# Solution of Nonlinear Partial Differential Equations by New Laplace Variational Iteration Method

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Additional information is available at the end of the chapter

**Abstract**

Nonlinear equations are of great importance to our contemporary world. Nonlinear phenomena have important applications in applied mathematics, physics, and issues related to engineering. Despite the importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics, there is still the daunting problem of finding new methods to discover new exact or approximate solutions. The purpose of this chapter is to impart a safe strategy for solving some linear and nonlinear partial differential equations in applied science and physics fields, by combining Laplace transform and the modified variational iteration method (VIM). This method is founded on the variational iteration method, Laplace transforms and convolution integral, such that, we put in an alternative Laplace correction functional and express the integral as a convolution. Some examples in physical engineering are provided to illustrate the simplicity and reliability of this method. The solutions of these examples are contingent only on the initial conditions.

**Keywords:** nonlinear partial differential equations, Laplace transform, modified variational iteration method

## 1. Introduction

In the recent years, many authors have devoted their attention to study solutions of nonlinear partial differential equations using various methods. Among these attempts are the Adomian decomposition method, homotopy perturbation method, variational iteration method (VIM) [1–5], Laplace variational iteration method [6–8], differential transform method and projected differential transform method.

Many analytical and numerical methods have been proposed to obtain solutions for nonlinear PDEs with fractional derivatives such as local fractional variational iteration method [9], local fractional Fourier method, Yang-Fourier transform and Yang-Laplace transform and other methods. Two Laplace variational iteration methods are currently suggested by Wu in [10–13]. In this chapter, we use the new method termed He's variational iteration method, and it is employed in a straightforward manner.

Also, the main aim of this chapter is to introduce an alternative Laplace correction functional and express the integral as a convolution. This approach can tackle functions with discontinuities as well as impulse functions effectively. The estimation of the VIM is to build an iteration method based on a correction functional that includes a generalized Lagrange multiplier. The value of the multiplier is chosen using variational theory so that each iteration improves the accuracy of the result.

In this chapter, we have applied the modified variational iteration method (VIM) and Laplace transform to solve convolution differential equations.

## 2. Combine Laplace transform and variational iteration method to solve convolution differential equations

In this section, we combine Laplace transform and modified variational iteration method to figure out a new case of differential equation called convolution differential equations; it is possible to obtain the exact solutions or better approximate solutions of these equivalences. This method is utilized for solving a convolution differential equation with given initial conditions. The results obtained by this method show the accuracy and efficiency of the method.

**Definition (2.1)**

Let $f(x), g(x)$ be integrable functions, then the convolution of $f(x), g(x)$ is defined as:

$$f(x)^*g(x) = \int_0^x f(x-t)g(t)dt$$

and the Laplace transform is defined as:

$$\ell[f(x)] = F(s) = \int_0^\infty e^{-sx}f(x)dx$$

where Re $s > 0$, where s is complex valued and $\ell$ is the Laplace operator.

Further, the Laplace transform of first and second derivatives are given by:

$$(i)\,\ell[f'(x)] = s\ell[f(x)] - f(0)$$
$$(ii)\,\ell[f''(x)] = s^2\ell[f(x)] - sf(0) - f'(0)$$

More generally:

$$\ell\left[f^{(n)}(x)\right] = s^n \ell[f(x)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

and the one-sided inverse Laplace transform is defined by:

$$\ell^{-1}[F(s)] = f(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{sx}ds$$

where the integration is within the regions of convergence. The region of convergence is half-plane $\alpha < \mathrm{Re}\{s\}$.

**Theorem (2.2) (Convolution Theorem)**

If

$$\ell[f(x)] = F(s), \quad \ell[g(x)] = G(s),$$

then:

$$\ell[f(x)^*g(x)] = \ell[f(x)g(x)] = f(s)g(s)$$

or equivalently,

$$\ell^{-1}[F(s)G(s)] = f(x)^*g(x)$$

Consider the differential equation,

$$L[y(x)] + R[y(x)] + N[y(x)] + N^*[y(x)] = 0 \tag{1}$$

With the initial conditions

$$y(0) = h(x), \quad y'(0) = k(x) \tag{2}$$

where $L$ is a linear second-order operator, $R$ is a linear first-order operator, $N$ is the nonlinear operator and $N^*[y(x)]$ is the nonlinear convolution term which is defined by:

$$N^*[y(x)] = f\left(y, y', y'', \ldots, y^{(n)}\right)^* g\left(y, y', y'', \ldots, y^{(n)}\right)$$

According to the variational iteration method, we can construct a correction functional as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left[Ly_n(\xi) + R\tilde{y}_n(\xi) + N\tilde{y}_n(\xi) + N^*\tilde{y}_n(\xi)\right]d\xi \tag{3}$$

$Ry_n(\xi)$, $N\tilde{y}_n(\xi)$ and $N^*\tilde{y}_n(\xi)$ are considered as restricted variations, that is,

$$\delta R\tilde{y}_n = 0,\ \delta N\tilde{y}_n = 0 \text{ and } \delta N^*\tilde{y}_n = 0,\ \ \lambda = -1$$

Then, the variational iteration formula can be obtained as:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[Ly_n(\xi) + Ry_n(\xi) + Ny_n(\xi) + N^*\tilde{y}_n(\xi)\right]d\xi \tag{4}$$

Eq. (4) can be solved iteratively using $y_0(x)$ as the initial approximation.

Then, the solution is $y(x) = \lim_{n\to\infty} y_n(x)$.

Now, we assume that $L = \frac{d^2}{dx^2}$ in Eq. (1).

Take Laplace transform ($\ell$) of both sides of Eq. (1) to find:

$$\ell\left[Ly(x)\right] + \ell\left[Ry(x)\right] + \ell\left[Ny(x)\right] + \ell\left[N^*y(x)\right] = 0 \tag{5}$$

$$s^2\ell y - sy(0) - y'(0) = -\ell\{Ry(x) + Ny(x) + N^*y(x)\} = 0 \tag{6}$$

By using the initial conditions and taking the inverse Laplace transform, we have:

$$y(x) = p(x) - \ell^{-1}\left[\frac{1}{s^2}Ry(x) + Ny(x) + N^*y(x)\right] = 0 \tag{7}$$

where $p(x)$ represents the terms arising from the source term and the prescribed initial conditions.

Now, the first derivative of Eq. (7) is given by:

$$\frac{dy(x)}{dx} = \frac{dp(x)}{dx} - \frac{d}{dx}\ell^{-1}\left[\frac{1}{s^2}\ell\{Ry(x) + Ny(x) + N^*y(x)\}\right] = 0 \tag{8}$$

By the correction functional of the irrational method, we have:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left\{ \left(y_n(\xi)\right)_\xi - \frac{d}{d\xi}p(\xi) - \frac{d}{d\xi}\ell^{-1}\left[\frac{1}{s^2}\ell\{Ry(\xi) + Ny(\xi) + N^*y(\xi)\}\right]\right\}d\xi$$

Then, the new correction functional (new modified VIM) is given by:

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left[\frac{1}{s^2}\ell\left\{Ry_n(x) + Ny_n(x) + N^*y_n(x)\right\}\right],\ \ n\geq 0 \tag{9}$$

Finally, we find the answer in the strain; if inverse Laplace transforms exist, Laplace transforms exist.

In particular, consider the nonlinear ordinary differential equations with convolution terms,

$$1 - y''(x) - 2 + 2y'^*y'' - y'^*(y'')^2 = 0, y(0) = y'(0) = 0 \tag{10}$$

Take Laplace transform of Eq. (10), and making use of initial conditions, we have:

$$s^2 \ell y(x) - \frac{2}{s} = \ell\left[y'^*(y'')^2 - 2y'^*y''\right]$$

The inverse Laplace transform of the above equation gives that:

$$y(x) = x^2 + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[y'^*(y'')^2 - 2y'^*y''\right]\right\}$$

By using the new modified (Eq. (9)), we have the new correction functional,

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[y'^*(y'')^2 - 2y'^*y''\right]\right\}$$

or

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s^2}\left[\ell(y')^*\ell(y'')^2 - 2\ell(y')^*\ell(y'')\right]\right\} \tag{11}$$

Then, we have:

$$y_0(x) = x^2$$

$$y_1(x) = x^2 + \ell^{-1}\left\{\frac{1}{s^2}\left[\ell(4)\ell(2x) - 2\ell(2x)\ell(2)\right]\right\} = x^2$$

$$y_2(x) = x^2, \quad y_3(x) = x^2, \quad .........., y_n(x) = x^2$$

This means that:

$$y_0(x) = y_1(x) = y_2(x) = ....... = y_n(x) = x^2$$

Then, the exact solution of Eq. (10) is $y(x) = x^2$.

$$2 - y' - (y')^2 - 2x + y'^*(y'')^2 = 0, y(0) = 1 \tag{12}$$

Take Laplace transform of Eq. (12), and using the initial condition, we obtain:

$$s\ell y - 1 - \frac{2}{s^2} = \ell\left[(y')^2 - y'^*(y'')^2\right]$$

Take the inverse Laplace transform to obtain

$$y(x) = 1 + x^2 + \ell^{-1}\left\{\frac{1}{s}\ell\left[(y')^2 - y'^*(y'')^2\right]\right\}$$

Using Eq. (9) to find the new correction functional in the form

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s}\ell\left[(y')^2 - y_n'^*(y_n'')^2\right]\right\}$$

or

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s}\left[\ell\left[(y')^2\right] - \ell[y_n']\ell\left[(y_n'')^2\right]\right]\right\} \tag{13}$$

Then, we have:

$$y_0(x) = 1 + x^2$$

$$y_1(x) = 1 + x^2 + \ell^{-1}\frac{1}{s}\left\{\ell\left(4x^2\right) - \ell(2x)\ell(4)\right\} = 1 + x^2 + \ell^{-1}\frac{1}{s}\left\{\frac{8}{s^3} - \left(\frac{2}{s^2}\right)\left(\frac{4}{s}\right)\right\} = 1 + x^2$$

$$y_0(x) = y_1(x) = y_2(x) = \ldots\ldots = y_n(x) = 1 + x^2$$

Then, the exact solution of Eq. (12) is:

$$y(x) = 1 + x^2$$

## 3. Solution of nonlinear partial differential equations by the combined Laplace transform and the new modified variational iteration method

In this section, we present a reliable combined Laplace transform and the new modified variational iteration method to solve some nonlinear partial differential equations. The analytical results of these equations have been obtained in terms of convergent series with easily computable components. The nonlinear terms in these equations can be handled by using the new modified variational iteration method. This method is more efficient and easy to handle such nonlinear partial differential equations.

In this section, we combined Laplace transform and variational iteration method to solve the nonlinear partial differential equations.

To obtain the Laplace transform of partial derivative, we use integration by parts, and then, we have:

$$\ell\left(\frac{\partial f(x,t)}{\partial t}\right) = sF(x,s) - f(x,0), \tag{14}$$

$$\ell\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = s^2 F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}$$

$$\ell\left(\frac{\partial f(x,t)}{\partial t}\right) = \frac{d}{dx}[F(x,s)],$$

$$\ell\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = \frac{d^2}{dx^2}[F(x,s)].$$

where $f(x,s)$ is the Laplace transform of $(x,t)$.

We can easily extend this result to the nth partial derivative by using mathematical induction.

To illustrate the basic concept of He's VIM, we consider the following general differential equations,

$$\ell[Lu(x,t)] + \ell[Nu(x,t)] = \ell[g(x,t)] \tag{15}$$

with the initial condition

$$u(x,0) = h(x) \tag{16}$$

where $L$ is a linear operator of the first-order, $N$ is a nonlinear operator and $g(x,t)$ is inhomogeneous term. According to variational iteration method, we can construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda[Lu_n(x,s) + N\tilde{u}_n(x,s) - g(x,s)]ds \tag{17}$$

where $\lambda$ is a Lagrange multiplier ($\lambda = -1$), the subscripts $n$ denotes the $nth$ approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$.

Eq. (17) is called a correction functional.

The successive approximation $u_{n+1}$ of the solution $u$ will be readily obtained by using the determined Lagrange multiplier and any selective function $u_0$; consequently, the solution is given by:

$$u = \lim_{u \to \infty} u_n$$

In this section, we assume that $L$ is an operator of the first-order $\frac{\partial}{\partial t}$ in Eq. (15).

Taking Laplace transform on both sides of Eq. (15), we get:

$$\ell[Lu(x,t)] + \ell[Nu(x,t)] = \ell[g(x,t)] \tag{18}$$

Using the differentiation property of Laplace transform and initial condition (16), we have:

$$s\ell[u(x,t)] - h(x) = \ell[g(x,t)] - \ell[Nu(x,t)] \tag{19}$$

Applying the inverse Laplace transform on both sides of Eq. (19), we find:

$$u(x, t) = G(x, t) - \ell^{-1}\left\{\frac{1}{s}Nu[x, t]\right\},\tag{20}$$

where $G(x, t)$ represents the terms arising from the source term and the prescribed initial condition.

Take the first partial derivative with respect to $t$ of Eq. (20) to obtain:

$$\frac{\partial}{\partial t}u(x, t) - \frac{\partial}{\partial t}G(x, t) + \frac{\partial}{\partial t}\ell^{-1}\left\{\frac{1}{s}\ell[Nu(x, t)]\right\}\tag{21}$$

By the correction functional of the variational iteration method

$$u_{n+1} = u_n - \int_0^t \left\{(u_n)_\xi(x, \xi) - \frac{\partial}{\partial \xi}G(x, \xi) + \frac{\partial}{\partial \xi}\ell^{-1}\left\{\frac{1}{\xi}\ell[Nu(\xi, t)]\right\}\right\}d\xi$$

or

$$u_{n+1} = G(x, t) - \ell^{-1}\left\{\frac{1}{s}\ell[Nu_n(x, t)]\right\}\tag{22}$$

Eq. (22) is the new modified correction functional of Laplace transform and the variational iteration method, and the solution $u$ is given by:

$$u(x, t) = \lim_{u \to \infty} u_n(x, t)$$

In this section, we solve some nonlinear partial differential equations by using the new modified variational iteration Laplace transform method; therefore, we have:

**Example (3.1)**

Consider the following nonlinear partial differential equation:

$$u_t + uu_x = 0 \quad , \qquad u(x, 0) = -x\tag{23}$$

Taking Laplace transform of Eq. (23), subject to the initial condition, we have:

$$\ell[u(x, t)] = -\frac{x}{s} - \frac{1}{s}\ell[uu_x]$$

The inverse Laplace transform implies that:

$$u(x, t) = -x - \ell^{-1}\left\{\frac{1}{s}\ell[uu_x]\right\}$$

By the new correction functional, we find:

$$u_{n+1}(x,t) = -x - \ell^{-1}\left\{\frac{1}{s}\ell\left[u_n(u_n)_x\right]\right\}$$

Now, we apply the new modified variational iteration Laplace transform method:

$$u_0(x,t) = -x$$

$$u_1(x,t) = -x - \ell^{-1}\left\{\frac{1}{s}\ell[x]\right\} = -x - \ell^{-1}\left[\frac{x}{s^2}\right] = -x - xt$$

$$u_2(x,t) = -x - \ell^{-1}\left[x\left(\frac{1}{s^2} + \frac{2}{s^3} + \frac{2}{s^4}\right)\right] = -x - xt - xt^2 - \frac{1}{3}xt^3$$

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\end{array}
$$

Therefore, we deduce the series solution to be:

$$u(x,t) = -x(1 + t + t^2 + t^3 + \ldots) = \frac{x}{t-1},$$

which is the exact solution.

**Example (3.2)**

Consider the following nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} \quad , \quad u(x.0) = x^2 \tag{24}$$

Taking Laplace transform of Eq. (24), subject to the initial condition, we have:

$$\ell[u(x,t)] = \frac{x^2}{s} + \frac{1}{s}\ell\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$$

Take the inverse Laplace transform to find that:

$$u(x,t) = x^2 + \ell^{-1}\left\{\frac{1}{s}\ell\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]\right\}$$

The new correction functional is given as

$$u_{n+1}(x,t) = x^2 + \ell^{-1}\left\{\frac{1}{s}\ell\left[\left(\frac{\partial u_n}{\partial x}\right)^2 + u_n\frac{\partial^2 u_n}{\partial x^2}\right]\right\}$$

This is the new modified variational iteration Laplace transform method.

The solution in series form is given by:

$$u_0(x, t) = x^2$$

$$u_1(x, t) = x^2 + \ell^{-1}\left\{\frac{6x^2}{s^2}\right\} = x^2 + 6x^2 t$$

$$u_2(x, t) = x^2\left(1 + 6t + 36t^2 + 72t^3\right)$$

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\begin{matrix}
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& \cdot & & \cdot & & \cdot \\
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\end{matrix}
$$

The series solution is given by:

$$u(x, t) = x^2\left(1 + 6t + 36t^2 + 72t^3 + \ldots\right) = \frac{x^2}{1 - 6t}$$

**Example (3.3)**

Consider the following nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} = 2u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} \quad , \quad u(x.0) = \frac{x+1}{2} \tag{25}$$

Using the same method in the above examples to find the new correction functional in the form:

$$u_{n+1}(x, t) = \frac{x+1}{2} + \ell^{-1}\left\{\frac{1}{s}\ell\left[2u_n\left(\frac{\partial u_n}{\partial x}\right)^2 + u^2{}_n\frac{\partial^2 u_n}{\partial x^2}\right]\right\}$$

Then, we have:

$$u_0(x, t) = \frac{x+1}{2}$$

$$u_1(x, t) = \frac{x+1}{2} + \ell^{-1}\left\{\frac{x+1}{4}\frac{1}{s^2}\right\} = \frac{x+1}{2}\left[1 + \frac{t}{2}\right]$$

$$u_2(x, t) = \frac{x+1}{2}\left(1 + \frac{t}{2} + \frac{3}{8}t^2 + \frac{1}{8}t^3 + \frac{1}{64}t^4\right)$$

$$
\begin{matrix}
\cdot & & \cdot & & \cdot \\
& \cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot
\end{matrix}
$$

The series solution is given by:

$$u(x, t) = \frac{x+1}{2}\left(1 + \frac{t}{2} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^2 + \ldots\right) = \frac{x+1}{2}(1 - t)^{-\frac{1}{2}},$$

which is the exact solution of Eq. (25).

**Example (3.4)**

Consider the following nonlinear partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u - u^2 = te^{-x}, \quad u(x.0) = 0, \frac{\partial u}{\partial t} = e^{-x} \tag{26}$$

Taking the Laplace transform of the Eq. (26), subject to the initial conditions, we have:

$$s^2 \ell[u(x,t)] - e^{-x} = \ell\left[te^{-x} + u^2 - \left(\frac{\partial u}{\partial x}\right)^2 - u\right]$$

Take the inverse Laplace transform to find that:

$$u(x,t) = te^{-x} + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[te^{-x} + u^2 - \left(\frac{\partial u}{\partial x}\right)^2 - u\right]\right\}$$

The new correct functional is given as:

$$u_{n+1}(x,t) = te^{-x} + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[te^{-x} + u_n^2 - \left(\frac{\partial u_n}{\partial x}\right)^2 - u_n\right]\right\}$$

This is the new modified variational iteration Laplace transform method.

The solution in series form is given by:

$$u_0(x,t) = te^{-x}$$
$$u_1(x,t) = te^{-x} \tag{27}$$
$$u_2(x,t) = te^{-x}$$
$$.$$
$$.$$

The series solution is given by:

$$u(x,t) = te^{-x}$$

# 4. New Laplace Variational iteration method

To illustrate the idea of new Laplace variational iteration method, we consider the following general differential equations in physics.

$$L[u(x,t)] + N[u(x,t)] = h(x,t) \tag{28}$$

where $L$ is a linear partial differential operator given by $\frac{\partial^2}{\partial t^2}$, $N$ is nonlinear operator and $h(x,t)$ is a known analytical function. According to the variational iteration method, we can construct a correction functional for Eq. (28) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \overline{\lambda}(x,\varsigma)[Lu_n(x,\varsigma) + N\tilde{u}_n(x,\varsigma) - h(x,\varsigma)]d\varsigma, \tag{29}$$

$$n \geq 0,$$

where $\lambda$ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript $n$ denotes the nth approximation, $N\tilde{u}_n(x,\varsigma)$ is considered as a restricted variation, that is, $\delta N\tilde{u}_n(x,\varsigma) = 0$.

Also, we can find the Lagrange multipliers, by using integration by parts of Eq. (28), but in this chapter, the Lagrange multipliers are found to be of the form $\lambda = \overline{\lambda}(x,t-\varsigma)$ and in such a case, the integration is basically the single convolution with respect to $t$, and hence, Laplace transform is appropriate to use.

Take Laplace transform of Eq. (29); then the correction functional will be constructed in the form:

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)]$$

$$+ \ell\left[\int_0^t \overline{\lambda}(x,\varsigma)[Lu_n(x,\varsigma) + N\tilde{u}_n(x,\varsigma) - h(x,\varsigma)]d\varsigma\right], n \geq 0, \tag{30}$$

Therefore

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)]$$

$$+ \ell\left[\overline{\lambda}(x,t)*[Lu_n(x,t) + N\tilde{u}_n(x,t) - h(x,t)]\right] \tag{31}$$

$$= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\ell[Lu_n(x,t) + N\tilde{u}_n(x,t) - h(x,t)]$$

where * is a single convolution with respect to $t$.

To find the optimal value of $\overline{\lambda}(x,t-\varsigma)$, we first take the variation with respect to $u_n(x,t)$. Thus:

$$\frac{\delta}{\delta u_n}\ell[u_{n+1}(x,t)] = \frac{\delta}{\delta u_n}\ell[u_n(x,t)] +$$

$$\frac{\delta}{\delta u_n}\ell\left[\overline{\lambda}(x,t)\right]\ell[Lu_n(x,t) + N\tilde{u}_n(x,t) - h(x,t)] \tag{32}$$

Then, Eq. (32) becomes

$$\ell\left[\delta u_{n+1}(x,t)\right] = \ell\left[\delta u_n(x,t)\right] + \delta\ell\left[\overline{\lambda}(x,t)\right]\ell\left[Lu_n(x,t)\right] \tag{33}$$

In this chapter, we assume that $L$ is a linear partial differential operator given by $\dfrac{\partial^2}{\partial t^2}$, then, Eq. (33) can be written in the form:

$$\ell\left[\delta u_{n+1}(x,t)\right]=\ell\left[\delta u_n(x,t)\right]+\ell\left[\overline{\lambda}(x,t)\right]\left[s^2\ell\delta u_n(x,t)\right] \tag{34}$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t)=0$. This means that the right hand side of Eq. (34) should be set to zero; then, we have the following condition:

$$\ell\left[\overline{\lambda}(x,t)\right]=\frac{-1}{s^2} \quad\Rightarrow\quad \overline{\lambda}(x,t)=-t \tag{35}$$

Then, we have the following iteration formula

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)]$$
$$-\ell\left[\int_0^t (t-\varsigma)[Lu_n(x,\varsigma) + N\tilde{u}_n(x,\varsigma) - h(x,\varsigma)]d\varsigma\right],\, n\geq 0, \tag{36}$$

## 5. Applications

In this section, we apply the Laplace variational iteration method to solve some linear and nonlinear partial differential equations in physics.

**Example (5.1)**

Consider the initial linear partial differential equation

$$u_{tt}(x,t)-u_{xx}(x,t)+u(x,t)=0 \quad,\quad u(x,0)=0 \quad,\quad \frac{\partial u(x,0)}{\partial t}=x \tag{37}$$

The Laplace variational iteration correction functional will be constructed in the following manner:

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)]$$
$$+\ell\left[\int_0^t \overline{\lambda}(x,t-\varsigma)\left[(u_n)_{tt}(x,\varsigma) - (u_n)_{xx}(x,\varsigma) + u_n(x,\varsigma)\right]d\varsigma\right] \tag{38}$$

or

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)]$$
$$+\ell\left[\overline{\lambda}(x,t)*\left[(u_n)_{tt}(x,t) - (u_n)_{xx}(x,t) + u_n(x,t)\right]\right]$$
$$= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\ell\left[(u_n)_{tt}(x,t) - (u_n)_{xx}(x,t) + u_n(x,t)\right]$$
$$= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\begin{bmatrix} s^2\ell u_n(x,t) - su_n(x,0) - \dfrac{\partial u_n}{\partial t}(x,0) \\ -\ell(u_n)_{xx}(x,t) + \ell u_n(x,t) \end{bmatrix} \tag{39}$$

Taking the variation with respect to $u_n(x,t)$ of Eq. (39), we obtain:

$$\frac{\delta}{\delta u_n} \ell[u_{n+1}(x,t)] = \frac{\delta}{\delta u_n} \ell[u_n(x,t)]$$

$$+ \frac{\delta}{\delta u_n} \ell[\overline{\lambda}(x,t)] \begin{bmatrix} s^2 \ell u_n(x,t) - s u_n(x,0) - \dfrac{\partial u_n}{\partial t}(x,0) \\ -\ell(u_n)_{xx}(x,t) + \ell u_n(x,t) \end{bmatrix} \tag{40}$$

Then, we have.

$$\ell\big[\delta u_{n+1}(x,t)\big] = \ell\big[\delta u_n(x,t)\big] + \ell\big[\overline{\lambda}(x,t)\big]\big[s^2\ell\delta u_n(x,t) + \ell\delta u_n(x,t)\big]$$
$$= \ell\big[\delta u_n(x,t)\big]\big\{1 + \ell\big[\overline{\lambda}(x,t)\big]\big(s^2 + 1\big)\big\}$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t) = 0$. Hence, we have:

$$1 + (s^2 + 1)\ell\overline{\lambda}(x,t) = 0 \quad , \quad and \quad \overline{\lambda}(x,t) = \ell^{-1}\left[\frac{-1}{s^2+1}\right] = -\sin t \tag{41}$$

Substituting Eq. (41) into Eq. (38), we obtain:

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)]$$

$$- \ell\left[\int_0^t \sin(t-\varsigma)\big[(u_n)_{tt}(x,\varsigma) - (u_n)_{xx}(x,\varsigma) + u_n(x,\varsigma)\big]d\varsigma\right] \tag{42}$$

$$= \ell[u_n(x,t)] - \ell[\sin t]\ell\big[(u_n)_{tt}(x,t) - (u_n)_{xx}(x,t) + u_n(x,t)\big]$$

Let $u_0(x,t) = u(x,0) + t\dfrac{\partial u}{\partial t}(x,0) = xt$, then, from Eq. (42), we have:

$$\ell[u_1(x,t)] = \ell[xt] - \ell[\sin t]\ell[xt] = \frac{x}{s^2} - \frac{x}{s^2(s^2+1)}$$

The inverse Laplace transforms yields:

$$u_1(x,t) = x\,\sin t \tag{43}$$

Substituting Eq. (43) into Eq. (38), we obtain:

$$\ell[u_2(x,t)] = \ell[x\,\sin t] - \ell[\sin t]\ell[0] \quad then \quad u_2(x,t) = x\,\sin t$$

Then, the exact solution of Eq. (37) is:

$$u(x,t) = x\,\sin t \tag{44}$$

**Example (4.2)**

Consider the nonlinear partial differential equation:

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2 t^2 \quad , \quad u(x,0) = 0 \quad , \quad \frac{\partial u(x,0)}{\partial t} = x \tag{45}$$

The Laplace variational iteration correction functional will be constructed as follows:

$$\ell\left[u_{n+1}(x,t)\right] = \ell\left[u_n(x,t)\right] - \ell\left[\int_0^t (t-\varsigma)\begin{bmatrix}(u_n)_{tt}(x,\varsigma) - (u_n)_{xx}(x,\varsigma) \\ +u_n^2(x,\varsigma) - x^2\varsigma^2\end{bmatrix}d\varsigma\right] \tag{46}$$

or

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t) * \begin{bmatrix}(u_n)_{tt}(x,t) - (u_n)_{xx}(x,t) \\ +u_n^2(x,t) - x^2t^2\end{bmatrix}\right]$$

$$= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\ell\left[(u_n)_{tt}(x,t) - (u_n)_{xx}(x,t) + u_n^2(x,t) - x^2t^2\right] \tag{47}$$

$$= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\begin{bmatrix}s^2\ell u_n(x,t) - su_n(x,0) - \dfrac{\partial u_n}{\partial t}(x,0) \\ -\ell(u_n)_{xx}(x,t) + \ell u_n^2(x,t) - \ell(x^2t^2)\end{bmatrix}$$

Taking the variation with respect to $u_n(x,t)$ of Eq. (47) and making the correction functional stationary we obtain:

This implies that:

$$\ell + s\ell\overline{\lambda}(x,t) = 0 \quad , \quad and \quad \overline{\lambda}(x,t) = \ell^{-1}\left[\frac{-1}{s}\right] = -1 \tag{48}$$

Substituting Eq. (21) into Eq. (19), we obtain:

$$\ell\left[u_{n+1}(x,t)\right] = \ell\left[u_n(x,t)\right] - \ell\left[\int_0^t (t-\varsigma)\begin{bmatrix}(u_n)_{tt}(x,\varsigma) - (u_n)_{xx}(x,\varsigma) \\ +u_n^2(x,\varsigma) - x^2\varsigma^2\end{bmatrix}d\varsigma\right] \tag{49}$$

or

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)] + \ell[-t]\ell\begin{bmatrix}(u_n)_{tt}(x,t) - (u_n)_{xx}(x,t) \\ +u_n^2(x,t) - x^2t^2\end{bmatrix} \tag{50}$$

Let $u_0(x,t) = u(x,0) + t\dfrac{\partial u}{\partial t}(x,0) = xt$, then, from Eq. (50), we have:

$$\ell\left[u_1(x,t)\right] = \ell\left[xt\right] + \ell\left[-t\right]\ell\left[0 - 0 + x^2t^2 - x^2t^2\right]$$

$$u_1(x,t) = xt$$

Then, the exact solution of Eq. (45) is: $u(x,t)=xt$

Again, the exact solution is obtained by using only few steps of the iterative scheme.

**Example (4.3)**

Consider the physics nonlinear boundary value problem,

$$u_t - 6uu_x + u_{xxx} = 0 \quad , \quad u(x,0) = \frac{6}{x^2}, \quad x \neq 0$$

The Laplace variational iteration correction functional is

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)]$$
$$+\ell\left[\int_0^t \overline{\lambda}(x,t-\varsigma)\begin{bmatrix}(u_n)_t(x,\varsigma) - 6u_n(x,\varsigma)(u_n)_x(x,\varsigma)\\ +(u_n)_{xxx}(x,\varsigma)\end{bmatrix}d\varsigma\right] \qquad (52)$$

or

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)*[(u_n)_t(x,t) - 6(u_n)(x,t)(u_n)_x(x,t) + (u_n)_{xxx}(x,t)]\right]$$
$$= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\ell\left[(u_n)_t(x,t) - 6(u_n)(x,t)(u_n)_x(x,t) + (u_n)_{xxx}(x,t)\right]$$
$$= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\left[s\ell u_n(x,t) - u_n(x,0) - \ell\left[6(u_n)(x,t)(u_n)_x(x,t) - (u_n)_{xxx}(x,t)\right]\right]$$

Taking the variation with respect to $u_n(x,t)$ of the last equation and making the correction functional stationary we obtain:

$$\ell\left[\delta u_{n+1}(x,t)\right] = \ell\left[\delta u_n(x,t)\right] + \ell\left[\overline{\lambda}(x,t)\right]\left[s\ell \delta u_n(x,t)\right]$$
$$= \ell\left[\delta u_n(x,t)\right]\left\{\ell + s\ell\left[\overline{\lambda}(x,t)\right]\right\}$$

This implies that:

$$1 + s\,\ell\overline{\lambda}(x,t) = 0 \quad , \quad and \quad \overline{\lambda}(x,t) = \ell^{-1}\left[\frac{-1}{s}\right] = -t \qquad (53)$$

Substituting Eq. (53) into Eq. (52), we obtain:

$$\ell\left[u_{n+1}(x,t)\right] = \ell\left[u_n(x,t)\right] + \ell\left[\int_0^t (-1)\begin{bmatrix}(u_n)_t(x,\varsigma) - 6(u_n)(x,\varsigma)(u_n)_x(x,\varsigma)\\ +(u_n)_{xxx}(x,\varsigma)\end{bmatrix}d\varsigma\right]$$

or

$$\ell\left[u_{n+1}(x,t)\right] = \ell\left[u_n\right] + \ell\left[-1\right]\ell\left[(u_n)_t - (u_n)(u_n)_x + (u_n)_{xxx}\right] \qquad (54)$$

Let $u_0(x,t) = u(x,0) = \frac{6}{x^2}$, then, from Eq. (54), we have:

$$\ell[u_1(x,t)] = \ell\left[\frac{6}{x^2}\right] + \ell[-1]\ell\left[\frac{288}{x^5}\right] = \frac{6}{x^2} - \frac{288}{x^5}t$$

$$u_2(x,t) = \frac{6}{x^2} - \frac{288}{x^5}t - \frac{6048}{x^8}t^2 ,\ldots\ldots$$

Then, the exact solution of Eq. (51) is: $u(x,t) = \dfrac{6x\left(x^3 - 24t\right)}{\left(x^3 - 12t\right)^2}$

### Exercises

Solve the following nonlinear partial differential equations by new Laplace variational iteration method:

$1)\, u_t + uu_x = 1 - e^{-x}(t + e^{-x}), \qquad u(x,0) = e^{-x}$

$2)\, u_t + uu_x = 2t + x + t^3 + xt^2, \quad u(x,0) = 0$

$3)\, u_t + uu_x = 2x^2t + 2xt^2 + 2x^3t^4, \quad u(x,0) = 1$

$4)\, u_t + uu_x = 1 + t\cos x + \dfrac{1}{2}\sin 2x, \ u(x,0) = \sin x$

$5)\, u_t + uu_x = 0, \qquad u(x,0) = -x$

$6)\, u_t + uu_x - u = e^t, \quad u(x,0) = 1 + x$

$7)\, u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2, \qquad u(x,0) = 1, u_t(x,0) = x$

$8)\, u_{tt} - u_{xx} + u^2 = 1 + 2xt + x^2t^2, \qquad u(x,0) = 1, u_t(x,0) = x$

$9)\, u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6, \quad u(x,0) = 0, u_t(x,0) = 0$

$10)\, u_{tt} - u_{xx} + u^2 = \left(x^2 + t^2\right)^2, \qquad u(x,0) = x^2, u_t(x,0) = 0$

$11)\, u_{tt} - u_{xx} + u + u^2 = x^2\cos^2 t, \qquad u(x,0) = x, u_t(x,0) = 0$

$12)\, u_t + uu_x = 0, \quad u(x,0) = x$

$13)\, u_t + uu_x = 0, \quad u(x,0) = -x$

$14)\, u_t + uu_x = 0, \quad u(x,0) = 2x$

$15)\, u_t + uu_x = u_{xx}, \quad u(x,0) = -x$

$16)\, u_t + uu_x = u_{xx}, \quad u(x,0) = 2x$

$17)\, u_t + uu_x = u_{xx}, \quad u(x,0) = 4\tan 2x$

## 6. Conclusions

The method of combining Laplace transforms and variational iteration method is proposed for the solution of linear and nonlinear partial differential equations. This method is applied in a direct way without employing linearization and is successfully implemented by using the initial conditions and convolution integral. But this method failed to solve the singular differential equations.

## Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this chapter.

## Answers

$1)\, u(x,t) = t + e^{-x}, \quad 2)\, u(x,t) = t^2 + xt, \quad 3)\, u(x,t) = 1 + x^2 t^2, \quad 4)\, u(x,t) = t + \sin x$

$5)\, u(x,t) = \dfrac{x}{t-1}, \quad 6)\, u(x,t) = x + e^t, \quad 7)\, u(x,t) = 1 + xt, \quad 8)\, u(x,t) = 1 + xt$

$9)\, u(x,t) = x^3 t^3, \quad 10)\, u(x,t) = t^2 + x^2, \quad 11)\, u(x,t) = x\cos t, \quad 12)\, u(x,t) = \dfrac{x}{1+t}$

$13)\, u(x,t) = \dfrac{x}{t-1}, \quad 14)\, u(x,t) = \dfrac{2x}{1+2t}, \quad 15)\, u(x,t) = \dfrac{x}{t-1}, \quad 16)\, u(x,t) = \dfrac{2x}{1+2t}$

$17)\, u(x,t) = 4\tan 2x$

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