Chapter 5

Nature of Phyllotaxy and Topology of H-matrix

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/intechopen.74676

Abstract

The main purpose of this chapter is to introduce a new type of regular matrix generated by Fibonacci numbers and we shall investigate its various topological properties. The concept of mathematical regularity in terms of Fibonacci numbers and phyllotaxy have been discussed.

Keywords: sequence spaces, infinite matrices, Fibonacci numbers, phyllotaxy
AMS Mathematical Subject Classification (2010); 46A45; 11B39d; 40C05

1. Preliminaries, background and notation

In several branches of analysis, for instance, the structural theory of topological vector spaces, Schauder basis theory, summability theory, and the theory of functions, the study of sequence spaces occupies a very prominent position. There is an ever-increasing interest in the theory of sequence spaces that has made remarkable advances in enveloping summability theory via unified techniques effecting matrix transformations from one sequence space into another.

Thus, we have several important applications of the theory of sequence spaces, and therefore, we attempt to present a survey on recent developments in sequence spaces and their different kinds of duals.

In many branches of science and engineering, we deal with different kinds of sequences and series, and when we deal with these, it is important to check their convergence. The use of infinite matrices is of great importance, we can bring even the bounded or divergent sequences and series in the domain of convergence. So we can say that the theory of sequence spaces and their matrix maps is the bigger scale to measure the convergence property. Summability can be roughly considered as the study of linear transformations on sequence spaces. The theory originated from the attempts of mathematicians to assign limits to divergent sequences. The classical summability theory deals with the generalization of the convergence of sequences or series of real or complex numbers. The idea is to assign a limit of some sort to divergent sequences or series by considering a transform of a sequence or series rather than the original sequence or series.

The earliest idea of summability theory was perhaps contained in a letter written by Leibnitz to C. Wolf (1713) in which he attributed the sum 1/2 to the oscillatory series $-1 + 1 - 1 + \ldots$. Frobenius in (1880) introduced the method of summability by arithmetic means, which was generalized by Cesàro in (1890) as the (C,K) method of summability. Toward the end of the nineteenth century, study of the general theory of sequences and transformations on them attracted mathematicians, who were chiefly motivated by problems such as those in summability theory, Fourier series, power series and system of equations with infinitely many variables.

Presenting some basic definitions and notations that are involved in the present work, the author proposes to give a brief resume of the hitherto obtained results against the background of which the main results studied in the present chapter suggest themselves.

2. Notations and symbols

Here, we state a few conventions which will be used throughout the chapter.

2.1. Symbols $\mathbb{N}$, $\mathbb{C}$, $\mathbb{R}$ and A

The symbols are denoted as follows:

$\mathbb{N}$: Set of non-negative integers.

$\mathbb{C}$: Set of complex numbers.

$\mathbb{R}$: Set of real numbers.

A: The infinite matrix (a_{nk}), $(n, k = 1, 2, \ldots)$.

2.2. Summation convention

By $\sum_{\alpha}^{\beta} f(n)$, we mean the sum of all values of $f(n)$ for which $\alpha \leq n \leq \beta$. In the case $\beta < \alpha$, then we take this to be zero.

Summations are over $0, 1, 2, \ldots$, when there is no indication to the contrary. If $(x_k) = (x_1, x_2, \ldots)$ is a sequence of terms, then, by $\sum_k x_k$ we mean $\sum_{k=1}^{\infty} x_k$ and we shall sometimes write as $\sum x_k$ incase where no possible confusion arises.

2.3. The spaces ω, l_∞, c, c_0, l_p

A sequence space is a set of scalar sequences (real or complex) which is closed under coordinate-wise addition and scalar multiplication. In other words, a sequence space is a linear subspace of the space ω of all complex sequences, that is,

$$\omega = \{\ x = (x_k) : x_k \in \mathbb{R}\ or\ \mathbb{C}\ \}.$$

The space l_∞: The space l_∞ of bounded sequences is defined by

$$\left\{\ x = (x_k) : \sup_k |x_k| < \infty \right\}$$

The spaces c: The spaces c and c_0 of convergent and null sequences are given by

$$\left\{\ x = (x_k) : \lim_k x_k = l, l \in \mathbb{C} \right\}$$

The space c_0: The space c_0 of all sequences converging to 0 is given by

$$\left\{\ x = (x_k) : \lim_k x_k = 0\ \right\}$$

The space l_p: The space l_p of absolutely p-summable sequences is defined by

$$\left\{\ x = (x_k) : \sum_k |x_k|^p < \infty \right\}, (0 < p < \infty)$$

The spaces l_∞, c, and c_0 are Banach spaces with the norm,

$$\|x\|_\infty = \sup_k |x_k|$$

The space l_p is a Banach space with the norm,

$$\|x\|_p = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}, 1 \le p < \infty$$

2.4. Cauchy sequence

A sequence $x = (x_k)$ is called a Cauchy sequence if and only if $|x^n - x^m| \to 0\ (m, n \to \infty)$ that is for any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that $|x^n - x^m| < \epsilon$ for all $n, m \ge N$. By $\mathfrak{C}$, we denote the space of all Cauchy sequences, that is,

$$\mathfrak{C} : \{x = (x_k) : |x^n - x^m| \to 0 \text{ as } n, m \to \infty\}$$

2.5. FK-space

A sequence space X is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbf{C}$ defined by $p_n(x) = x_n$ for all $x \in X$ and every $n \in \mathbf{N}$ [1, 2].

2.6. BK-space

A BK-space is a normed FK-space, that is, a BK-space is a Banach space with continuous coordinates [3–6].

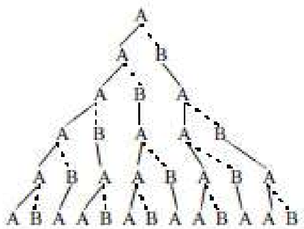
2.7. Fibonacci numbers

In the 1202 AD, Leonardo Fibonacci wrote in his book Liber Abaci of a simple numerical sequence that is the foundation for an incredible mathematical relationship behind phi. This sequence was known as early as the sixth century AD by Indian mathematicians, but it was Fibonacci who introduced it to the west after his travels throughout the Mediterranean world and North Africa. He is also known as Leonardo Bonacci, as his name is derived in Italian from words meaning son of (the) Bonacci.

The Fibonacci numbers have been introduced [7–14]. The Fibonacci numbers are the sequence of numbers $\{f_n\}, n \in \mathbf{N}$ defined by recurrence relations

$$f_0 = 0, f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}; n \geq 2$$

First derived from the famous rabbit problem of 1228, the Fibonacci numbers were originally used to represent the number of pairs of rabbits born of one pair in a certain population. Let us assume that a pair of rabbits is introduced into a certain place in the first month of the year. This pair of rabbits will produce one pair of offspring every month, and every pair of rabbits will begin to reproduce exactly 2 months after being born. No rabbit ever dies, and every pair of rabbits will reproduce perfectly on schedule.



Month	Pairs	Number of pairs of adults (*A*)	Number of pairs of babies (*B*)	Total pairs
January 1		1	0	1
February 1		1	1	2
March 1		2	1	3
April 1		3	2	5
May 1		5	3	8
June 1		8	5	13
July 1		13	8	21
August 1		21	13	34
September 1		34	21	55
October 1		55	34	89
November 1		89	55	144
December 1		144	89	233
January 1		233	144	377

The number of pairs of mature rabbits living each month determines the Fibonacci sequence (column 1): 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

http://dx.doi.org/10.5772/intechopen.74676

So, in the first month, we have only the first pair of rabbits. Likewise, in the second month, we again have only our initial pair of rabbits. However, by the third month, the pair will give birth to another pair of rabbits, and there will now be two pairs. Continuing on, we find that in month 4, we will have 3 pairs, then 5 pairs in month 5, then 8, 13, 21, 34, …, etc., continuing in this manner. It is quite apparent that this sequence directly corresponds with the Fibonacci sequence introduced above, and indeed, this is the first problem ever associated with the now-famous numbers.

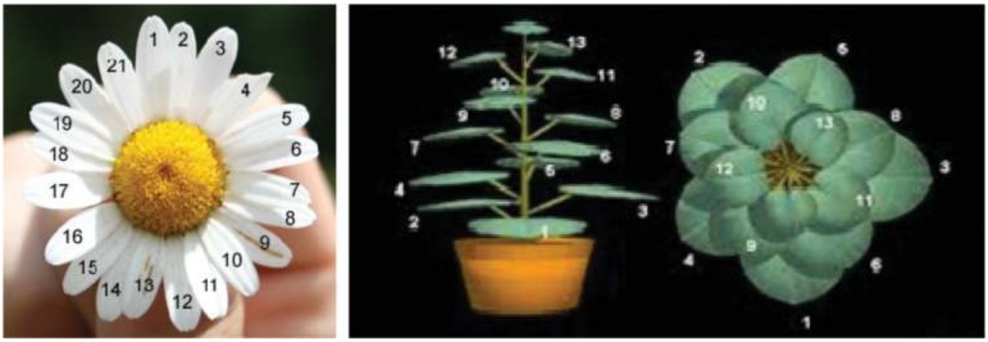
Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. Also, following [7], some basic properties are as follows

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1; n \in \mathbf{N},$$

and

$$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}; n \in \mathbf{N}$$

Everything in Nature is subordinated to stringent mathematical laws. Prove to be that leaf's disposition on plant's stems also has stringent mathematical regularity and this phenomenon is called phyllotaxis in botany. An essence of phyllotaxis consists in a spiral disposition of leaves on plant's stems of trees, petals in flower baskets, seeds in pine cone and sunflower head, etc.



This phenomenon, known already to Kepler, was a subject of discussion of many scientists, including Leonardo da Vinci, Turing, Veil, and so on. In phyllotaxis phenomenon, more complex concepts of symmetry, in particular, a concept of helical symmetry, are used. The phyllotaxis phenomenon reveals itself especially brightly in inflorescences and densely packed botanical structures such as pine cones, pineapples, cacti, heads of sunflower and cauliflower, and many other objects [11].

On the surfaces of such objects, their bio-organs (seeds on the disks of sunflower heads and pine cones, etc.) are placed in the form of the left-twisted and right-twisted spirals. For such phyllotaxis objects, it is used usually the number ratios of the left-hand and right-hand spirals observed on the surface of the phyllotaxis objects. Botanists proved that these ratios are equal to the ratios of the adjacent Fibonacci numbers, that is,

$$\frac{f_{i+1}}{f_i} : \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots = \frac{1+\sqrt{5}}{2}$$

By using hyperbolic Fibonacci functions, he had developed an original geometric theory of phyllotaxis and explained why Fibonacci spirals arise on the surface of the phyllotaxis objects namely, pine cones, cacti, pine apple, heads of sunflower, and so on, in process of their growths. Bodnar's geometry [15] confirms that these functions are 'natural' functions of the nature, which show their value in the botanic phenomenon of phyllotaxis. This fact allows us to assert that these functions can be attributed to the class of fundamental mathematical discoveries of contemporary science because they reflect natural phenomena, in particular, phyllotaxis phenomenon.

From above discussion, it gave us motivation to see the behavior of the infinite matrices generated by Fibonacci numbers.

In the present chapter, we have introduced a new type of matrix $H = \left(h^u_{nk}\right)$ $n, k \in \mathbb{N}$ by using Fibonacci numbers f_n and we call it as H-matrix generated by Fibonacci numbers f_n and introduce some new sequence spaces related to matrix domain of H in the sequence spaces l_p, l_∞, c and c_0, where $1 \le p < \infty$.

2.8. The space $r^q(u, p)$

Sheikh and Ganie [16] introduced the Riesz sequence space $r^q(u, p)$ and studied its various topological properties where $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$ and $\left(q_k\right)$ the sequence of positive numbers and

$$Q_n = \sum_{k=0}^{n} q_k, \forall n \in \mathbb{N}$$

Then, the matrix $R^q_u = \left(r^q_{nk}\right)$ of the Riesz mean $\left(R_u, q_n\right)$ is given by

$$r^q_{nk} = \begin{cases} \dfrac{u_k q_k}{Q_n} & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

The Riesz mean $\left(R_u, q_n\right)$ is regular if and only if $Q_n \to \infty$ as $n \to \infty$.

3. H-matrix generated by Fibonacci numbers

Let X and Y be two subsets of ω. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the A-transformation from X into Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y where

$$(Ax)_n = \sum_k a_{nk} x_k.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By (X, Y), we denote the class of all such matrices. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}, \tag{1}$$

which is a sequence space.

An infinite matrix $A = (a_{nk})$ is said to be regular if and only if the following conditions (or Toplitz conditions) hold [17–19]:

i. $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1,$

ii. $\lim_{n\to\infty} a_{nk} = 0, \quad (k = 0, 1, 2, \ldots),$

iii. $\sum_{k=0}^{\infty} |a_{nk}| < M, \quad (M > 0, j = 0, 1, 2, \ldots).$

In the present paper, we introduce H-matrix with $H = (h^u_{nk})$ $n, k \in \mathbb{N}$ as follows:

$$h^u_{nk} = \begin{cases} \dfrac{u_k {f_k}^2}{f_n f_{n+1}} & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Thus, for $u_k = 1$ and for all $k \in \mathbf{N}$, we have

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/6 & 1/6 & 4/6 & 0 & 0 & \cdots \\ 1/15 & 1/15 & 4/15 & 9/15 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

It is obvious that the matrix H is a triangle, that is, $h^u_{nn} \neq 0$ and $h^u_{nk} = 0$ for $k > n$ and for all $n \in \mathbb{N}$. Also, since it satisfies the conditions of Toeplitz matrix and hence it is regular matrix.

Note that if we take $q_k = f_k^2$, then the matrix H is special case of the matrix R^q_u, where

$$Q_n = \sum_{k=0}^{n} f_k^2 = f_n f_{n+1},$$

introduced by Sheikh and Ganie [16].

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors [17–26].

Throughout the text of the chapter, X denotes any of the spaces l_∞, c, c_0 and l_p $(1 \leq p < \infty)$. Then, the Fibonacci sequence space $X(H)$ is defined by

$$X(H) = \{x = (x_k) \in \omega : y = (y_k) \in X\},$$

where the sequence $y = (y_k)$ is the H-transform of the sequence $x = (x_k)$ and is given by

$$y_k = H_k(x) = \frac{1}{f_k f_{k+1}} \sum_{i=0}^{k} f_i^2 u_i x_i \text{ for all } k \in \mathbb{N}. \tag{2}$$

With the definition of matrix domain given by Eq. (1), we can redefine the space $X(H)$ as the matrix domain of the triangle H in the space X, that is,

$$X(H) = X_H.$$

Theorem 1: The space $X(H)$ is a BK-space with the norm given by

$$\|x\| = |H(x)\|_X = \|y\|_X = \begin{cases} \left[\sum_{k=0}^{\infty} |y_k|^p\right]^{\frac{1}{p}} & \text{for } \quad \textit{for } X \in \{l_p\}. \\ \sup_k y_k & \text{for } X \in \{l_\infty, c, c_0\}. \end{cases} \tag{3}$$

Proof: Since the matrix $H = (h_{nk}^u)$ is a triangle, that is, $h_{nn}^u \neq 0$ and $h_{nk}^u = 0$ for $k > n$ for all n. We have the result by Eq. (3) and Theorem 4.3.2 of Wilansky [6] gives the fact that $X(H)$ is a BK-space.◊

Theorem 2: The space $X(H)$ is isometrically isomorphic to the space X.

Proof: To prove the result, we should show the linear bijection between the spaces $X(H)$ and X. For that, consider the transformation T from $X(H)$ to X by $x \to y = Tx$. Then, the linearity of T follows from Eq. (2). Further, we see that $x = 0$ whenever $Tx = 0$ and consequently T is injective.

Moreover, let $y = (y_k) \in X$ be given and define the sequence $x = (x_k)$ by

$$x_k = \frac{f_{k+1}}{u_k f_k} y_k - \frac{f_{k-1}}{u_k f_k} y_{k-1}; k \in \mathbf{N}. \tag{4}$$

Then, by using (2) and (4), we have for every $k \in \mathbf{N}$ that

$$\begin{aligned} H(x) &= \frac{1}{f_k f_{k+1}} \sum_{i=0}^{k} f_i^2 u_i x_i \\ &= \frac{1}{f_k f_{k+1}} \sum_{i=0}^{k} f_i (f_{i+1} y_i - f_{i-1} y_{i-1}) \\ &= y_k. \end{aligned}$$

This shows that $H(x) = y$ and since $y \in X$, we conclude that $H(x) \in X$. Thus, we deduce that $x \in X(H)$ and $Tx = y$. Hence, T is surjective.

Furthermore, for any $x \in X(H)$, we have by (3) that

$$\|T(x)\| = \|y\| = \|H(x)\|_X = \|x\|_X$$

which shows that T is norm preserving. Hence, T is isometry. Consequently, the spaces $X(H)$ and X are isometrically isomorphic. Hence, the proof of the Theorem is complete.◊

Theorem 3: Let $\{f_j\}$ be Fibonacci number sequences. Then, we have

$$\sup_i \left(f_i^2 \sum_{j=i}^{\infty} \frac{1}{f_j f_{j+1}} \right) < \infty.$$

Proof: We have,

$$\sum_{k=n}^{\infty} \left(\frac{1}{f_k} - \frac{1}{f_{k+1}} \right) = \frac{1}{f_n}$$

This gives,

$$\begin{aligned} 1 &= f_n \sum_{k=n}^{\infty} \left(\frac{1}{f_k} - \frac{1}{f_{k+1}} \right) \\ &= f_n^2 \frac{1}{f_n} \sum_{k=n}^{\infty} \left(\frac{f_{k+1} - f_k}{f_k f_{k+1}} \right) \\ &= f_n^2 \frac{1}{f_n} \sum_{k=n}^{\infty} \left(\frac{f_{k-1}}{f_k f_{k+1}} \right) \\ &\geq f_n^2 \frac{f_{n-1}}{f_n} \sum_{k=n}^{\infty} \left(\frac{1}{f_k f_{k+1}} \right) \end{aligned}$$

and the conclusion follows because $f_n f_{n-1}$ is bounded since it converges to $\frac{\sqrt{5}+1}{2}$.◊

Theorem 4: $X \subset X(H)$ holds.

Proof: It is obvious that $c_0 \subset c_0(H)$ and $c \subset c(H)$, since the matrix H is regular matrix. Now, let $x \in l_\infty$. Then, there is a constant $K > 0$ such that $|x_j| < \frac{K}{|u_j|}$ for all $j \in \mathbf{N}$. Thus, we have for every $i \in \mathbf{N}$ that

$$\begin{aligned} |H_i(x)| &\leq \frac{1}{f_i f_{i+1}} \sum_{j=0}^{i} f_j^2 |u_j x_j| \\ &\leq \frac{K}{f_i f_{i+1}} \sum_{j=0}^{i} f_j^2 = K \end{aligned}$$

which shows that $H(x) \in l_\infty$. Therefore, we deduce that $x \in l_\infty$ implies $x \in l_\infty(H)$.

We now consider the case $1 \leq p < \infty$. We only consider the case $1 < p < \infty$ and by similar argument will follow for $p = 1$. So, let $x \in l_p$. Then, for every $i \in \mathbf{N}$ and by Holder's inequality, we have

$$|H_i(x)|^p \leq \left(\sum_{j=0}^{i} \frac{f_j^2}{f_i f_{i+1}} |u_j x_j| \right)^p$$

$$\leq \left(\sum_{j=0}^{i} \frac{f_j^2}{f_i f_{i+1}} |u_j x_j| \right)^p \left(\sum_{j=0}^{i} \frac{f_j^2}{f_i f_{i+1}} \right)^{p-1}$$

$$= \frac{1}{f_i f_{i+1}} \sum_{j=0}^{i} f_j^2 |u_j x_j|^p.$$

Hence, we have

$$\sum_{i=0}^{\infty} |H_i(x)|^p \leq \sum_{i=0}^{\infty} \frac{1}{f_i f_{i+1}} \sum_{j=0}^{i} f_j^2 |u_j x_j|^p$$

$$= \sum_{i=0}^{\infty} |x_j|^p |u_j|^p f_j^2 \sum_{i=j}^{\infty} \frac{1}{f_i f_{i+1}}.$$

Hence, the right-hand side of above inequality can be made arbitrary small, since, $\sup_j \left(f_j^2 \sum_{i=j}^{\infty} \frac{1}{f_i f_{i+1}} \right) < \infty$ by Theorem 3 (above) and $x \in l_p$. This shows that $x \in l_p(H)$. This completes the proof of the theorem.◊

Acknowledgements

The author would like to express his sincere thanks for the refree(s) for the kind remarks that improved the presentation of the chapter.

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