Chapter

Transformation Groups of the Doubly-Fed Induction Machine

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Abstract

Three-phase, doubly-fed induction (*DFI*) machines are key constituents in energy conversion processes. An ideal *DFI* machine is modeled by inductance matrices that relate electric and magnetic quantities. This work focuses on the algebraic properties of the mutual (rotor-to-stator) inductance matrix $\mathbf{L}_{sr}$: its kernel, range, and left zero divisors are determined. A formula for the differentiation of $\mathbf{L}_{sr}$ with respect to the rotor angle θ_r is obtained. Under suitable hypotheses $\mathbf{L}_{sr}$ and its derivative are shown to admit an exponential representation. A recurrent formula for the powers of the corresponding infinitesimal generator $\mathbf{A}_0$ is provided. Historically, magnetic decoupling and other requirements led to the Blondel-Park transformation which, by mapping electric quantities to a suitable reference frame, simplifies the *DGI* machine equations. Herewith the transformation in exponential form is axiomatically derived and the infinitesimal generator is related to $\mathbf{A}_0$. Accordingly, a formula for the product of matrices is derived which simplifies the proof of the Electric Torque Theorem. The latter is framed in a Legendre transform context. Finally, a simple, "realistic" machine model is outlined, where the three-fold rotor symmetry is broken: a few properties of the resulting mutual inductance matrix are derived.

Keywords: mutual inductance matrix, Blondel-Park transformation, exponential representation, infinitesimal generator, zero divisors, circulants, broken symmetry

1. Introduction

Three-phase, doubly-fed induction (*DFI*) machines have a long history [1–4] and continue to be key constituents in energy conversion processes [5, 6]. Motivation for modeling a *DFI* generator comes from the need to deal with intermittency in the primary energy supply (e.g., the wind field) and with uncertainty in the load (i.e., the grid). Similarly, the modeling and control of a *DFI* motor can improve the efficiency and reliability of electric-to-mechanical work conversion. The equations modeling the ideal *DFI* machine have been studied for more than a century. Results in modeling and control [7–13], including those which derive from numerical simulation, demonstrate how attention to the *DFI* machine is being continuously paid. In essence, the ideal three-phase machine model centers on two matrices, on which this work focuses: the rotor-to-stator mutual inductance matrix, $\mathbf{L}_{sr}[.]$, which depends on the "rotor angle" θ_r and characterizes the machine itself, and the Blondel-Park transformation matrix, $\mathbf{K}[.]$, which depends on another angle and describes a change of variables, from the $\{abc\}$ reference frame (Section 2) to the $\{dq0\}$ reference frame (Section 4). Both matrices, $\mathbf{L}_{sr}[.]$ and $\mathbf{K}[.]$, appear in the Electric Torque Theorem (*ETT*) which relates mechanical to electrical variables and as such represents the *raison d'être* of the *DFI* machine. Stated in the $\{abc\}$ frame, the *ETT* is a straightforward application of energy balance, once a Legendre transformation (Section 5.2) has been introduced and co-energy accordingly defined. Instead, the proof, in fact the translation of the *ETT* in the $\{dq0\}$ frame (Section 5.3), requires all relevant properties of $\mathbf{L}_{sr}[.]$ and $\mathbf{K}[.]$ to be known. For this reason, in Section 3 the kernel, the range (Proposition 1), the classical adjoint and the left zero divisors (Proposition 3) of $\mathbf{L}_{sr}[.]$ are determined. Derivation benefits from $\mathbf{L}_{sr}[.]$ being a circulant matrix [14] (Lemma 1) and from its eigenvalues representing the discrete Fourier transform of a 3-sequence [15] (Proposition 2). Special attention need two constant matrices, $\mathbf{A}_0$ and its square (Lemma 2), because they relate differentiation of $\mathbf{L}_{sr}$ with respect to θ_r to multiplication (Theorem 1). In a suitable subspace of $\mathbb{R}^3$, $\mathbf{L}_{sr}$ admits an exponential representation (Theorem 2) with $\mathbf{A}_0$ as infinitesimal generator. Section 4 is devoted to $\mathbf{K}[.]$: its structure, as well as its exponential representation with generator $-\mathbf{A}_0$, is inferred by satisfying, in sequence, a list of requirements (Proposition 4 and Theorem 4). The key formula for the product of matrices (Theorem 5) is then applied to prove the *ETT* in the $\{dq0\}$ frame in one step (Theorem 6). An attempt is finally made in Section 6 to deal with a "realistic" machine model, where the three-fold rotor symmetry is broken: the b and c rotor axes are misaligned by angles ϵ_b and ϵ_c. To second-order in ϵ_b and ϵ_c there exists a constant $\mathbf{B}$ which relates differentiation to multiplication of the approximate inductance matrix (Proposition 7).

2. The ideal doubly-fed induction machine

Definition 1. (*Three-phase, ideal DFI machine*) [3, 4]. A three-phase *DFI* machine is said ideal whenever its stator and rotor windings exhibit three-fold symmetry. Moreover, magnetomotive forces and flux waves created by the windings are sinusoidally distributed and windings give rise to a linear electric network.

Remark 1. (*Neglected phenomena*). Higher harmonics, hysteresis, and eddy currents are excluded by the model. Deviations from three-fold symmetry will be addressed in Section 6.

Notation. (*$\{abc\}$ frames*). The most natural frames where three-phase stator and rotor voltages and currents can be represented are the $\{abc\}$ frames. For example, the stator currents are an ordered triple which one agrees to represent as a vector

$$\overrightarrow{j}_{\{abc\}s} = \begin{bmatrix} j_{as} & j_{bs} & j_{cs} \end{bmatrix}^{\mathrm{Trs}} \in \mathbb{R}^3, \tag{1}$$

whose components are functions of time $t \in \mathfrak{T}$. A similar notation will hold for other electric quantities. Unless otherwise specified, all vectors are understood in $\mathbb{R}^3$.

Hp. (*Function class*). Dependence of all quantities of interest on time is assumed as smooth as required.

Definition 2. (*Balanced triple*). An $\{abc\}$ current triple is balanced or is a trivial zero sequence, whenever

$$j_a + j_b + j_c = 0, \quad \forall t \in \mathfrak{T}. \tag{2}$$

Such sequences define the subspace $\mathfrak{B} \subset \mathbb{R}^3$, a plane through the origin; the corresponding notation is $\overrightarrow{j}_{\mathfrak{B}} \in \mathfrak{B}$.

Notation.

$\theta_r[t] \in [0, 2\pi)$ is the electric rotor angle at a time t, formed by the rotor ar axis with respect to the stator as axis.

$\beta_r[t] \in [0, 2\pi)$ is the electric rotor angle at a time t, formed by d axis with respect to the rotor ar axis (Section 4).

$\beta_s[t] \in [0, 2\pi)$ is the electric rotor angle at a time t, formed by d axis with respect to the stator as axis (Section 4).

$\overrightarrow{j}'_{\{abc\}r} := \frac{N_r}{N_s} \overrightarrow{j}_{\{abc\}r}$ is the stator-referred (*ständer-bezogen*) vector of rotor currents, where N_r and N_s are the rotor and stator turns.

A boldface, roman capital denotes a matrix in $\mathcal{M}[N_{\text{row}}, N_{\text{col}}]$ of $N_{\text{row}}\ (\geq 2)$ rows $\times N_{\text{col}}\ (\geq 2)$ columns.

$a^*_{m,n}$ is the cofactor of entry $\{m, n\}$ in $\mathbf{A} \in \mathcal{M}[N, N]$, where $N_{\text{row}} = N_{\text{col}} = N\ (\geq 2)$.

$\left[a^*_{m,n}\right]$ is the corresponding matrix.

$\text{adj}[\mathbf{A}]$ is the matrix adjoint to $\mathbf{A} \in \mathcal{M}[N, N]$, obtained by transposing $\left[a^*_{m,n}\right]$.

$\vec{u})(\vec{v}$ is the dyadic product of the column vector $\vec{u}$ by the row vector $\vec{v}$, both $\in \mathbb{R}^3$.

$\mathbf{1}_3$ is the 3×3 identity matrix.

The end of a *Proof* is marked by $\triangleright\triangleleft$, that of a general statement or of a Remark by $\Diamond$.

The electrical network equations which describe the dynamics of the three-phase, ideal *DFI* machine are well-known [3, 4], as a consequence they are omitted.

3. Group properties of the mutual inductance matrix

By letting $\varphi \equiv \frac{2}{3}\pi$, the rotor-referred (*läufer-bezogene*) form of the mutual inductance matrix is [3, 4]

$$\mathbf{L}_{sr}[\theta_r] = \begin{bmatrix} \cos[\theta_r] & \cos[\theta_r + \varphi] & \cos[\theta_r - \varphi] \\ \cos[\theta_r - \varphi] & \cos[\theta_r] & \cos[\theta_r + \varphi] \\ \cos[\theta_r + \varphi] & \cos[\theta_r - \varphi] & \cos[\theta_r] \end{bmatrix} \tag{3}$$

Given $\mathbf{L}_{sr}[\theta_r]$ one defines the stator-referred (*ständer-bezogen*) rotor-to-stator mutual inductance matrix

$$\mathbf{L}'_{sr}[\theta_r] := \frac{N_r}{N_s} L_{ms} \mathbf{L}_{sr}[\theta_r] \ , \tag{4}$$

where $L_{ms}(>0)$ is a constant parameter. Hereinafter, the dependence of the involved matrices and of related quantities on θ_r will be shown only if mandatory. The following properties hold because rows two and three of $\mathbf{L}_{sr}$ are right shift-circular permutations of the first row and because all row-wise (and column-wise) sums of $\mathbf{L}_{rs}$ vanish.

Proposition 1. (*Eigenvalues of* $\mathbf{L}_{sr}$ *and their implications*).

- The eigenvalues of $\mathbf{L}_{sr}$ are $\mu_0 = 0$, $\mu_\pm = \frac{3}{2}e^{\pm i\theta_r}$.

- $\det[\mathbf{L}_{sr}] = 0, \forall\theta_r$,
- $\dim[\mathrm{Ker}[\mathbf{L}_{sr}]] = 1$ and $\mathrm{Ker}[\mathbf{L}_{sr}] = \left\{ \overrightarrow{j} \in \mathbb{R}^3 | \; j_1 = j_2 = j_3 \right\} := \mathfrak{K}_{sr}, \forall\theta_r$.
- $\dim[\mathrm{range}[\mathbf{L}_{sr}]] = 2$ and $\mathrm{range}[\mathbf{L}_{sr}] = \mathfrak{B}, \forall\theta_r$,

Remark 2. (*Orthogonal decomposition of* $\mathbb{R}^3$). The last two properties in the list translate the orthogonal decomposition

$$\begin{array}{rcccc} \mathbb{R}^3 & = & \mathrm{Ker}[\mathbf{L}_{sr}] & \oplus & \mathrm{range}[\mathbf{L}_{sr}] \\ & = & \mathfrak{K}_{sr} & \oplus & \mathfrak{B} \quad , \\ \overrightarrow{j} & = & \overrightarrow{j}_{\mathfrak{K}_{sr}} & + & \overrightarrow{j}_{\mathfrak{B}} \end{array} \tag{5}$$

where the straight line $\mathfrak{K}_{sr} : \{ j_1 = j_2 = j_3 \}$ is the normal to the plane $\mathfrak{B} : \{ j_1 + j_2 + j_3 = 0 \}$ of Eq. (2).

Lemma 1 (*Eigenvalues of a permutation matrix*, pp. 65–66 of M. Marcus and H. Minc's textbook [14]). For a general $N(\geq 2)$ and for an $N \times N$ matrix $\mathbf{P}$, a.k.a. "circulant", which results from the right shift-circular permutation of the first row $[c_0 \;\; c_{N-1} \;\; c_{N-2} \;\; \dots \;\; c_1]$, one denotes $\epsilon := \mathrm{e}^{\mathrm{i}2\pi/N}$ and introduces the polynomial $\psi[.]$ of degree $N-1$ in the complex variable ζ

$$\psi[\zeta] := \sum_{n=0}^{N-1} c_n \zeta^n \quad . \tag{6}$$

The possibly multiple eigenvalues $\{\mu_k | 0 \leq k \leq N-1\}$ of $\mathbf{P}$ are obtained by letting $\zeta = \epsilon^m$ and evaluating $\psi[\epsilon^m]$ for $m = 1, 2, \dots, N$. Since $\epsilon^m := \mathrm{e}^{\mathrm{i}2\pi m/N}$, there exists only one value of m, denoted by ℓ, at which all powers of ϵ appearing in $\psi[.]$ are equal: $\epsilon^\ell = \epsilon^{2\ell} = \epsilon^{3\ell} = \dots = \epsilon^{(N-1)\ell} = 1$. Such value is $\ell = N$. Therefore

$$\psi\left[\epsilon^N\right] = \sum_{n=1}^{N-1} c_n = \psi\left[\epsilon^0\right] \;\; \text{when} \;\; \ell = N \quad . \tag{7}$$

If the additional property

$$\sum_{n=1}^{N-1} c_n = 0 \tag{8}$$

is exhibited by the rows of $\mathbf{P}$, then

$$\psi\left[\epsilon^N\right] = 0 = \mu_0 \quad . \tag{9}$$

Such eigenvalue is algebraically (and geometrically) simple.

Remark 3. (*Features of* $\psi[.]$, *k and n*). The polynomial $\psi[.]$ shall not be confused with any of the polynomials annihilated by $\mathbf{P}$. There is no correspondence between the eigenvalue label k and the ordering of powers induced by m.

Proof of Proposition 1. Since $\mathbf{L}_{sr}[\theta_r]$ is a circulant matrix of the sequence $\{c_0 \;\; c_2 \;\; c_1\}$

$$\mathbf{L}_{sr}[\theta] = \begin{bmatrix} c_0 & c_2 & c_1 \\ c_1 & c_0 & c_2 \\ c_2 & c_1 & c_0 \end{bmatrix} \tag{10}$$

and

$$c_0 + c_1 + c_2 = 0 \ , \ \forall \theta_r \in [0, 2\pi] \ , \tag{11}$$

then Lemma 1 applies with $N = 3$. Hence $\ell = 3$. In the first place, μ_0 is independent of θ_r. Next, one verifies the other two eigenvalues, $\mu_\pm = \frac{3}{2} e^{\pm i\theta_r}$, are respectively returned by $\psi[\epsilon]$ and by $\psi[\epsilon^2]$ and do instead depend on θ_r. The property $\dim[\mathrm{Ker}[\mathbf{L}_{sr}]] = 1$ derives from the algebraic simplicity of $\mu_0 = 0$. From Eq. (11) one deduces $\mathrm{Ker}[\mathbf{L}_{sr}] = \mathfrak{K}_{sr}$. The properties of range$[\mathbf{L}_{sr}]$ are not independent of those of $\mathrm{Ker}[\mathbf{L}_{sr}]$: namely, they follow from orthogonality, as highlighted by Eq. (5). ⊳⊲

Proposition 2. (*Eigenvalues of a circulant as the discrete* Fourier transform of a 3-sequence [15]). Let ϵ be as in Lemma 1. The discrete Fourier transform $\mathbf{b}^{(3)} := \{b_0 \ b_1 \ b_2\}$ of a 3-sequence $\mathbf{c}^{(3)} := \{c_0 \ c_1 \ c_2\}$ is obtained, in terms of row vectors, by

$$(b_0 \ b_1 \ b_2) = (c_0 \ c_1 \ c_2) \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon \end{bmatrix} := (c_0 \ c_1 \ c_2) \cdot \mathbf{T} \ . \tag{12}$$

The circulant $\mathbf{P}\left[\mathbf{c}^{(3)}\right]$ assembled from $\mathbf{c}^{(3)}$ is diagonalized by $\mathbf{T}$ according to

$$\mathbf{P}\left[\mathbf{c}^{(3)}\right] = \mathbf{T}^{-1} \cdot \begin{bmatrix} b_0 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_2 \end{bmatrix} \cdot \mathbf{T} \ . \tag{13}$$

Application to $\mathbf{L}_{sr}$ of Eq. (10) requires a permutation of the first row:

$$(\mu_0 \ \mu_1 \ \mu_2) = (c_0 \ c_2 \ c_1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \mathbf{T} \ . \tag{14}$$

Lemma 2 (*The matrix* $\mathbf{A}_0$ *and its properties*). Let the matrix $\mathbf{A}_0$ be defined by

$$\mathbf{A}_0 := \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \ . \tag{15}$$

Its properties are the following.

- (*Determinant, rank, eigenvalues, eigenspaces*).

$$\det \mathbf{A}_0 = 0 \ ; \ \mathrm{rank}[\mathbf{A}_0] = 2 \ , \tag{16}$$

$$\lambda_0 = 0 \ , \lambda_1 = -i \ , \ \lambda_2 = +i \ . \tag{17}$$

There is an eigenspace of dimension one, $\mathcal{X}_0[\mathbf{A}_0]$, which corresponds to λ_0:

$$\mathcal{X}_0[\mathbf{A}_0] = \mathrm{Ker}[\mathbf{A}_0] = \mathrm{Ker}[\mathbf{L}_{sr}] = \mathfrak{K}_{sr}. \tag{18}$$

- (*Left zero divisors*). The constant, nontrivial left zero divisors of $\mathbf{A}_0$ are given by dyads

$$\mathbf{Z}\left[\vec{c}\right] = \vec{c})(\vec{1} \tag{19}$$

where c_k, $k = 1, 2, 3$, are real constants with $c_k \neq 0$ for at least one k.

- (*Dyadic representation*). The matrix $\mathbf{A}_0$ admits **no** “algebraic” dyadic representation of the form

$$\mathbf{A}_0 = \vec{a})(\vec{b} \ \text{(false)} \tag{20}$$

with constant $a_k, b_k \in \mathbb{R}$, $k = 1, 2, 3$.

- (*Sign reversal*). There exists no left zero divisors of $\mathbf{A}_0$ which, added to $\mathbf{A}_0$, reverses its sign.

- (*Recurrent formula for powers of* $\mathbf{A}_0$). Given $\mathbf{A}_0$ and

$$\mathbf{A}_0^2 = -\mathbf{1}_3 + \frac{1}{3}\vec{1})(\vec{1}, \tag{21}$$

the powers of $\mathbf{A}_0$ are obtained from

$$\mathbf{A}_0^n = (-1)^{1+((n-3)\%4)/2}\mathbf{A}_0^{2-(n-2)\%2}, \quad n \geq 3, \tag{22}$$

where $(n-3)\%4$ stands for the remainder from integer division of $(n-3)$ by 4 and the “/” (slash) denotes division between integers; similarly, $(n-2)\%2$ stands for the remainder from integer division of $(n-2)$ by 2.

Proof of Lemma 2. The properties described by Eqs. (16)-(19) are immediately verified, as well as the nonexistence of an algebraic dyadic representation. The statement about sign reversal is proved by contradiction. To obtain the recurrent formula for powers of $\mathbf{A}_0$ one computes $\mathbf{A}_0^3$ $(= -\mathbf{A}_0)$ and $\mathbf{A}_0^4$ $(=-\mathbf{A}_0^2)$, then one examines the higher powers $\mathbf{A}_0$ and the sequence formed by their signs. Since the sequence has period 4 and reads $- - + + - - + + \cdots$, then the exponents of both $\mathbf{A}_0$ and (-1) in Eq. (22) can be determined. ⊳⊲

Remark 4, to Lemma 2.

- Let $\vec{a}$) in Eq. (20) be replaced by ∇), then there exists a $\mathcal{C}^1$, divergence-free vector field $\vec{f}$ giving rise to the “differential” dyadic representation of $\mathbf{A}_0$

$$\mathbf{A}_0 = \frac{1}{\sqrt{3}}\nabla)(\vec{f}. \tag{23}$$

The system of first-order linear partial differential equations to which $\vec{f}$ is the solution is obtained by comparing like terms in the arrays.

- As already noticed in the proof, $\mathbf{A}_0$ cannot be a left zero divisor of itself. (In fact, $\mathbf{A}_0^2$ is given by Eq. (21)).

- Obviously, there is no way of including $n = 0$ in any recurrent formula for the powers of $\mathbf{A}_0$, of which Eq. (22) is an example because $\det[\mathbf{A}_0] = 0$.

- As one can easily verify, the eigenvalues of $\mathbf{A}_0^2$ are $\lambda_0 = 0$, $\lambda_1 = -1$. The latter has algebraic multiplicity $\alpha_1 = 2$ and geometric multiplicity $\gamma_1 = 1$. As a consequence, its eigenspace, $\mathcal{X}_1[\mathbf{A}_0^2]$, not only has a dimension $\alpha_1 - \gamma_1 + 1 = 2$ but complies with

$$\mathcal{X}_1[\mathbf{A}_0^2] = \mathfrak{B} \tag{24}$$

as well. In other words, by recalling Eqs. (5), (18), and (21),

$$\mathbf{A}_0^2 = -\mathbf{1}_3\restriction_{\mathfrak{B}} \text{ or } \mathbf{A}_0^2 \cdot \vec{\psi} = -\vec{\psi}_{\mathfrak{B}}, \quad \forall \vec{\psi} \in \mathbb{R}^3 \tag{25}$$

i.e., $\mathbf{A}_0^2$ coincides with $-\mathbf{1}_3$ restricted to the subspace $\mathfrak{B}$. ◊

Proposition 3. (*The matrix* $\mathbf{L}_{sr}[.]$: *trigonometric decomposition and classical adjoint; the left zero divisors of* $\mathbf{L}_{sr}[.]$, *their kernel and range*).

- (*The matrices C and S*). $\mathbf{L}_{sr}[\theta_r]$ is a linear combination of trigonometric functions according to

$$\mathbf{L}_{sr}[\theta_r] = \mathbf{C}\cos[\theta_r] + \mathbf{S}\sin[\theta_r], \tag{26}$$

where $\mathbf{C}$ and $\mathbf{S}$ are the constant, 3×3 matrices

$$\mathbf{C} = \frac{3}{2}\mathbf{1}_3 - \frac{1}{2}\vec{1})(\vec{1} \quad ; \quad \mathbf{S} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \frac{\sqrt{3}}{2}. \tag{27}$$

- (*Relations between* $\mathbf{A}_0$, $\mathbf{C}$ *and* $\mathbf{S}$).

$$\mathbf{C} = -\frac{3}{2}\mathbf{A}_0^2, \quad \mathbf{S} = \frac{3}{2}\mathbf{A}_0. \tag{28}$$

- (*The classical adjoint matrix*). The classical adjoint to $\mathbf{L}_{sr}[\theta_r]$ ($\equiv$ transpose of the cofactor matrix) is

$$\operatorname{adj}[\mathbf{L}_{sr}[\theta_r]] = \frac{3}{4}\vec{1})(\vec{1} \quad , \quad \forall \theta \in [0, 2\pi] \quad . \tag{29}$$

- (*Left divisors as dyads*). If $f_k[.]$, $k = 1, 2, 3$, denote real-valued functions of class $\mathcal{C}^M([0, 2\pi])$ (for some M), one of which, at least, does not vanish identically, then a left zero divisor $\mathbf{Z}$ (*linker Nullteiler*) of $\mathbf{L}_{sr}$ is a rank one dyad

$$\mathbf{Z}\left[\vec{f}[\theta]\right] = \vec{f}[\theta])(\vec{1} \tag{30}$$

forming an algebra $\{\mathbf{Z}\} \equiv \mathfrak{Z}$.

- (*Kernel of the* $\mathbf{Z}$*'s*).

$$\mathrm{Ker}[\mathbf{Z}] = \mathfrak{B} \ , \ \forall \mathbf{Z} \in \mathfrak{Z}. \tag{31}$$

Proof of Proposition 3. The identification of **C** and **S** follows from expanding the $\cos\left[. \pm \frac{2}{3}\pi\right]$ entries in $\mathbf{L}_{sr}$. In order to determine $\mathrm{adj}[\mathbf{L}_{sr}]$ one starts from a relation which is one of the many formulas due to Laplace [16]

$$\mathrm{adj}[\mathbf{A}] \cdot \mathbf{A} = \det[\mathbf{A}]\mathbf{1}_N = \mathbf{A} \cdot \mathrm{adj}[\mathbf{A}] \tag{32}$$

and holds for a general $\mathbf{A} \in \mathcal{M}[N \times N]$; then one recalls $\det[\mathbf{L}_{sr}] = 0$ and the θ_r-invariance of μ_0 of Eq. (9): one can thus compute the classical adjoint to either constant matrix, **C** or **S**, whichever is simpler to deal with; the result is the dyad on the right side of Eq. (29), a result which holds $\forall \theta_r$. The search for left zero divisors of $\mathbf{L}_{sr}$ as dyads like that of Eq. (30) is suggested by Eqs. (29) and (32) because $\mathrm{adj}[\mathbf{L}_{sr}]$ must be a zero divisor of $\mathbf{L}_{sr}$. The most general form of a left zero divisor **Z** is inferred from Eq. (11): since all column-wise sums of $\mathbf{L}_{sr}$ vanish, the columns of Z must be equal. Therefore such divisor, if non-trivial, has rank one and is obtained from the dyadic product of Eq. (30). Finally, Eq. (32) and the orthogonal decomposition Eq. (5) imply

$$\mathrm{Ker}[\mathrm{adj}[\mathbf{L}_{sr}]] = \mathrm{Ker}[\mathbf{Z}] = \mathrm{range}[\mathbf{L}_{sr}] \ \ (= \mathfrak{B}) \ . \tag{33}$$

$$\mathrm{Ker}[\mathbf{L}_{sr}] = \mathrm{range}[\mathrm{adj}[\mathbf{L}_{sr}]] = \mathrm{range}[\mathbf{Z}] \ \ (= \mathfrak{K}_{sr}) \ . \tag{34}$$

⊳⊲

Remark 5. *(Duality; divisors; other properties of* $\mathbf{L}_{sr}$*)*.

- From Eqs. (32–34) one says $\mathbf{L}_{sr}$ and $\mathrm{adj}[\mathbf{L}_{sr}]$ are <u>dual</u> to each other.
- The <u>constant</u>, nontrivial left zero divisors of $\mathbf{L}_{sr}$ are those of Eq. (19). By consistency, the matrices **C** and **S** of Eq. (26) not only have the same classical adjoint as $\mathbf{L}_{sr}[\theta_r]$ has but have all and the same zero divisors, because Eq. (26) holds $\forall \theta_r$.
- Right zero divisors are obtained by transposing the left ones.
- Nonexistence of the representation of Eq. (20) prevents $\mathbf{A}_0$ it from being a left zero divisor of $\mathbf{L}_{sr}[.]$. Nor can $\mathbf{A}_0$ be, as Eqs. (28) and (22) show, a divisor of either **C** or **S** taken separately.
- If $\mathfrak{L}_{sr}$ stands for the subspace of functions $\vec{f}[.] \in \left(\mathcal{C}^0([0, 2\pi])\right)^3$ complying with $\oint \mathbf{L}_{sr}[\theta_r] \cdot \vec{f}[\theta_r] \mathrm{d}\theta_r = \vec{0}$, and if $a_m^{(k)}$, $m = 0, 1, 2, \cdots$, $b_m^{(k)}$, $m = 1, 2, \cdots$, are the cosine and, respectively, the sine Fourier coefficients of the (real-valued) components $f_k[.]$, $k = 1, 2, 3$ of $\vec{f}[.]$, then

$$\vec{f} \in \mathfrak{L}_{sr} \Leftrightarrow \left\{ \left\{ a_1^{(1)} = a_1^{(2)} = a_1^{(3)} \right\} \quad \text{and} \quad \left\{ b_1^{(1)} = b_1^{(2)} = b_1^{(3)} \right\} \right\}. \tag{35}$$

◊

Remark 6. (*The physical meaning of* $\mathbf{L}_{sr}$, **C** *and* **S**). As Eq. (5) suggests, given any instantaneous current vector $\vec{j}[t] \in \mathbb{R}^3$, left multiplication by $\mathbf{L}_{sr}[\theta_r[t]]$ returns a balanced current triple $\vec{j}_{\mathfrak{B}}[t] \in \mathfrak{B}$. This follows from the three-fold symmetry of the ideal

DFI machine, mirrored by the structure of $\mathbf{L}_{sr}[.]$. Moreover, the $\mathbf{C}$ term of Eq. (26) represents the opposite of reactive torque, whereas the $\mathbf{S}$ term represents active torque.◊

Theorem 1. (*Differentiation of* $\mathbf{L}_{sr}$ *with respect to* θ_r). Let $\mathbf{A}_0$ and $\mathbf{Z}\left[\overrightarrow{c}\right]$ be respectively given by Eqs. (15) and (19). Then the derivative of $\mathbf{L}_{sr}$ with respect to θ_r is the set-valued map

$$\left(\mathbf{A}_0 + \mathbf{Z}\left[\overrightarrow{c}\right]\right) \cdot \mathbf{L}_{sr}[\theta_r] \in \left(\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r}\right)[\theta_r]. \tag{36}$$

Notation: a convenient notation is $\left(\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r}\right)[\theta_r] \equiv \mathbf{M}[\theta_r]$.

Proof of Theorem 1. Differentiation of $\mathbf{L}_{sr}[\theta_r]$ as represented by Eq. (26), and the use of Eq. (28) yield

$$\left(\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r}\right)[\theta_r] = \frac{3}{2}\mathbf{A}_0 \cos[\theta_r] + \frac{3}{2}\mathbf{A}_0^2 \sin[\theta_r]. \tag{37}$$

One seeks for a constant matrix $\mathbf{B}_1$ which complies with

$$\frac{3}{2}\mathbf{A}_0 \cos[\theta_r] + \frac{3}{2}\mathbf{A}_0^2 \sin[\theta_r] = \mathbf{B}_1 \cdot \mathbf{C} \cos[\theta_r] + \mathbf{B}_1 \cdot \mathbf{S} \sin[\theta_r]. \tag{38}$$

The application of Eq. (22) leads to

$$\mathbf{B}_1 = \mathbf{A}_0. \tag{39}$$

Then, the whole set of constant matrices $\mathbf{B}$ complying with $\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r} \mathbf{B} \cdot \mathbf{L}_{sr}$ is obtained by adding to $\mathbf{B}_1$ a left zero divisor $\mathbf{Z}\left[\overrightarrow{c}\right]$ as of Eq. (19)

$$\mathbf{B} = \mathbf{A}_0 + \mathbf{Z}\left[\overrightarrow{c}\right]. \tag{40}$$

The result justifies the notation for $\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r}$ of Eq. (36) as a set-valued map. ⊳⊲

The above Eq. (36) means $\frac{\partial \mathbf{L}_{sr}[\theta_r]}{\partial \theta_r} \cdot \overrightarrow{j} = \left(\mathbf{A}_0 + \mathbf{Z}\left[\overrightarrow{c}\right]\right) \cdot \mathbf{L}_{sr}[\theta_r] \cdot \overrightarrow{j}, \forall \overrightarrow{j} \in \mathbb{R}^3$. In spite of this last relation, the search for an exponential representation of $\mathbf{L}_{sr}[.]$ and for a one-parameter (θ_r) group acting on the whole of $\mathbb{R}^3$ is ill-posed. Namely, $\mathbf{L}_{sr}[0]$ is not invertible, hence one cannot normalize $\mathbf{L}_{sr}$ by $\mathbf{L}_{sr}[0]$ and no unit element of the group can be defined. To a greater extent, the search for a generator for the group would make no sense. Nonetheless, an exponential representation is obtained in the subspace $\mathfrak{B}$.

Theorem 2. (*Exponential representations on* $\mathfrak{B}$). If $\overrightarrow{j} \in \mathfrak{B}$ then the following hold.

$$\mathbf{L}_{sr}[\theta_r] \cdot \overrightarrow{j} = \frac{3}{2} e^{\theta_r \mathbf{A}_0} \cdot \overrightarrow{j} \quad , \quad \forall \overrightarrow{j} \in \mathfrak{B} \tag{41}$$

and

$$\mathbf{M}[\theta_r] \cdot \overrightarrow{j} = \mathbf{M}[0] \cdot e^{\theta_r \mathbf{A}_0} \cdot \overrightarrow{j} \quad , \quad \forall \overrightarrow{j} \in \mathfrak{B} \tag{42}$$

with

$$\mathbf{M}[0] = \frac{3}{2}\mathbf{A}_0. \tag{43}$$

Proof of Theorem 2. A matrix $\mathbf{J}[\theta_r]$ is sought for, which, like $\mathbf{L}_{sr}[\theta_r]$, splits into a $\cos[.]$ and a $\sin[.]$ term as in Eq. (26) and, unlike $\mathbf{L}_{sr}[.]$, satisfies $\mathbf{J}[0] = \mathbf{1}_3$. As Eqs. (28) suggest, one solution is

$$\mathbf{J}[\theta_r] = \mathbf{1}_3 \cos\theta_r + \frac{2}{3}\mathbf{S}\sin\theta_r = \frac{2}{3}\left(\mathbf{C} + \frac{1}{2}\vec{1}\right)\left(\vec{1}\right)\cos\theta_r + \frac{2}{3}\mathbf{S}\sin\theta_r \ . \tag{44}$$

Next, one requires θ_r-differentiation to coincide with the multiplication of $\mathbf{J}[.]$ by a constant matrix $\mathbf{H}$

$$\left(\frac{\partial\mathbf{J}}{\partial\theta_r}\right)[\theta_r] = \mathbf{H}\cdot\mathbf{J}[\theta_r] \ . \tag{45}$$

By identifying terms like trigonometric functions one obtains the pair

$$\left\{\mathbf{H} = \frac{2}{3}\mathbf{S} \ ; \ \frac{2}{3}\mathbf{H}\cdot\mathbf{S} = -\mathbf{1}_3\right\} \text{ i.e. } \left\{\mathbf{H}^2\cdot\vec{j} = -\mathbf{1}_3\cdot\vec{j} \ , \ \forall\vec{j}\in\mathfrak{B}\right\} \ . \tag{46}$$

One solution, *modulo* left zero divisors, comes from the properties of $\mathbf{A}_0^2$ in Eq. (25):

$$\mathbf{H} = \mathbf{A}_0 \ . \tag{47}$$

Hence $\mathbf{J}[\theta_r] = \mathrm{e}^{\theta_r\mathbf{A}_0}$. The proposed representation of $\mathbf{L}_{sr}[.]$ is

$$\mathbf{L}_{sr}[\theta_r] = \frac{3}{2}\mathrm{e}^{\theta_r\mathbf{A}_0} - \frac{1}{2}\vec{1})(\vec{1}\cos\theta_r \ . \tag{48}$$

Consistency with $\frac{\partial\mathbf{L}_{sr}}{\partial\theta_r}[.]$ implies

$$\left(\frac{\partial\mathbf{L}_{sr}}{\partial\theta_r}\right)[\theta_r] = \frac{3}{2}\mathbf{A}_0\cdot\mathbf{J}[\theta_r] + \frac{1}{2}\vec{1})(\vec{1}\sin\theta_r \ . \tag{49}$$

Since $\vec{j}\in\mathfrak{B}$, then the rightmost dyads in Eqs. (48) and (49) return $\vec{0}$ when right multiplied by $\vec{j}$. Replacing $\mathbf{A}_0$ in Eq. (47) by the $\mathbf{B}$ of Eq. (40) does not change the results (Eqs. (48) and (49)) because

$$\mathrm{e}^{\theta_r\mathbf{Z}[\vec{c}]}\cdot\vec{j} = \vec{0} \ , \ \forall\vec{j}\in\mathfrak{B} \ , \ \forall\mathbf{Z}\left[\vec{c}\right]\in\mathfrak{Z} \ . \tag{50}$$

$\rhd\lhd$

Theorem 3. (*Symmetry properties*).

$$\mathbf{L}_{sr}^{\mathrm{Trs}}[\theta_r] = \mathbf{L}_{sr}[-\theta_r] \tag{51}$$

$$\mathbf{M}^{\mathrm{Trs}}[\theta_r] = -\mathbf{M}[-\theta_r] \tag{52}$$

$$[\mathbf{L}_{sr}, \quad \mathbf{A}_0] = \mathbf{0}. \tag{53}$$

Proof of Theorem 3. The first two relations are immediate. The third one follows from the representation of $\mathbf{L}_{sr}$ as linear combination of powers of $\mathbf{A}_0$ according to Eqs. (26) and (28). ▷◁

4. The Blondel-Park transformation and the rotation group

4.1. Axiomatics of the transformation

The Kirchhoff voltage and current laws bring redundancy into the $\{abc\}$ frame representations. In order to remove said redundancy, another frame, called $\{dq0\}$, is introduced, where only two components of a vector shall matter, the direct one, d, and the quadrature component, q.

Definition 3. (*The dq*0 *frame*). Let $\{dq0\}$ denote a reference frame for electric quantities of axes d and q, subject to five specifications.

d.1) The new frame shall be suitable to represent both stator-referenced and rotor-referenced quantities.

d.2) The component of a stator-referenced quantity with respect to both the *direct* or d axis and the stator *as* axis shall be represented by the same function of angle, evaluated at arguments which differ by β_s. Similarly for variables pertaining to the rotor *ar*: the phase difference shall be β_r.

d.3) The above angles are related by

$$\beta_s = \beta_r + \theta_r. \tag{54}$$

d.4) The *quadrature* or q axis shall be orthogonal to d in the $L^2([0, 2\pi])$ sense: if $w_d[.]$ and $w_q[.]$ are the d- and q-components of a (generally complex-valued) signal $\vec{w}[.]$ which depends on η, then: $\int_0^{2\pi} w_d[\eta]^* w_q[\eta] \mathrm{d}\eta = 0$.

d.5) The third entry $w_0[.]$ of a vector $\vec{w}[.]$ in the $\{dq0\}$ frame shall be equal to the sum of its $\{abc\}$ components. (For this reason, such a sum is called "zero sequence", or *Nullfolge*, and may be trivial or not).

Problem 1. (*The* $\{abc\}$ *to* $\{dq0\}$ *transformation problem* [3, 4, 6]). Find a transformation $\mathbf{K}[.]$ that maps a vector $\vec{w}_{abc}$ (a physical quantity) from the $\{abc\}s$ and, respectively, the $\{abc\}r$ frames to a vector $\vec{w}_{dq0}$ in the $\{dq0\}$ frame, as specified by Definition 3 and

K.1) is invertible and linear;

K.2) conserves instantaneous electric power;

K.3) has the same functional form for both stator and rotor quantities,

K.4) depends at most on one real parameter, an "electric angle", which may be different for stator or rotor quantities;

K.5) is of class C^1 at least with respect to that parameter;

K.6) magnetically decouples flux linkages [6].

Proposition 4. (*Matrix representation*). A solution to Problem 1 which applies to a three-phase machine exists and is the Blondel-Park [1, 2, 7] transformation $\mathbf{K}[.]$

$$\mathbf{K}[\eta] := \sqrt{\frac{2}{3}} \begin{bmatrix} \cos[\eta] & \cos[\eta-\varphi] & \cos[\eta+\varphi] \\ -\sin[\eta] & -\sin[\eta-\varphi] & -\sin[\eta+\varphi] \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \tag{55}$$

where η stands for an electrical angle. One has

$$\vec{w}_{dq0}[t] = \mathbf{K}[\eta[t]] \cdot \vec{w}_{abc}[t] \tag{56}$$

and $w_0[t] = (w_a + w_b + w_c)[t], \forall t$ and, if $\vec{w}[t]$ has a period 2π, $\oint w_d[t] w_q[t] \mathrm{d}t = 0$.

Proof of Proposition 4. The structure of $\mathbf{K}[.]$ can be inferred by satisfying, in sequence, requirements $K.1$, $K.2$, $K.6$, $d.4$, $d.5$. The Ansatz

$$\mathbf{K}[\eta] = \mathbf{K}_0 \cdot \mathrm{e}^{\eta \mathbf{F}}, \tag{57}$$

where $\mathbf{K}_0 \equiv \mathbf{K}[0]$ and $\mathbf{F}$ is a constant matrix, is shown to be consistent with all requirements, hence the entries of $\mathbf{K}_0$ and $\mathbf{F}$ can be identified. No further details can be provided for reasons of space. ⊳⊲

Remark 7 to Proposition 4. (*On the exponential representation of* $\mathbf{K}[\eta]$). As a result of work at proving Proposition 4, $\mathbf{K}[\eta]$ defines a one-parameter (η) group of unitary (power preserving) transformations, represented by Eq. (57). Since $\mathbf{K}_0$ is invertible, then

$$\mathbf{R}[\eta] := \mathbf{K}_0^{-1} \cdot \mathbf{K}[\eta] \tag{58}$$

and $\mathbf{R}[0]$ is the unit element. The existence of the composition law is implied by the Ansatz. Obviously, $\det[\mathbf{K}[\eta]] = 1$ implies $\det[\mathbf{R}[\eta]] = 1, \forall \eta$. ◊.

Theorem 4. (*Infinitesimal generator*). The infinitesimal generator $\mathbf{F}$ of $\mathbf{K}[.]$ is $\mathbf{K}_0$-similar to the opposite of the infinitesimal generator $\mathbf{A}_3$ of rotations about the $\hat{\mathbf{x}}_3$ axis of $\mathbb{R}^3$ according to

$$\mathbf{F} = -\mathbf{K}[0]^{-1} \cdot \mathbf{A}_3 \cdot \mathbf{K}[0] \tag{59}$$

and is related to the $\mathbf{A}_0$ of Eq. (41) by

$$\mathbf{F} = -\mathbf{A}_0. \tag{60}$$

Proof of Theorem 4. From Eq. (55)

$$\frac{\mathrm{d}\mathbf{K}[\eta]}{\mathrm{d}\eta} = \sqrt{\frac{2}{3}} \begin{bmatrix} -\sin[\eta] & -\sin\left[\eta - \frac{2\pi}{3}\right] & -\sin\left[\eta + \frac{2\pi}{3}\right] \\ -\cos[\eta] & -\cos\left[\eta - \frac{2\pi}{3}\right] & -\cos\left[\eta + \frac{2\pi}{3}\right] \\ 0 & 0 & 0 \end{bmatrix} \tag{61}$$

and the constant matrix $\mathbf{B}$ satisfying $\frac{\mathrm{d}\mathbf{K}[\eta]}{\mathrm{d}\eta} = \mathbf{B} \cdot \vec{\mathbf{K}}[\eta]$ reads

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -\mathbf{A}_3 \ . \tag{62}$$

Next, the infinitesimal generator of $\mathbf{R}[.]$ Eq. (58) is identified according to

$$\frac{\mathrm{d}\mathbf{R}[\eta]}{\mathrm{d}\eta} = \mathbf{K}_0^{-1} \cdot \mathbf{B} \cdot \mathbf{K}[\eta] = \mathbf{K}_0^{-1} \cdot \mathbf{B} \cdot \mathbf{K}_0 \cdot \mathbf{R}[\eta] \coloneqq \mathbf{F} \cdot \mathbf{R}[\eta]. \tag{63}$$

In other words, the matrix $\mathbf{F} \coloneqq \mathbf{K}_0^{-1} \cdot \mathbf{B} \cdot \mathbf{K}_0$ is the sought for infinitesimal generator of the group $\mathbf{R}[.]$. This proves Eq. (59). The relation between $\mathbf{F}$ and $\mathbf{B}$ is to be expected (e.g., § 2.5 of Altmann's textbook [17]). Finally, Eq. (60) follows from direct verification. ⊳⊲

4.2. The product of matrices formula

Theorem 5. (*The formula*). Equations (54), (57) and (42) imply

$$\mathbf{K}[\beta_s] \cdot \left(\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r}\right)[\theta_r] \cdot \mathbf{K}[\beta_r]^{-1} = \mathbf{K}_0 \cdot \mathbf{M}[0] \cdot \mathbf{K}_0^{-1} = \frac{3}{2}\mathbf{A}_3 \quad . \tag{64}$$

Proof of Theorem 5. The proof branches out according to which current triple is being dealt with.

- (*Balanced current triple* ≡ *trivial zero sequence*). Let $\overrightarrow{j}_{\{abc\}} \in \mathfrak{B}$, then, by Eqs. (42), (57), and (60) and applying transposition

$$\mathbf{K}[\beta_s] \cdot \left(\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r}\right)[\theta_r] \cdot \mathbf{K}[\beta_r]^{-1} = \mathbf{K}[\beta_s] \cdot \mathbf{M}[\theta_r] \cdot \mathbf{K}[\beta_r]^{-1} = \mathbf{K}_0 \cdot \mathrm{e}^{-\beta_s \mathbf{A}_0} \cdot \mathbf{M}[\beta_r + \theta_r] \cdot \mathbf{K}_0^{-1} =$$
$$= \mathbf{K}_0 \cdot \mathbf{M}[-\beta_s + \beta_r + \theta_r] \cdot \mathbf{K}_0^{-1} = \mathbf{K}_0 \cdot \mathbf{M}[0] \cdot \mathbf{K}_0^{-1} = \frac{3}{2}\mathbf{A}_3. \tag{65}$$

- (*General current triple*). For general $\overrightarrow{j}_{\{abc\}} \in \mathbb{R}^3$ no exponential representation is available. In analogy with Eq. (26) one identifies the constant matrices $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ giving rise to $\mathbf{K}[\eta] = \sqrt{\frac{2}{3}}\mathbf{P}\cos\eta + \sqrt{\frac{2}{3}}\mathbf{Q}\sin\eta + \frac{1}{\sqrt{3}}\mathbf{R}$. The <u>product</u> $\mathbf{M}[\theta_r] \cdot \mathbf{K}^{-1}[\beta_r]$, after simplification, turns out to be an affine function of $\cos[\theta_r + \beta_r]$ and $\sin[\theta_r + \beta_r]$ which in turn depend on angle <u>sums</u>: products of the involved matrices hide an <u>addition formula</u> for angles on which trigonometric functions depend. Taking Eq. (54) into account, left multiplication by $\mathbf{K}[\beta_s]$ leads to a polynomial in $\cos[\beta_s]$ and $\sin[\beta_s]$ with coefficients like $\mathbf{P} \cdot \mathbf{C} \cdot \mathbf{Q}^{\mathrm{Trs}}$, $\mathbf{R} \cdot \mathbf{C} \cdot \mathbf{P}^{\mathrm{Trs}}$ and so forth. All β_s-dependent terms in the polynomial disappear. Eventually, the only non-zero term is $\frac{2}{3}\mathbf{P} \cdot \mathbf{C} \cdot \mathbf{Q}^{\mathrm{Trs}} = \frac{3}{2}\mathbf{A}_3$, a constant. ⊳⊲

Remark 8. (*Prior results*). To the best of the authors' knowledge, the role of the Blondel-Park transformation in realization theory was pointed out by J.L. Willems [8], who derived the exponential representation of $\mathbf{K}[.]$ while obtaining a time-invariant system from time-varying electric machine equations. The group properties of $\mathbf{K}[.]$ have been known for some time (e.g., [9], p. 1060). Instead, the relation of $\mathbf{F}$ to $\mathbf{A}_0$, at least in the form of Eq. (60), the relation between exponential representations of $\mathbf{K}[.]$ and $\mathbf{L}_{sr}[.]$, and the roles played by the left zero divisors of $\mathbf{L}_{sr}[.]$ and by the subspace $\mathfrak{B}$, seem to have been overlooked so far.◊

5. Electric torque

5.1. The electric torque law in the $\{abc\}$ frame

From the principles of analytical mechanics, the following relation can be deduced [3, 4] for the ideal *DFI* machine in generator mode. The relation involves previously defined quantities, namely stator and rotor currents and a machine parameter, the $\mathbf{L}'_{sr}[.]$ of Eq. (4), and a quantity, the electric torque $\mathcal{T}_{el,g}$, which has not yet been mentioned herewith. As a consequence, the relation can be regarded as the physical "law" which defines $\mathcal{T}_{el,g}$.

Definition 4. (*Electric torque in the* $\{abc\}$ *frame*). Let the ideal *DFI* machine have P poles and be described by current vectors $\overrightarrow{j}_{\{abc\}s}$ and $\overrightarrow{j}'_{\{abc\}r}$. The electric torque in generator mode is defined by

$$\mathcal{T}_{el,g} = -\frac{P}{2} \quad \overrightarrow{j}_{\{abc\}s}^{\mathrm{Trs}} \cdot \frac{\partial \mathbf{L}'_{sr}[\theta_r]}{\partial \theta_r} \cdot \overrightarrow{j}'_{\{abc\}r} \quad . \tag{66}$$

The relevance of Eq. (66) sits in the link it establishes between electric quantities and a mechanical one: in generator mode, it is the torque produced by, usually a working fluid, on the *DFI* machine shaft which, through a suitably excited rotor, gives rise to electric currents in the stator coils; in motor mode power flow from machine coils to the shaft is reversed. All machine control laws rely on Eq. (66) in order to be implemented.

5.2. Co-energy and the Legendre transform

Ansatz. (*Internal energy*). If S is entropy and T is temperature, then the first differential of internal energy U of an electric machine with one mechanical degree of freedom, θ_m, at constant volume V and numbers of moles $\overrightarrow{N}$, is

$$\mathrm{d}U = T\mathrm{d}S + \overrightarrow{j} \cdot \mathrm{d}\overrightarrow{\lambda} - \mathcal{T}_{el,m}\mathrm{d}\theta_m, \tag{67}$$

where $\overrightarrow{\lambda}$ is the vector of flux linkages, $\mathcal{T}_{el,m}$ is mechanical torque in motor mode (opposite to that in generator mode), and θ_m is the shaft angle.

A consequence of the Ansatz is the following.

Proposition 5. (*Relation between motor torque and internal energy*).

$$\mathcal{T}_{el,m} = -\left(\frac{\partial U}{\partial \theta_m}\right)_{S,V,\overrightarrow{N},\overrightarrow{\lambda}} \quad . \tag{68}$$

Definition 5. (Legendre *transform of energy with respect to flux linkage*). Let $\Lambda \subset IR^3$ be a subset where U is at least twice differentiable and convex with respect to $\overrightarrow{\lambda}$ and let $\overrightarrow{p}$ denote the variable conjugate to $\overrightarrow{\lambda}$. Then the Legendre transform $\mathcal{Y}$ of energy U with respect to $\overrightarrow{\lambda}$ is defined by

$$\mathcal{Y}\left[S, V, \overrightarrow{N}, \overrightarrow{p}, \theta_m\right] := \sup_{\overrightarrow{\lambda} \in \Lambda} \quad \left(\overrightarrow{p} \cdot \overrightarrow{\lambda} - U\left[S, V, \overrightarrow{N}, \overrightarrow{\lambda}, \theta_m\right]\right). \tag{69}$$

Remark 9. (*Conjugate variables; motor torque; differential geometric setting*).

- $\vec{p}$ coincides with $\vec{j}$ and one has

$$\mathcal{Y} + U = \vec{j} \cdot \vec{\lambda}. \tag{70}$$

- Motor torque can thus be rewritten as

$$\mathcal{T}_{el,m} = \left(\frac{\partial \mathcal{Y}}{\partial \theta_m}\right)_{S,V,\vec{N},\vec{j}}. \tag{71}$$

- The extensive variables on which *energy* depends are S, $\vec{N}$, $\vec{\lambda}$ and θ_m, and as such are coordinates of the dynamical system's manifold $\mathcal{N}$. Instead, the intensive variables T, $\vec{\mu}$ (vector of chemical potentials), $\vec{j}$ and $\mathcal{T}_{el,m}$ belong to the system's co-tangent bundle $T^*\mathcal{N}$ [11, 13]. Upon a multivariate Legendre transformation, as many extensive variables can be replaced by their conjugates, which are intensive variables.◊

Remark 10. ($\mathcal{Y}$ *vs.* W'_{fld}). By identifying U with "the energy W_{fld} stored in the coupling fields" of an electric machine having P poles, the rotor of which forms the mechanical angle θ_m in the stator frame, one has the following relations:

- the electrical angle θ_r is related to the mechanical angle θ_m by $\theta_r = \frac{P}{2}\theta_m$ (multiplier effect of P);
- usually [3, 4] $\mathcal{T}_{el,m}$ is related to "co-energy" $W'_{fld}\left[\vec{j}_{\{abc\}s}, \theta_r\right]$ by

$$\mathcal{T}_{el,m} = \frac{\partial W'_{fld}\left[\vec{j}_{\{abc\}s}, \theta_r\right]}{\partial \theta_m} = \frac{P}{2}\frac{\partial W'_{fld}\left[\vec{j}_{\{abc\}s}, \theta_r\right]}{\partial \theta_r}. \tag{72}$$

In other words,

$$W'_{fld} = \mathcal{Y}\Big|_{S,V,\vec{N},\vec{j}}. \tag{73}$$

◊

Remark 11. (*Models of real machines*). The relation between $\mathcal{Y}$ and torque applies to any machine and can, in principle, deal with any functional dependence between $\vec{\lambda}$ and $\vec{j}$. Nonlinear $\vec{\lambda}\left[\vec{j}\right]$ relations [9, 10, 12] become of interest when saturation of the magnetic circuit has to be modeled. Hysteresis and the related energy losses pose further difficulties. ◊

5.3. The electric torque theorem in the $\{dq0\}$ frame

Translating Eq. (66) into the $\{dq0\}$ relies on relations between $\mathbf{K}[.]$-transformed current vectors which involve all three angles, β_s , β_r , θ_r. Translation is made remarkably simpler by Theorem 5.

Theorem 6. (*Electric torque in the dq0 frame*). For an ideal *DFI* machine, the electric torque in generator mode and in the $\{dq0\}$-frame is the following bilinear form for the matrix $\mathbf{A}_3$:

$$\mathcal{T}_{el,g} = -\frac{P}{2}\ \frac{3}{2}\ \frac{N_r}{N_s} L_{ms} \begin{bmatrix} j_{ds} & j_{qs} & \checkmark \end{bmatrix} \cdot \mathbf{A}_3 \cdot \begin{bmatrix} j'_{dr} \\ j'_{qr} \\ \checkmark \end{bmatrix} \tag{74}$$

which simplifies to

$$\mathcal{T}_{el,g} = +\frac{P}{2}\ \frac{3}{2}\ \frac{N_r}{N_s} L_{ms} \left(j_{ds}\ \ j'_{qr} - j_{qs}\ \ j'_{dr} \right). \tag{75}$$

Proof of Theorem 6. It suffices to combine Eqs. (66), (56) and (64). The matrix $\mathbf{A}_3$ makes the 3$^{\text{rd}}$ entries of current vectors not relevant (✓). ⊳⊲

6. A "realistic" machine model

Real machines deviate from the hypotheses which have led to the relatively simple form of the equations discussed so far. A satisfactory model shall account for one or more of the following features:

(a) the effects of tooth saliency and slots on the linked fluxes,

(b) deviations from three-fold symmetry,

(c) the instantaneous dependence of self-and mutual inductances on current, when the magnetic material is not linear,

(d) memory effect in a non-linear, hysteretic magnetic material.

Models which, step-wise, account for features (a) to (d) are "realistic" in the sense of Fitzgerald and Kingsley [3]. Features listed under (a) are relatively simple to model if three-fold symmetry is assumed: very briefly, higher harmonics are introduced which, because of linearity, can be dealt with separately. Instead, broken symmetry may be of some interest: the model outlined herewith focuses on feature (b) and consists of constructing a "realistic" mutual inductance matrix, then determining its algebraic (determinant, eigenvalues) and analytical (θ_r-derivative) properties.

Definition 6. (*Broken symmetry in the rotor*). At fixed θ_r the rotor *ar* axis forms angles θ_r, $\theta_r - \varphi$ and $\theta_r + \varphi$ with the *as*, *bs* and *cs* axes respectively. With ϵ_b and ϵ_c satisfying

$$0 \le \left|\frac{3\epsilon_b}{2\pi}\right| \ ,\ \left|\frac{3\epsilon_c}{2\pi}\right| < < 1 \tag{76}$$

the rotor *br* axis forms angles $\theta_r + \varphi + \epsilon_b$, $\theta_r + \epsilon_b$ and $\theta_r - \varphi + \epsilon_b$ with the *as*, *bs* and *cs* axes respectively. Similar relations hold for the rotor *cr* axis in terms of ϵ_c.

As a consequence the mutual inductance matrix is

$$\mathbf{L}_{sr}[\theta_r;\epsilon_b,\epsilon_c] = \begin{bmatrix} \cos[\theta_r] & \cos[\theta_r+\varphi+\epsilon_b] & \cos[\theta_r-\varphi+\epsilon_c] \\ \cos[\theta_r-\varphi] & \cos[\theta_r+\epsilon_b] & \cos[\theta_r+\varphi+\epsilon_c] \\ \cos[\theta_r+\varphi] & \cos[\theta_r-\varphi++\epsilon_b] & \cos[\theta_r+\epsilon_c] \end{bmatrix} . \tag{77}$$

Because of broken symmetry, $\mathbf{L}_{sr}[\theta_r;\epsilon_b,\epsilon_c]$ is no longer circulant. However, its column-wise entries add to zero and the following properties hold.

Proposition 6. (*Kernel, adjoint, zero divisors for general* ϵ_b *and* ϵ_c).

$$\begin{gathered} \mathrm{Ker}[\mathbf{L}_{sr}[\theta_r;\epsilon_b,\epsilon_c]] = \mathfrak{K}_{sr} \ , \ \mathbf{Z}\left[\overrightarrow{f}[\theta]\right] = \overrightarrow{f}[\theta])(\overrightarrow{1} \ , \\ \mathrm{adj}[\mathbf{L}_{sr}[\theta_r;\epsilon_b,\epsilon_c]] \in \mathfrak{Z} \ , \ \mathrm{Ker}[\mathrm{adj}[\mathbf{L}_{sr}[\theta_r;\epsilon_b,\epsilon_c]]] = \mathfrak{B} \ . \end{gathered} \tag{78}$$

To second order in ϵ_b and ϵ_c, $\mathbf{L}_{sr}[\theta_r;\epsilon_b,\epsilon_c]$ is approximated by

$$\begin{gathered} \mathbf{L}_{sr}[\theta_r;\epsilon_b,\epsilon_c] \simeq \mathbf{G}^{(2)}_{\epsilon_b,\epsilon_c} = \\ = \mathbf{C}^{(2)}_{\epsilon_b,\epsilon_c}\cos[\theta_r] + \mathbf{S}^{(2)}_{\epsilon_b,\epsilon_c}\sin[\theta_r] + \mathbf{C}^{(1)}_{\epsilon_b,\epsilon_c}\cos[\theta_r] + \mathbf{S}^{(1)}_{\epsilon_b,\epsilon_c}\sin[\theta_r] \ , \end{gathered} \tag{79}$$

where the four new matrices have to be defined. $\mathbf{C}^{(2)}_{\epsilon_b,\epsilon_c}$ and $\mathbf{S}^{(2)}_{\epsilon_b,\epsilon_c}$ are obtained from $\mathbf{C}$ and $\mathbf{S}$ of Eq. (27) when their second columns are multiplied by $\left(1-\frac{1}{2}\epsilon_b^2\right)$ and their third columns are multiplied by $\left(1-\frac{1}{2}\epsilon_c^2\right)$. Similarly, $\mathbf{C}^{(1)}_{\epsilon_b,\epsilon_c}$ and $\mathbf{S}^{(1)}_{\epsilon_b,\epsilon_c}$ derive from splitting the $\cos[\theta_r]$ and $\sin[\theta_r]$ terms in the following matrix

$$\begin{bmatrix} 0 & -\epsilon_b\sin[\theta_r+\varphi] & -\epsilon_c\sin[\theta_r-\varphi] \\ 0 & -\epsilon_b\sin[\theta_r] & -\epsilon_c\sin[\theta_r+\varphi] \\ 0 & -\epsilon_b\sin[\theta_r-\varphi] & -\epsilon_c\sin[\theta_r] \end{bmatrix} \mathbf{C}^{(1)}_{\epsilon_b,\epsilon_c}\cos[\theta_r] + \mathbf{S}^{(1)}_{\epsilon_b,\epsilon_c}\sin[\theta_r]. \tag{80}$$

As a consequence, a property can be stated about the derivative of $\mathbf{G}^{(2)}_{\epsilon_b,\epsilon_c}$.

Proposition 7. (*Differentiation as multiplication*). At least to second order in ϵ_b and ϵ_c, there exists a matrix $\mathbf{B}$, independent of θ_r, by which the differentiation of $\mathbf{G}^{(2)}_{\epsilon_b,\epsilon_c}$ is represented as multiplication

$$\left(\frac{\partial \mathbf{G}^{(2)}_{\epsilon_b,\epsilon_c}}{\partial\theta_r}\right)[\theta_r]\cdot\overrightarrow{j} = \mathbf{B}\cdot\mathbf{G}^{(2)}_{\epsilon_b,\epsilon_c}[\theta_r]\cdot\overrightarrow{j} \ , \ \forall\overrightarrow{j}\in\mathbb{R}^3 \ . \tag{81}$$

Such matrix complies with

$$\mathbf{B}^2\cdot\left(\mathbf{C}^{(2)}_{\epsilon_b,\epsilon_c}+\mathbf{C}^{(1)}_{\epsilon_b,\epsilon_c}\right) = -\left(\mathbf{C}^{(2)}_{\epsilon_b,\epsilon_c}+\mathbf{C}^{(1)}_{\epsilon_b,\epsilon_c}\right) \ . \tag{82}$$

In particular, to 1st order in ϵ_b and ϵ_c

$$\mathbf{B}^2\cdot\left(\mathbf{C}+\mathbf{C}^{(1)}_{\epsilon_b,\epsilon_c}\right) = -\left(\mathbf{C}+\mathbf{C}^{(1)}_{\epsilon_b,\epsilon_c}\right) \ . \tag{83}$$

7. Conclusion

In view of the large amounts of power converted from electric to mechanical or *vice-versa*, mathematical methods for electric machinery have to undergo continuous investigation and, possibly, improvement. Model errors, although “small” in relative terms, may translate into large amounts of mishandled power. To date, control methods and the corresponding algorithms are satisfactory in the low frequency (tens of Hz) range: better performance is needed to deal with the higher (thousands of Hz) frequency components of a transient [18]. This work has focused on the basics of the ideal *DFI* machine model, where linearity and three-fold symmetry are the main features. As a result, the electric torque theorem has been stated in the $\{dq0\}$ frame without any restriction on the $\vec{j}$'s. The product of matrices formula has accordingly simplified the proof. Some of the properties derived in the ideal case have been shown to hold even if symmetry is broken.

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