Chapter

Fundamentals of Narrowband Array Signal Processing

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Abstract

Array signal processing is an actively developing research area connected to the progress in optimization theory, and remains the key technological development that attracts prevalent attention in signal processing. This chapter provides an overview of the fundamental concepts and essential terminologies employed in narrowband array signal processing. We first develop a general signal model for narrowband adaptive arrays and discuss the beamforming operation. We next introduce the basic performance parameters of adaptive arrays and the second order statistics of the array data. We then formulate various optimal weigh vector solution criteria. Finally, we discuss various types of adaptive filtering algorithms. Besides, this chapter emphasizes the theory of narrowband array signal processing employed in narrowband beamforming and direction-of-arrival (DOA) estimation algorithms.

Keywords: Adaptive algorithms, Adaptive arrays, Array signal processing, Beamforming

1. Introduction

Array signal processing [1, 2] is an indispensable technique in signal processing with ubiquitous applications. The fundamental principles and techniques of array signal processing are applicable in various fields such as sonar, radar, and wireless communications etc. Antenna array processing manipulate and process each sensor output according to a certain algorithm to achieve better system performance than just a single antenna, and estimate the signal parameters from the data accumulated over the spatial aperture of an antenna array. [3, 4]. These parameters of interest include the signal content itself, their DOAs, and power. To get this information, the sensor array data is processed using statistical and adaptive signal processing techniques. These techniques include parameter estimation and adaptive filtering applied to array signal processing. Meanwhile, it also plays an important role in the multi-input multi-output (MIMO) communication system and a waveform diversity MIMO radar system, by improving its performance, reducing the clutter, and increasing the array resolution [1–4].

All in all, there are numerous potential advantages of array signal processing techniques, such as improved system capacity, signal bandwidth, the space division multiple access (SDMA), high signal-to-noise ratio (SNR), frequency reuse factor, side-lobe offsets or nulls, degree of freedoms, and the resolution of the antenna array [5]. In this chapter, we introduce the basic principle of array signal processing techniques to further understand its implementation process and applications. We begin by formulating the signal mathematical model used as a basis for discussing array signal processing in beamforming and direction-of-arrival (DOA) estimation algorithms. We also provide some introductory materials about beamforming techniques, performance analysis parameters, and a brief overview of some basic beamforming algorithms.

2. Adaptive array signal model

Since the real signal transmission environment is complex, so a strict mathematical model is the basis for adaptive beamforming and lays the groundwork for the discussion of beamforming algorithms. To simplify the analysis of the model, the signal source used in this chapter is a narrowband signal, that is, the bandwidth of the received array signal is much smaller than the carrier frequency of the signal, assuming that [6]:

a. Each array element is an ideal omnidirectional point source, and the inter-element spacing is less than or equal to half-a-wavelength.

b. The number of received signals is known, and less than the number of array elements.

c. The signal sources are assumed to be in the far-field so that the signals impinging on the array can be regarded as a plane wave;

d. The spacing between array elements are equal, i.e., evenly spaced array;

e. The noise is zero-mean Gaussian white noise, and uncorrelated with the signal source.

f. The effect of mutual coupling between array elements is assumed to be negligible, i.e., the different element receives the same signal amplitude.

Although the above assumptions are not valid for wideband signal source, the fundamental model used for them is very similar. Therefore, this chapter focuses on the mathematical model based on narrowband signal beamforming principle.

Adaptive antenna arrays may have different geometrical configurations. Different spatial distribution of array elements leads to different array configurations, such as linear arrays, circular arrays, rectangular arrays, and triangular arrays etc. [7, 8]. For an arbitrary array structure with M-elements as shown in **Figure 1**, θ and ϕ denote the elevation angle and the azimuth angle, respectively. Vector $\boldsymbol{a}$ and $\boldsymbol{p}_i$ respectively denote the direction vector of the signal and the coordinates of the $i-th$ array element. Since the signal received by each array element has a certain delay relative to the origin of the coordinates, the delay time [9] for the signal received at the $i-th$ array element is

$$\tau_i = \frac{\boldsymbol{a}^T \boldsymbol{p}_i}{c} \tag{1}$$

where c is the speed of light, and

$$\boldsymbol{a} = \begin{bmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{bmatrix}. \tag{2}$$

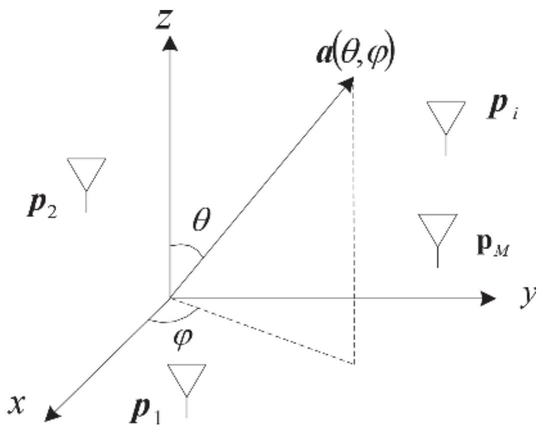

Figure 1.
Geometry of array.

$$p_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}, \quad i = 1, 2, \cdots, M. \tag{3}$$

The signal received by the first sensor located at the origin of the coordinates is

$$\tilde{x}_1(t) = x_1(t)e^{j\omega t}. \tag{4}$$

The overall signal received by the array can be expressed as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_M(t) \end{bmatrix} = \begin{bmatrix} x_1(t-\tau_1)e^{j\omega(t-\tau_1)} \\ x_2(t-\tau_2)e^{j\omega(t-\tau_2)} \\ \cdots \\ x_M(t-\tau_M)e^{j\omega(t-\tau_L)} \end{bmatrix}. \tag{5}$$

If the received signal is a narrowband, we can ignore its amplitude changes for different elements. Consider the phase change only [10], the array received signal is simplified to

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_M(t) \end{bmatrix} = x_1(t) \begin{bmatrix} e^{j\omega(t-\tau_1)} \\ e^{j\omega(t-\tau_2)} \\ \cdots \\ e^{j\omega(t-\tau_M)} \end{bmatrix} \tag{6}$$

Let us consider a uniform linear array (ULA) composed of M elements with inter-element spacing d as shown in **Figure 2**. Assume the first array element located at origin of coordinate as a reference element. Consider the far field source with P signals $s_0(t), s_1(t), \dots, s_{P-1}(t)$, having the same center carrier frequency f_c, the narrowband signal $s_i(t)$ impinges on the array at an angle θ_i relative to the broadside, which refers to the direction normal to the array, where $i = 0, \dots, P-1$ (without taking into account the azimuth angle, consider only the elevation angle).

Due to multipath propagation, each element receive the same signal with a different time delay. Due to the fact that the incident signal is a narrowband signal,

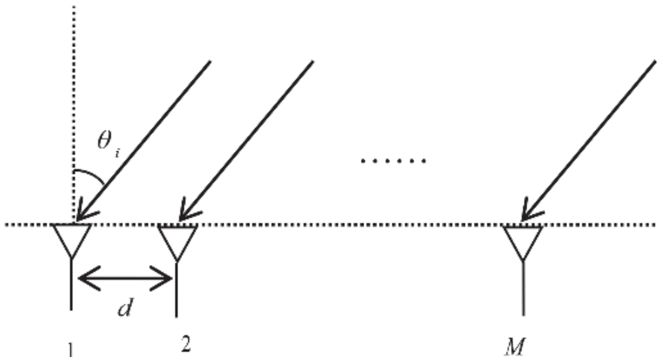

Figure 2.
Structure of uniform linear array antenna.

the amplitude variation is negligible, and only phase delay is considered. This delay is determined by the array element spacing d and the elevation angle of incidence. Consider the signal received by the first array element as a reference signal, then the analytical expression for the $i-th$ signal received with respect to the reference array element is

$$s_i(t) = m_i(t)e^{j2\pi f_c t}, \quad i = 0, \dots, P-1 \tag{7}$$

where $m_i(t)$ is the complex envelope of the $i-th$ modulated signal, and f_c is the carrier frequency.

The propagation delay of the received signal from reference array element to the $m-th$ array element can be expressed as

$$\tau_m(\theta_i) = \frac{d}{c}(m-1)\sin\theta_i, \quad m = 1, \dots, M. \tag{8}$$

According to Eq. (7), the signal received at the $m-th$ array element can be expressed as the superposition of all the signals, that is

$$x_m(t) = \sum_{i=0}^{P-1} m_i(t-\tau_m(\theta_i))e^{j2\pi f_c(t-\tau_m(\theta_i))} + n_m(t), \tag{9}$$

where $n_m(t)$ is the Gaussian noise signal received at the $m-th$ array element having zero mean and variance σ^2.

Since we consider a narrowband signal source located in the far-field, the bandwidth B of the signal satisfy the condition $B < < f_c$, and $m_i(t)$ changes relatively slowly because the signal delay is $\tau_m(\theta_i) < < \frac{1}{B}$. Therefore, complex envelope of the signal can be approximated as $m_i(t-\tau_m(\theta_i)) \approx m_i(t)$, that is, the difference in the array received signal complex envelope can be neglected. Thus, Eq. (9) is simplified as

$$x_m(t) = \sum_{i=0}^{P-1} m_i(t)e^{j2\pi f_c(t-\tau_m(\theta_i))} + n_m(t). \tag{10}$$

Since the carrier component in the system does not affect the analysis, and the adaptive algorithm is often carried out in the baseband (complex envelope), so the carrier part $e^{j2\pi f_c t}$ in the Eq. (10) can be ignored. Eq. (10) can then be expressed as

$$x_m(t) \approx \sum_{i=0}^{P-1} m_i(t) e^{-j(m-1)kd \sin\theta_i} + n_m(t). \tag{11}$$

where k is the free-space wave number given by [11].

$$k = 2\pi f_c / c. \tag{12}$$

At time t, the overall received signal can be expressed as

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_M(t) \end{bmatrix} = \boldsymbol{Am}(t) + \boldsymbol{n}(t)$$
$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-jkd\sin\theta_0} & e^{-jkd\sin\theta_1} & \cdots & e^{-jkd\sin\theta_{P-1}} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-j(M-1)kd\sin\theta_0} & e^{-j(M-1)kd\sin\theta_1} & \cdots & e^{-j(M-1)kd\sin\theta_{P-1}} \end{bmatrix} \begin{bmatrix} m_1(t) \\ m_2(t) \\ \vdots \\ m_P(t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_M(t) \end{bmatrix} \tag{13}$$

where $\boldsymbol{A} = [\boldsymbol{a}(\theta_0) \ \ \boldsymbol{a}(\theta_1) \ \cdots \ \boldsymbol{a}(\theta_{P-1})]$ is the direction matrix (also called the array manifold matrix), $\boldsymbol{a}(\theta_i)$ is the direction vector for the $i - th$ signal $s_i(t)$, and $\boldsymbol{n}(t)$ is the noise vector, expressed as

$$\boldsymbol{a}(\theta_i) = \begin{bmatrix} 1 & e^{-j2\pi f_c \tau_1(\theta_i)} & \cdots & e^{-j2\pi f_c \tau_M(\theta_i)} \end{bmatrix}^T \tag{14}$$

$$\mathbf{n}(t) = [n_1(t) \quad n_2(t) \quad \cdots \quad n_M(t)]^T \tag{15}$$

where the sign $[]^T$ denotes the transpose operation.

3. Adaptive beamforming

Beamforming is a concept originating in array signal processing. The fundamental aim of beamforming is to estimate the desired signal properties by adjusting the complex weights at each sensor applied to the received signal which result in enhancement of desired signal and place nulls in the direction of interference. Adaptive arrays are capable to adjust its weights automatically according to the environment.

The beamforming can be classified into two types that are analog beamforming and digital beamforming [12].

The analog beamforming is performed in the analog domain. The block diagram of an analog beamformer is shown in **Figure 3**. The analog RF signal received by the antenna array is converted to an intermediate frequency by the RF front end, which is the analog intermediate frequency signal. The weight vector is calculated by the weights update algorithm. The weighted sum of the analog IF signal is obtained, and the array received signal is synthesized. At this point the signal is still analog signal; then by analog-to-digitical converter (A/D) the analog signal is sampled and quantized, and the analog IF signal is converted to a digital intermediate frequency signa. Then the digital IF signal is given to the next - level processing.

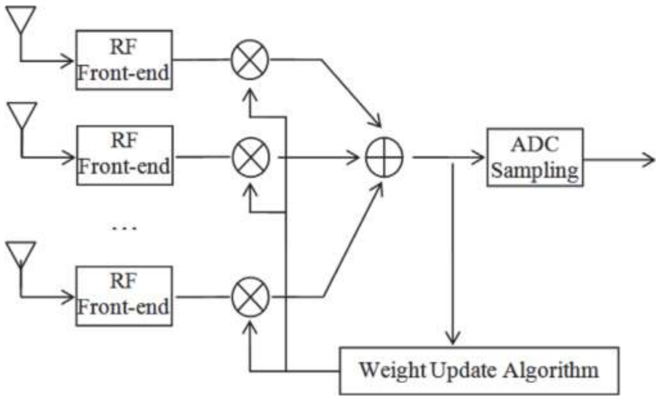

Figure 3.
The structure of analog beamforming.

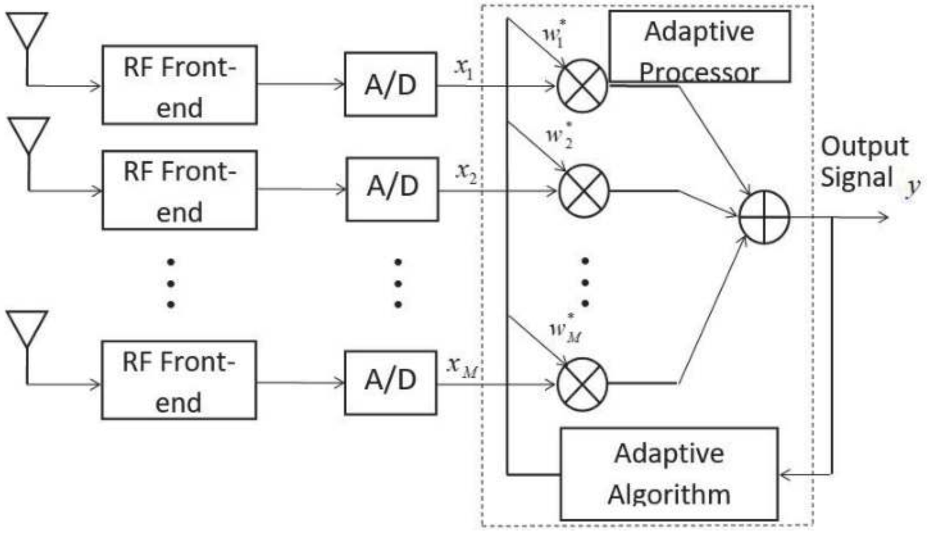

Figure 4.
Structure diagram of adaptive beamforming.

The digital beamforming is carried out in the digital domain, which is shown in the **Figure 4**.

Adaptive beamforming is a subclass of digital beamforming. Usually adaptive beamformer [13] comprises of RF Front-end, A/D converter module, and the signal processing (beam-control formation) module. A basic adaptive beamformer is shown in **Figure 4** which is composed of antenna array elements and an adaptive signal processor.

The antenna array elements receive the spatially-propagating desired signal and interference signal at the array aperture. In the RF Front-end, the received signal is down-converted to baseband signal [14], and then transformed into a digital signal through A/D converter, which is then processed by the adaptive processor. In adaptive processor, suitable adaptive filtering algorithm according to the requirements is applied to get the optimal weight vector. The weights are applied to the received signal at each array element to obtain a weighted sum of the signal. After the adaptive processing, the weighted signals are combined to get the output of the beamformer, which direct the main lobe in the direction of the desired signal and nulls in the directions of the interferers. The interference and noise are suppressed, and the output signal-to-interference-plus-noise ratio (SINR) of beamformer is thus improved.

Clearly, based on the adaptive beamformer structure shown in **Figure 4**, the output of each element is multiplied by a complex weight and summed to form the array output, $y(t)$, expressed as

$$y(t) = \mathbf{w}^H \mathbf{x}(t) = \sum_{m=1}^{M} w_m^* x_m(t) \tag{16}$$

where the symbol $[]^H$ represents the Hermitian (complex conjugate) transpose, $()^*$ indicates the conjugate, and $\mathbf{w}$ is the $M \times 1$ dimensional optimal weight vector computed by an adaptive filtering algorithm, given as

$$\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_M \end{bmatrix}^T. \tag{17}$$

In this way, the array output, $y(t)$, is obtained by combining the weighted sum of each of the sensor signals. The different weight vectors for beamforming of signals from different directions have different response, thus pointing to the desired signal and suppress the interference signal.

Array output signal power is expressed as

$$P_{out} = E[y(t)^* y(t)] = \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w}. \tag{18}$$

where

$$\boldsymbol{R} = E\left[\boldsymbol{x}(t)\boldsymbol{x}^H(t)\right], \tag{19}$$

is the covariance matrix of the received signal, and $E[]$ denotes the expectation operator. Substitute Eq. (13) into Eq. (19), the covariance matrix can be expressed as

$$\mathbf{R} = \sum_{i=0}^{P-1} p_i \boldsymbol{a}(\theta_i)\boldsymbol{a}(\theta_i)^H + \sigma^2 \mathbf{I}, \tag{20}$$

where p_i is the power of signal $s_i(t)$, and $\mathbf{I}$ represents a identity/unit matrix. If the input signal in space has only one desired signal $s_0(t)$, and $P-1$ interference signals, then the covariance matrix can be expressed as

$$\begin{aligned} \mathbf{R} &= p_0 a(\theta_i) a(\theta_i)^H + \sum_{i=1}^{P-1} p_i a(\theta_i) a(\theta_i)^H + \sigma^2 \mathbf{I} \\ &= \boldsymbol{R}_s + \mathbf{R}_i + \mathbf{R}_n, \end{aligned} \tag{21}$$

where $\boldsymbol{R}_s$ is the covariance matrix of the desired signal, $\mathbf{R}_i$ is an interference signal covariance matrix, and $\mathbf{R}_n$ is the covariance matrix of the noise. Substitute Eq. (21) into Eq. (18), the output signal power can be expressed as a sum of desired signal power P_{os}, interference power as P_{oi} and noise power P_{on}.

$$P_{os} = \boldsymbol{w}^H \boldsymbol{R}_s \boldsymbol{w} \tag{22}$$

$$P_{oi} = \boldsymbol{w}^H \boldsymbol{R}_i \boldsymbol{w} \tag{23}$$

$$P_{on} = \boldsymbol{w}^H \boldsymbol{R}_n \boldsymbol{w} \tag{24}$$

The output SINR, a performance parameter of the beamformer, is defined as the ratio of the output desired signal power and the output power due to interference-plus-noise, and can be expressed as

$$SINR_{out} = \frac{P_{os}}{P_{oi} + P_{on}} = \frac{\mathbf{w}^H \mathbf{R}_s \mathbf{w}}{\mathbf{w}^H \mathbf{R}_i \mathbf{w} + \mathbf{w}^H \mathbf{R}_n \mathbf{w}}, \tag{25}$$

Adaptive antenna array takes the output SINR as an index to compute the optimal weights by maximizing the output SINR [15].

The most important performance indicator of the beamforming is the direction of the beampattern. It can be quite obvious to determine whether the resolution of any beamforming method is enough to enhance the desired signal and the extent of the suppression of interference signal is large enough. Array beampattern is defined as

$$B(\theta) = \left| \boldsymbol{w}^H \boldsymbol{a}(\theta) \right|. \tag{26}$$

When using analog beamforming, the hardware circuit is very complex, and the accuracy is low. In digital beamforming, the operations of phase shifting and amplitude scaling for each antenna element, and summation of received signals, are done digitally through a general-purpose DSP or dedicated beamforming chips. Therefore, digital beamforming is more flexible and do not require modification of the hardware structure.

Compared with analog beamforming, the digital beamforming has the following advantages:

a. Under the condition that the output SNR is not reduced and the hardware is not increased, digital beamforming can track multiple signals and form multi-beam.

b. The digital beamforming can make full use of the information received by the array antenna, real-time optimization of system performance, and achieve the real-time tracking of the desired signal.

c. In theory, digital beamforming can be achieved by implementing various algorithms.

d. Digital beamforming can achieve independent beamforming for each signal, and each beamforming can be optimized.

4. Basic parameters of adaptive array antenna

The performance parameters of an adaptive array antenna are basically the same as that of a single antenna, but because of the weight of the array, the specific values of each parameter depend on the array element characteristics, the weight vector, and geometry of the array [16].

4.1 Array pattern

The array pattern is the visual performance parameter of an antenna array. According to the pattern multiplication theorem of array antenna, the overall array pattern is the product of the element pattern $P_E(\varphi, \theta)$ and the array pattern $P_A(\varphi, \theta)$, that is

$$P(\varphi, \theta) = P_E(\varphi, \theta) P_A(\varphi, \theta). \tag{27}$$

Generally, it is assumed that the array elements are identical and omni-directional, hence

$$P_E(\varphi, \theta) = 1. \tag{28}$$

Thus, mostly adaptive array antenna patterns defined in the literature refers to the array factor part only, and the relationship between the received signal and the output signal is given as

$$y(t) = \boldsymbol{w}^{\mathrm{H}} \boldsymbol{x}(t). \tag{29}$$

Let's assume a single array element with the input signal power 1, the output signal power can be expressed as

$$P(\varphi, \theta) = \left| \boldsymbol{w}^{\mathrm{H}} \boldsymbol{a}(\varphi, \theta) \right|^2. \tag{30}$$

The above expression defines the power pattern of the array antenna. As can be seen from Eq. (30), the antenna beampattern is determined by the value of the weight vector; on the other hand, it also depends on the direction vector which is determined by the array geometry. Since we define the power pattern $P(\varphi, \theta)$ as the squared magnitude of the beampattern, therefore

$$B(\varphi, \theta) = \left| \boldsymbol{w}^{\mathrm{H}}(\theta_0) \boldsymbol{a}(\theta) \right|. \tag{31}$$

4.2 Array directivity and directivity index

The directivity of an adaptive array is closely related to the pattern of the array, which can be expressed as follows

$$D = \frac{4\pi P_{\max}(\varphi_0, \theta_0)}{\int_0^{\pi} d\theta \int_0^{2\pi} \sin\theta P(\varphi, \theta) d\varphi}, \tag{32}$$

where $P_{\max}(\varphi_0, \theta_0)$ is the maximum pattern that points to the direction of the main lobe.

The directivity is usually expressed in dB and is called array directivity index (DI) given by

$$DI = 10 \log_{10} D. \tag{33}$$

4.3 Array gain

The purpose of antenna array is to improve the G/T (gain of an antenna divided by its system temperature) ratio of the antenna. Array gain G is the main parameter to measure the SNR of the array, which is defined as the ratio of the output signal to noise ratio SNR_o and the input signal to noise ratio SNR_i.

$$G = \frac{SNR_o}{SNR_i}. \tag{34}$$

4.4 Sensitivity

The array beampattern is a function of weight vector and direction vector. However, due to the influence of various errors, the weight vector and the direction vector will have some errors, such as sensor position errors, covariance matrix estimation errors, inconsistent channel errors, and the mutual coupling between the array elements cause weight vector errors. Suppose the error-free weight vector $\boldsymbol{w}^0$ of the $m - th$ element is

$$\mathbf{w}_m^0 = g_m^0 e^{j\varphi_m^0}. \tag{35}$$

The $m - th$ element weight vector with error is

$$\mathbf{w}_m = \left(g_m^0 + \Delta g_m\right) e^{j\left(\varphi_m^0 + \Delta\varphi_m\right)}, \tag{36}$$

where the error Δg_m and $\Delta\varphi_m$ are zero mean Gauss random variables, and the variance is

$$Var\left(\Delta g_m\right) = \sigma_g^2 \tag{37}$$

$$Var(\Delta\varphi_m) = \sigma_\varphi^2 \tag{38}$$

For the direction vector, the error is mainly derived from the array element position errors. For the $m - th$ element, if there is no error in the array element position coordinates, then

$$\mathbf{p}_m^0 = \begin{bmatrix} p_{mx} & p_{my} & p_{mz} \end{bmatrix}^{\mathrm{T}}. \tag{39}$$

While the coordinate with the error can be expressed as

$$\mathbf{p}_m = \begin{bmatrix} p_{mx} + \Delta p_{mx} & p_{my} + \Delta p_{my} & p_{mz} + \Delta p_{mz} \end{bmatrix}^{\mathrm{T}}, \tag{40}$$

where the error quantity is Gauss random variable, which are zero mean, and the variance is

$$Var(\Delta p_{mx}) = Var\left(\Delta p_{my}\right) = Var(\Delta p_{mz}) = \sigma_p^2. \tag{41}$$

The array pattern at this instant is

$$P(\varphi, \theta) = P^0 e^{-\left(\sigma_\varphi^2 + \sigma_\lambda^2\right)} + \sum_{m=1}^{M} \left(g_m^0\right)^2 \left(1 + \sigma_g^2 - e^{-\left(\sigma_\varphi^2 + \sigma_\lambda^2\right)}\right), \tag{42}$$

where λ is the wavelength, and P^0 denotes the error-free pattern given by

$$P^0 = \left|\mathbf{w}^0 \mathbf{a}^0\right|^2, \tag{43}$$

and the variance is

$$\sigma_\lambda^2 = \left(\frac{2\pi}{\lambda}\right)^2 \sigma_p^2. \tag{44}$$

From Eq. (42), it is seen that the actual pattern consists of two parts. The first part is the error free pattern, i.e., the first term of the equation, and the error in the second term. In the second term, the coefficient g_m^0 is used to amplify the error, so the sensitivity of the array is defined as

$$T_s = \sum_{m=1}^{M} \left(g_m^0\right)^2. \tag{45}$$

5. Optimal beamforming

In beamforming, the weight vector is computed by solving the optimization of the cost function. The different cost functions corresponds to different criteria. Some of the most frequently used performance criteria's include minimum mean squared error (MMSE), maximum signal-to-interference-and noise ratio (MSINR), maximum likelihood (ML), minimum noise variance (MV), minimum output power (MP), and maximum gain, etc. [17]. These criteria's are often expressed as cost functions which are typically inversely associated with the quality of the signal at the array output. As the weights are iteratively adjusted, the cost function becomes smaller and smaller. When the cost function is minimized, the performance criterion is met and the algorithm is said to have converged.

5.1 Maximum signal-to-interferer-noise ratio

As can be seen from Eq. (21), the array output signal power consists of the desired signal power, interference power and noise power, and they are mutually uncorrelated. Since the interference signal and the noise is independent i.e. mutually uncorrelated and zero mean, so, $\mathbf{R}_\mathrm{i} + \mathbf{R}_n$ is a full rank and Hermite positive definite matrix. By unitary transformation it can be converted into unitary matrix as

$$\begin{aligned}\mathbf{U}^*(\mathbf{R}_\mathrm{i}+\mathbf{R}_n)\mathbf{U}^T &= \mathbf{U}^* E\left[\left(\sum_{i=0}^{P-1} m_i(t)\boldsymbol{a}(\theta_i)\right)\left(\sum_{i=0}^{P-1} m_i(t)a(\theta_i)\right)^H\right]\mathbf{U}^T + \sigma^2\mathbf{I} \\ &= E\left[\left(\mathbf{U}\sum_{i=0}^{P-1} m_i(t)\boldsymbol{a}(\theta_i)\right)^*\left(\mathbf{U}\sum_{i=0}^{P-1} m_i(t)\boldsymbol{a}(\theta_i)\right)^T\right] + \sigma^2\mathbf{I} \\ &= \sigma^2\mathbf{I}\end{aligned} \tag{46}$$

If we make

$$\boldsymbol{w} = \boldsymbol{U}^T\hat{\boldsymbol{w}}, \tag{47}$$

the output SINR will be

$$\begin{aligned}SINR_{out} &= \frac{\hat{\mathbf{w}}^H E\left[(\mathbf{U}m_0\boldsymbol{a}(\theta_0))^*(\mathbf{U}m_0\boldsymbol{a}(\theta_0))^T\right]\hat{\mathbf{w}}}{\hat{\mathbf{w}}^H E\left[\left(\mathbf{U}\sum_{\mathrm{i}=0}^{P-1} m_i\boldsymbol{a}(\theta_i)\right)^*\left(\mathbf{U}\sum_{\mathrm{i}=0}^{P-1} m_i\boldsymbol{a}(\theta_i)\right)^T\right]\hat{\mathbf{w}} + \sigma^2\mathbf{I}} \\ &= \frac{\hat{\mathbf{w}}^H E\left[(m_0\mathbf{U}\boldsymbol{a}(\theta_0))^*(m_0\mathbf{U}`\boldsymbol{a}(\theta_0))^T\right]\hat{\mathbf{w}}}{\|\hat{\mathbf{w}}\|^2}.\end{aligned} \tag{48}$$

According to Cauchy-Schwartz inequality

$$SINR_o \leq \|m_0 \mathbf{U}\boldsymbol{a}(\theta_0)\|^2 = E\left[|m_0|^2\right] \cdot \|\mathbf{U}^* \boldsymbol{a}(\theta_0)\|^2. \tag{49}$$

When the equality holds, then

$$\hat{\mathbf{w}} = \mathbf{U}^* \boldsymbol{a}(\theta_0). \tag{50}$$

The optimal solution for the weight vector

$$\mathbf{w}_{MSINR} = \mathbf{U}^T \mathbf{U}^* \boldsymbol{a}(\theta_0) = (\mathbf{R}_i + \mathbf{R}_n)^{-1} \boldsymbol{a}(\theta_0). \tag{51}$$

The optimal weight vector solution of the MSINR has the following advantages: only the DOA of the desired signal is required, and the DOA information for the interference signals is not needed; $\boldsymbol{R}_i + \boldsymbol{R}_n$ can be obtained through sampling and estimating the signal of each array element when the desired signal is interrupted; taking into account the constraints of the interference and noise signal, the output has a maximum SINR.

5.2 Minimum mean square error

Mean squared error refers to the mean squared difference between the beamformer output and the desired signal. The MMSE algorithm minimizes the error with respect to a reference signal $d(t)$. If the signal prior knowledge is known, the receiver can generate a local reference signal which has a strong correlation with the desired signal. The main idea of MMSE is to adjust the weight vector in real time, so that the mean squared error between the array output signal and the reference signal can be minimized. The estimator is of the form

$$y = \boldsymbol{w}^H \boldsymbol{x}. \tag{52}$$

The cost function, i.e., the mean square value of the error signal is

$$J(\boldsymbol{w}) = E\left[\left|\boldsymbol{w}^H \boldsymbol{x} - d\right|^2\right]. \tag{53}$$

Expanding the right-side of Eq. (53) and $\boldsymbol{w}$ should be taken out of the expectation operator, $E[\cdot]$, because it is not a statistical variable, we get

$$J(\boldsymbol{w}) = \boldsymbol{w}^H E\left[\boldsymbol{x}\boldsymbol{x}^H\right]\boldsymbol{w} - E\left[d\boldsymbol{x}^H\right]\boldsymbol{w} - \boldsymbol{w}^H E[\boldsymbol{x}d^*] + E[dd^*]. \tag{54}$$

According to the Lagrange multiplier method, in order to minimize the mean squared error function, taking the derivative with respect to $\boldsymbol{w}$ of the above expression

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{w}} J(\boldsymbol{w}) &= 2E\left[\boldsymbol{x}\boldsymbol{x}^H\right]\boldsymbol{w} - 2E[\boldsymbol{x}d^*] \\ &= 2\boldsymbol{R}\,\boldsymbol{w} - 2\boldsymbol{r}_{xd}, \end{aligned} \tag{55}$$

where $\boldsymbol{r}_{xd}$ is the cross-correlation vector between the input signal and the reference signal. Set the above result equal to 0 and solve for $\boldsymbol{w}$, the optimal MMSE weights are

$$\boldsymbol{w}_{MMSE} = \boldsymbol{R}^{-1} \boldsymbol{r}_{xd}. \tag{56}$$

Since the reference signal is only related to the desired signal, and is not related to the interference signal and noise, therefore

$$\mathbf{r}_{xd} = E[\mathbf{x}d^*] = E[m_0\boldsymbol{a}(\theta_0)d^*] = E[m_0 d^*]\boldsymbol{a}(\theta_0) = p_0\boldsymbol{a}(\theta_0), \tag{57}$$

and according to the matrix inversion formula

$$\mathbf{R}^{-1} = \frac{(\mathbf{R}_i + \mathbf{R}_n)^{-1}}{1 + p_0\boldsymbol{a}(\theta_0)^H(\mathbf{R}_i + \mathbf{R}_n)^{-1}\boldsymbol{a}(\theta_0)}. \tag{58}$$

Substitute Eq. (57) and Eq. (58) into Eq. (56), we get

$$\mathbf{w}_{MMSE} = \frac{p_0}{1 + p_0\boldsymbol{a}(\theta_0)^H(\mathbf{R}_i + \mathbf{R}_n)^{-1}\boldsymbol{a}(\theta_0)}\mathbf{w}_{MSINR}. \tag{59}$$

From the above analysis, it can be seen that the received signal is correlated with the desired signal. Therefore, it is not required to decompose the received signal into the desired signal and interference signal, and the correlation of the received signal and the reference signal can be estimated by sampling, so it is not difficult to determine.

On the other hand, from Eq. (59) it can be shown that the MMSE beamformer $\mathbf{w}_{MMSE}$ is a scalar multiple of the Max-SINR beamformer $\mathbf{w}_{MSINR}$ in Eq. (51), i.e., the adaptive weights obtained by using the MMSE and Max-SINR criteria are proportional to each other. Since the multiplicative constants in adaptive weights do not matter, these two techniques are therefore equivalent.

5.3 Minimum variance

In the signal received by the array, the desired signal is the content of cooperative communication, and the interference is often unpredictable, so the form of the desired signal and DOA of the signal should be known. In this case, in order to detect the desired signal more efficiently, it is necessary to eliminate the clutter background. From Eq. (22)-(24) it is shown that the array output power includes three parts: desired signal power, interference power and noise power, while the interference and noise power can be considered as the variance of the desired signal error. The smaller the variance is, the more close is it to the expectation. Interference and noise power can be expressed as

$$P_{oi} + P_{on} = \boldsymbol{w}^H\boldsymbol{R}_i\boldsymbol{w} + \boldsymbol{w}^H\boldsymbol{R}_n\boldsymbol{w} \tag{60}$$

For array main-lobe (desired look direction), the unit gain is considered, that is

$$\begin{cases} \min\limits_{w} \quad \boldsymbol{w}^H\boldsymbol{R}_{i+n}\boldsymbol{w} \\ s.t. \quad \boldsymbol{w}^H\boldsymbol{a}(\theta_0) = 1 \end{cases} \tag{61}$$

Therefore, the minimum interference and noise variance is the choice of the appropriate **w**, using the Eq. (61) constraints, so that the Eq. (60) is minimized. The weight vector **w** that minimizes Eq. (60) subject to the constraint in Eq. (61) can be selected by using a vector Lagrange multiplier to form the modified performance measure. According to Lagrange multiplier method, the objective function is

$$L(\mathbf{w}) = \mathbf{w}^H\mathbf{R}_i\mathbf{w} + \mathbf{w}^H\mathbf{R}_n\mathbf{w} + \lambda\left(\mathbf{w}^H\boldsymbol{a}(\theta_0) - 1\right) \tag{62}$$

Setting the derivative of the above expression Eq. (62) with respect to $\mathbf{w}$ equal to zero to obtain optimal weight vector $\mathbf{w}_{MV}$based on minimum variance criteria, requiring $\mathbf{w}_{MV}$ to satisfy the constraint in Eq. (61) to evaluate μ, and substituting the resulting value of μ into $\mathbf{w}_{MV}$ gives the minimum variance weight vector solution

$$\frac{\partial}{\partial \mathbf{w}} L(\mathbf{w}) = 2(\mathbf{R}_i + \mathbf{R}_n)\mathbf{w} + \lambda \boldsymbol{a}(\theta_0) = 0 \tag{63}$$

Solution of the above equation yields the optimal weights vector by the minimum interference and the noise variance criterion.

$$\mathbf{w}_{MV} = \mu(\mathbf{R}_i + \mathbf{R}_n)^{-1}\boldsymbol{a}(\theta_0) = \mu \mathbf{w}_{MSINR} \tag{64}$$

According to the constraint conditions of the main beam, using the property that $(\mathbf{R}_i + \mathbf{R}_n)$ is the Hermitian matrix, can be obtained as

$$\mu = \frac{1}{\boldsymbol{a}^H(\theta_0)(\mathbf{R}_i + \mathbf{R}_n)^{-1}\boldsymbol{a}(\theta_0)} \tag{65}$$

When the snapshot data used to estimate $\mathbf{R}$ contains only the noise and interference environment, this processor is referred to as minimum variance distortionless response (MVDR). In the event, the desired signal is also present in the snapshot data, the same solution for the weight vector results, but is sometimes referred to as minimum power distortionless response (MPDR) to indicate the difference in the observed data [2]. In practice, the distinction makes a significant difference in terms of the required snapshot support to achieve good performance [18].

5.4 Minimum power

The formulation of the MV can be derived by minimizing the total output power of the array subject to the similar constraint of distortion-less response of Eq. (61). The total power of the output signal is considered, if the gain of the desired signal is kept fixed, that is the same as the constraint condition of Eq. (61), which is equivalent to the received power of the signal under the condition of ensuring the normal receiving of the desired signal while suppressing interference and noise power, the resultant criterion is defined as the minimum total output power of the array (MP). The cost function is

$$\begin{cases} \min\limits_{w} \quad \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w} \\ s.t. \quad \boldsymbol{w}^H \boldsymbol{a}(\theta_0) = 1 \end{cases} \tag{66}$$

Also using the method of Lagrange multiplier, the objective function to be minimized is

$$L(\boldsymbol{w}) = \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w} + \lambda\left(\boldsymbol{w}^H \boldsymbol{a}(\theta_0) - 1\right) \tag{67}$$

Taking the complex gradient with respect to $\boldsymbol{w}$ and setting to zero

$$\frac{\partial}{\partial \boldsymbol{w}} L(\boldsymbol{w}) = 2\boldsymbol{R}\boldsymbol{w} + \lambda \boldsymbol{a}(\theta_0) = 0 \tag{68}$$

Under this criterion, the optimal weight vector is

$$\boldsymbol{w}_{MP} = \kappa \boldsymbol{R}^{-1} \boldsymbol{a}(\theta_0) \tag{69}$$

where the constant (normalize the array main beam gain to unity) is

$$\kappa = \frac{1}{\boldsymbol{a}^H(\theta_0)\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_0)} \tag{70}$$

This criterion (MP) compared with the previously defined criterion (MV) is almost equivalent, since minimizing the total output power of the beamformer while preserving the desired signal is equivalent to minimizing the output power due to interference-plus-noise. The difference is only in the optimal weight vector of the MP criterion, and it is not necessary to separate the interference and noise, and only the covariance matrix of the received signal is estimated and thus the two optimization problems in Eq. (61) and Eq. (66) are equivalent.

5.5 Maximum likelihood criterion

Assume the space has only one desired signal and number of interference signals, the input signals can be expressed as

$$\boldsymbol{x} = m_0 \boldsymbol{a}_0 + \sum_{i=1}^{M} m_i \boldsymbol{a}_i + \boldsymbol{n} = m_0 \boldsymbol{a}_0 + \left(\sum_{i=1}^{M} m_i \boldsymbol{a}_i + \boldsymbol{n} \right) \tag{71}$$

If the interference signal and noise are zero mean Gaussian random process, the above equation is a Gaussian random process, and its mean is the desired signal $m_0\mathbf{a}_0$. The output signal is defined as the likelihood function vector

$$L(\boldsymbol{x}) = -\ln \left(P\left(\boldsymbol{x} \middle| \boldsymbol{x} = \sum_{i=1}^{M} m_i \boldsymbol{a}_i + \boldsymbol{n} \right) \right) \tag{72}$$

The expression of the conditional probability can be further changed to

$$L(\boldsymbol{x}) = c(\boldsymbol{x} - m_0\boldsymbol{a}_0)^{\mathrm{H}} (\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1} (\boldsymbol{x} - m_0\boldsymbol{a}_0) \tag{73}$$

where c is a constant independent of $\boldsymbol{x}$ and $m_0\boldsymbol{a}_0$. Taking derivative of the above expression with respect to m_0 and set the result equal to zero, we will get the maximum likelihood estimation m_0

$$\frac{\partial}{\partial m_0} L(\boldsymbol{x}) = -2\boldsymbol{a}_0^{\mathrm{H}} (\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1} \boldsymbol{x} + 2m_0 \boldsymbol{a}_0^{\mathrm{H}} (\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1} \boldsymbol{a}_0 = 0 \tag{74}$$

$$m_{0ML}(t) = \frac{\boldsymbol{a}_0^{\mathrm{H}} (\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1}}{\boldsymbol{a}_0^{\mathrm{H}} (\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1} \boldsymbol{a}_0} \boldsymbol{x} \tag{75}$$

The optimal weight vector is obtained by the above equation of the maximum likelihood criterion.

$$\boldsymbol{w}_{ML} = \frac{(\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1} \boldsymbol{a}_0^{\mathrm{H}}}{\boldsymbol{a}_0^{\mathrm{H}} (\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1} \boldsymbol{a}_0} \tag{76}$$

Compared with the weight vector solution under the Maximum Signal-to--Interferer-Noise Ratio (MSINR) criterion, the above expression can be rewritten as

$$\boldsymbol{w}_{ML} = \frac{1}{\boldsymbol{a}_0^{\mathrm{H}}(\boldsymbol{R}_i + \boldsymbol{R}_n)^{-1}\boldsymbol{a}_0}\boldsymbol{w}_{MSINR} \tag{77}$$

From Eq. (77) it is clear that, the ML beamformer $\mathbf{w}_{ML}$ is a scalar multiple of the Max-SINR beamformer $\mathbf{w}_{MSINR}$ in Eq. (51). i.e., the adaptive weights obtained using the ML and Max SINR criteria are proportional to each other. Since multiplicative constants in the adaptive weights have no impact on the array beampattern, these two techniques have no essential difference and are therefore equivalent.

6. Adaptive filtering algorithms

The expression of the optimal weight vector is obtained by solving the equations based on the optimization theory. In practical engineering, the optimal weight vector is obtained by the adaptive filtering algorithms. When there is a reference signal available, the reference signal may be the training sequence of the desired signal or the DOA information of the desired signal, the resultant technique is categorized as a non-blind adaptive spatial filtering. These classical adaptive algorithms include Direct Matrix Inversion (DMI) [19], Least Mean Square (LMS) [20–22], Recursive Least Square (RLS) [23–25], Conjugate Gradient (CG) and its improved algorithms [26, 27]. When there is no reference signal available, the optimal weight vector solution can be obtained by using other characteristics of the signal, the resultant techniques are categorized as blind adaptive spatial filtering. Blind algorithm mainly includes Constant Modulus (CM) algorithm [28–30], smooth circulation (Cyclo-stationary) algorithm [31], and High Order Cumulant (HOC) algorithm [32].

6.1 Direct matrix inversion algorithm

The basic idea of DMI algorithm is to compute the optimal weight vector directly instead of calculating it iteratively, based on an estimate of the correlation matrix $\boldsymbol{R} = E\left[\boldsymbol{x}(t)\boldsymbol{x}^H(t)\right]$ of the adaptive array output samples [33]. In communication systems, the signal source consists of a desired signal, interference and noise, therefore, the maximum SINR criterion, the minimum mean square error (MMSE) criterion, the minimum variance (MV) criterion and the maximum likelihood (ML) criterion need to know the covariance matrix of the interference signal and the noise signal, and do not contain the covariance matrix of the desired signal. So these criteria are not suitable for communication systems, and are suitable for radar systems, because it is easy to realize the interference and noise superimposed signal as long as the radar does not transmit the signal but only receives the signal.

For the MP criterion, the solution also needs the desired signal DOA, which is based on Eqs. (68) and (69), thus obtaining the desired signal direction vector $\boldsymbol{a}(\theta_0)$. On the other hand, unlike the MV criterion, the signal covariance matrix of MP criterion is the sum of the covariance matrices of the desired signal, the interference and the noise. Therefore, the MP criterion is suitable for the communication system.

Assume that there are P signals in the space, wherein, the desired signal is $\mathbf{s}_0 = m_0\boldsymbol{a}(\theta_0)$, the power is p_0, and the interference signals are $\mathbf{s}_1 = m_1\boldsymbol{a}(\theta_1), \dots, \mathbf{s}_P = m_P\boldsymbol{a}(\theta_{P-1})$ with power $p_1, \dots, p_{P-1}$, respectively. The noise vector is $\boldsymbol{n}$, and power is σ^2. According to the definition of covariance matrix

$$\begin{aligned}\mathbf{R} &= E\left[\left(\sum_{i=0}^{P-1}\mathbf{s}_i + \mathbf{n}\right)\left(\sum_{i=0}^{P-1}\mathbf{s}_i + \mathbf{n}\right)^H\right] \\ &= E\left[\left(\sum_{i=0}^{P-1} m_i\boldsymbol{a}(\theta_i) + \mathbf{n}\right)\left(\sum_{i=0}^{P-1} m_i\boldsymbol{a}(\theta_i) + \mathbf{n}\right)^H\right]\end{aligned} \tag{78}$$

Because the spatial separation between signal and interference is large enough, they are spatially uncorrelated. When sources are uncorrelated

$$E\left[\boldsymbol{a}(\theta_i)\boldsymbol{a}\left(\theta_j\right)^H\right] = 0 \quad i \neq j \tag{79}$$

At the same time

$$E\left[m_i^2\boldsymbol{a}(\theta_i)\boldsymbol{a}(\theta_i)^H\right] = p_i \tag{80}$$

$$E\left[\boldsymbol{n}\boldsymbol{n}^H\right] = \sigma^2 \tag{81}$$

Obviously, in practical applications, it is very difficult to estimate the covariance matrix by the respective amount of power, instead it can be estimated from samples of the received signal. DMI algorithm assumes that the covariance matrix has been estimated, and the expression $\mathbf{R}^{-1}$ is obtained by matrix inversion, combine with the known DOA, calculate the direction vector $\boldsymbol{a}(\theta_0)$, and the optimal weight vector solution is obtained by MP criterion.

Because the actual covariance matrix is not ideal, the performance of the DMI algorithm is affected by the eigen-value spread of the covariance matrix. The divergence is determined by the temporal and spatial correlation between the desired signal and the interference or between the interference and interference.

The optimal weight vector by DMI algorithm can be computed as:

The K snapshots constitute data matrix $\boldsymbol{X}$, the covariance matrix $\boldsymbol{R}$ is given as

$$\boldsymbol{R} = \frac{\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}}{K} \tag{82}$$

Directly estimate the covariance matrix and then by matrix inversion, obtain the inverse matrix $\boldsymbol{R}^{-1}$ combined with the desired signal direction vector, and the optimal weight vector is calculated according to Eq. (69).

$$\boldsymbol{w} = \frac{\boldsymbol{R}^{-1}\boldsymbol{a}_0}{\boldsymbol{a}_0^{\mathrm{H}}\boldsymbol{R}^{-1}\boldsymbol{a}_0} \tag{83}$$

DMI algorithm needs to choose suitable number of sampling snapshots K. When the number of snapshots K is sufficiently large, the covariance matrix $\boldsymbol{R}$ is more accurate, but larger number of sampling snapshots increases the computing load [34]. The major disadvantage of DMI algorithm is its computational complexity which makes it difficult to implement on FPGA and DSP. On the other hanf, the truncated finite number of computation makes the matrix inverse operation instable.

extremely simple and numerically robust.

6.2 Least mean square algorithm

The least mean square (LMS) algorithm proposed by Widrow et al. [20] is the most classical algorithm in signal processing. The LMS algorithm is extremely simple and numerically robust. More detailed description about the LMS algorithm is given in Ref. [18, 35]. The LMS algorithm is based on the method of steepest descent, and therefore sometime it is referred to as a Stochastic Gradient Descent (SGD) algorithm. The unconstrained LMS algorithm is a training sequence based adaptive spatial filtering algorithm which recursively compute and update the optimal weight vector. It uses the gradient search method to solve the weight vector, thus avoiding the direct matrix inversion of the covariance matrix. Its iterative equation is given as

$$\boldsymbol{w}(k+1) = \boldsymbol{w}(k) + \mu \boldsymbol{g}(\boldsymbol{w}(k)) \tag{84}$$

where $\boldsymbol{w}(k+1)$ represents the new weight vector computed at the $(k+1)^{th}$ iteration, $\boldsymbol{g}(\boldsymbol{w}(k))$ is the gradient vector of the squared error (objective function) with respect to the weight vector $\boldsymbol{w}(k)$, and the scalar constant μ is the step size parameter which controls the rate of convergence [33]. The gradient vector is given by

$$\boldsymbol{g}(\boldsymbol{w}(k)) = -2\boldsymbol{x}(k+1)\varepsilon^*(\boldsymbol{w}(k)) \tag{85}$$

where $\boldsymbol{x}(k+1)$ is the $k+1$ array snapshots, namely the $k+1$ array sample, and $\varepsilon^*(\boldsymbol{w}(k))$ is the error between the array output and the reference signal [33]. Thus, the estimated gradient vector is the product of the error between the array output and the reference signal, and the array signal received at the $k-th$ iteration. The error $\varepsilon^*(\boldsymbol{w}(k))$ can be expressed as

$$\varepsilon(\boldsymbol{x}(k)) = d(k+1) - \boldsymbol{w}^H(k)\boldsymbol{x}(k+1) \tag{86}$$

where $d(k+1)$ is the reference signal at the $(k+1)^{th}$ iteration. As one of the most classical adaptive filtering algorithms, ULMS has the advantage of computational simplicity and simple hardware requirement, but its convergence speed is relatively slow. In order to ensure the convergence of the algorithm, the iterative step size must meet the following condition [18, 20, 33–37].

$$0 < \mu < \frac{2}{\lambda_{\max}} \tag{87}$$

where $\lambda_{\max}$ denoted the largest eigenvalue of the received signal covariance matrix.

The algorithm is based on the gradient of the adaptive algorithm, which is an important feature of the gradient of the average value problem. The mean of the gradient estimate is expressed as

$$\overline{\boldsymbol{g}}(\boldsymbol{w}(k)) = 2\boldsymbol{R}\boldsymbol{w} - \boldsymbol{2r}_{xd} \tag{88}$$

In the iterative process of the algorithm, the gradient vector can be obtained by estimation. From the mean or expected value of the gradient estimate, the estimate is unbiased. At the same time, the estimation of the variance has also an effect on the performance of the algorithm. The variance is defined as

$$\xi(\boldsymbol{w}(k)) = E\left\{\left|d(k) - \boldsymbol{w}^H(k)\boldsymbol{x}(k+1)\right|^2\right\} \tag{89}$$

whose value is the error between the reference signal and the array output signal. From this, we can see that the Misadjustment of LMS algorithm is

$$MA = \mu tr\left\{[\boldsymbol{I} - \mu\boldsymbol{R}]^{-1}\boldsymbol{R}\right\} \tag{90}$$

The misadjustment defined as a ratio provides a measure of how close an adaptive algorithm is to optimality in the mean-square-error sense. The smaller the misadjustment, the more accurate is the steady-state solution of the algorithm. In other words, the difference between the weights estimated by the adaptive algorithm and optimal weights is further characterized by the ratio of the average excess steady-state MSE and the MMSE. It is referred to as the misadjustment. It is a dimensionless parameter and measures the performance of the algorithm. The misadjustment is a kind of noise and is caused by the use of noisy estimate of the gradient [38, 39].

From the above analysis, we can see that the LMS algorithm has different performance when choosing different steps and different covariance matrix estimation methods.

The basic steps of the LMS algorithm are as follows:

1. First initialize, $\boldsymbol{w}(0) = 0$, $k = 0$;

2. Iterative updates, so that $k = k + 1$;

$$e(k+1) = d(k+1) - \mathbf{w}^{\mathrm{T}}(k)\mathbf{x}(k+1)$$
$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu\mathbf{x}(k+1)e(k+1)$$

3. Stop iteration after the weight vector $\boldsymbol{w}(k)$ is convergent, so this time define$k = K$,$\boldsymbol{w}(K)$ is the desired weight vector.

Figure 5 shows the learning curve of the LMS algorithm with different step size parameters. It can be seen that when the step size parameter μ is small, the algorithm converges slowly, while the large value of step size parameter μ make the algorithm converge faster.

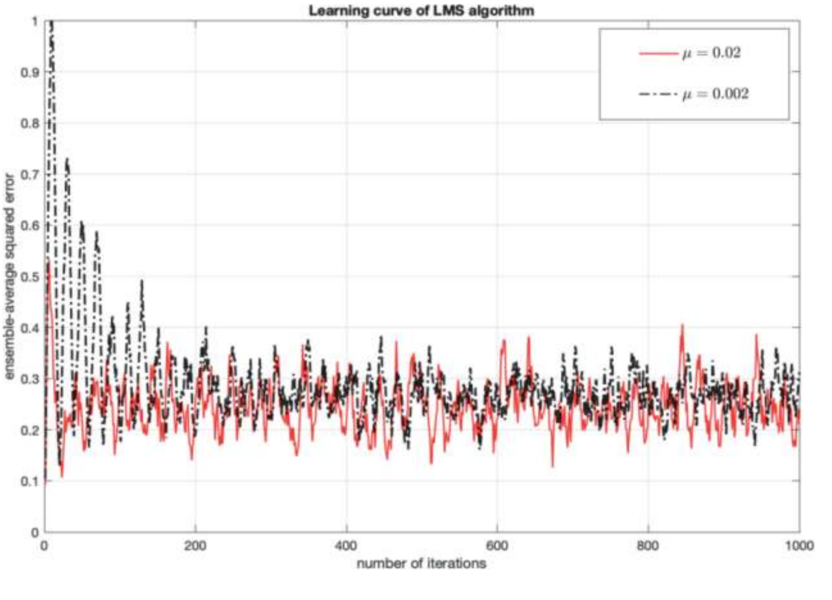

Figure 5.
Learning curve of the LMS algorithm.

The least mean square algorithm requires the training sequence, if the training sequence in the LMS algorithm is replaced by the DOA information of the desired signal, the Frost LMS algorithm can be obtained [40].

Iterative equation of the Frost LMS algorithm is

$$\boldsymbol{w}(k+1) = \boldsymbol{P}\{\boldsymbol{w}(k) - \mu\mathbf{g}(\boldsymbol{w}(k))\} + \frac{\boldsymbol{a}_0}{L} \tag{91}$$

where the matrix

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{a}_0\left(\boldsymbol{a}_0^{\mathrm{H}}\boldsymbol{a}_0\right)^{-1}\boldsymbol{a}_0^{\mathrm{H}} \tag{92}$$

and $\boldsymbol{g}(\boldsymbol{w}(k))$ is the gradient vector of the output signal power with respect to the weight vector $\boldsymbol{w}(k)$, and is given by

$$\boldsymbol{g}(\boldsymbol{w}(k)) = \boldsymbol{x}(k+1)y^*(k+1) \tag{93}$$

In the above equation, the output signal is given as

$$y(k+1) = \boldsymbol{w}^{\mathrm{H}}(k)\boldsymbol{x}(k+1) \tag{94}$$

Moreover, the initial value of the weights is given as

$$\boldsymbol{w}(0) = \frac{\boldsymbol{a}_0}{L} \tag{95}$$

In order to ensure the convergence of the iterative algorithm, the iterative step size still needs to meet the following conditions $\mu < 2/\lambda_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of the covariance matrix of the received signal.

Basic steps for the Frost LMS algorithm are as follows:

1. First initialize，$\boldsymbol{w}(0) = \frac{\boldsymbol{a}_0}{L}$，$k = 0$

2. Iterative updates, so that $k = k + 1$;

$$y(k+1) = \mathbf{w}^{\mathrm{H}}(k)\mathbf{x}(k+1);$$

$$\mathbf{w}(k+1) = \left(\mathbf{I} - \mathbf{a}_0\left(\mathbf{a}_0^{\mathrm{H}}\mathbf{a}_0\right)^{-1}\mathbf{a}_0^{\mathrm{H}}\right)\{\mathbf{w}(k) - \mu\mathbf{x}(k+1)y^*(k+1)\} + \tfrac{\mathbf{a}_0}{L};$$

3. Stop iteration after the weight vector $\boldsymbol{w}(k)$ is convergent, so this time define $k = K$, $\boldsymbol{w}(K)$is the desired weight vector.

The convergence rate of both the LMS algorithm and Frost LMS algorithm is associated with the step size parameter. Since, the eigenvalues of the received signal covariance matrix are not easy to obtain, the appropriate step size parameter cannot be chosen easily.

If the step size is too larger than twice the reciprocal of the maximum eigenvalue of the covariance matrix of the received signal, the weight vector diverges. Large μ's (step-size) speed up the convergence of the algorithm but also lower the precision of the steady-state solution of the algorithm. It should be noted that value of the step size must be less than twice the reciprocal of the maximum eigenvalue. Similarly, when the step-size is much less than twice the reciprocal of the maximum eigenvalue of the covariance matrix of received signals, the offset (steady state error) is small but the weight vector converges slowly.

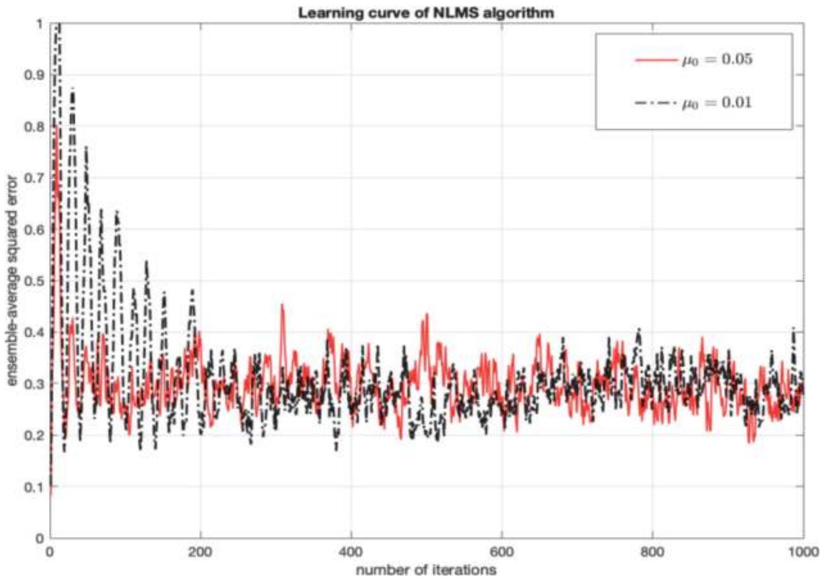

Figure 6.
Learning curve of the NLMS algorithm.

Another variant of the LMS family is the normalized LMS (NLMS) algorithm. This algorithm replaces the constant-step-size of conventional LMS algorithm with a data-dependent normalized step size at each iteration. At the k-th iteration, the step size is given by

$$\mu(k) = \frac{\mu_0}{x^H(k)x(k)} \tag{96}$$

where μ_0 is a constant. The .convergence of the NLMS algorithm is faster as compared to the LMS algorithm due to the data-dependent step size. **Figure 6** shows the convergence behavior of the NLMS algorithm with different μ_0.

One major advantage of the LMS algorithm is its simplicity, and when the step size is selected appropriately, the algorithm is stable (converged properly) and easy to be realized [21]. However, the LMS algorithm is sensitive to eigenvalues of the covariance matrix of received signals, and the convergence of the algorithm is poor when the eigenvalues are dispersed.

Various other variants of LMS algorithm are briefly discusses in [21]. In recent years, adaptive filtering algorithms have been extended into DOA estimation. DOA estimation based on adaptive filtering algorithms can be found in [41, 42].

6.3 Conjugate Gradient Method

The Conjugate Gradient Method (CGM) [43–45] proposed by Hestenes and Stiefel in 1952 (as direct method), is generally applied to the symmetric positive definite linear systems equations of the form $\boldsymbol{Aw} = \boldsymbol{b}$. In application of antenna arrays, the the weight vector computation by conjugate gradient method is discussed in [46]. Here, we have briefly outlined the conjugate gradient method (CGM) in application to beamforming [47].

In array signal processing, $\boldsymbol{w}$ represent the array weight vector, $\boldsymbol{A}$ is a matrix whose columns are corresponded to the consecutive samples obtained from array elements, while $\boldsymbol{b}$ is a vector containing consecutive samples of the desired signal. Thus, a residual vector

$$\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{w} \tag{97}$$

refers to the error between the desired signal and array output at each sample, with the sum of the squared error given by $\boldsymbol{r}^H\boldsymbol{r}$.

The process is started with weight vector $\boldsymbol{w}(0)$ as an initial guess, to get a residual

$$\boldsymbol{r}(0) = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{w}(0) \tag{98}$$

and the initial direction vector can be expressed as

$$\boldsymbol{g}(0) = \boldsymbol{A}^H\boldsymbol{r}(0) \tag{99}$$

Then moves the weights in this direction to yield a weight update equation

$$\boldsymbol{w}(k+1) = \boldsymbol{w}(k) + \mu(k)\boldsymbol{g}(k) \tag{100}$$

where the step size $\mu(k)$ is

$$\mu(k) = \frac{\left|\boldsymbol{A}^H\boldsymbol{r}(k)\right|^2}{\left|\boldsymbol{A}^H\boldsymbol{g}(k)\right|^2} \tag{101}$$

The residual $\boldsymbol{r}(k)$ and the direction vector $\boldsymbol{g}(k)$ are updated using

$$\boldsymbol{r}(k+1) = \boldsymbol{r}(k) + \mu(k)\boldsymbol{A}\boldsymbol{g}(k) \tag{102}$$

and

$$\boldsymbol{g}(k+1) = \boldsymbol{A}^H\boldsymbol{r}(k+1) - \alpha(k)\boldsymbol{g}(k) \tag{103}$$

with

$$\alpha(k) = \frac{\left|\boldsymbol{A}^H\boldsymbol{r}(k+1)\right|^2}{\left|\boldsymbol{A}^H\boldsymbol{r}(k)\right|^2} \tag{104}$$

A pre-determined threshold level is defined and the algorithm is stopped when the residual falls below the threshold level.

It should be noted that the direction vector points in the direction of error surface gradient $\boldsymbol{r}^H(k)\boldsymbol{r}(k)$ at the $k-th$ iteration, which the algorithm is trying to minimize. The method converges to the error surface minimum within at most K iterations for a K-rank matrix equation, and thus provides the fastest convergence of all iterative methods [46, 48].

6.4 Recursive least square algorithm

In order to further improve the convergence rate, a more sophisticated algorithm is recursive least square algorithm. RLS algorithm is based on the Recursive Least Squares Estimation (RLSE), which uses time average instead of statistical (ensemble) average or stochastic expectations. The RLS algorithm work well even when the eigenvalue spread of the input signal correlation matrix is large [49, 50]. So RLS algorithm has an advantage of insensitivity to variations in eigenvalue spread of the input correlation matrix [49, 50]. These algorithms have excellent

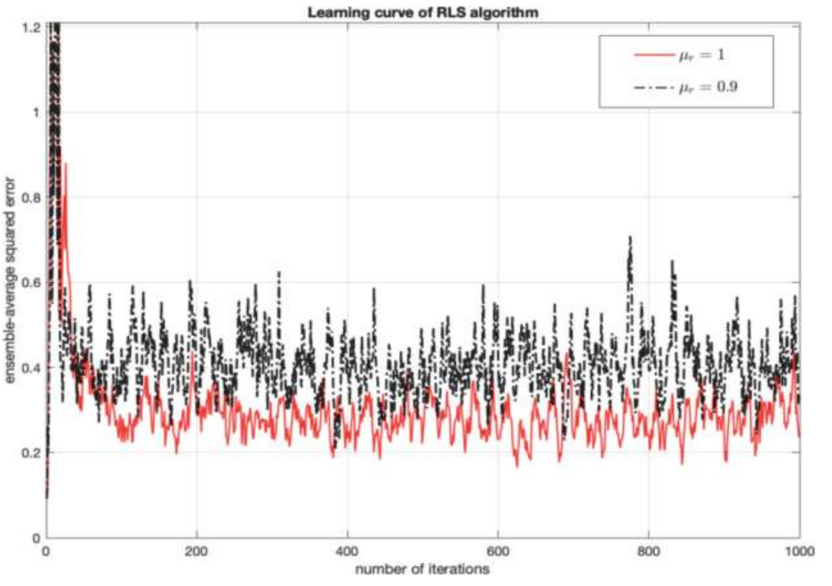

Figure 7.
Learning curve of the RLS algorithm.

performance when working in time-varying environments [49, 50]. Therefore, in the practical application, the forgetting factor μ is usually taken into account, and the optimal weight vector solution is slightly different. According to the optimal weight vector solution of MP criterion, the covariance matrix estimation is defined as

$$\boldsymbol{\Phi}(k) = \sum_{k=1}^{K} \mu^{K-k} \boldsymbol{x}(k)\boldsymbol{x}^H(k) \tag{105}$$

where the parameter μ should be chosen in the range $0 \ll \mu \leq 1$.

The above equation can also be expressed as

$$\boldsymbol{\Phi}(k) = \mu\boldsymbol{\Phi}(k-1) + \boldsymbol{x}(k)\boldsymbol{x}^H(k) \tag{106}$$

Using Matrix Inversion Lemma [14, 36, 51–54] (See Appendix A)

$$\begin{aligned} \boldsymbol{P}(k) &= \boldsymbol{\Phi}^{-1}(k) \\ &= \mu^{-1}\boldsymbol{\Phi}^{-1}(k-1) - \frac{\mu^{-2}\boldsymbol{\Phi}^{-1}(k-1)\boldsymbol{x}(k)\boldsymbol{x}^H(k)\boldsymbol{\Phi}^{-1}(k-1)}{1+\mu^{-1}\boldsymbol{x}^H(k)\boldsymbol{\Phi}^{-1}(k-1)\boldsymbol{x}(k)} \end{aligned} \tag{107}$$

Let

$$\mathbf{g}(k) = \frac{\mu^{-1}\boldsymbol{\Phi}^{-1}(k-1)\boldsymbol{x}(k)}{1+\mu^{-1}\boldsymbol{x}^H(k)\boldsymbol{\Phi}^{-1}(k-1)\boldsymbol{x}(k)} \tag{108}$$

then Eq. (106) can be expressed as

$$\boldsymbol{P}(k) = \mu^{-1}\boldsymbol{P}(k-1) - \mu^{-1}\boldsymbol{g}(k)\boldsymbol{x}^H(k)\boldsymbol{P}(k-1) \tag{109}$$

The iterative formula of the algorithm can be expressed as.

$$\begin{aligned}\boldsymbol{w}(k) &= \Lambda(k)\left[\mu^{-1}\boldsymbol{P}(k-1) - \mu^{-1}\boldsymbol{g}(k)\boldsymbol{x}^H(k)\boldsymbol{P}(k-1)\right]\boldsymbol{a}(\theta_0) \\ &= \left\{\frac{\Lambda(k)}{\mu\Lambda(k)-1}\left[\boldsymbol{I} - \boldsymbol{g}(k)\boldsymbol{x}^H(k)\right]\right\}\boldsymbol{w}(k-1)\end{aligned} \tag{110}$$

By taking different values of the K, the optimal weight vector recursion expression can be obtained. Compared with the LMS algorithm, RLS has a faster convergence rate, which is also a closed-loop adaptive algorithm.

The implementation of the RLS algorithm is carried out with different values of the forgetting factor μ. **Figure 7** shows the learning curves of the RLS algorithm. With the forgetting factor $\mu = 1$, the algorithm requires only 50 iterations to converge to its steady-state. It takes only 25 adaptation cycles to converge the RLS algorithm with a lower forgetting factor of $\mu = 0.9$.

7. Conclusion

In this chapter, we have introduced the basic principles and theoretical background of narrowband array signal processing. In particular, this chapter emphasized the fundamentals of narrowband signal processing exclusively used for the narrowband beamforming and DOA estimation. Furthermore, we reviewed the geometry of adaptive array antennas, the mathematical approaches for the development of signal models of the receiver array, and the selection criteria of the received signal processing technique, i.e. the criteria and guidelines related to adaptive filtering algorithms for solving the optimal weights. Considering the far-field narrowband signal using a uniform linear array as an example, the mathematical model is established in this chapter for the adaptive array antenna beamforming system. The basic theory of this chapter also laid a foundation for the theory of the wideband signal beamforming, which is then convenient for us to understand.

Appendix A

Matrix Inversion Lemma [52]: Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two positive-definite $N \times N$ matrices, $\boldsymbol{C}$ a $N \times M$ matrix, and $\boldsymbol{D}$ a positive definite $M \times M$ matrix. If they are related by

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}\boldsymbol{D}^{-1}\boldsymbol{C}^T,$$

then the inverse of the matrix $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1} - \boldsymbol{B}^{-1}\boldsymbol{C}\left(\boldsymbol{D} + \boldsymbol{C}^T\boldsymbol{B}^{-1}\boldsymbol{C}\right)^{-1}\boldsymbol{C}^T\boldsymbol{B}^{-1}$$

DOI: http://dx.doi.org/10.5772/intechopen.98702

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