On the Long-Run Equilibria of a Class of Large Supergames[*]

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Additional information is available at the end of the chapter

1. Introduction

This paper studies a broad class of large supergames, i.e.,infinitely repeated games played by (infinitely) many players. By relating each supergame to a relevant stochastic process of strategy configuration, we first investigate the existence, uniqueness and stability of invariant measures which represent the long-run equilibrium plays. Then we study the relationship between those invariant measures and the solution concepts evolved from the game theory literature.

In our stylized class of supergames, game players are located on the vertex set of a graph, typically the d-dimensional integer lattice, Z^d (see Fig.1-3). The spatial arrangement and the location of a particular player have no tangible restriction other than offering a convenient way to establish a neighborhood structure, since in the class of games which we study the individual players are assumed to be identical.

Each player plays a continent stage game, only with her neighbor, in each and every period of discrete times. Players may or may not have chances to change their strategy simultaneously at every period of time. We endow the graph with a pre-specified ordering according to which is the global updating rule. Supergames differ from one another possibly because of differences in the stage games or differences in the orderings.
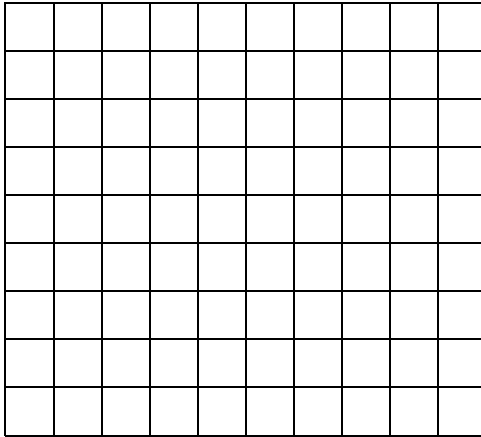
When a player is on her turn to update her strategy, the following features of bounded rationality are assumed:

(1) Although all players may employ mixed strategy, each player can only observes an track of the history of pure strategies of her relevant neighbors. In a series of studies Kalai and
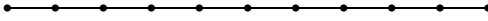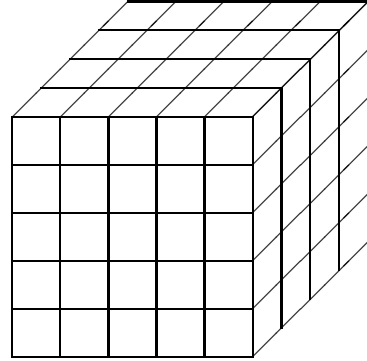

(a)   one dimensional lattice Z

(b) 2-dimensional rectangle lattice Z^2

(c) 3-dimensional cubic lattice Z^3

Figure 1. Lattices Z^d

Labra(1990[6],1991a[7],1991b[8]) investigated, in a finite-player supergame, the possibility for the players to learn to play Nash equilibria by keeping track of the history players, engaging in Baysian learning and taking best response policy. In this paper we do not intend to extend their results to large supergames although it may be an interesting topic. Relevant problems have been investigated in the literature of Cellular Automata (for example,see [4,10,11]).

(2) To make a strategy choice, the player only takes into account her neighbors' plays in the previous period. She is not sensitive enough to instaneously make response to the change in their neighbors' plays and to play based on her inference of her neighbors' current and future plays.

(3) The player may or may not have full control over her choices in the sense that she may or may not be able to use the best response strategy. The possibility of choosing strategy *a* versus strategy *b* depends on the difference in average payoffs accrued by playing *a* against her neighbors versus by playing *b*.

In summary, the class of games which we study in this paper is a class of infinitely repeated games with infinite numbers of players. Each player is directly connected only with her finite neighbors. The information flow is very close to the traditional open-loop setting.

In Section 2 we describe the general formulation of the class of supergames by offering the ingredients needed. We relate each game setting to a stochastic process which represents the evolution of the strategy configuration of the game. This enables us to investigate the evolution of the supergame by looking at the invariant measure of the relevant stochastic process. For the whole class of games we first prove the existence of invariant measures and then study the relationship among the concepts of invariant measures, reversible measures, ergodic measures, Gibbs measures and Markovian random fields.

In Section 3, we derive invariant measures for some special supergames. We assume that each person located on the vertices of the rectangle lattice plays two-person two strategies games with her 4 neighbors simultaneously. The payoff function is symmetric. We prove some results on the existence of ergodic measures. In this Section, we also investigate another class of supergames. We assume that each person located on the vertices of the rectangle lattice plays four 4-person team games or 4-person 2-pairs team games simultaneously with her neighbors. The formula of invariant measure which represents the long-run equilibrium plays with symmetric payoffs is obtained.

Section 4 is the conclusion. Some speculations on possible future research of other type supergames on different lattices are discussed.

2. General formulation of a class of large supergames

This section is devoted to describe the general formulation of a class of supergames. Subsection 2.1 presents the ingredients. Subsection 2.2 introduces the strategy evolution process(SEP). Subsection 2.3 depicts all the subclass of games. Subsection 2.4 contains general results on the existence of invariant distribution, reversibility and ergodicity of the SEP.

2.1. Ingredients

The class of large supergames which we investigate in this paper has the following ingredients:

The players:

We assume that players are located on the sites of a graph $G = \{V, E\}$, where V is the vertex set and E the edge set of the graph. In this work we assume that V is a lattice, usually Z^d or its finite sublattice. We also assume that all the players are identical.

Neighborhood:

A neighborhood system $\mathcal{N} = \{N_i; \ i \in V\}$ is assigned with each specific model. $\mathcal{N}$ is a collection of nonempty subsets of the vertices of V such that

(i) i does not belong to N_i for all $i \in V$;

(ii) $i \in N_j$ if and only if $j \in N_i$, for all $i, j \in V$;

N_i is called the neighborhood of i. We define the set $W_i = N_i \cup \{i\}$. A clique C of V is either a single vertex or a nonempty subset of vertex set of V such that all vertices that belong to C are neighbors of each other.

To simplify the model we assume that the neighborhood structure is translation invariant. For example, in Z^d cases, the invariance of the neighborhood system means that $i \in N_j$ if and only if $i + k \in N_{j+k}$. Since $N_i = (i - j) + N_j$ for any $i, j \in Z^d$, from $N_i = i + N_0$, which 0 is the origin vertex of Z^d, we deduce that all the information of the neighborhood structure is contained in N_0. For $V \subset Z$, the neighborhood N_0 is usually taken as $N_0 = \{-s, -s+1, \cdots, -1, +1, \cdots, s\}$ for some $s > 0$. For $V \subset Z^2$, the two commonest choices for N_0 are the following

$N_0^N = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$ the von Neumann neighborhood,

$N_0^M = N_0^N \cup \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ the Moore Neumann neighborhood.

Stage games:

In our class of supergames, some stage games are played over discrete time $t \in Z^d$. At each discrete time every player plays a finite-strategy n-person game simultaneously with his neighbors. Mixed strategy is used in general. At the end of each stage game, every player obtains information about the pure strategies that his neighbors took in the finished game and then may revise his strategy, under some global ordering of updating, for the next game according to these information and the payoff that he received. Then the game is repeated again.

Strategy and payoff:

Let A_i be the finite set of all possible pure strategies that the player i can take and assume that $A_i = A$ for all $i \in V$. In a n-person game, let $Q(x_1, \cdots, x_n), x_1, \cdots, x_n \in A$ be the payoff to the player who plays $x_1 \in A$, when his relevent neighbors play $(x_2, \cdots, x_n) \in A^{n-1}$. We assume $Q(x_1, \cdots, x_n)$ that is invariant under any permutation of $(x_2, \cdots, x_n)$.

Ordering of strategy changes:

At each period of time, all players may or may not update their strategies simultaneously. Instead, associated with each game is an ordering according to which the players change their strategies. Such an ordering over V which is pre-specified as in extensive form games, is represented by the global updating rule.

The global updating rule will be called *synchronous* if all the players change their strategies simultaneously at the same time; *sequential*, if they change their strategies one by one under a fixed ordering; *group − sequential*, if the players within a group change their strategies simultaneously at the same time, but different groups change their strategies one group at a time under a fixed ordering; and *asynchronous* if at a given time only one player - selected by random with uniform probability - updates his strategy. The *sequential* and *asynchronous* updating rules are applicable only for the case with finite players(i.e., V is finite).

2.2. Strategy evolution process(SEP)

The dynamics of a supergame is characterized by a stochastic process which is called strategy evolution process(SEP) in this paper. Technically, the SEP for a large supergame is a Markov chain whose state at time t is denoted by $\mathbf{X_t} = \{X_{t,i}; i \in V\}$. It takes value over $\Omega_t = \Omega = A^V$ which is called configuration space of the SEP at time t. $\mathbf{x_t} = \{x_{t,i}; i \in V\}$ is the realization of $\mathbf{X_t}$. Equivalently we may model the state of SEP at time t by a probability distribution μ_t

on A^V. Suppose that the configuration $\mathbf{x_{t-1}}$ determines the strategy of player i at time t with probability(called local transition probability)

$$p_i(x_{t,i}|\mathbf{x_{t-1}}) = p_i(x_{t,i}|x_{t-1,j}; j \in W_i) \tag{2.1}$$

Note that

$$\sum_{x_{t,i}\in A} p_i(x_{t,i}|x_{t-1,j}; j \in W_i) = 1, \ for \ all \ i \in V. \tag{2.2}$$

Let $P(\mathbf{y}|\mathbf{x})$ be the global one-step transition probabilities from $\mathbf{x}$ to $\mathbf{y}$. They are defined for different global updating modes as follows, respectively.

(i)Synchronous: the global transition probabilities of the SEP are defined by

$$P(\mathbf{x}_t|\mathbf{x}_{t-1}) = \prod_{i\in V} p_i(x_{t,i}|x_{t-1,j}; \ j \in W_i) \tag{2.3}$$

(ii)Group-sequential:

In this work we discuss a specific mode of group-sequential rules, say even-odd sequential rule for Z^d model. We partition the lattice Z^d into two disjoint equivalent sublattices, say V^E and V^O, arranged so that the nearest neighbor sites lie in different sublattices. The player $i \in V^E$ updates his strategy at even time t with the local transition probability given by (2.1). Then the global transition probabilities for the SEP is given by

$$P(\mathbf{x}_t|\mathbf{x}_{t-1}) = \begin{cases} \displaystyle\prod_{i\in V^E} p_i(x_{t,i}|x_{t-1,j}; j \in W_i), \ x_{t,i} = x_{t-1,i}, \ \text{for any } i \in V^O \\ 0, \hspace{5cm} \text{otherwise} \end{cases} \tag{2.4}$$

If t is odd, the updating rule is obtained by reversing the rules of V^E and V^O.

(iii)Asynchronous:

In this case, we assume that V is finite, $\|V\| = M$. We denote by $\mathbf{x}(i,y)$ the configuration that is identical to $\mathbf{x}$, except the strategy of player i is $y \in A$. Then

$$P(\mathbf{x}|\mathbf{y}) = P(\mathbf{X}_t = \mathbf{x}|\mathbf{X}_{t-1} = \mathbf{y})$$

$$= \begin{cases} \frac{1}{M}\sum_{i\in V} P_i(X_{t,i} = x_i|\mathbf{X}_{t-1} = \mathbf{x}), & \text{if } \mathbf{y} = \mathbf{x} \\ \frac{1}{M}P_i(X_{t,i} = x_i|\mathbf{X}_{t-1} = \mathbf{x}(i,y)), & \text{if } \mathbf{y} = \mathbf{x}(i,y) \neq \mathbf{x} \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

2.3. Subclasses

In the remaining of this section, we prove some results which apply to the whole class. The class of supergames that we study is rather broad. All the subclasses can be represented in the following chart:

The ordering pattern of strategy updating	Number of persons in a stage game		
	two	four	four in two pairs
Synchronous	yes	yes	yes
Asynchronous	yes	yes	yes
Group-sequential	yes	yes	yes

Each cell in the chart can be further divided into four subcells according to homogeneity and symmetry of the payoff. For detail see the next section.

2.4. Invariant measures, ergodicity and reversibility

We are interested in the condition on the local transition probability for the existence and the uniqueness of the invariant measure, the ergodicity and reversibility of the SEP. For the same local transition rule given by (2.1), what are the differences between the invariant measures for the SEP with different type of global updating rules? In certain cases there may exist multiple invariant measures. This phenomena is called phase transition.

We are also interested in the inverse problem-for a given distribution π on A^V find all the SEP with π as their invariant measures, specifically when π is Gibbsian.

The answers of these problems vary for different types of transition and updating rules. Finite V or infinite V will imply different results too. We will discuss them separately in the next section.

The global transition probabilities (2.3), (2.4) or (2.5) defines a discrete-time Markov process on the configuration space A^V. Given a measure ρ_{t-1} on the configuration $\mathbf{x}_{t-1}$ (2.3), (2.4) or (2.5) defines a probability measure $\rho_t = \rho_{t-1}P$ on $\mathbf{x}_t$.

$$\rho_t(d\mathbf{x}_t) = \int \rho_{t-1}(d\mathbf{x}_{t-1})P(d\mathbf{x}_t|\mathbf{x}_{t-1})$$

We say that a measure ν is *stationary* or time *invariant* if $\nu = \nu P$. The following result is well known.

Lemma 2.1. *The invariant measures for the time evolution form a nonempty convex set.*

Proof: See [9].

For the SEP with certain type of updating rule we define the following. A SEP is *ergodic* if the chain is regular, i.e., it has a unique invariant measure which almost surely describes the limit behavior of the SEP. A SEP will be called *Gibbsian*, if its invariant measure corresponds to the probability distribution of a Markov random field(MRF) on A^V, i.e.,

$$\lim_{t \longrightarrow \infty} Pr(\mathbf{X}_t = \mathbf{x}) = \pi(\mathbf{x}) = \frac{1}{\Lambda}exp[-U(\mathbf{x})], \tag{2.6}$$

with $\pi(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega$, Λ the normalization constant, and $U(\mathbf{x}) = \sum_C v_C(\mathbf{x})$, where the summation is taken over the cliques of V, and where the function $v_C(.)$ is called potential

function. We call a SEP *reversible* if the corresponding chain is reversible. It is well known that the reversibility is equivalent to the detailed balance condition.

$$\pi(\mathbf{y})P(\mathbf{x}|\mathbf{y}) = \pi(\mathbf{x})P(\mathbf{y}|\mathbf{x}). \tag{2.7}$$

Any reversible probability distribution is invariant since (2.7) implies that

$$\pi(\mathbf{x}) = \sum_{\mathbf{y}} \pi(\mathbf{y})P(\mathbf{x}|\mathbf{y}). \tag{2.8}$$

Theorem 2.1. *A SEP is Gibbsian if and only if it is reversible.*

Proof: See [9].

3. Invariant measures for some special models

In this section we study the above-mentioned problems in detail and depth for some special game setting. The discussion is organized according to the number of players in a game.

3.1. Two-person game

In this subsection we discuss the case that the players are located on V which may be Z^d or its finite sublattice with the neighborhood structure of von Neuman type, i.e., $N_0 = \{\mathbf{j} = (j_1, \cdots, j_d); |j| = \sum_{k=1}^{d} j_k^2 = 1\}$. Every player plays a q-strategy two person game simultaneously with each of his nearest neighbors (Z^2 case see Fig. 2). We denote by $\mathbf{Q}_{ij} = \{Q_{ij}(x,y); x, y \in A\}$ the payoff matrix of player i playing with player j. $\mathbf{Q}_{ij}$ is called symmetric if $Q_{ij}(x,y) = Q_{ji}(y,x)$ for all $x, y \in A$.
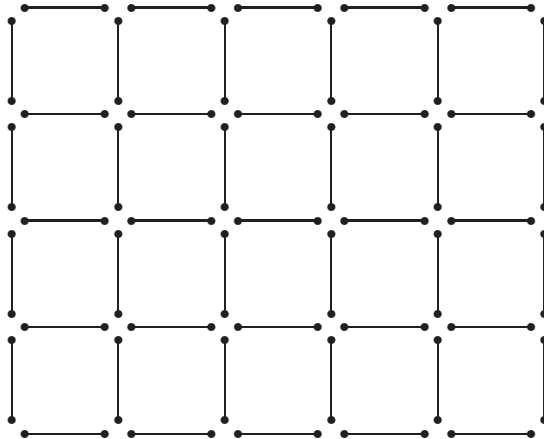

Figure 2. Supergame based on basic two-person games on Z^2

A game played by all nearest pairs on V is called homogeneous if the same payoff function is assigned for all pair players. Otherwise, it is called nonhomogeneous. We will discuss them separately.

At the end of each game, player i receives payoff $Q_{ij}(y,x)$ if he plays strategy y while his neighbor j plays strategy x; so his total payoff from playing strategy y is the sum of the payoffs received from playing y against each of his neighbors. Then player i may revises his strategy from y to z, under the given global updating rule, for the next game with probability

$$p_i(z|\mathbf{x}(i,y)) = \frac{1}{\lambda}exp\{\beta\frac{1}{\|N_i\|}\sum_{j\in N_i}[Q_{ij}(z,x_j) - Q_{ij}(y,x_j)]\}$$

$$= \frac{1}{\lambda'}exp\{\beta\frac{1}{\|N_i\|}\sum_{j\in N_i}Q_{ij}(z,x_j)\}. \tag{3.1}$$

where

$$\lambda = \sum_{z\in A}exp\{\beta\frac{1}{\|N_i\|}\sum_{j\in N_i}[Q_{ij}(z,x_j) - Q_{ij}(y,x_j)]$$

$$\lambda' = \sum_{z\in A}exp\{\beta\frac{1}{\|N_i\|}\sum_{j\in N_i}Q_{ij}(z,x_j)\}$$

$\|N\|$ is the cardinality of set N. Note that both λ and λ' depend on $x^{N_i} = \{x_j : j \in N_i\}$. We may write them by $\lambda(x^{N_i})$ and $\lambda'(x^{N_i})$. Roughly speaking, the probability the player i switches his strategy from y to z is proportional to the utility differences for these two strategies. We will discuss three cases.

3.1.1. Homogeneous game with symmetric payoff

In this case all the payoff functions equals to $\mathbf{Q}$ which is symmetric. The local transition probability will read as

$$p_i(z|\mathbf{x}(i,y)) = \frac{1}{\lambda}\exp\{\beta\frac{1}{\|N_i\|}\sum_{j\in N_i}[Q_i(z,x_j) - Q_i(y,x_j)]\}$$

$$= \frac{1}{\lambda'}\exp\{\beta\frac{1}{\|N_i\|}\sum_{j\in N_i}Q_i(z,x_j)\}. \tag{3.2}$$

To discuss different global updating rule we denote $V_n = \{\mathbf{j} = (j_1,\cdots,j_d) \in Z^d; |j_k| \leq n$ for $1 \leq k \leq d\}$ and assume that V is some finite cubic with this type. $\|V\| = M$.

(i)Asynchronous and even-odd sequential cases

Theorem 3.1. *Consider a large homogeneous supergame with finite players located on a lattice V. The payoff matrix of a two-person game is symmetric. Then the SEP whose asynchronous global transition probability is given by (2.5) with the above local transition rule (3.2) and the SRP whose even-odd sequential global transition probability is given by (2.4) with the above local transition rule (3.2) have the following distribution on A^V as their reversible invariant measure*

$$\pi(\mathbf{x}) = \frac{1}{\Lambda}exp\{\beta\frac{1}{\|N_0\|}\sum_{<i,j>}Q(x_i,x_j)\} \tag{3.3}$$

where the summation is taken over all nearest neighboring pairs of players, and

$$\Lambda = \sum_{\mathbf{x}} \exp\{\beta \frac{1}{\|N_0\|} \sum_{<i,j>} Q(x_i, x_j)\}$$

Proof: We only need to check (2.7) for $\mathbf{y} = \mathbf{x}(i, y)$ for the asynchronous case and $\mathbf{y}$ whose even (or odd) coordinates are different from those of $\mathbf{x}$ for even-odd sequential case.

(a)Asynchronous cases

$$\pi(\mathbf{x})P(\mathbf{x}(i,y)|\mathbf{x})$$
$$= \frac{1}{\Lambda} \exp\{\frac{\beta}{\|N_0\|} [\sum_{<j,k>,j,k\neq i} Q(x_j, x_k) + \sum_{j\in N_i} Q(x_i, x_j)]\} \cdot \frac{1}{M\lambda'(x^{N_i})} \exp\{\frac{\beta}{\|N_0\|} \sum_{j\in N_i} Q(y, x_j)\}$$
$$= \frac{1}{\Lambda} \exp\{\frac{\beta}{\|N_0\|} [\sum_{<j,k>,j,k\neq i} Q(x_j, x_k) + \sum_{j\in N_i} Q(y, x_j)]\} \cdot \frac{1}{M\lambda'(x^{N_i})} \exp\{\frac{\beta}{\|N_0\|} \sum_{j\in N_i} Q(x_i, x_j)\}$$
$$= \pi(\mathbf{x}(i,y))P(\mathbf{x}|\mathbf{x}(i,y))$$

(b)Even-odd sequential cases

We denote by $\mathbf{x}(E, y^E)(\mathbf{x}(O, y^O))$ for the configuration whose even components equals $y^E = \{y_j, j \in V^E\}$, while the odd components equal $y^O = \{x_i, i \in V^O\}$(vice versa for $\mathbf{x}(O, y^O)$). We need to prove (2.7) for $\mathbf{y} = \mathbf{x}(E, y^E)$(or $\mathbf{x}(O, y^O)$). In fact

$$\pi(\mathbf{x})P(\mathbf{x}(E, y^E)|\mathbf{x})$$
$$= \frac{1}{\Lambda} \exp\{\frac{\beta}{\|N_0\|} \sum_{<j,k>} Q(x_j, x_k)\} \prod_{i\in V^E} \frac{1}{\lambda'(x^{N_i})} \exp\{\frac{\beta}{\|N_0\|} \sum_{j\in N_i} Q(y, x_j)\}$$
$$= \frac{1}{\Lambda} \exp\{\frac{\beta}{\|N_0\|} \sum_{<i,j>,i\in V^E, j\in V^O} Q(y_i, x_j)\} \prod_{i\in V^E} \frac{1}{\lambda'(x^{N_i})} \exp\{\frac{\beta}{\|N_0\|} \sum_{j\in N_i} Q(x_i, x_j)\}$$
$$= \pi(\mathbf{x}(E, y^E))P(\mathbf{x}|\mathbf{x}(E, y^E)).$$

(ii)Synchronous cases

The global transition probability is given by (2.3) with the above local transition rule (3.2). The above distribution $\pi(\cdot)$ in (3.3) is not the invariant distribution of this SEP.

For even-odd sequential and asynchronous models, it has been realized that there is an intimate relation between d-dimensional time evolution and equilibrium statistical model(ESM) in $(d + 1)$-dimension, the extra dimension being the discrete time ([3], [9]). In fact, we can consider $\underline{\mathbf{x}} = \{\mathbf{x}_t\}_{t\in Z}$ as a configuration on the space-time lattice Z^{d+1}.

It is easy to see that if the transition probability $p_i(x_{t,i}|\mathbf{x}_{t-1})$ are all strictly positive, then μ_ν is a Gibbsian measure on $A^{Z^{d+1}}$. When V is infinite there are various ways to define finite-volume Gibbs states in the thermodynamical limit, yielding the space-time measure μ_ν of the time evolution as an infinite-volume Gibbs measure. From the theory of ESM, it is important to

note that there may exist more than one Gibbs measure on $A^{Z^{d+1}}$ which indicates the existence of more than one stationary or periodic measure ν for the time evolution as a phase transition.

Example : binary strategy game

We assume that each player has only two choices of strategies which may be identified as {-1,+1} for simplicity and convenience. The payoff matrix is given by

$$Q = \begin{pmatrix} Q(+1,+1) & Q(+1,-1) \\ Q(-1,+1) & Q(-1,-1) \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \tag{3.4}$$

Or we may write $Q(x,y)$ in the following form

$$Q_i(x,y) = Q(x,y) = Jxy + K(x+y) + L \qquad \text{for all } i \in I \tag{3.5}$$

where J, K and L are uniquely determined by a, b and d; and vice versa.

$$Q(+1,+1) = a = J + 2K + L$$

$$Q(+1,-1) = Q(-1,+1) = b = -J + L$$

$$Q(-1,-1) = d = J - 2K + L;$$

or

$$J = \frac{1}{4}(a - 2b + d)$$

$$K = \frac{1}{4}(a - d)$$

$$L = \frac{1}{4}(a + 2b + d).$$

(i)Asynchronous case

The invariant probability measure can be written as the following form

$$\pi(\mathbf{x}) = \frac{1}{\Lambda} \exp\{\beta \frac{1}{\|N_0\|} \sum_{<i,j>} Q(x_i, x_j)\}$$

$$= \frac{1}{\Lambda} \exp\{\beta \frac{1}{\|N_0\|} \sum_{<i,j>} [Jx_i x_j + K(x_i + x_j) + L]\}$$

$$= \frac{1}{\Lambda'} \exp\{\tilde{J} \sum_{<i,j>} x_i x_j + \tilde{K} \sum_{i \in V} x_i\} \tag{3.6}$$

where $\tilde{J} = \frac{\beta J}{\|N_0\|}$ and $\tilde{K} = \frac{4\beta K}{\|N_0\|}$. This is the well known Ising model. From standard point of view $\tilde{J}$ represents local pair interaction, and $\tilde{K}$ represents the global interactions. It is interesting to notice that the payoff matrix **Q** contains information of both local and global

interactions. This is not surprising because the game is homogeneous, i.e., the same payoff matrix is assigned for all two-person games.

(ii)Even-odd sequential case

The invariant measure has same form of (3.6). It is well known that this model exhibits the phenomena of phase transition in certain cases when V is infinite. To see this we first consider the SRP with even-odd sequential updating rule for each V_n and denote by $\pi_n(\cdot)$ the corresponding invariant measures. Then let $n \longrightarrow \infty$, V_n spreads to cover the whole lattice Z^d. It is well known that there is a unique limiting distribution of sequence $\{\pi_n\}$ when $d = 1$. While $d = 2$ the situation becomes complex but more interesting.

There is no phase transition whenever $\tilde{K} \neq 0$, i.e., there exists only one invariant measure. When $\tilde{K} = 0$(i.e., $a = d$) there is a value $\tilde{J}_c > 0$ called critical value such that, for $0 \leq \tilde{J} \leq \tilde{J}_c$, no phase transition occurs; but if $\tilde{J} > \tilde{J}_c$, phase transition does occur. The critical value is about $\tilde{J}_c = 0.44$(see [1],[9]). Notice that $a = d$ means $Q(+1, +1) = Q(-1, -1)$, and $J = \frac{1}{2}(a - b)$. So when a is sufficiently larger than b, phase transition occurs.

Furthermore, there exist only two extreme distributions π^+ and π^- which are all transition invariant in 2-dimension such that all other invariant measure π can be expressed as convex combination of them, i.e.,

$$\pi = p\pi^+ + (1 - p)\pi^-$$

where $0 \leq p \leq 1$(see [1]). The marginal distributions of π^+ and π^- at single site satisfy

$$\pi^+(+1) = \pi^-(-1) > 2/3$$

(see [9]). This means that under $\pi^+(\pi^-)$, all players favor strategy +1(-1). Note that there are two Nash Equilibrium states for which all players play strategy +1 or -1.

For $d \geq 3$, the phase diagrams are more complex and not completely known.

(iii)Synchronous case

To find the invariant measure is, in general, a difficult task.

Remarks: For the SEP of a large homogeneous supergame with asymmetric payoff under different global updating rule with the local transition probability given by (3.1), the invariant Gibbsian measure may not exist, in general. But for some special payoff it exists. Also for binary strategy game it always exists.

3.1.2. Nonhomogeneous game with symmetric payoff

Now we consider the case of nonhomogeneous game with symmetric payoff. We assign different payoff function $Q_{ij}(x, y)$ for different pairs of players i and j, but assume that all payoff are symmetric, i.e., $Q_{ij}(x, y) = Q_{ji}(y, x)$. Then we have similar results. For the SEP with synchronous and even-odd sequential global updating rule associated with the local transition probability (3.1) the invariant measure is then given by

$$\pi(\mathbf{x}) = \frac{1}{\Lambda} \exp\{\frac{\beta}{\|N_0\|} \sum_{<i,j>} Q_{ij}(x_i, x_j)\}$$

3.2. Four-person team game

We return to the model on Z^2. This time we assume that the four players located on the vertices of a basic square $\square = \{(i_1, i_2), (i_1, i_2 + 1), (i_1 + 1, i_2 + 1), (i_1 + 1, i_2)\}$ form a team to play a four-person team game. Let $\square(i, j, k, l)$ to be the square with vertices i, j, k and l in clockwise order. Denote by S_i the set of basic squares which contain vertex i, and S the set of all basic square.
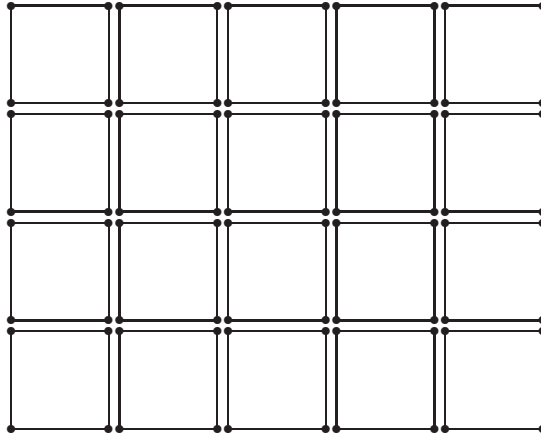

Figure 3. Supergame based on 4-person team games

Each player is a member of four four-person teams consisting of his neighbors. At each time every player plays finite-strategy four-person team games with four neighboring teams simultaneously(see Fig. 3).

Suppose that the payoff function $\mathbf{Q} = \{Q(x, y, z, u); x, y, z, u \in A\}$ is symmetric. The local transition probability for the SEP is given by

$$
p_i(z|\mathbf{x}(i, y)) = \frac{1}{\lambda} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l) \in S_i} [Q_i(z, x_j, x_k, x_l) - Q_i(y, x_j, x_k, x_l)]\}
$$

$$
= \frac{1}{\lambda'} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l) \in S_i} Q_i(z, x_j, x_k, x_l)\} \tag{3.7}
$$

where.

$$
\lambda = \sum_{z \in A} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l) \in S} [Q_i(z, x_j, x_k, x_l) - Q_i(y, x_j, x_k, x_l)]\}
$$

$$
\lambda' = \sum_{z \in A} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l) \in S_i} Q_i(z, x_j, x_k, x_l)\}.
$$

We discuss homogeneous game with symmetric payoff only and claim that for the SEP with asynchronous global updating rule associated with the local transition probability (3.7). The

invariant measure is given by

$$\pi(\mathbf{x}) = \frac{1}{\Lambda} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l)\in S} Q(x_i, x_j, x_k, x_l)\}. \tag{3.8}$$

Example: binary strategy game

We assume that each player has only two choices of strategies which may be identified as $\{+1, -1\}$. The payoff function is symmetric. We denote

$$Q(x,y,z,u) = Hxyzu + I(xyz + yzu + zux + xyu)$$
$$+ J(xy + xz + xu + yz + yu + zu) + K(x + y + z + u) + L$$

where H, I, J, K and L are uniquely determined by a, b, c, d and e and vice versa.

$$Q(+1, +1, +1, +1) = a = H + 4I + 6J + 4K + L$$
$$Q(+1, +1, +1, -1) = b = -H - 2I + 2K + L$$
$$Q(+1, +1, -1, -1) = c = H - 2J + L$$
$$Q(+1, -1, -1, -1) = d = -H + 2I - 2K + L$$
$$Q(-1, -1, -1, -1) = e = H - 4I + 6J - 4K + L$$

or

$$H = \frac{1}{16}(a - 4b + 6c - 4d + e)$$
$$I = \frac{1}{16}(a - 2b + 2d - e)$$
$$J = \frac{1}{16}(a - 2c + e)$$
$$K = \frac{1}{16}(a + 2b - 2d - e)$$
$$L = \frac{1}{16}(a + 4b + 6c + 4d + e)$$

The invariant measure $\pi(\mathbf{x})$ is a Ising model with one-site, two-site, three-site and four-site intersections.

$$\pi(\mathbf{x}) = \frac{1}{\Lambda} \exp\{\tilde{H} \sum_{\square(i,j,k,l)\in S} x_i x_j x_k x_l + \tilde{I} \sum_{\Delta(i,j,k)\in T} x_i x_j x_k + \tilde{J} \sum_{<i,j>} x_i x_j + \tilde{K} \sum_{i\in I} x_i\} \tag{3.9}$$

where $\tilde{H} = \frac{\beta H}{4}$, $\tilde{I} = \frac{4\beta I}{4}$, $\tilde{J} = \frac{6\beta J}{4}$ and $\tilde{K} = \frac{4\beta K}{4}$; $\Delta(i, j, k)$ the triangle with the vertices i, j and k.

3.3. Four-person two-pair game

We consider another type of 4-person game for Z^2 model. This time we assume that four players within a basic square form two pairs along the diagonal lines (example includes

bridge-a card play). Therefore each player plays four 4-person 2-pairs games simultaneously every time (see Fig. 4).
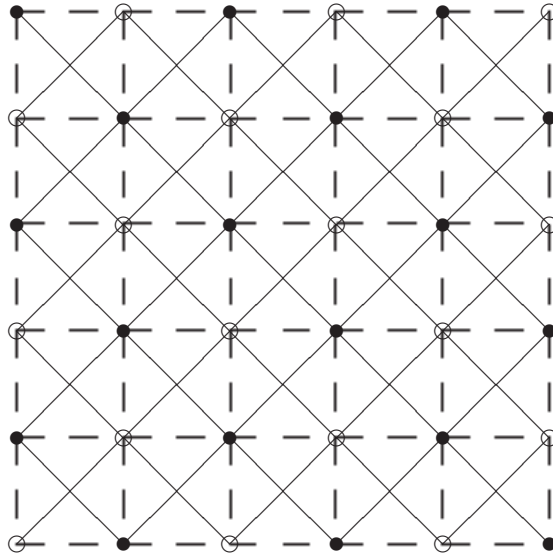

Figure 4. Supergame based on 4-person 2-pairs games on Z^2

If the strategies of four players located on the sites of a basic square $\square(i,j,k,l)$ are (x,y,z,u), then the payoff function is defined by $Q((x,z),(y,u)), x,y,z,u \in A$. Again we assume that each player has only two choices of strategies which may be identified as $A = \{+1,-1\}$, and the payoff function Q is symmetric between pairs (x,z) and (y,u) and within each pair, i.e.,

$$Q((x,z),(y,u)) = Q((z,x),(y,u)) = Q((y,u),(x,z))$$

It can be represented as the following:

$$Q(x,y,z,u) = Hxyzu + I(xyz + yzu + zux + xyu)$$

$$+ J_1(xz + yu) + J_2(xy + yz + zu + ux) + K(x + y + z + u) + L$$

where H, I, J_1, J_2, K and L are uniquely determined by a, b, c, d, e and f and vice versa.

$$Q((+1,+1),(+1,+1)) = a = H + 4I + 2J_2 + 4J_2 + 4K + L$$

$$Q((+1,+1),(+1,-1)) = b = -H - 2I + 2K + L$$

$$Q((+1,+1),(-1,-1)) = c = H + 2J_1 - 4J_2 + L$$

$$Q((+1,-1),(+1,-1)) = d = H - 2J_1 + L$$

$$Q((+1,-1),(-1,-1)) = e = -H + 2I - 2K + L$$

$$Q((-1,-1),(-1,-1)) = f = H - 4I + 2J_1 + 4J_2 - 4K + L$$

or

$$H = \frac{1}{16}(a - 4b + 2c + 4d - 4e + f)$$

$$I = \frac{1}{16}(a - 2b + 2e - f)$$

$$J_1 = \frac{1}{16}(a + 2c - 4d + f)$$

$$J_2 = \frac{1}{16}(a - 2c + f)$$

$$K = \frac{1}{16}(a + 2b - 2e - f)$$

$$L = \frac{1}{16}(a + 4b + 2c + 4d + 4e + f)$$

We define synchronous, even-odd sequential and asynchronous global updating rules as before. The local transition probability is defined by

$$P_i(z|\mathbf{x}(i,y)) = \frac{1}{\Lambda} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l)\in S_i} Q((z,x_k),(x_j,x_l))\}. \tag{3.10}$$

where

$$\lambda = \sum_{z \in A} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l)\in S_i} Q((z,x_k),(x_j,x_l))\}.$$

For the SEP with asynchronous and even-odd sequential global updating rules we can prove that the invariant measure is given by

$$\pi(\mathbf{x}) = \frac{1}{\Lambda} \exp\{\tilde{H} \sum_{\square(i,j,k,l)\in S} x_i x_j x_k x_l + \tilde{I} \sum_{\Delta(i,j,k)\in T} x_i x_j x_k + \tilde{J}_1 \sum_{diagonal<i,k>} x_i x_k$$

$$+ \tilde{J}_2 \sum_{horizonal\ or\ vertical<i,j>} x_i x_j + \tilde{K} \sum_{i \in I} x_i\}. \tag{3.11}$$

where

$$\Lambda = \sum_{\mathbf{x}} \exp\{\frac{\beta}{4} \sum_{\square(i,j,k,l)\in S} Q((x_i,x_k),(x_j,x_l))\}.$$

and $\tilde{H} = \frac{\beta H}{4}$, $\tilde{I} = \frac{\beta I}{4}$, $\tilde{J}_1 = \frac{\beta J_1}{4}$, $\tilde{J}_2 = \frac{2\beta J_2}{4}$, $\tilde{K} = \frac{4\beta K}{4}$. Notice that the pair interaction for diagonal pairs is different from that for horizontal and vertical pairs. When $c = d$, i.e., $J_1 = J_2$, $(\tilde{J}_1 = \tilde{J}_2)$ this reduces the model of 4-person game. The invariant measure is another new type of Ising model with one-site, two-site, three-site and four-site interactions. We also conjecture that for some set of parameters there exists phase transition. This is again an interesting problem needed to be pursued.

4. Further researches

We conclude with a few remarks about the possible problems for future research along this line.

(i) In Section 3 we have discussed some reversible SEP with certain types of global and local transition probability. We treated symmetric payoff. For asymmetric payoff, the situation becomes more complicated. A famous example is so-called Prisoner's Dilemma. The payoff of this two-person two-strategies game is given by

$$Q = \begin{pmatrix} Q(+1,+1) & Q(+1,-1) \\ Q(-1,+1) & Q(-1,-1) \end{pmatrix} = \begin{pmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{pmatrix} \tag{4.1}$$

The long run behavior exhibits complex dynamics.

(ii) For q-strategies ($q > 2$) with the general payoff function which is not necessarily symmetric, the process may not be reversible. A simple example is called Rock-Paper-Scissors game, in which there are three strategies: rock(R), paper(P) and scissor(S). The payoff is given by

$$Q = \begin{pmatrix} Q(R,R) & Q(R,P) & Q(R,S) \\ Q(P,R) & Q(P,P) & Q(P,S) \\ Q(S,R) & Q(S,P) & Q(S,S) \end{pmatrix} = \begin{pmatrix} (0,0) & (-1,1) & (1,-1) \\ (1,-1) & (0,0) & (-1,1) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix} \tag{4.2}$$

This is also an example of zero-sum game. It is interesting to find other or possible all asymmetric payoff functions with which the SEP could be reversible.

(iii) For synchronous global updating rule, it seems more difficult to find the invariant measure.

(iv) We may consider various types of team games. For example we may consider the 2-dimensional Union Jack lattice (see Figure 5). The player located on the vertex of the lattice plays three-person team game for players located on the triangle$((i_1,i_2),(i_1+1,i_2),(i_1,i_2+1)),((i_1,i_2),(i_1-1,i_2),(i_1,i_2-1)),((i_1,i_2),(i_1+1,i_2),(i_1,i_2-1)),((i_1,i_2),(i_1-1,i_2),(i_1,i_2+1))$ for Z^2 model, or five-person star-team game for site $((i_1,i_2),(i_1+1,i_2),(i_1,i_2+1),(i_1-1,i_2),(i_1,i_2-1))$. They may deduce different results, specifically, different behavior of phase transition are expected.

(v) We may consider models on other lattices. Besides the above-mensioned 2-dimensional Union Jack lattice, there is a rich theory on the lattices including tree, other two and three-dimensional lattice models, such as 2-dimensional triangle lattice, 2-dimensional honeycomb lattice, 2-dimensional Kagome lattice, 3-dimensional cubic lattice, 3-dimensional face-centered lattice, 3-dimensional body-centered cubic lattice. Also the supergame on small world network is worth to be pursued. Different behavior of phase transition of the invariant measures for these lattices has been found. we have treated the supergame on trees and the result is reported in [12].

(vi) It is also interesting to pursue the inverse problem-for a given Gibbsian invariant measure what is the sufficient and necessary conditions on the local transition probabilities(or on the payoff) for Gibbsian SEP with different global updating rules.
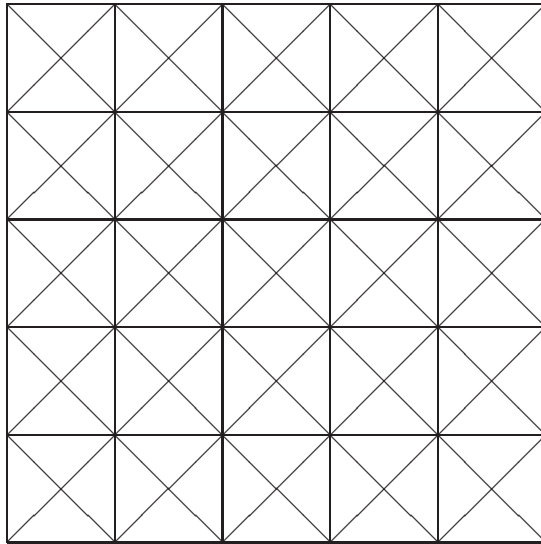
Figure 5. 2-dimensional Union Jack lattice

(vii) For those models for which the theoretical analysis is difficult, the numerical simulation could be of great help. We will report some numerical results elsewhere.

Overall this work is the first step to treat the SEP for large supergame over discrete time. More and deeper results are expected when we pursue the above problems.

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