Chapter 3

# Trajectory Tracking Error Using Fractional Order PID Control Law for Two-Link Robot Manipulator via Fractional Adaptive Neural Networks

Joel Perez P., Jose Paz Perez P. and
Martha S. Lopez de la Fuente

Additional information is available at the end of the chapter

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**Abstract**

The problem of trajectory tracking of unknown nonlinear systems of fractional order is solved using fractional order dynamical neural networks. For this purpose, we obtained control laws and laws of adaptive weights online, obtained using the Lyapunov stability analysis methodology of fractional order. Numerical simulations illustrate the obtained theoretical results.

**Keywords:** fractional order PID control, fractional adaptive neural networks, fractional Lyapunov functions, fractional nonlinear systems, trajectory tracking

## 1. Introduction

The fractional calculus is a branch of mathematics that attracted attention since G.W. Leibnitz proposed it in the seventeenth century. However, the researchers were not attracted to this area because of the lack of applications and analytical results of the fractional calculus.

On the contrary, the fractional calculus currently attracts the attention of a large number of scientists for their applications in different fields of science, engineering, chemistry, and so on.

This chapter presents the design of a fractional order nonlinear identifier modeled by a dynamic neural network of fractional order.

Although PID controllers are introduced long time ago, they are widely used in industry because of their advantages such as low price, design simplicity, and suitable performance. While three parameters of design including proportional (Kp), integral (Ki), and derivative

(Kd) are available in PID controllers, two more parameters exist in FOPID controllers for adjustment. These parameters are integral fractional order and derivative fractional order. In comparison with PID controllers, FOPID controllers have more flexible design that results in more precise adjustment of closed-loop system. FOPID controllers are defined by FO differential equations. It is possible to tune frequency response of the control system by expanding integral and derivative terms of the PID controller to fractional order case. This characteristic feature results in a more robust design of control system, but it is not easily possible. According to nonlinearity, uncertainty, and confusion behaviors of robot arms, they are highly recommended for experimenting designs of control systems. Despite nonlinear behavior of robot arm, it is demonstrable that a linear proportional derivative controller can stabilize the system using Lyapunov. But, classic PD controller itself cannot control robot to reach suitable condition. Several papers and wide researches in optimizing performance of the robot manipulator show the importance of this issue.

There are several ways of defining the derivative and fractional integral, for example, the derivative of Grunwald-Letnikov given by Eq. (1)

$$aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{(t-\alpha)}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh) \tag{1}$$

where [.] is a flooring operator, while the RL definition is given by:

$$aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{2}$$

For $(n-1 < \alpha < n)$, $\Gamma(x)$ is the well-known Euler's Gamma function.

Similarly, the notation used in ordinary differential equations, we will use the following notation, Eq. (3), when we are referring to the fractional order differential equations where $\alpha_k \in \mathbb{R}^+$.

which is:

$$g(t,\ x, aD_t^{\alpha_1}\ x,\ aD_t^{\alpha_2}\ x, \dots\ ) = 0 \tag{3}$$

The Caputo's definition can be written as

$$aD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{4}$$

For

$$(n-1 < \alpha < n).$$

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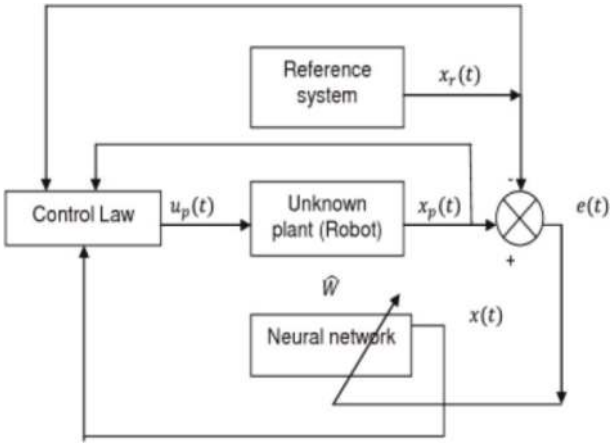


**Figure 1.** Recurrent neural network scheme.

Trajectory tracking, synchronization, and control of linear and nonlinear systems are a very important problem in science and control engineering. In this chapter, we will extend these concepts to force the nonlinear system (plant) to follow any linear and nonlinear reference signals generated by fractional order differential equations.

The proposed adaptive control scheme is composed of a recurrent neural identifier and a controller (**Figure 1**).

We use the above scheme to model the unknown nonlinear system by means of a dynamic recurrent neural network of adaptable weights; the above is modeled by differential equations of fractional order. Also, the scheme allows us to determine the control actions, the error of approach of trajectories, as well as the laws of adaptation of adaptive weights and the interconnection of such systems.

## 2. Modelling of the plant

The nonlinear system (Eq. (5)) is forced to follow a reference signal:

$$aD_t^{\alpha} x_p = F_p(x_p, u) \triangleq f_p(x_p) + g_p(x_p)u_p \tag{5}$$

$$x_p,\ f_p. \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ g_p \in \mathbb{R}^{nxn}.$$

The differential equation will be modeled by:

$$aD_t^{\alpha} x_p = \mathrm{A}(x) + \mathrm{W}^{*}\Gamma_z(x) + \Omega_u.$$

The tracking error between these two systems:

$$w_{per} = x - x_p \tag{6}$$

We use the next hypotheses.

$$aD_t^{\alpha} w_{per} = -k w_{per} \tag{7}$$

In this research, we will use $k = 1$, so that, Eq. (6), $aD_t^{\alpha} w_{per} = aD_t^{\alpha} x - aD_t^{\alpha} x_p$,; so

$$aD_t^{\alpha} x_p = \ aD_t^{\alpha} x + w_{per}$$

The nonlinear system is [1]:

$$aD_t^{\alpha} x_p = aD_t^{\alpha} x + w_{per} = \mathrm{A}(x) + \mathrm{W}^{*}\Gamma_{\mathrm{z}}(x) + w_{per} + \Omega_{\mathrm{u}} \tag{8}$$

Where the $\mathrm{W}^{*}$ is the matrix weights.

## 3. Tracking error problem

In this part, we will analyze the trajectory tracking problem generated by

$$aD_t^{\alpha} x_r = f_r(x_r, u_r), \quad w_r, \ x_r \in \mathbb{R}^n \tag{9}$$

Are the state space vector, input vector, and $f_r$, is a nonlinear vectorial function.

To achieve our goal of trajectory tracking, we propose

$$e = x_p - x_r \tag{10}$$

The time derivative of the error is:

$$aD_t^{\alpha} e = aD_t^{\alpha} x_p - aD_t^{\alpha} x_r = \mathrm{A}(x) + \mathrm{W}^{*}\Gamma_{\mathrm{z}}(x) + w_{per} + \Omega_{\mathrm{u}} - f_r(x_r, u_r) \tag{11}$$

The Eq. (11) can be rewritten as follows, adding and subtracting the next terms $\widehat{W}\Gamma_{\mathrm{z}}(x_r)$, $\alpha_r\left(\mathrm{t}, \widehat{W}\right)$, $Ae$ and $w_{per} = x - x_p$,; then,

$$\begin{gathered} aD_t^{\alpha} e = \mathrm{A}(x) + \mathrm{W}^{*}\Gamma_{\mathrm{z}}(x) + x - x_p + \Omega_{\mathrm{u}} - f_r(x_r, u_r) + \widehat{W}\Gamma_{\mathrm{z}}(x_r) \\ -\widehat{W}\Gamma_{\mathrm{z}}(x_r) + \Omega\alpha_r\left(\mathrm{t}, \widehat{W}\right) - \Omega\alpha_r\left(\mathrm{t}, \widehat{W}\right) + Ae - Ae \\ aD_t^{\alpha} e = \mathrm{A}e + \mathrm{W}^{*}\Gamma_{\mathrm{z}}(x) + \Omega_{\mathrm{u}} - f_r(x_r, u_r) + \widehat{W}\Gamma_{\mathrm{z}}(x_r) + \Omega\alpha_r\left(\mathrm{t}, \widehat{W}\right) \\ -\widehat{W}\Gamma_{\mathrm{z}}(x_r) - \Omega\alpha_r\left(\mathrm{t}, \widehat{W}\right) - Ax_r - x_r + x + A(x) \end{gathered} \tag{12}$$

The unknown plant will follow the fractional order reference signal, if:

$Ax_r + \widehat{W}\Gamma_{\mathrm{z}}(x_r) + x_r - x_p + \Omega\alpha_r\left(\mathrm{t}, \widehat{W}\right) = f_r(x_r, u_r)$, where

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$$\Omega\alpha_r\left(\mathrm{t},\widehat{W}\right)=f_r(x_r,u_r)-Ax_r-\widehat{W}\Gamma_z(x_r)-x_r+x_p \tag{13}$$

$$aD_t^{\alpha}e=\mathrm{A}e+\mathrm{W}^{*}\Gamma_z(x)-\widehat{W}\Gamma_z(x_r)-\mathrm{A}e+(A+I)(x-x_r)+\Omega\left(u-\alpha_r\left(\mathrm{t},\widehat{W}\right)\right) \tag{14}$$

Now, $\widehat{W}$ is part of the approach, given by $\mathrm{W}^{*}$. The Eq. (14) can be expressed as Eq. (15), adding and subtracting the term $\widehat{W}\Gamma_z(x)$ and if $\Gamma_z(x)=\Gamma(z(x)-z(x_r))$

$$aD_t^{\alpha}e=\mathrm{A}e+\left(\mathrm{W}^{*}-\widehat{W}\right)\Gamma_z(x)+\widehat{W}\Gamma(z(x)-z(x_r))+(A+I)(x-x_r)-\mathrm{A}e+\Omega\left(u-\alpha_r\left(\mathrm{t},\widehat{W}\right)\right) \tag{15}$$

If

$$\widetilde{W}=\mathrm{W}^{*}-\widehat{W}\text{ and }\widetilde{u}=u-\alpha_r\left(\mathrm{t},\widehat{W}\right) \tag{16}$$

And by replacing Eq. (16) in Eq. (15), we have:

$$aD_t^{\alpha}e=\mathrm{A}e+\widetilde{W}\Gamma_z(x)+\widehat{W}\Gamma(z(x)-z(x_r))+(A+I)(x-x_r)-\mathrm{A}e+\Omega\widetilde{u}$$

$$\begin{aligned}aD_t^{\alpha}e=\mathrm{A}e+\widetilde{W}\Gamma_z(x)+\widehat{W}\Gamma\big(z(x)-z\big(x_p\big)+z\big(x_p\big)-z(x_r)\big)+\\(A+I)\big(x-x_p+x_p-x_r\big)-\mathrm{A}e+\Omega\widetilde{u}\end{aligned} \tag{17}$$

And:

$$\widetilde{u}=u_1+u_2 \tag{18}$$

So, the result for $\Omega u_1$ is

$$\Omega u_1=-\widehat{W}\Gamma\big(z(x)-z\big(x_p\big)\big)-(A+I)\big(x-x_p\big), \tag{19}$$

and Eq. (17) is simplified:

$$aD_t^{\alpha}e=\mathrm{A}e+\widetilde{W}\Gamma_z(x)+\widehat{W}\Gamma\big(z\big(x_p\big)-z(x_r)\big)+(A+I)\big(x_p-x_r\big)-\mathrm{A}e+\Omega\widetilde{u}$$

Taking into account that $e=x_p-x_r$, the equation for $aD_t^{\alpha}\mathrm{e}$ is

$$\begin{aligned}aD_t^{\alpha}e&=(\mathrm{A}+\mathrm{I})e+\tilde{W}\Gamma_z(x)+\widehat{W}\Gamma(z(e+x_r)-z(x_r))+\Omega u_2\\&=(A+I)e+\widetilde{W}\sigma(x)+\widehat{W}(\sigma(e+x_r)-\sigma(x_r))+\Omega u_2\end{aligned}$$

If $\phi(e)=(\sigma(e+x_r)-\sigma(x_r))$, then

$$aD_t^{\alpha}e=(A+I)e+\widetilde{W}\sigma(x)+\widehat{W}\phi(e)+\Omega u_2 \tag{20}$$

Now, the problem is to find the control law $\Omega u_2$, in which it stabilizes to the system Eq. (20). We will obtain the control law using the fractional order Lyapunov methodology.

## 4. Asymptotic stability of the approximation error

From Eq. (20), we consider the stability of the tracking error, for which we first observe that $\left(e, \widehat{W}\right) = 0$, is an equilibrium state of dynamical system from Eq. (20), and we consider a particular case when $A = -\lambda I, \lambda > 0$

For such stability analysis of the trajectory tracking (Eq. (20)), we propose the following FOPID control law [2]:

$$\Omega u_2 = K_p e + K_i {}_a D_t^{-\alpha} e + K_v {}_a D_t^{\alpha} e - \gamma \left( \frac{1}{2} + \frac{1}{2} \|\widehat{W}\|^2 L_{\phi}^2 \right) e \tag{21}$$

Our objective is to find $K_p$, $K_i$, $K_v$,$\widehat{W}$, $L_{\phi_z}^2$, and this guarantees that the tracking error given by Eq. (20) is asymptotically stable, for which we will later propose a Lyapunov function, with $\gamma > 0$ ; this control law (Eq. (21)) is similar to [3].

A FOPID controller, also known as a $PI^{\lambda}D^{\alpha}$ controller, takes on the form [4]:

$$u(t) = K_p e(t) + K_i {}_a D_t^{-\lambda} e(t) + K_d {}_a D_t^{\alpha} e(t)$$

where $\lambda$ and $\alpha$ are the fractional orders of the controller and $e(t)$ is the system error, where $\lambda = \alpha$. Note that the system error $e(t)$ replaces the general function $f(t)$.

We will show that the feedback system is asymptotically stable. Replacing Eq. (21) in Eq. (20), we have

$a D_t^{\alpha} e = (A + I)e + \widetilde{W}\sigma(x) + \widehat{W}\phi(e) + K_p e + K_i {}_a D_t^{-\alpha} e + K_v {}_a D_t^{\alpha} e - \gamma \left( \frac{1}{2} + \frac{1}{2} \|\widehat{W}\|^2 L_{\phi}^2 \right) e$, then

$(1 - K_v) a D_t^{\alpha} e = (A + I)e + \widetilde{W}\sigma(x) + \widehat{W}\phi(e) + K_p e + K_i {}_a D_t^{-\alpha} e - \gamma \left( \frac{1}{2} + \frac{1}{2} \|\widehat{W}\|^2 L_{\phi}^2 \right) e$. If

$a = (1 - K_v)$, then

$$a D_t^{\alpha} e = \frac{1}{a}(A + I)e + \frac{1}{a}\widetilde{W}\sigma(x) + \frac{1}{a}\widehat{W}\phi(e) + \frac{1}{a}K_p e + \frac{1}{a}K_i {}_a D_t^{-\alpha} e - \frac{\gamma}{a}\left( \frac{1}{2} + \frac{1}{2}\widehat{W}^2 L_{\phi}^2 \right) e \tag{22}$$

$$a D_t^{\alpha} e = \frac{-1}{a}(\lambda - 1 + K_p)e + \frac{1}{a}\widetilde{W}\sigma(x) + \frac{1}{a}\widehat{W}\phi(e) + \frac{1}{a}K_i {}_a D_t^{-\alpha} e - \frac{\gamma}{a}\left( \frac{1}{2} + \frac{1}{2}\widehat{W}^2 L_{\phi}^2 \right) e \tag{23}$$

And if $w = \frac{1}{a}K_i {}_a D_t^{-\alpha} e$, then $a D_t^{\alpha} w = \frac{1}{a}K_i e(t)$ [5]; then, we rewrite Eq. (23) as:

$$a D_t^{\alpha} e = \frac{-1}{a}(\lambda - 1 + K_p)e + \frac{1}{a}\widetilde{W}\sigma(x) + \frac{1}{a}\widehat{W}\phi(e) + w - \frac{\gamma}{a}\left( \frac{1}{2} + \frac{1}{2}\widehat{W}^2 L_{\phi}^2 \right) e \tag{24}$$

We will show that the new state $(e, w)^T$ is asymptotically stable and the equilibrium point is $(e, w)^T = (0, 0)^T$, when $\widetilde{W}\sigma(x_r) = 0$, as an external disturbance.

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Let $V$ be, the next candidate Lyapunov function as [6]:

$$V = \frac{1}{2}\left(e^T, w^T\right)(e,w)^T + \frac{1}{2a}tr\left\{\widetilde{W}^T\widetilde{W}\right\} \tag{25}$$

The fractional order time derivative of Eq. (25) along with the trajectories of Eq. (24) is

$$aD_t^\alpha V = e^T aD_t^\alpha e + w^T aD_t^\alpha w + \frac{1}{a}tr\left\{aD_t^\alpha\ \widetilde{W}^T\widetilde{W}\right\} \tag{26}$$

$$\begin{aligned} aD_t^\alpha V = {} & e^T\left(\frac{-1}{a}\left(\lambda - 1 + K_p\right)e + \frac{1}{a}\widetilde{W}\sigma(x) + \frac{1}{a}\widehat{W}\phi(e) + w - \frac{\gamma}{a}\left(\frac{1}{2} + \frac{1}{2}\widehat{W}^2 L_\phi^2\right)e\right) \\ & + \frac{1}{a}\widetilde{W}^T K_i e + \frac{1}{a}tr\left\{aD_t^\alpha\ \widetilde{W}^T\widetilde{W}\right\} \end{aligned} \tag{27}$$

In this part, we select the next learning law from the neural network weights as in [7] and [8]:

$$tr\left\{aD_t^\alpha\ \widetilde{W}^T\widetilde{W}\right\} = -e^T\widetilde{W}\sigma(x) \tag{28}$$

Then, Eq. (27) is reduced to

$$aD_t^\alpha V = \frac{-1}{a}\left(\lambda - 1 + K_p\right)e^T e + \frac{e^T}{a}\widehat{W}\,\phi(e) + \left(1 + \frac{K_i}{a}\right)e^T w - \frac{\gamma}{a}\left(\frac{1}{2} + \frac{1}{2}\|\widehat{W}\|^2 L_\phi^2\right)e^T e \tag{29}$$

Next, lets consider the following inequality proved in [9]

$$X^T Y + Y^T X \le X^T \Lambda X + Y^T \Lambda^{-1} Y \tag{30}$$

which holds for all matrices $X, Y \in \mathbb{R}^{nxk}$ and $\Lambda \in \mathbb{R}^{nxn}$ with $\Lambda = \Lambda^T > 0$. Applying (30) with $\Lambda = I$ to the term $\frac{e^T}{a}\widehat{W}\phi(e)$ from Eq. (29), where

$$e^T\widehat{W}\phi(e) \le \frac{1}{2}\|\mathrm{e}\|^2 + \frac{1}{2}L_\phi^2\|\widehat{W}\|^2\mathrm{e}^2 = \frac{1}{2}\left(1 + L_\phi^2\|\widehat{W}\|^2\right)\|\mathrm{e}\|^2$$

we get

$$aD_t^\alpha V \le \frac{-1}{a}\left(\lambda - 1 + K_p\right)e^T e + \frac{1}{a}\left(\frac{e^T e}{2} + \frac{1}{2}\|\widehat{W}\|^2 L_\phi^2\right)e^T e + \left(1 + \frac{K_i}{a}\right)e^T w - \frac{\gamma}{a}\left(\frac{1}{2} + \frac{1}{2}\|\widehat{W}\|^2 L_\phi^2\right)e^T e \tag{31}$$

Here, we select $\left(1 + \frac{K_i}{a}\right) = 0$ and $K_v = K_i + 1$, with $K_v \ge 0$ then $K_i \ge -1$, with this selection of the parameters from Eq. (31) is reduced to:

$$aD_t^\alpha V \le \frac{-1}{a}\left(\lambda - 1 + K_p\right)e^T e - \frac{(\gamma - 1)}{a}\left(\frac{1}{2} + \frac{1}{2}\|\widehat{W}\|^2 L_\phi^2\right)e^T e \tag{32}$$

Of the previous inequality, Eq. (32), we need that the fractional order Lyapunov derivative, $aD_t^{\alpha}V \le 0$, to ensure that the trajectory tracking error is asymptotically stable, that is, $\lim_{t\to\infty} e(t) = 0$, which means that the nonlinear system follows the reference signal.

To achieve this purpose, we select:

$$\lambda - 1 + K_p > 0, a > 0, (\gamma - 1) > 0, aD_t^{\alpha}V \le 0, \forall\, e,\ w,\ \widehat{W} \neq 0, e \neq 0$$

With the above Eq. (32), the control law that guarantees asymptotic stability of the tracking error is given by Eq. (33)

$$\begin{aligned} u = \Omega^{\dagger}[-\widehat{W}\Gamma\big(z(x) - z\big(x_p\big)\big) - (A + I)\big(x - x_p\big) + K_p e + K_i aD_t^{-\alpha}e + K_v aD_t^{\alpha}e \\ -\gamma\left(\frac{1}{2} + \frac{1}{2}\|\widehat{W}\|^2 L_{\phi}^2\right)e + f_r(x_r, u_r) - Ax_r - \widehat{W}\Gamma_z(x_r) - x_r + x_p \end{aligned} \tag{33}$$

**Theorem:** The control laws (Eq. (33)) and the adaptive weights (Eq. (28)) ensure that the trajectory tracking error between the fractional nonlinear system (Eq. (8)) and the fractional reference signal (Eq. (9)) satisfies $\lim_{t\to\infty} e(t) = 0$

**Remark 2:** *From Eq. (32), we have*

$aD_t^{\alpha}V \le \frac{-1}{a}\big(\lambda - 1 + K_p\big)e^T e - \frac{(\gamma - 1)}{a}\left(\frac{1}{2} + \frac{1}{2}\|\widehat{W}\|^2 L_{\phi}^2\right)e^T e < 0, \forall\, e \neq 0,\ \forall\, \widehat{W}$, where $V$ is decreasing and bounded from below by $V(0)$, and:

$V = \frac{1}{2}\big(e^T, w^T\big)(e, w)^T + \frac{1}{2a}tr\left\{\widetilde{W}^T\widetilde{W}\right\},$; then we conclude that $e,\ \widetilde{W} \in L_1$; this means that the weights remain bounded.

## 5. Simulation

The manipulator used for simulation is a two revolute joined robot (planar elbow manipulator), as shown in **Figure 2**.

The dynamics of the robot is established by [10, 11], $M_{ij}(q),\ \ i,\ j = 1, 2$ of the inertia matrix $M(q)$ as

$$M_{11}(q) = m_1 l_{c1}^2 + m_2\big(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos\big(q_2\big)\big) + I_1 + I_2;$$

$$M_{12}(q) = m_2\big(l_{c2}^2 + l_1 l_{c2}\cos\big(q_2\big)\big) + I_2;$$

$$M_{21}(q) = m_2\big(l_{c2}^2 + l_1 l_{c2}\cos\big(q_2\big)\big) + I_2;$$

$$M_{22}(q) = m_2 l_{c2}^2 + I_2.$$

$$C_{11}(q, \dot{q}) = -m_2 l_1 l_{c2}\sin\big(q_2\big)\dot{q}_2;$$

$$C_{21}(q, \dot{q}) = -m_2 l_1 l_{c2}\sin\big(q_2\big)\big(\dot{q}_1 + \dot{q}_2\big);$$

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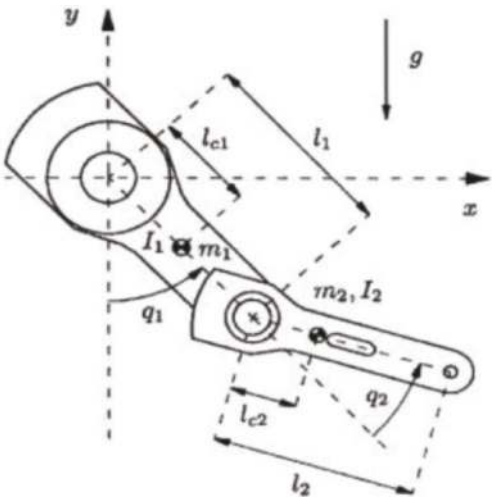


**Figure 2.** Diagram of the prototype planar robot with two degrees of freedom.

$$C_{21}(q,\dot{q}) = -m_2 l_1 l_{c2} \sin\left(q_2\right)\dot{q}_1$$

$$C_{22}(q,\dot{q}) = 0$$

And the torque vector $g(q)$:

$$g_1(q) = (m_1 l_{c1} + m_2 l_1) g sin\left(q_1\right) + m_2 l_{c2} g sin\left(q_1 + q_2\right);$$

$$g_2(q) = m_2 l_{c2} g sin\left(q_1 + q_2\right);$$

Thus, it is possible to write the equations of motion using the Lagrange equations for fractional manipulator system as [12]:

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos\left(\theta_2 - \theta_1\right) - m_2 l_1 l_2 \dot{\theta}_2^2 \sin\left(\theta_2 - \theta_1\right) + (m_1 + m_2) g l_1 \sin\left(\theta_1\right)$$
$$+ \frac{(\alpha - 1)}{(t - \tau)}\left[(m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_2 - \theta_1)\right] = Q_1$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\left(\theta_2 - \theta_1\right) + m_2 l_1 l_2 \dot{\theta}_1^2 sin(\theta_2 - \theta_1) + m_2 g l_2 \sin\left(\theta_2\right)$$
$$+ \frac{(\alpha - 1)}{(t - \tau)}\left[m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 cos(\theta_2 - \theta_1)\right] = Q_2$$

The terms containing $\alpha$ indicate the additional terms resulting from the fractional order model and the right-hand sides denote the generalized force terms resulting from the forcing functions, and there is a specific set of values for $Q_1$ and $Q_2$ for each case.

With the end of supporting the effectiveness of the proposed controller, we have used a Duffing equation.

The fractional order neural network is modelling by the differential equation:

$aD_t^{\alpha}x_p = \mathrm{A}(x) + \mathrm{W}^{*}\Gamma_z(x) + \Omega_{\mathrm{u}}$, with $A = -\lambda I, I \in \mathbb{R}^{4x4}$, and $\lambda = 20$, $\mathrm{W}^{*}$ is estimated using the learning law given in Eq. (28).

$\Gamma_z(x) = (\tanh(x_1),\ \tanh(x_2), \ldots, \tanh(x_n))^{\mathrm{T}}$, $\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$ and the $u$ is calculated using Eq. (33). The plant is stated in [3] and [13], and it is given by:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

We try to force this manipulator to track a reference signal [14] given by undamped Duffing equation:

$$\ddot{x} - x + x^3 = 0.114\cos(1.1t) : x(0) = 1, \dot{x}(0) = 0.114$$

To get the fractional order Duffing's system, this equation can be rewritten as a system of the first-order autonomous differential equations in the form [15]:

$$\frac{x(t)}{dt} = y(t)$$

$$\frac{y(t)}{dt} = x(t) - x^3(t) - \alpha y(t) + \delta\cos(\omega t)$$

Here, the conventional derivatives are replaced by the fractional derivatives as follows:

$$aD_t^{\alpha}x(t) = y(t)$$

$$aD_t^{\alpha}x(t) = x(t) - x^3(t) - \alpha y(t) + \delta\cos(\omega t)$$

where $\alpha$ is the fractional orders and $\alpha,\ \delta,\ \omega$ are the system parameters.

Illustrated, the response in the time, angular position and torque applied to the fractional nonlinear system are shown in **Figures 3–7**. As can be observed, the trajectory tracking objective is obtained

$$\alpha = 1, \quad \beta = 1$$

Its phase space trajectory is given in **Figure 8**, and the time evolution for the position angles and applied torque are shown in **Figures 9–12**. As can be seen in **Figures 9** and **10**, the trajectory tracking is successfully obtained where plant and reference signals are the same.

$$\alpha = 0.99, \quad \beta = 0.99$$

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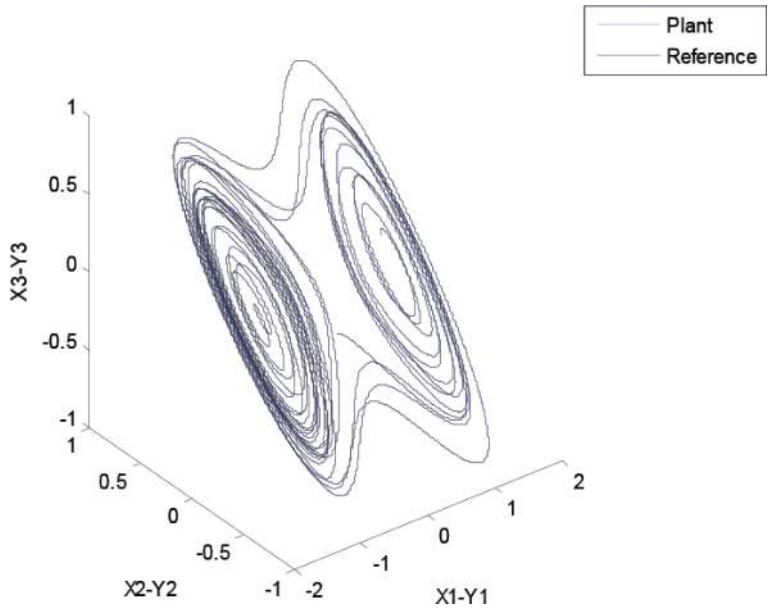


**Figure 3.** A phase space trajectory of Duffing equation.

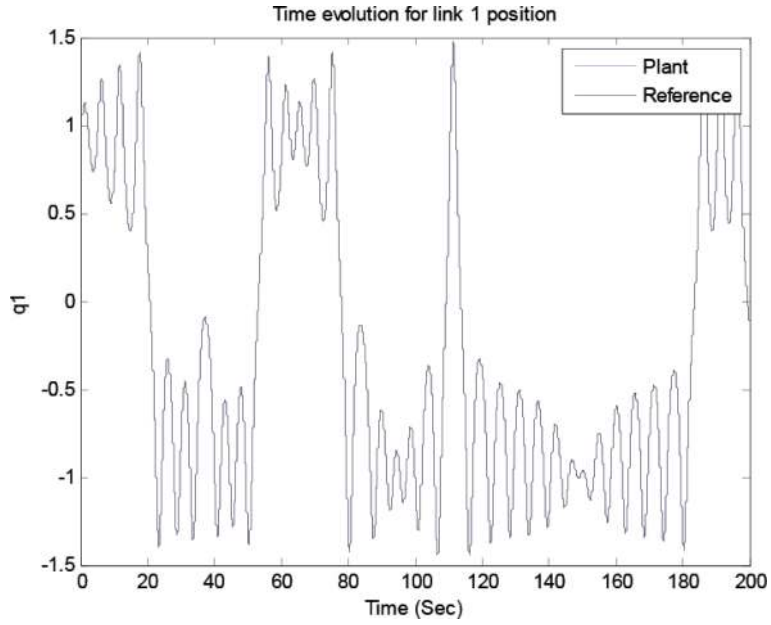


**Figure 4.** Time evolution for the angular position $q_1$ (rad) of link 1.

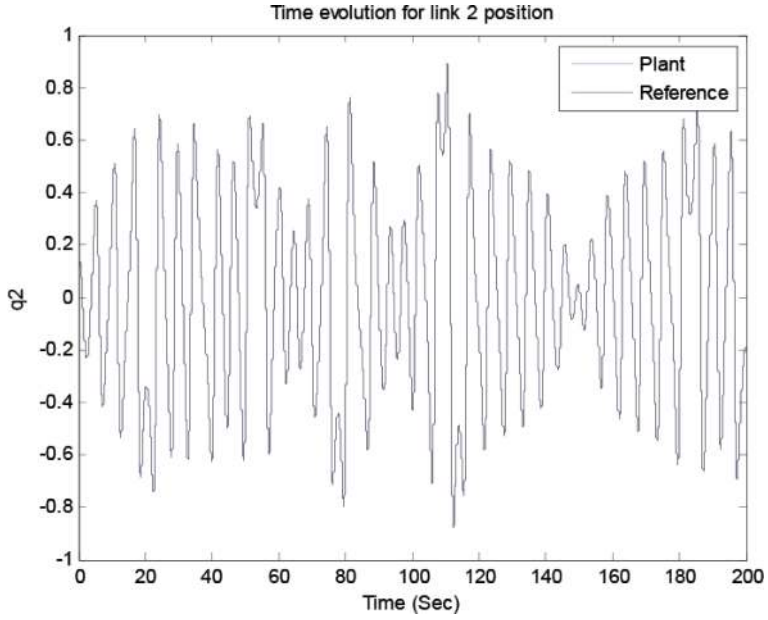


**Figure 5.** Time evolution for the angular position $q_2$ (rad) of link 2.

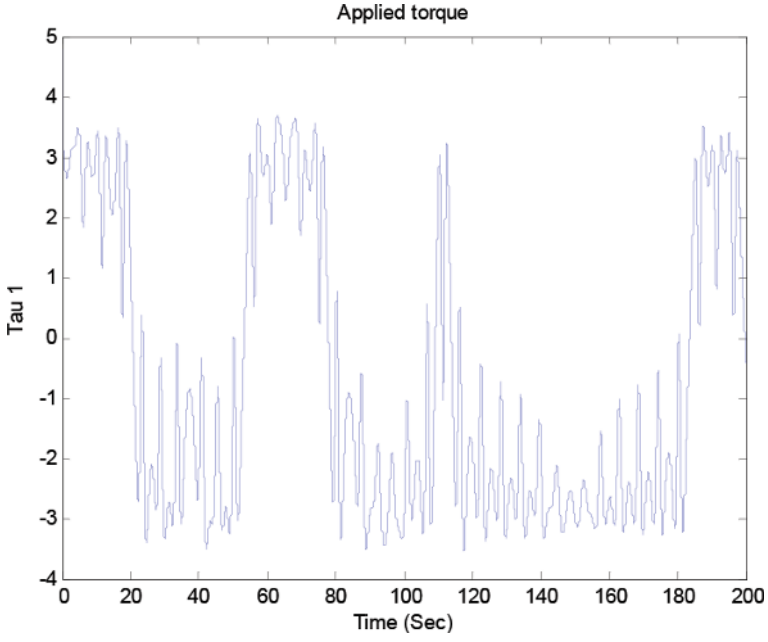


**Figure 6.** Torque (Nm) applied to link 1.

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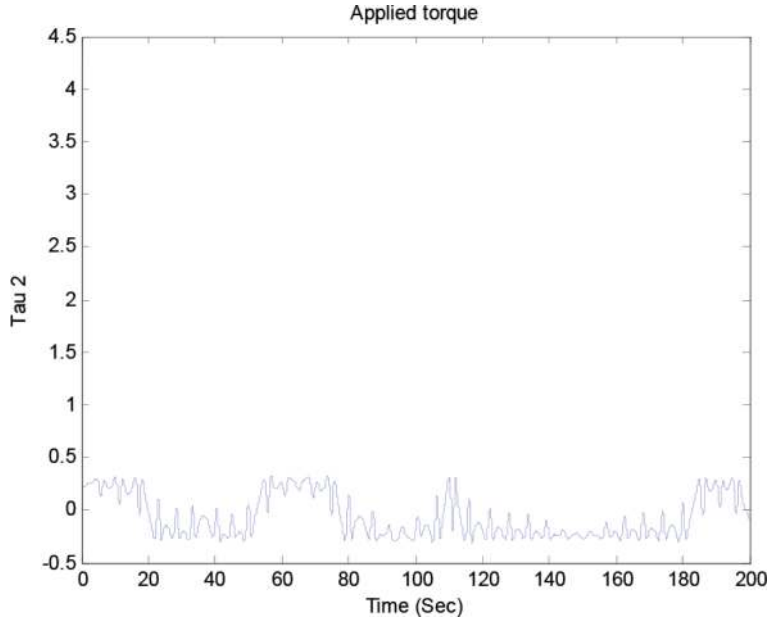


**Figure 7.** Torque (Nm) applied to link 2.

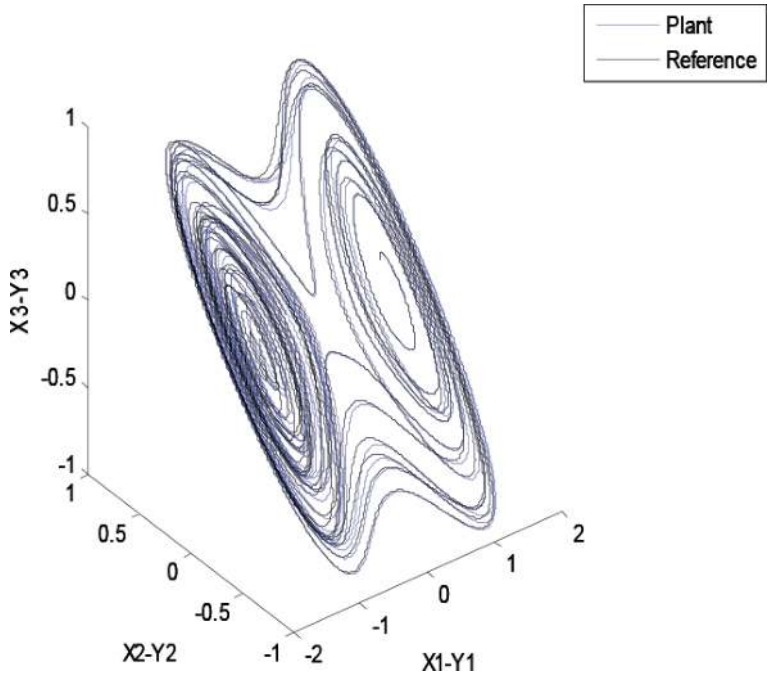


**Figure 8.** A phase space trajectory of Duffing equation.

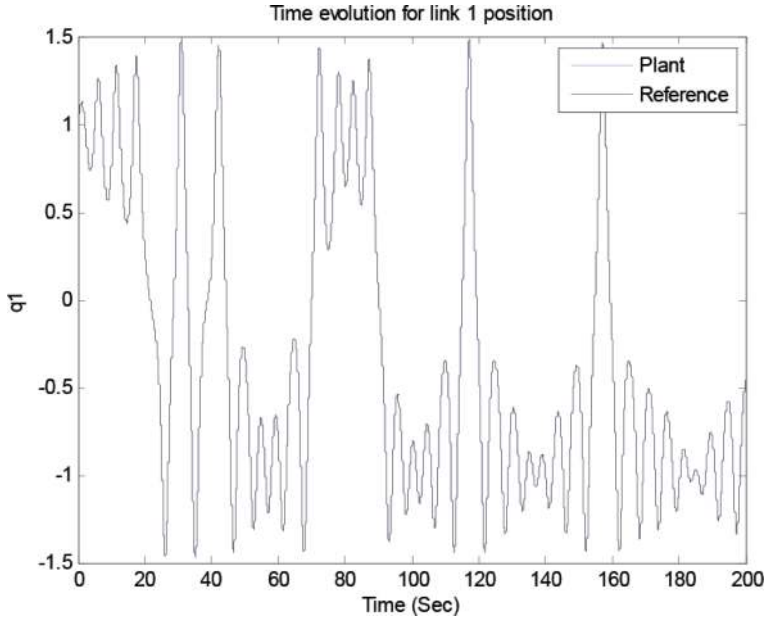


**Figure 9.** Time evolution for the angular position $q_1$ (rad) of link 1.

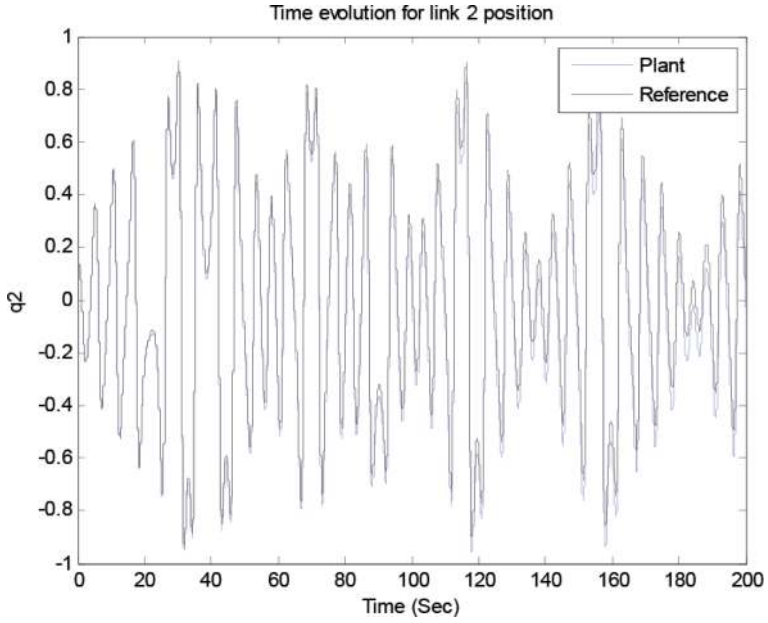


**Figure 10.** Time evolution for the angular position $q_2$ (rad) of link 2.

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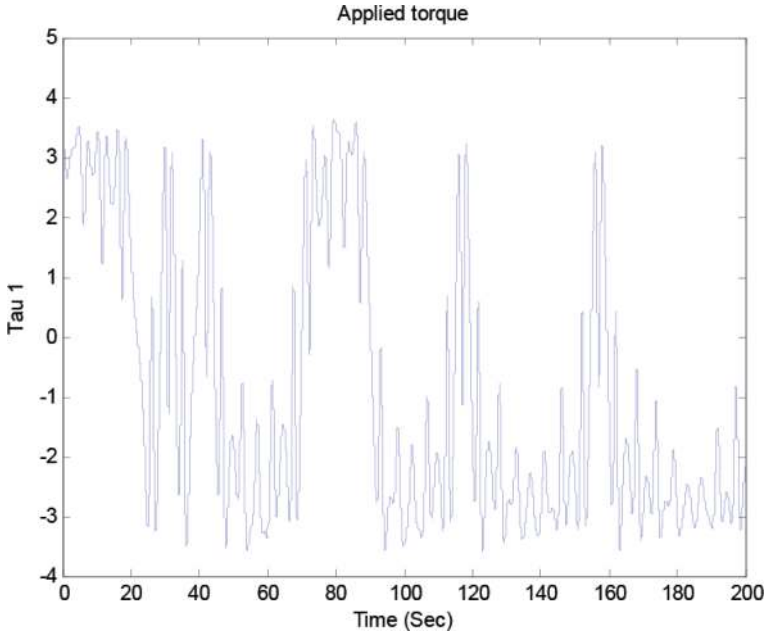


**Figure 11.** Torque (Nm) applied to link 1.

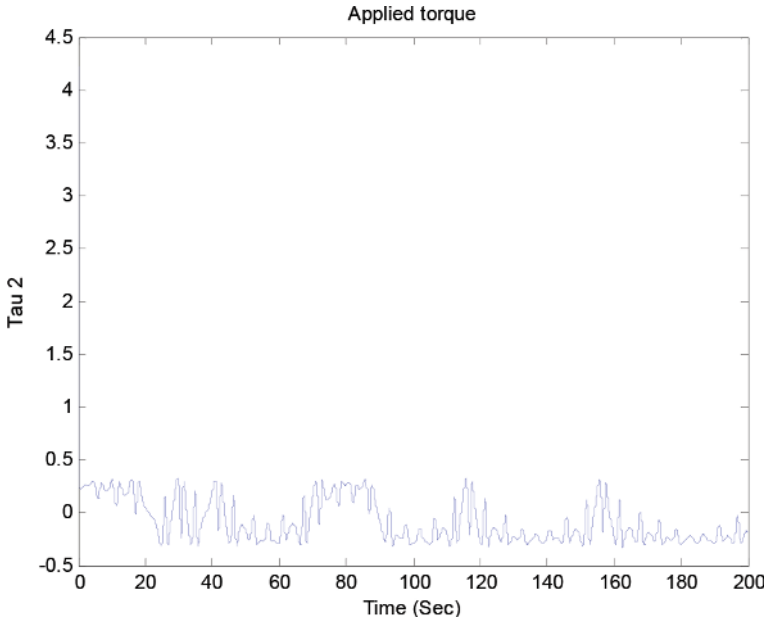


**Figure 12.** Torque (Nm) applied to link 2.

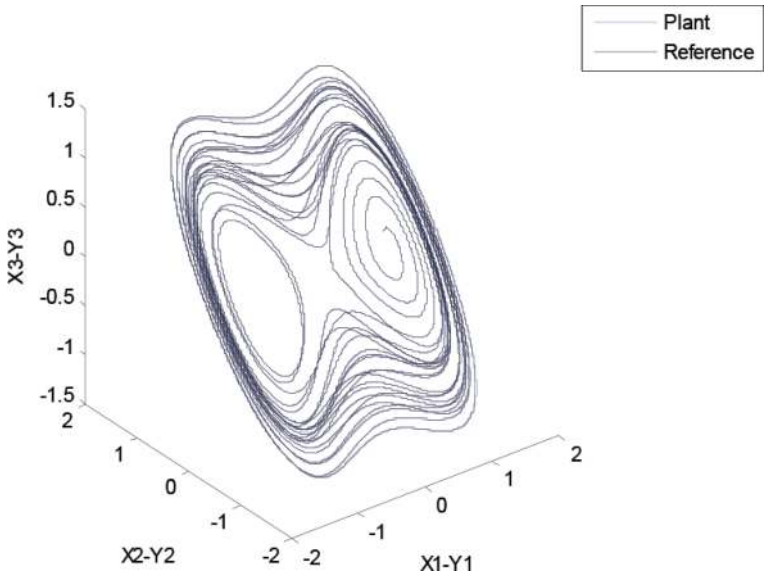


**Figure 13.** A phase space trajectory of Duffing equation.

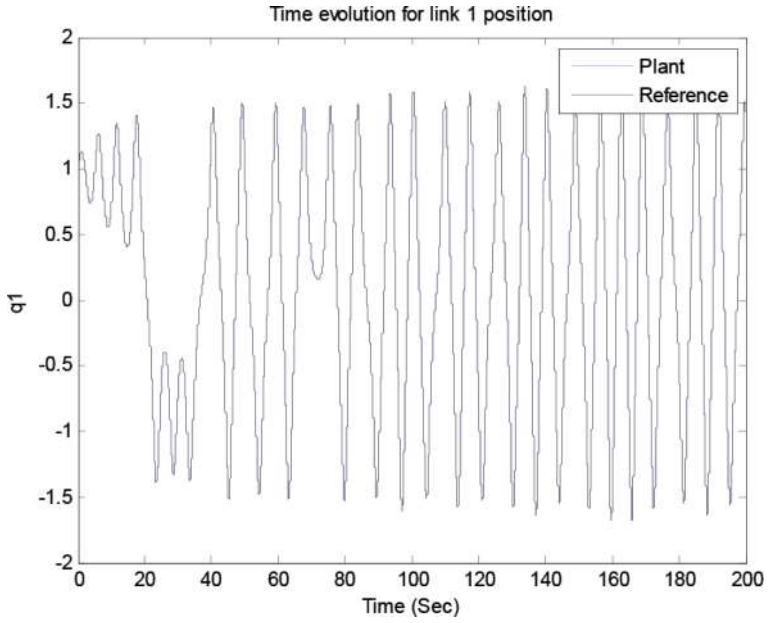


**Figure 14.** Time evolution for the angular position $q_1$ (rad) of link 1.

http://dx.doi.org/10.5772/intechopen.70020

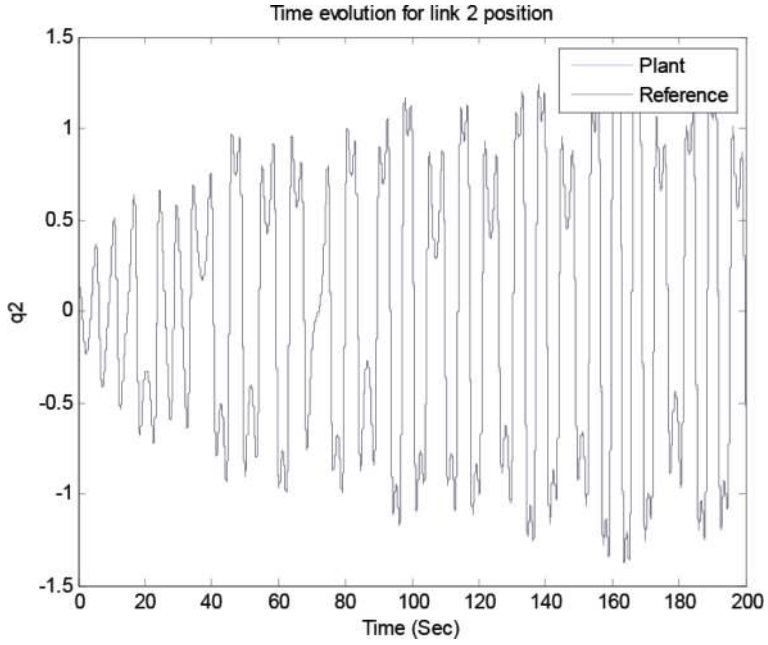


**Figure 15.** Time evolution for the angular position $q_2$ (rad) of link 2.

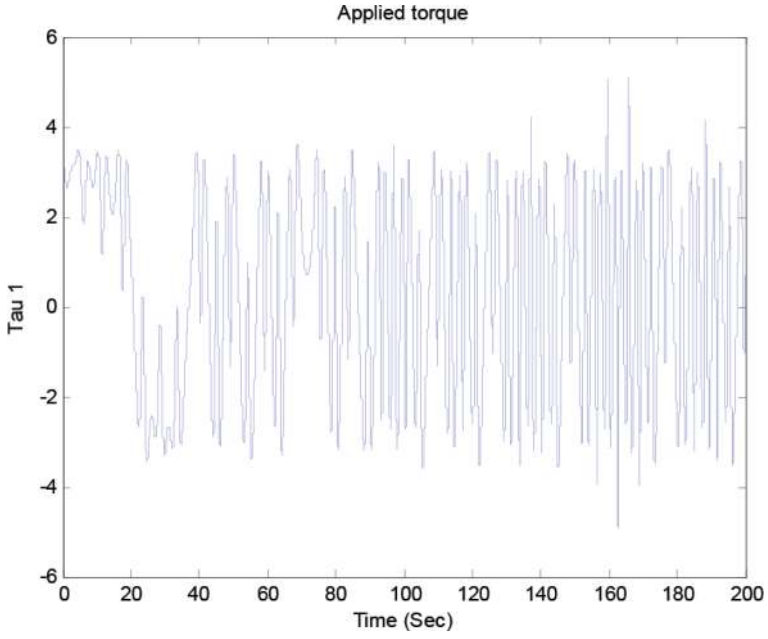


**Figure 16.** Torque (Nm) applied to link 1.

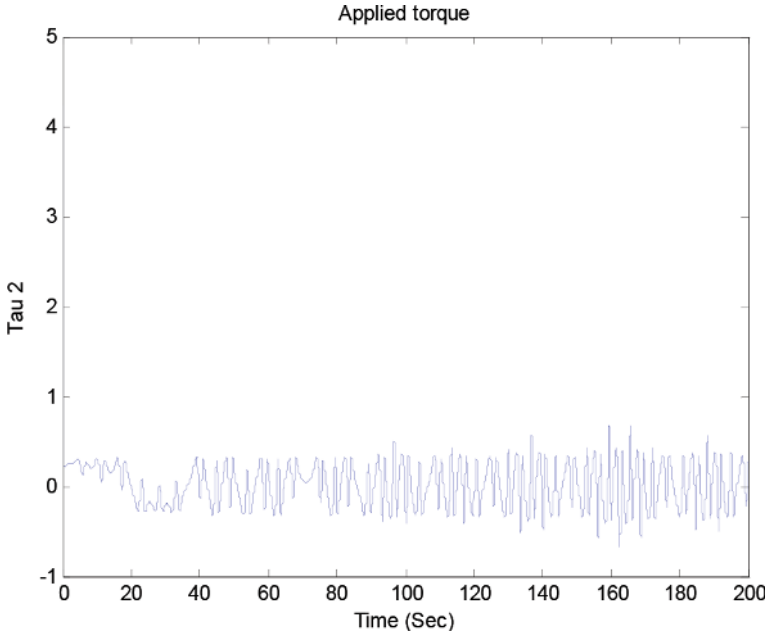


**Figure 17.** Torque (Nm) applied to link 2.

Its phase space trajectory is given in **Figure 13**, and the time evolution for the position angles and applied torque are shown in **Figures 14–17**. As can be seen in **Figures 14** and **15**, the trajectory tracking is successfully obtained where plant and reference signals are the same.

$$\alpha = 0.001, \quad \beta = 0.001$$

As can be observed, in the graphs of the trajectory tracking, the experimental results obtained in this chapter show a good experimental performance. The laws of control are obtained online, as well as the laws of adaptive weights in the fractional order neural network.

The control laws obtained are robust to modeling errors and nonmodeled dynamics (unknown nonlinear systems).

## 6. Conclusions

We have discussed the application of the stability analysis by Lyapunov of fractional order to follow trajectories of nonlinear systems whose mathematical model is unknown. The convergence of the tracking error is established by means of a Lyapunov function, as well as a control law based on Lyapunov and laws of adaptive weights of fractional order dynamical neural networks.

The results show a satisfactory performance of the fractional order dynamical neural network with online learning.

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## Author details

Joel Perez P.[1]*, Jose Paz Perez P.[1] and Martha S. Lopez de la Fuente[2]

*Address all correspondence to: joel.perezpd@uanl.edu.mx

1 Department of Physical and Mathematical Sciences, Autonomous University of Nuevo Leon (FCFM, UANL), San Nicolas de los Garza, Nuevo Leon, Mexico

2 University of Monterrey (UDEM), Mexico

3 Department of Electrical Engineering, Autonomous University of Nuevo Leon (FIME, UANL), San Nicolas de los Garza, Nuevo Leon, Mexico

4 Bilingual Progressive Bachelor, No. 16, Autonomous University of Nuevo Leon (Prepa 16, UANL), San Nicolas de los Garza, Nuevo Leon, Mexico

5 Department of Physical and Mathematical Sciences, Autonomous University of Nuevo Leon (UANL), San Nicolas de los Garza, Nuevo Leon, Mexico

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