vehicle after an accident has occurred. Share your findings on the discussion board.

## ANSWERS

1.  $3|x|$

2.

3.  $2y$

4.

5.  $2|x|$

6.

7.  $6a^2$

8.

9.  $2|a^3|$

10.

11.

12.

13.

14.

15.  $|5x-4|$

16.

17.  $|x-3|$

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18.

19.  $|2x+3|$

20.

21.  $7a$

22.

23.  $xy$

24.

25.

26.

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31.

32.

33.  $x+1$

34.

35.  $2(3x-1)$

36.

37.

38.

39.

40.

41.

42.

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43.  $3a$

44.

45.

46.

47.  $4xy^2z^3$

48.

49.

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51.

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56.

57.  $3x$

58.

59.  $2xy^2$

60.

61.

62.

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66.

67.  $-6x$

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68.

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73.  $4x^3$

74.

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85.

86.

87.  $\pi$  seconds; 3.1 seconds

88.

89. seconds; 0.8 seconds

90.

91. seconds; 1.7 seconds

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- 92.
93. 2seconds; 3.5 seconds
- 94.
95. 25 miles per hour
- 96.
97. (0,2)
- 98.
99. (0,-2)
- 100.
- 101.
- 102.
103.  $5\sqrt{\text{units}}$
- 104.
105.  $210--\sqrt{\text{units}}$
- 106.
107.  $310--\sqrt{\text{units}}$
- 108.
109.  $35\sqrt{\text{units}}$
- 110.
111. units
- 112.
113. Right triangle
- 114.
115. Not a right triangle
- 116.

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- 117. Right triangle
- 118.
- 119. Right triangle
- 120.
- 121. Answer may vary
- 122.
- 123. Answer may vary

## 5.3 Adding and Subtracting Radical Expressions

### LEARNING OBJECTIVES

1. Add and subtract like radicals.
2. Simplify radical expressions involving like radicals.'

### Adding and Subtracting Like Radicals

Adding and subtracting radical expressions is similar to adding and subtracting like terms. Radicals are considered to be like radicals, or similar radicals, when they share the same index and radicand. For example, the terms  $\sqrt{2}$  and  $3\sqrt{2}$  contain like radicals and can be added using the distributive property as follows:

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Typically, we do not show the step involving the distributive property and simply write,

When adding terms with like radicals, add only the coefficients; the radical part remains the same.

### **Example 1**

Add:

Solution:

The terms are like radicals; therefore, add the coefficients.

Answer:

Subtraction is performed in a similar manner.

### **Example 2**

Subtract:

Solution:

Answer:

If the radicand and the index are not exactly the same, then the radicals are not similar and we cannot combine them.

### **Example 3**

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Simplify:

Solution:

We cannot simplify any further because  $5\sqrt{\phantom{x}}$  and  $2\sqrt{\phantom{x}}$  are not like radicals; the radicands are not the same.

Answer:

**Caution:** It is important to point out that We can verify this by calculating the value of each side with a calculator.

In general, note that

### Example 4

Simplify:

Solution:

We cannot simplify any further, because  $10\sqrt[3]{\phantom{x}}$  and  $10\sqrt{\phantom{x}}$  are not like radicals; the indices are not the same.

Answer:

## Adding and Subtracting Radical Expressions

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Often, we will have to simplify before we can identify the like radicals within the terms.

### **Example 5**

Subtract:

Solution:

At first glance, the radicals do not appear to be similar. However, after simplifying completely, we will see that we can combine them.

Answer:

### **Example 6**

Simplify:

Solution:

Begin by looking for perfect cube factors of each radicand.

Answer:

Try this! Simplify:

Answer:

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Next, we work with radical expressions involving variables. In this section, assume all radicands containing variable expressions are nonnegative.

### **Example 7**

Simplify:

Solution:

Combine like radicals.

We cannot combine any further because the remaining radical expressions do not share the same radicand; they are not like radicals. Note:

Answer:

We will often find the need to subtract a radical expression with multiple terms. If this is the case, remember to apply the distributive property before combining like terms.

### **Example 8**

Simplify:

Solution:

Answer:

Until we simplify, it is often unclear which terms involving radicals are similar. The general steps for simplifying radical expressions are outlined in the following example.

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## Example 9

Simplify:

Solution:

Step 1: Simplify the radical expression. In this case, distribute and then simplify each term that involves a radical.

Step 2: Combine all like radicals. Remember to add only the coefficients; the variable parts remain the same.

Answer:

## Example 10

Simplify:

Solution:

Answer:

Try this!

Answer:

## Tip

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Take careful note of the differences between products and sums within a radical. Assume both  $x$  and  $y$  are nonnegative.

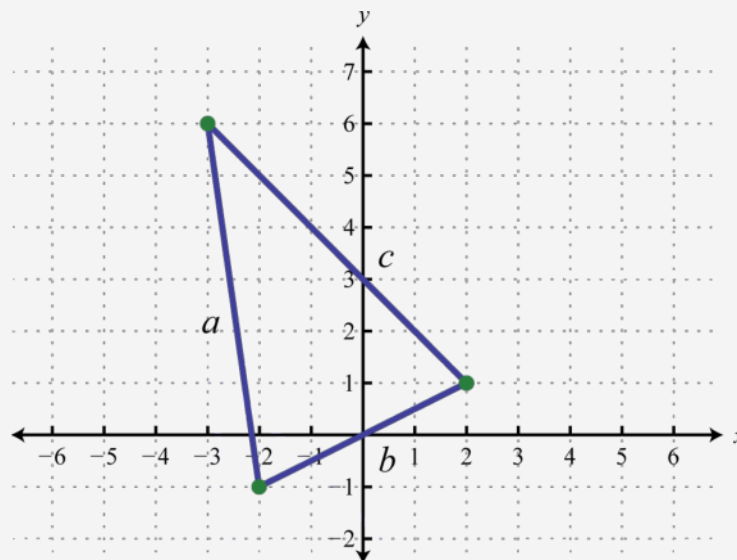
The property says that we can simplify radicals when the operation in the radicand is multiplication. There is no corresponding property for addition.

### Example 11

Calculate the perimeter of the triangle formed by the points  $(-2,-1)$ ,  $(-3,6)$ , and  $(2,1)$ .

Solution:

The formula for the perimeter of a triangle is  $P = a + b + c$  where  $a$ ,  $b$ , and  $c$  represent the lengths of each side. Plotting the points we have,



Use the distance formula to calculate the length of each side.



Similarly we can calculate the distance between  $(-3, 6)$  and  $(2, 1)$  and find that  $c = 5\sqrt{2} - \sqrt{5}$  units. Therefore, we can calculate the perimeter as follows:

Answer: units

## KEY TAKEAWAYS

Add and subtract terms that contain like radicals just as you do like terms. If the index and radicand are exactly the same, then the radicals are similar and can be combined. This involves adding or subtracting only the coefficients; the radical part remains the same.

Simplify each radical completely before combining like terms.

## TOPIC EXERCISES

### PART A: ADDING AND SUBTRACTING LIKE RADICALS

Simplify

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- 2.
- 3.
- 4.
- 5.
- 6.
- 7.

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18.

Simplify. (Assume all radicands containing variable expressions are positive.)

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- 22.

## PART B: ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Simplify.

- 41.
- 42.
- 43.
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- 45.
- 46.
- 47.
- 48.
- 49.
- 50.

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64.

Simplify. (Assume all radicands containing variable expressions are positive.)

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48.

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Calculate the perimeters of the triangles formed by the following sets of vertices.

41.  $\{(-4, -5), (-4, 3), (2, 3)\}$

42.  $\{(-1, 1), (3, 1), (3, -2)\}$

43.  $\{(-3, 1), (-3, 5), (1, 5)\}$

44.  $\{(-3, -1), (-3, 7), (1, -1)\}$

45.  $\{(0,0), (2,4), (-2,6)\}$

46.  $\{(-5,-2), (-3,0), (1,-6)\}$

47. A square garden that is 10 feet on each side is to be fenced in. In addition, the space is to be partitioned in half using a fence along its diagonal. How much fencing is needed to do this? (Round to the nearest tenth of a foot.)

48. A garden in the shape of a square has an area of 150 square feet. How much fencing is needed to fence it in? (Hint: The length of each side of a square is equal to the square root of the area. Round to the nearest tenth of a foot.)

### PART C: DISCUSSION BOARD

105. Choose values for  $x$  and  $y$  and use a calculator to show that

106. Choose values for  $x$  and  $y$  and use a calculator to show that

### ANSWERS

1.

2.

3.

4.

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6.

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7.

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17.  $69 - \sqrt{3} - 3 - \sqrt{3}$

18.

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94.

95.

96.

97. 24 units

98.

99. units

100.

101. units

102.

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103. 54.1 feet
- 104.
105. Answer may vary

## 5.4 Multiplying and Dividing Radical Expressions

### LEARNING OBJECTIVES

1. Multiply radical expressions.
2. Divide radical expressions.
3. Rationalize the denominator.

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## Multiplying Radical Expressions

When multiplying radical expressions with the same index, we use the product rule for radicals. Given real numbers and ,

### Example 1

Multiply:

Solution:

Apply the product rule for radicals, and then simplify.

Answer:

Often, there will be coefficients in front of the radicals.

### Example 2

Multiply:

Solution:

Using the product rule for radicals and the fact that multiplication is commutative, we can multiply the coefficients and the radicands as follows.

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Typically, the first step involving the application of the commutative property is not shown.

Answer:

### **Example 3**

Multiply:

Solution:

Answer:

Use the distributive property when multiplying rational expressions with more than one term.

### **Example 4**

Multiply:

Solution:

Apply the distributive property and multiply each term by

Answer:

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### Example 5

Multiply:

Solution:

Apply the distributive property, and then simplify the result.

Answer:

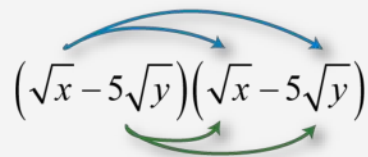
The process for multiplying radical expressions with multiple terms is the same process used when multiplying polynomials. Apply the distributive property, simplify each radical, and then combine like terms.

### Example 6

Multiply:

Solution:

Begin by applying the distributive property.

$$(\sqrt{x} - 5\sqrt{y})(\sqrt{x} - 5\sqrt{y})$$


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Answer:

The binomials are called conjugates. When multiplying conjugate binomials the middle terms are opposites and their sum is zero.

### **Example 7**

Multiply:

Solution:

Apply the distributive property, and then combine like terms.

Answer: 7

It is important to note that when multiplying conjugate radical expressions, we obtain a rational expression. This is true in general

Alternatively, using the formula for the difference of squares we have,

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Try this! Multiply: (Assume  $y$  is positive.)

Answer:

## Dividing Radical Expressions

To divide radical expressions with the same index, we use the quotient rule for radicals.

Given real numbers and ,

### Example 8

Divide:

Solution:

In this case, we can see that 6 and 96 have common factors. If we apply the quotient rule for radicals and write it as a single cube root, we will be able to reduce the fractional radicand.

Answer:

### Example 9

Divide:

Solution:

Write as a single square root and cancel common factors before simplifying.

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Answer:

## **Rationalizing the Denominator**

When the denominator (divisor) of a radical expression contains a radical, it is a common practice to find an equivalent expression where the denominator is a rational number. Finding such an equivalent expression is called rationalizing the denominator.

To do this, multiply the fraction by a special form of 1 so that the radicand in the denominator can be written with a power that matches the index. After doing this, simplify and eliminate the radical in the denominator. For example:

Remember, to obtain an equivalent expression, you must multiply the numerator and denominator by the exact same nonzero factor.

### **Example 10**

Rationalize the denominator:

Solution:

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The goal is to find an equivalent expression without a radical in the denominator. The radicand in the denominator determines the factors that you need to use to rationalize it. In this example, multiply by 1 in the form

Answer:

Sometimes, we will find the need to reduce, or cancel, after rationalizing the denominator.

### **Example 11**

Rationalize the denominator:

Solution:

In this example, we will multiply by 1 in the form

Notice that  $b$  does not cancel in this example. Do not cancel factors inside a radical with those that are outside.

Answer:

Try this! Rationalize the denominator:

Answer:

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Up to this point, we have seen that multiplying a numerator and a denominator by a square root with the exact same radicand results in a rational denominator. In general, this is true only when the denominator contains a square root. However, this is not the case for a cube root. For example,

Note that multiplying by the same factor in the denominator does not rationalize it. In this case, if we multiply by 1 in the form of  $\frac{1}{\sqrt[3]{x}}$ , then we can write the radicand in the denominator as a power of 3. Simplifying the result then yields a rationalized denominator.

Therefore, to rationalize the denominator of a radical expression with one radical term in the denominator, begin by factoring the radicand of the denominator. The factors of this radicand and the index determine what we should multiply by. Multiply the numerator and denominator by the  $n$ th root of factors that produce  $n$ th powers of all the factors in the radicand of the denominator.

### **Example 12**

Rationalize the denominator:

Solution:

The radical in the denominator is equivalent to  $\sqrt[3]{x}$ . To rationalize the denominator, we need:  $\sqrt[3]{x^2}$ . To obtain this, we need one more factor of  $x$ . Therefore, multiply by 1 in the form of  $\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$

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Answer:

### **Example 13**

Rationalize the denominator:

Solution:

In this example, we will multiply by 1 in the form .

Answer:

### **Example 14**

Rationalize the denominator:

Solution:

In this example, we will multiply by 1 in the form .

Answer:

When two terms involving square roots appear in the denominator, we can rationalize it using a very special technique. This technique involves multiplying the numerator and the denominator of the fraction by the conjugate of the denominator. Recall that multiplying a radical expression by its conjugate produces a rational number.

### **Example 15**

Rationalize the denominator:

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Solution:

In this example, the conjugate of the denominator is  $a + b\sqrt{c}$ . Therefore, multiply by 1 in the form  $\frac{a + b\sqrt{c}}{a + b\sqrt{c}}$

Answer:

Notice that the terms involving the square root in the denominator are eliminated by multiplying by the conjugate. We can use the property to expedite the process of multiplying the expressions in the denominator.

### Example 16

Rationalize the denominator:

Solution:

Multiply by 1 in the form  $\frac{a + b\sqrt{c}}{a + b\sqrt{c}}$  . .

Answer:

### Example 17

Rationalize the denominator:

Solution:

In this example, we will multiply by 1 in the form  $\frac{a + b\sqrt{c}}{a + b\sqrt{c}}$

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Answer:

Try this! Rationalize the denominator:

Answer:

## KEY TAKEAWAYS

To multiply two single-term radical expressions, multiply the coefficients and multiply the radicands. If possible, simplify the result.

Apply the distributive property when multiplying a radical expression with multiple terms. Then simplify and combine all like radicals.

Multiplying a two-term radical expression involving square roots by its conjugate results in a rational expression.

It is common practice to write radical expressions without radicals in the denominator. The process of finding such an equivalent expression is called rationalizing the denominator.

If an expression has one term in the denominator involving a radical, then rationalize it by multiplying the numerator and denominator by the  $n$ th root of factors of the radicand so that their powers equal the index.

If a radical expression has two terms in the denominator involving square roots, then rationalize it by multiplying the numerator and denominator by the conjugate of the denominator.

## TOPIC EXERCISES

### PART A: MULTIPLYING RADICAL EXPRESSIONS

Multiply. (Assume all variables represent non-negative real numbers.)

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56.

57. What is the perimeter and area of a rectangle with length measuring centimeters and width measuring centimeters?

58. What is the perimeter and area of a rectangle with length measuring centimeters and width measuring centimeters?

59. If the base of a triangle measures meters and the height measures meters, then calculate the area.

60. If the base of a triangle measures meters and the height measures meters, then calculate the area.

## PART B: DIVIDING RADICAL EXPRESSIONS

Divide. (Assume all variables represent positive real numbers.)

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## PART C: RATIONALIZING THE DENOMINATOR

Rationalize the denominator. (Assume all variables represent positive real numbers.)

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120.

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122.

123. The radius of the base of a right circular cone is given by where  $V$  represents the volume of the cone and  $h$  represents its height. Find the radius of a right circular cone with volume 50 cubic centimeters and height 4 centimeters. Give the exact answer and the approximate answer rounded to the nearest hundredth.

124. The radius of a sphere is given by where  $V$  represents the volume of the sphere. Find the radius of a sphere with volume 135 square centimeters. Give the exact answer and the approximate answer rounded to the nearest hundredth.

## PART D: DISCUSSION

125. Research and discuss some of the reasons why it is a common practice to rationalize the denominator.

126. Explain in your own words how to rationalize the denominator.

## ANSWERS

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4.

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7. 7

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51.

52.

53. 10

54.

55.

56.

57. Perimeter: centimeters; area: square centimeters

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58.

59. 18 square meters

60.

61. 5

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109.

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111.

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117.

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120.

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122.

123.       centimeters; 3.45 centimeters

124.

125.       Answer may vary

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## 5.5 Rational Exponents

### LEARNING OBJECTIVES

1. Write expressions with rational exponents in radical form.
2. Write radical expressions with rational exponents.
3. Perform operations and simplify expressions with rational exponents.
4. Perform operations on radicals with different indices.

### Rational Exponents

So far, exponents have been limited to integers. In this section, we will define what rational (or fractional) exponents mean and how to work with them. All of the rules for exponents developed up to this point apply. In particular, recall the product rule for exponents. Given any rational numbers  $m$  and  $n$ , we have

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For example, if we have an exponent of  $1/2$ , then the product rule for exponents implies the following:

Here is one of two equal factors of 5; hence it is a square root of 5, and we can write

Furthermore, we can see that is one of three equal factors of 2.

Therefore, is a cube root of 2, and we can write

This is true in general, given any nonzero real number  $a$  and integer ,

In other words, the denominator of a fractional exponent determines the index of an  $n$ th root.

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### Example 1

Rewrite as a radical.

- 1.
- 2.

Solution:

- 1.
- 2.

### Example 2

Rewrite as a radical and then simplify.

- 1.
- 2.

Solution:

- 1.
- 2.

### Example 3

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Rewrite as a radical and then simplify.

1.

2.

Solution:

a.

b.

Next, consider fractional exponents where the numerator is an integer other than 1. For example, consider the following:

This shows that  $\sqrt[3]{27}$  is one of three equal factors of 27. In other words,  $\sqrt[3]{27}$  is a cube root of 27 and we can write:

In general, given any nonzero real number  $a$  where  $m$  and  $n$  are positive integers,

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An expression with a **rational exponent** is equivalent to a radical where the denominator is the index and the numerator is the exponent. Any radical expression can be written with a rational exponent, which we call **exponential form**.

### Example 4

Rewrite as a radical.

- 1.
- 2.

Solution:

- 1.
- 2.

### Example 5

Rewrite as a radical and then simplify.

- 1.
- 2.

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Solution:

We can often avoid very large integers by working with their prime factorization.

a.

b.

Given a radical expression, we might want to find the equivalent in exponential form. Assume all variables are positive.

### Example 6

Rewrite using rational exponents:

Solution:

Here the index is 5 and the power is 3. We can write

Answer:

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## Example 7

Rewrite using rational exponents: .

Solution:

Here the index is 6 and the power is 3. We can write

Answer:

It is important to note that the following are equivalent.

In other words, it does not matter if we apply the power first or the root first. For example, we can apply the power before the  $n$ th root:

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Or we can apply the  $n$ th root before the power:

The results are the same.

### Example 8

Rewrite as a radical and then simplify:

Solution:

Here the index is 3 and the power is 2. We can write

Answer: 4

**Try this!** Rewrite as a radical and then simplify:

Answer: 1,000

Some calculators have a caret button which is used for entering exponents. If so, we can calculate approximations for radicals using it and rational exponents. For example, to calculate  $\sqrt[3]{1000}$ , we make use of the parenthesis buttons and type

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To calculate , we would type

$$2 ^ { ( 2 \div 3 ) } =$$

## Operations Using the Rules of Exponents

In this section, we review all of the rules of exponents, which extend to include rational exponents. If given any rational numbers  $m$  and  $n$ , then we have

<i>Product rule for exponents:</i>	
------------------------------------	--



<i>Quotient rule for exponents:</i>	
<i>Power rule for exponents:</i>	
<i>Power rule for a product:</i>	

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<i>Power rule for a quotient:</i>	
<i>Negative exponents:</i>	
<i>Zero exponent:</i>	

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These rules allow us to perform operations with rational exponents.

### **Example 9**

Simplify:

Solution:

Answer:

### **Example 10**

Simplify: .

Solution:

Answer:  $x^{5/6}$

### **Example 11**

Simplify:

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Solution:

Answer:

### **Example 12**

Simplify: .

Solution:

Answer: .

### **Example 13**

Simplify:

Solution:

Answer:

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**Try this!** Simplify:

Answer:

## Radical Expressions with Different Indices

To apply the product or quotient rule for radicals, the indices of the radicals involved must be the same. If the indices are different, then first rewrite the radicals in exponential form and then apply the rules for exponents.

### Example 14

Multiply:

Solution:

In this example, the index of each radical factor is different. Hence the product rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents. Then apply the product rule for exponents.

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Answer:

### **Example 15**

Divide:

Solution:

In this example, the index of the radical in the numerator is different from the index of the radical in the denominator. Hence the quotient rule for radicals does not apply.

Begin by converting the radicals into an equivalent form using rational exponents and then apply the quotient rule for exponents.

Answer:

### **Example 16**

Simplify:

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Solution:

Here the radicand of the square root is a cube root. After rewriting this expression using rational exponents, we will see that the power rule for exponents applies.

Answer:

## KEY TAKEAWAYS

Any radical expression can be written in exponential form:  $a\sqrt[n]{m} = am/n$ .

Fractional exponents indicate radicals. Use the numerator as the power and the denominator as the index of the radical.

All the rules of exponents apply to expressions with rational exponents.

If operations are to be applied to radicals with different indices, first rewrite the radicals in exponential form and then apply the rules for exponents.

## TOPIC EXERCISES

### PART A: RATIONAL EXPONENTS

**Express using rational exponents.**

- 1.
- 2.
- 3.

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- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.

**Express in radical form.**

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.



**Write as a radical and then simplify.**

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

15.

16.

17.

18.

19.

20.

21.

22.

23.

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24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.

**Use a calculator to approximate an answer rounded to the nearest hundredth.**

1.

2.

3.

4.

5.

6.

7.

8.

9. Explain why  $(-4)^{(3/2)}$  gives an error on a calculator and  $-4^{(3/2)}$  gives an answer of  $-8$ .

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10. Marcy received a text message from Mark asking her age. In response, Marcy texted back " $125^{(2/3)}$  years old." Help Mark determine Marcy's age.

## PART B: OPERATIONS USING THE RULES OF EXPONENTS

**Perform the operations and simplify. Leave answers in exponential form.**

71.

72.

73.

74.

75.

76.

77.

78.

79.

80.

81.

82.

83.

84.

85.

86.

87.

88.

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- 89.
- 90.
- 91.
- 92.
- 93.
- 94.
- 95.
- 96.
- 97.
- 98.
- 99.
- 100.
- 101.
- 102.
- 103.
- 104.
- 105.
- 106.
- 107.
- 108.
- 109.
- 110.

## **PART C: RADICAL EXPRESSIONS WITH DIFFERENT INDICES**

**Perform the operations.**

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- 111.
- 112.
- 113.
- 114.
- 115.
- 116.
- 117.
- 118.
- 119.
- 120.
- 121.
- 122.
- 123.
- 124.
- 125.
- 126.
- 127.
- 128.

## PART D: DISCUSSION BOARD

129. Who is credited for devising the notation that allows for rational exponents? What are some of his other accomplishments?
130. When using text, it is best to communicate  $n$ th roots using rational exponents. Give an example.

## ANSWERS

- 1.





- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.
- 15.
- 16.
- 17.
- 18.
- 19.
- 20.
- 21.
- 22.
- 23.
- 24.
- 25.
- 26.
- 27.

8

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- 28.
- 29.
- 30.
- 31.
- 32.
- 33. 2
- 34.
- 35. 2
- 36.
- 37.
- 38.
- 39. -3
- 40.
- 41. 2
- 42.
- 43.
- 44.
- 45. 10
- 46.
- 47.
- 48.
- 49. 27
- 50.
- 51. 32
- 52.
- 53. 64



- 54.
- 55.
- 56.
- 57. 9
- 58.
- 59. -8
- 60.
- 61. 1.41
- 62.
- 63. 1.68
- 64.
- 65. 1.38
- 66.
- 67. Not a real number
- 68.
- 69. Answer may vary
- 70.
- 71. 25
- 72.
- 73.
- 74.
- 75.
- 76.
- 77. 125
- 78.
- 79.



80.

81. 2

82.

83.

84.

85.

86.

87.

88.

89.

90.

91.

92.

93.

94.

95.

96.

97.  $y$

98.

99.

100.

101.

102.

103.

104.

105.

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106.

107.

108.

109.

110.

111.

112.

113.

114.

115.

116.

117.

118.

119.

120.

121.

122.

123.

124.

125.

126.

127.

128.

129. Answer may vary

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## 5.6 Solving Radical Equations

### LEARNING OBJECTIVES

1. Solve equations involving square roots.
2. Solve equations involving cube roots.

### Radical Equations

A **radical equation** is any equation that contains one or more radicals with a variable in the radicand. Following are some examples of radical equations, all of which will be solved in this section:

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--	--	--

We begin with the **squaring property of equality**; given real numbers  $a$  and  $b$ , we have the following:

In other words, equality is retained if we square both sides of an equation.



The converse, on the other hand, is not necessarily true,



This is important because we will use this property to solve radical equations. Consider a very simple radical equation that can be solved by inspection,

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Here we can see that  $x = 2$  is a solution. To solve this equation algebraically, make use of the squaring property of equality and the fact that  $\sqrt{a^2} = a$  when  $a$  is nonnegative. Eliminate the square root by squaring both sides of the equation as follows:

As a check, we can see that  $x = -2$  is not a solution. Because the converse of the squaring property of equality is not necessarily true, solutions to the squared equation may not be solutions to the original. Hence squaring both sides of an equation introduces the possibility of **extraneous solutions**, which are solutions that do not solve the original equation. For example,

This equation clearly does not have a real number solution. However, squaring both sides gives us a solution:

As a check, we can see that  $x = 2$  is not a solution. For this reason, we must check the answers that result from squaring both sides of an equation.

### Example 1

Solve:

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Solution:

We can eliminate the square root by applying the squaring property of equality.

.

Next, we must check.



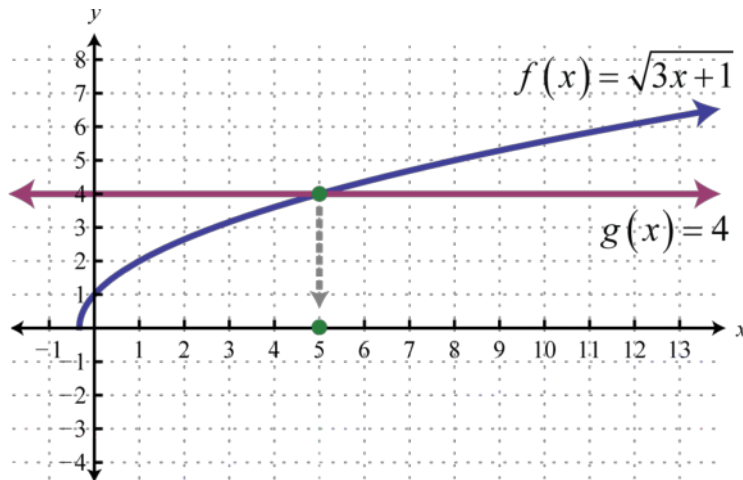
Answer: The solution is 5.

There is a geometric interpretation to the previous example. Graph the function defined by  $y = \sqrt{x}$  and determine where it intersects the graph defined by  $y = x - 5$ .

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As illustrated, where

### Example 2

Solve:

Solution:

Begin by squaring both sides of the equation.

The resulting quadratic equation can be solved by factoring.

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Checking the solutions after squaring both sides of an equation is not optional. Use the original equation when performing the check.

x	✓

After checking, you can see that is an extraneous solution; it does not solve the original radical equation. Disregard that answer. This leaves as the only solution.

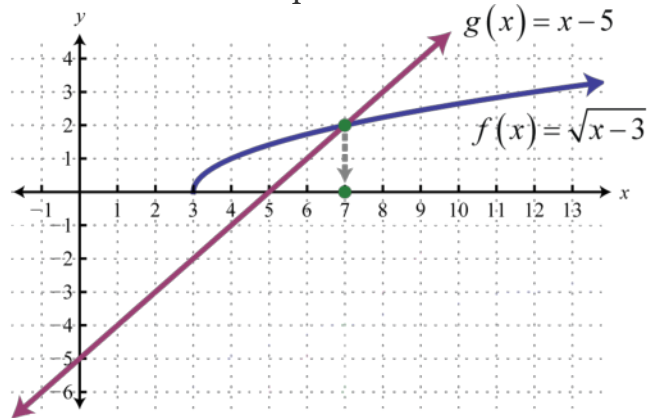
Answer: The solution is 7.

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Geometrically we can see that is equal to where



In the previous two examples, notice that the radical is isolated on one side of the equation. Typically, this is not the case. The steps for solving radical equations involving square roots are outlined in the following example.

### Example 3

Solve:

Solution:

**Step 1:** Isolate the square root. Begin by subtracting 2 from both sides of the equation.

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**Step 2:** Square both sides. Squaring both sides eliminates the square root.

**Step 3:** Solve the resulting equation. Here we are left with a quadratic equation that can be solved by factoring.

**Step 4:** Check the solutions in the original equation. Squaring both sides introduces the possibility of extraneous solutions; hence the check is required.

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x	✓

After checking, we can see that is an extraneous solution; it does not solve the original radical equation. This leaves as the only solution.

Answer: The solution is 5.

Sometimes there is more than one solution to a radical equation.

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#### Example 4

Solve:

Solution:

Begin by isolating the term with the radical.

Despite the fact that the term on the left side has a coefficient, we still consider it to be isolated. Recall that terms are separated by addition or subtraction operators.

Solve the resulting quadratic equation.

Since we squared both sides, we must check our solutions.

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✓	✓

After checking, we can see that both are solutions to the original equation.

Answer: The solutions are  $\pm 2$ .

Sometimes both of the possible solutions are extraneous.

### Example 5

Solve:

Solution:

Begin by isolating the radical.

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Since we squared both sides, we must check our solutions.

$x$	$x$

Since both possible solutions are extraneous, the equation has no solution.

Answer: No solution,  $\emptyset$

The squaring property of equality extends to any positive integer power  $n$ . Given real numbers  $a$  and  $b$ , we have the following:

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This is often referred to as the **power property of equality**. Use this property, along with the fact that when  $a$  is nonnegative, to solve radical equations with indices greater than 2.

### Example 6

Solve:

Solution:

Isolate the radical, and then cube both sides of the equation.

Check.

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✓	✓

Answer: The solutions are

**Try this!**

Answer: The solution is 33.

It may be the case that the equation has more than one term that consists of radical expressions.

### Example 7

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Solve:

Solution:

Both radicals are considered isolated on separate sides of the equation.

Check  $x=2$ .



Answer: The solution is 2.

### Example 8

Solve:

Solution:

Eliminate the radicals by cubing both sides.

Check.

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✓	

Answer: The solutions are  $\pm 8$ .

It may not be possible to isolate a radical on both sides of the equation. When this is the case, isolate the radicals, one at a time, and apply the squaring property of equality multiple times until only a polynomial remains.

### Example 9

Solve:

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Solution:

Begin by isolating one of the radicals. In this case, add to both sides of the equation.

Next, square both sides. Take care to apply the distributive property to the right side.

At this point we have one term that contains a radical. Isolate it and square both sides again.

Check to see if satisfies the original equation



Answer: The solution is

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*Note:* Because  $(a + b)^2 \neq a^2 + b^2$ , we cannot simply square each term. For example, it is incorrect to square each term as follows.

This is a common mistake and leads to an incorrect result. When squaring both sides of an equation with multiple terms, we must take care to apply the distributive property.

### **Example 10**

Solve:

Solution:

Begin by isolating one of the radicals. In this case, add  $\sqrt{2}$  to both sides of the equation.

Next, square both sides. Take care to apply the distributive property to the right side.

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At this point we have one term that contains a radical. Isolate it and square both sides again.

Check.

✓	✗

Answer: The solution is 3.

**Try this!** Solve:

Answer: The solution is 7.

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## KEY TAKEAWAYS

Solve equations involving square roots by first isolating the radical and then squaring both sides. Squaring a square root eliminates the radical, leaving us with an equation that can be solved using the techniques learned earlier in our study of algebra.

Squaring both sides of an equation introduces the possibility of extraneous solutions. For this reason, you must check your solutions in the original equation.

Solve equations involving  $n$ th roots by first isolating the radical and then raise both sides to the  $n$ th power. This eliminates the radical and results in an equation that may be solved with techniques you have already mastered.

When more than one radical term is present in an equation, isolate them one at a time, and apply the power property of equality multiple times until only a polynomial remains.

## TOPIC EXERCISES

### PART A: SOLVING RADICAL EQUATIONS

Solve

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
- 11.
- 12.

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- 13.
- 14.
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- 37.
- 38.

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- 39.
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- 50.
- 51. **6**
- 52.
- 53.
- 54.
- 55.
- 56.
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- 62.
- 63.
- 64.

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- 84.
- 85.
- 86.
- 87.
- 88.
- 89.
- 90.

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- 91.
- 92.
- 93.
- 94.
- 95.
- 96.
- 97.
- 98.
- 99.
- 100.
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- 103.
- 104.
- 105.
- 106.
- 107.
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- 110.
- 111.
- 112.
- 113.
- 114.
- 115.
- 116.



**Determine the roots of the given functions. Recall that a root is a value in the domain that results in zero. In other words, find  $x$  where**

- 1.
- 2.
- 3.
- 4.
- 5.
- 122.

**Solve for the indicated variable.**

1. Solve for  $P$ :
2. Solve for  $x$ :
3. Solve for  $s$ :
4. Solve for  $L$ :
5. Solve for  $R$ :
6. Solve for  $h$ :
7. Solve for  $V$ :
8. Solve for  $c$ :
9. The square root of 1 less than twice a number is equal to 2 less than the number. Find the number.
10. The square root of 4 less than twice a number is equal to 6 less than the number. Find the number.

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11. The square root of twice a number is equal to one-half of that number. Find the number.
12. The square root of twice a number is equal to one-third of that number. Find the number.
13. The distance  $d$  in miles a person can see an object on the horizon is given by the formula

where  $h$  represents the height in feet of the person's eyes above sea level. How high must a person's eyes be to see an object 5 miles away?

14. The current  $I$  measured in amperes is given by the formula

where  $P$  is the power usage measured in watts and  $R$  is the resistance measured in ohms. If a light bulb requires  $1/2$  amperes of current and uses 60 watts of power, then what is the resistance through the bulb?

**The period of a pendulum  $T$  in seconds is given by the formula**

**where  $L$  represents the length in feet. Calculate the length of a pendulum given the period. Give the exact value and the approximate value rounded to the nearest tenth of a foot.**

1. 1 second
2. 2 seconds
3. second
4. second

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The time  $t$  in seconds, an object is in free fall is given by the formula

where  $s$  represents the distance it has fallen, in feet. Calculate the distance an object will fall given the amount of time.

1. 1 second
2. 2 seconds
3. second
4. second

## PART B: DISCUSSION BOARD

145. Discuss reasons why we sometimes obtain extraneous solutions when solving radical equations. Are there ever any conditions where we do not need to check for extraneous solutions? Why or why not?

146. If an equation has multiple terms, explain why squaring all of them is incorrect. Provide an example.

## ANSWERS

1. 49
- 2.
3. 1
- 4.
5.  $\emptyset$
- 6.
- 7.
- 8.
9. 1

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- 10.
11. 3
- 12.
- 13.
- 14.
15. 0
- 16.
- 17.
- 18.
19.  $\emptyset$
- 20.
21. 27
- 22.
23. 9
- 24.
25. -3
- 26.
27. 6
- 28.
- 29.
- 30.
31. 2
- 32.
33. 7
- 34.
35. 2

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- 36.
- 37. -3
- 38.
- 39. 13
- 40.
- 41. 7
- 42.
- 43. 2, 6
- 44.
- 45. 2
- 46.
- 47. -1, 8
- 48.
- 49. 5
- 50.
- 51.  $\emptyset$
- 52.
- 53. -3, 3
- 54.
- 55. 2, 5
- 56.
- 57. -4, 4
- 58.
- 59.
- 60.
- 61. 2, 7

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- 62.
- 63.  $\emptyset$
- 64.
- 65. 10
- 66.
- 67. -6, -4
- 68.
- 69.
- 70.
- 71.  $\emptyset$
- 72.
- 73. -5, 5
- 74.
- 75. -9, 3
- 76.
- 77.
- 78.
- 79.
- 80.
- 81. -1, 1/2
- 82.
- 83. 5, 10
- 84.
- 85. -7, 7
- 86.
- 87.



- 88.
- 89. 1
- 90.
- 91. 10
- 92.
- 93.  $\emptyset$
- 94.
- 95. 3
- 96.
- 97. -1, 2
- 98.
- 99. 100
- 100.
- 101. -8
- 102.
- 103. 10
- 104.
- 105. -13
- 106.
- 107.
- 108.
- 109. -6, -4
- 110.
- 111. -2, 2
- 112.
- 113. 1

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- 114.
- 115. -2
- 116.
- 117. -1
- 118.
- 119. 14
- 120.
- 121. -9
- 122.
- 123.
- 124.
- 125.
- 126.
- 127.
- 128.
- 129.
- 130.
- 131. 5
- 132.
- 133. 0, 8
- 134.
- 135. feet
- 136.
- 137. feet; 0.8 feet
- 138.
- 139. feet; 0.2 feet



140.

141. 16 feet

142.

143. 4 feet

144.

145. Answer may vary

146.

## 5.7 Complex Numbers and Their Operations

### LEARNING OBJECTIVES

1. Define the imaginary unit and complex numbers.
2. Add and subtract complex numbers.
3. Multiply and divide complex numbers.

### Introduction to Complex Numbers

Up to this point the square root of a negative number has been left undefined. For example, we know that is not a real number.

There is no real number that when squared results in a negative number. We begin to resolve this issue by defining **the imaginary unit**,  $i$ , as the square root of  $-1$ .

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To express a square root of a negative number in terms of the imaginary unit  $i$ , we use the following property where  $a$  represents any non-negative real number:

With this we can write

If then we would expect that squared will equal  $-9$ :



In this way any square root of a negative real number can be written in terms of the imaginary unit. Such a number is often called an imaginary number.

### Example 1

Rewrite in terms of the imaginary unit  $i$ .

- 1.
- 2.
- 3.

Solution:

- 1.
- 2.



3.

Notation Note: When an imaginary number involves a radical, we place  $i$  in front of the radical. Consider the following:

Since multiplication is commutative, these numbers are equivalent. However, in the form  $\sqrt{a}i$  the imaginary unit  $i$  is often misinterpreted to be part of the radicand. To avoid this confusion, it is a best practice to place  $i$  in front of the radical and use  $i\sqrt{a}$ .

A complex number is any number of the form,

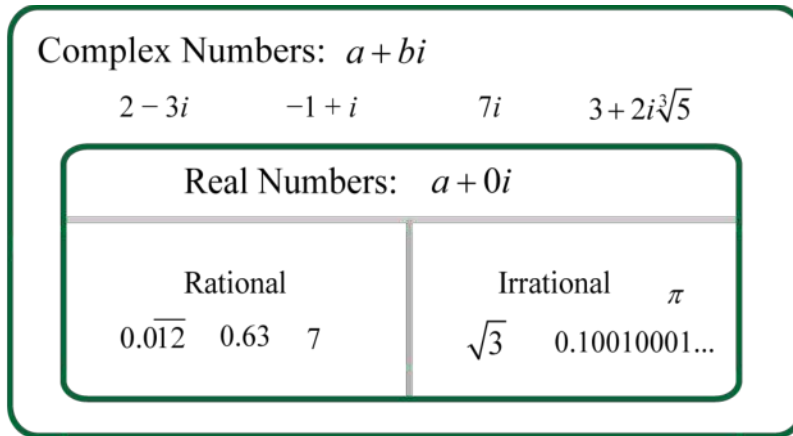
where  $a$  and  $b$  are real numbers. Here,  $a$  is called the real part and  $b$  is called the imaginary part. For example,  $3 - 4i$  is a complex number with a real part of 3 and an imaginary part of  $-4$ . It is important to note that any real number is also a complex number. For example, 5 is a real number; it can be written as  $5 + 0i$  with a real part of 5 and an imaginary part of 0. Hence, the set of real numbers, denoted  $\mathbb{R}$ , is a subset of the set of complex numbers, denoted  $\mathbb{C}$ .

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Complex numbers are used in many fields including electronics, engineering, physics, and mathematics. In this textbook we will use them to better understand solutions to equations such as For this reason, we next explore algebraic operations with them.

### Adding and Subtracting Complex Numbers

Adding or subtracting complex numbers is similar to adding and subtracting polynomials with like terms. We add or subtract the real parts and then the imaginary parts.

#### Example 2

Add:

Solution:

Add the real parts and then add the imaginary parts.

Answer:

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To subtract complex numbers, we subtract the real parts and subtract the imaginary parts. This is consistent with the use of the distributive property.

### Example 3

Subtract:

Solution:

Distribute the negative sign and then combine like terms.

Answer:

In general, given real numbers  $a$ ,  $b$ ,  $c$  and  $d$ :

### Example 4

Simplify:

Solution:

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Answer:

In summary, adding and subtracting complex numbers results in a complex number.

## Multiplying and Dividing Complex Numbers

Multiplying complex numbers is similar to multiplying polynomials. The distributive property applies. In addition, we make use of the fact that to simplify the result into standard form

### Example 5

Multiply:

Solution:

We begin by applying the distributive property.

Answer:

### Example 6

Multiply:

Solution:

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Answer:

In general, given real numbers  $a$ ,  $b$ ,  $c$  and  $d$ :

Try this! Simplify:

Answer:

Given a complex number  $z$ , its complex conjugate is  $\bar{z}$ . We next explore the product of complex conjugates.

### Example 7

Multiply:

Solution:

Answer: 29

In general, the product of complex conjugates follows:

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Note that the result does not involve the imaginary unit; hence, it is real. This leads us to the very useful property

To divide complex numbers, we apply the technique used to rationalize the denominator. Multiply the numerator and denominator by the conjugate of the denominator. The result can then be simplified into standard form

### **Example 8**

Divide:

Solution:

In this example, the conjugate of the denominator is  $a - bi$ . Therefore, we will multiply by 1 in the form

To write this complex number in standard form, we make use of the fact that  $a^2 + b^2$  is a common denominator.

Answer:

### **Example 9**

Divide:

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Solution:

Answer:

In general, given real numbers  $a$ ,  $b$ ,  $c$  and  $d$  where  $c$  and  $d$  are not both 0:

### Example 10

Divide:

Solution:

Here we can think of and thus we can see that its conjugate is

Because the denominator is a monomial, we could multiply numerator and denominator by 1 in the form of and save some steps reducing in the end.

Answer:

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Try this! Divide:

Answer:

When multiplying and dividing complex numbers we must take care to understand that the product and quotient rules for radicals require that both  $a$  and  $b$  are positive. In other words, if  $a$  and  $b$  are both real numbers then we have the following rules.

For example, we can demonstrate that the product rule is true when  $a$  and  $b$  are both positive as follows:



However, when  $a$  and  $b$  are both negative the property is not true.



Here  $a$  and  $b$  are not real numbers and the product rule for radicals fails to produce a true statement. Therefore, to avoid some common errors associated with this technicality, ensure that any complex number is written in terms of the imaginary unit  $i$  before performing any operations.

## Example 11

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Multiply:

Solution:

Begin by writing the radicals in terms of the imaginary unit  $i$ .

Now the radicands are both positive and the product rule for radicals applies.

Answer:

### Example 12

Multiply:

Solution:

Begin by writing the radicals in terms of the imaginary unit and then distribute.

Answer:

In summary, multiplying and dividing complex numbers results in a complex number.

Try this! Simplify:

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Answer:

## KEY TAKEAWAYS

The imaginary unit  $i$  is defined to be the square root of negative one. In other words, and

Complex numbers have the form  $a + bi$  where  $a$  and  $b$  are real numbers.

The set of real numbers is a subset of the complex numbers.

The result of adding, subtracting, multiplying, and dividing complex numbers is a complex number.

The product of complex conjugates,  $a + bi$  and  $a - bi$ , is a real number. Use this fact to divide complex numbers. Multiply the numerator and denominator of a fraction by the complex conjugate of the denominator and then simplify.

Ensure that any complex number is written in terms of the imaginary unit  $i$  before performing any operations.

## TOPIC EXERCISES

### PART A: INTRODUCTION TO COMPLEX NUMBERS

Rewrite in terms of imaginary unit  $i$ .

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.

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- 7.
- 8.
- 9.
- 10.
- 11.
- 12.
- 13.
- 14.

Write the complex number in standard form

- 1.
- 2.
- 3.
- 4.

5.

6.

7.

8.

Given that compute the following powers of  $i$ .

9.

10.

11.

12.

13.

14.

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## PART B: ADDING AND SUBTRACTING COMPLEX NUMBERS

Perform the operations.

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51.

52.

## PART C: MULTIPLYING AND DIVIDING COMPLEX NUMBERS

Perform the operations.

53.

54.

55.

56.

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- 73.
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- 97.
- 98.



99.

100.

101.

102.

103.

104.

105.

106.

Given that compute the following powers of

53.

54.

55.

56.

Perform the operations and simplify.

53.

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61.

62.

63.

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- 64.
- 65.
- 66.
67. Show that both and satisfy
68. Show that both and satisfy
69. Show that both and satisfy
70. Show that both and satisfy
71. Show that 3, , and are all solutions to
72. Show that  $-2$ , , and are all solutions to

### **PART D: DISCUSSION BOARD.**

131. Research and discuss the history of the imaginary unit and complex numbers.
132. How would you define and why?
133. Research what it means to calculate the absolute value of a complex number. Illustrate your finding with an example.
134. Explore the powers of  $i$ . Look for a pattern and share your findings.

### **ANSWERS**

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.



- 8.
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- 32.
- 33.

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59.

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22



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- 100.
- 101.
- 102.
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- 105.
- 106.
- 107.
- 108.
- 109.
- 110.

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111.

112.

113.

114.

115. 0

116.

117. -3

118.

119.

120.

121.

122.

123.

124.

125. Proof

126.

127. Proof

128.

129. Proof

130.

131. Answer may vary

132.

133. Answer may vary

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## 5.8 Review Exercises and Sample Exam

### REVIEW EXERCISES

### ROOTS AND RADICALS

Simplify.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.

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- 8.
9. Given  $a$  and  $d$
10. Given  $a$  and  $d$
11. Determine the domain of the function defined by
12. Determine the domain of the function defined by

Simplify.

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.

## SIMPLIFYING RADICAL EXPRESSIONS

Simplify.

- 21.
- 22.
- 23.
- 24.



Simplify. (Assume all variable expressions are nonzero.)

21.

22.

23.

24.

25.

26.

27.

28.

29.

30.

31. The period  $T$  in seconds of a pendulum is given by the formula  $T = 2\pi\sqrt{\frac{L}{g}}$  where  $L$  represents the length in feet of the pendulum. Calculate the period of a pendulum that is 16 feet long. Give the exact answer and the approximate answer to the nearest hundredth of a second.

32. The time in seconds an object is in free fall is given by the formula  $s = 16t^2$  where  $s$  represents the distance in feet the object has fallen. How long does it take an object to fall 28 feet? Give the exact answer and the approximate answer to the nearest tenth of a second.

33. Find the distance between  $(-5, 6)$  and  $(-3, -4)$ .

34. Find the distance between  $(-2, 3)$  and  $(4, -1)$ .



Determine whether or not the three points form a right triangle. Use the Pythagorean theorem to justify your answer.

21.  $(-4,5)$ ,  $(-3,-1)$ , and  $(3,0)$

22.  $(-1,-1)$ ,  $(1,3)$ , and  $(-6,1)$

## ADDING AND SUBTRACTING RADICAL EXPRESSIONS

Simplify. Assume all radicands containing variables are nonnegative.

41.

42.

43.

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53.

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55.

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56.

57.

58.

59.

60.

61. Calculate the perimeter of the triangle formed by the following set of vertices:

62. Calculate the perimeter of the triangle formed by the following set of vertices:

## MULTIPLYING AND DIVIDING RADICAL EXPRESSIONS

Multiply.

63.

64.

65.

66.

67.

68.

69.

70.

71.

72.

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Divide. Assume all variables represent nonzero numbers and rationalize the denominator where appropriate.

63.

64.

65.

66.

67.

68.

69.

70.

71.

72.

73.

74.

75.

76.

77.

78.

79.

80.

81.

82.

83. The base of a triangle measures  $\frac{1}{2}$  units and the height measures  $\frac{1}{3}$  units. Find the area of the triangle.

84. If each side of a square measures  $\frac{1}{4}$  units, find the area of the square.

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## RATIONAL EXPONENTS

Express in radical form.

95.

96.

97.

98.

Write as a radical and then simplify.

95.

96.

97.

98.

99.

100.

Perform the operations and simplify. Leave answers in exponential form.

95.

96.

97.

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98.

99.

100.

101.

102.

103.

104.

105.

106.

Perform the operations with mixed indices.

95.

96.

97.

98.

## SOLVING RADICAL EQUATIONS

Solve.

121.

122.

123.

124.

125.

126.

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127.

128.

129.

130.

131.

132.

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134.

135.

136.

137.

138.

139.

140.

141.

142.

143.

144.

145.

146.

147.

148.

149. Solve for  $g$ :

150. Solve for  $x$ :

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151. The period in seconds of a pendulum is given by the formula where  $L$  represents the length in feet of the pendulum. Find the length of a pendulum that has a period of seconds. Find the exact answer and the approximate answer rounded off to the nearest tenth of a foot.

152. The outer radius of a spherical shell is given by the formula where  $V$  represents the inner volume in cubic centimeters. If the outer radius measures 8 centimeters, find the inner volume of the sphere.

153. The speed of a vehicle before the brakes are applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed  $v$  in miles per hour can be estimated by the formula , where  $d$  represents the length of the skid marks in feet. Estimate the length of a skid mark if the vehicle is traveling 30 miles per hour before the brakes are applied.

154. Find the real root of the function defined by

## COMPLEX NUMBERS AND THEIR OPERATIONS

Write the complex number in standard form

155.

156.

157.

158.

Perform the operations.

155.

156.

157.

158.

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- 159.
- 160.
- 161.
- 162.
- 163.
- 164.
- 165.
- 166.
- 167.
- 168.
- 169.
- 170.
- 171.
- 172.
- 173.
- 174.
- 175.
- 176.
- 177.
- 178.
179. Show that both and satisfy
180. Show that both and satisfy

## ANSWERS

1. -11
- 2.



3.

4.

5. 5

6.

7.

8.

9. ;

10.

11.

12.

13.

14.

15.

16.

17. -9

18.

19.

21.

22.

23.

24.

25.

26.

27.

28.

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29.

30.

31.

32.

33.

34.

35. seconds; 1.76 seconds

36.

37. units

38.

39. Right triangle

40.

41.

42.

43.

44.

45.

46.

47.

48.

49.

50.

51.

52.

53.

54.

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55.

56.

57.

58.

59.

60.

61. units

62.

63.

64.

65.

66.

67. 22

68.

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80.

81.

82.

83.

84.

85.

86.

87.

88.

89.

90.

91.

92.

93. square units

94.

95.

96.

97.

98.

99. 4

100.

101. 4

102.

103.  $\frac{1}{27}$

104.

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105. 36  
106.  
107. 6  
108.  
109.  
110.  
111.  
112.  
113.  
114.  
115.  
116.  
117.  
118.  
119.  
120.  
121.  
121. 25  
122.  
123. 21  
124.  
125. 9  
126.  
127. 4  
128.

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- 129. 4
- 130.
- 131. 7
- 132.
- 133. 1
- 134.
- 135.
- 136.
- 137.
- 138.
- 139.
- 140.
- 141. 9
- 142.
- 143. 9
- 144.
- 145. 1
- 146.
- 147. 12
- 148.
- 149.
- 150.
- 151. feet; 1.8 feet
- 152.
- 153. 37.5 feet



- 154.
- 155.
- 156.
- 157.
- 158.
- 159.
- 160.
- 161.
- 162.
- 163.
- 164.
- 165.
- 166.
- 167.
- 168.
- 169. 13
- 170.
- 171.
- 172.
- 173.
- 174.
- 175.
- 176.
- 177.
- 178.
- 179.



180.

181.

182.

183. Answer may vary

## SAMPLE EXAM

Simplify. (Assume all variables are positive.)

1.

2.

3. Calculate the distance between and

4. The time in seconds an object is in free fall is given by the formula where  $s$  represents the distance in feet that the object has fallen. If a stone is dropped into a 36-foot pit, how long will it take to hit the bottom of the pit?

Perform the operations and simplify. (Assume all variables are positive and rationalize the denominator where appropriate.)

1.

2.

3.

4.

5.

6.

7.

8. Simplify:

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9. Express in radical form:

Simplify. Assume all variables are nonzero and leave answers in exponential form.

1.

2.

Solve.

1.

2.

3.

4.

5.

6. The time in seconds an object is in free fall is given by the formula  $s = 16t^2$  where  $s$  represents the distance in feet that the object has fallen. If a stone is dropped into a pit and it takes 4 seconds to reach the bottom, how deep is the pit?

7. The width in inches of a container is given by the formula  $W = \sqrt[3]{\frac{V}{4}}$  where  $V$  represents the inside volume in cubic inches of the container. What is the inside volume of the container if the width is 6 inches?

8.

Perform the operations and write the answer in standard form.

1.

2.

3.

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## ANSWERS

1.

2.

3. units

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

15.

16.

17.

18.

19.  $\emptyset$

20.

21. 256 feet

22.

23.

24.

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25.

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