

Cramer's Rules for the System of Two-Sided Matrix Equations and of Its Special Cases

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Abstract

Within the framework of the theory of row-column determinants previously introduced by the author, we get determinantal representations (analogs of Cramer's rule) of a partial solution to the system of two-sided quaternion matrix equations $A_1XB_1=C_1$, $A_2XB_2=C_2$. We also give Cramer's rules for its special cases when the first equation be one-sided. Namely, we consider the two systems with the first equation $A_1X=C_1$ and $XB_1=C_1$, respectively, and with an unchanging second equation. Cramer's rules for special cases when two equations are one-sided, namely the system of the equations $A_1X=C_1$, $XB_2=C_2$, and the system of the equations $A_1X=C_1$, $A_2X=C_2$ are studied as well. Since the Moore-Penrose inverse is a necessary tool to solve matrix equations, we use its determinantal representations previously obtained by the author in terms of row-column determinants as well.

Keywords: Moore-Penrose inverse, quaternion matrix, Cramer rule, system matrix equations

2000 AMS subject classifications: 15A15, 16 W10

1. Introduction

The study of matrix equations and systems of matrix equations is an active research topic in matrix theory and its applications. The system of classical two-sided matrix equations

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2. \end{cases} \quad (1)$$

over the complex field, a principle domain, and the quaternion skew field has been studied by many authors (see, e.g. [1–7]). Mitra [1] gives necessary and sufficient conditions of the system

(1) over the complex field and the expression for its general solution. Navarra et al. [6] derived a new necessary and sufficient condition for the existence and a new representation of (1) over the complex field and used the results to give a simple representation. Wang [7] considers the system (1) over the quaternion skew field and gets its solvability conditions and a representation of a general solution.

Throughout the chapter, we denote the real number field by \mathbb{R} , the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$ and by $\mathbb{H}_r^{m \times n}$, and the set of matrices over \mathbb{H} with a rank r . For $\mathbf{A} \in \mathbb{H}^{n \times m}$, the symbols \mathbf{A}^* stands for the conjugate transpose (Hermitian adjoint) matrix of \mathbf{A} . The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n \times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

Generalized inverses are useful tools used to solve matrix equations. The definitions of the Moore-Penrose inverse matrix have been extended to quaternion matrices as follows. The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m \times n}$, denoted by \mathbf{A}^\dagger , is the unique matrix $\mathbf{X} \in \mathbb{H}^{n \times m}$ satisfying (1) $\mathbf{AXA} = \mathbf{A}$, (2) $\mathbf{XAX} = \mathbf{X}$, (3) $(\mathbf{AX})^* = \mathbf{AX}$, and (4) $(\mathbf{XA})^* = \mathbf{XA}$.

The determinantal representation of the usual inverse is the matrix with the cofactors in the entries which suggests a direct method of finding of inverse and makes it applicable through Cramer's rule to systems of linear equations. The same is desirable for the generalized inverses. But there is not so unambiguous even for complex or real generalized inverses. Therefore, there are various determinantal representations of generalized inverses because of looking for their more applicable explicit expressions (see, e.g. [8]). Through the noncommutativity of the quaternion algebra, difficulties arise already in determining the quaternion determinant (see, e.g. [9–16]).

The understanding of the problem for determinantal representation of an inverse matrix as well as generalized inverses only now begins to be decided due to the theory of column-row determinants introduced in [17, 18]. Within the framework of the theory of column-row determinants, determinantal representations of various kinds of generalized inverses and (generalized inverses) solutions of quaternion matrix equations have been derived by the author (see, e.g. [19–25]) and by other reseachers (see, e.g. [26–29]).

The main goals of the chapter are deriving determinantal representations (analogs of the classical Cramer rule) of general solutions of the system (1) and its simpler cases over the quaternion skew field.

The chapter is organized as follows. In Section 2, we start with preliminaries introducing of row-column determinants and determinantal representations of the Moore-Penrose and Cramer's rule of the quaternion matrix equations, $\mathbf{AXB} = \mathbf{C}$. Determinantal representations of a partial solution (an analog of Cramer's rule) of the system (1) are derived in Section 3. In Section 4, we give Cramer's rules to special cases of (1) with 1 and 2 one-sided equations. Finally, the conclusion is drawn in Section 5.

2. Preliminaries

For $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$, we define n row determinants and n column determinants as follows. Suppose S_n is the symmetric group on the set $I_n = \{1, \dots, n\}$.

Definition 2.1. The i th row determinant of $\mathbf{A} \in \mathbb{H}^{n \times m}$ is defined for all $i = 1, \dots, n$ by putting

$$\text{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} \left(a_{i i_{k_1}} a_{i k_1 i_{k_1+1}} \dots a_{i k_1+i_1 i} \right) \dots \left(a_{i k_r i_{k_r+1}} \dots a_{i k_r+l_r i_{k_r}} \right),$$

$$\sigma = (i i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}),$$

with conditions $i_{k_2} < i_{k_3} < \dots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2, \dots, r$ and all $s = 1, \dots, l_t$.

Definition 2.2. The j th column determinant of $\mathbf{A} \in \mathbb{H}^{n \times m}$ is defined for all $j = 1, \dots, n$ by putting

$$\text{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} \left(a_{j k_r j_{k_r+l_r}} \dots a_{j k_r+1 i_{k_r}} \right) \dots \left(a_{j j_{k_1+1}} \dots a_{j k_1+1 j_{k_1}} a_{j k_1 j} \right),$$

$$\tau = (j_{k_r+l_r} \dots j_{k_r+1} j_{k_r}) \dots (j_{k_2+l_2} \dots j_{k_2+1} j_{k_2}) (j_{k_1+l_1} \dots j_{k_1+1} j_{k_1} j),$$

with conditions, $j_{k_2} < j_{k_3} < \dots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for $t = 2, \dots, r$ and $s = 1, \dots, l_t$.

Since $\text{rdet}_1 \mathbf{A} = \dots = \text{rdet}_n \mathbf{A} = \text{cdet}_1 \mathbf{A} = \dots = \text{cdet}_n \mathbf{A} \in \mathbb{R}$ for Hermitian $\mathbf{A} \in \mathbb{H}^{n \times n}$, then we can define the determinant of a Hermitian matrix \mathbf{A} by putting, $\det \mathbf{A} := \text{rdet}_i \mathbf{A} = \text{cdet}_j \mathbf{A}$, for all $i = 1, \dots, n$. The determinant of a Hermitian matrix has properties similar to a usual determinant. They are completely explored in [17, 18] by its row and column determinants. In particular, within the framework of the theory of the column-row determinants, the determinantal representations of the inverse matrix over \mathbb{H} by analogs of the classical adjoint matrix and Cramer's rule for quaternionic systems of linear equations have been derived. Further, we consider the determinantal representations of the Moore-Penrose inverse.

We shall use the following notations. Let $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. \mathbf{A}_β^α denotes the submatrix of $\mathbf{A} \in \mathbb{H}^{n \times m}$ determined by the rows indexed by α and the columns indexed by β . Then, \mathbf{A}_α^α denotes the principal submatrix determined by the rows and columns indexed by α . If $\mathbf{A} \in \mathbb{H}^{n \times n}$ is Hermitian, then $|\mathbf{A}_\alpha^\alpha|$ is the corresponding principal minor of $\det \mathbf{A}$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of k integers chosen from $\{1, \dots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 < \dots < \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Let \mathbf{a}_j be the j th column and \mathbf{a}_i be the i th row of \mathbf{A} . Suppose $\mathbf{A}_{\cdot j}(\mathbf{b})$ denotes the matrix obtained from \mathbf{A} by replacing its j th column with the column \mathbf{b} , then $\mathbf{A}_i(\mathbf{b})$ denotes the matrix obtained

from \mathbf{A} by replacing its i th row with the row \mathbf{b} . \mathbf{a}_j^* and \mathbf{a}_i^* denote the j th column and the i th row of \mathbf{A}^* , respectively.

The following theorem gives determinantal representations of the Moore-Penrose inverse over the quaternion skew field \mathbb{H} .

Theorem 2.1. [19] *If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then the Moore-Penrose inverse $\mathbf{A}^\dagger = (a_{ij}^\dagger) \in \mathbb{H}^{n \times m}$ possesses the following determinantal representations:*

$$a_{ij}^\dagger = \frac{\sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{a}_j^*))_\beta^\beta}{\sum_{\beta \in I_{r,n}} |\mathbf{A}^* \mathbf{A}|_\beta^\beta}, \quad (2)$$

or

$$a_{ij}^\dagger = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((\mathbf{A} \mathbf{A}^*)_{.j}(\mathbf{a}_i^*))_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |\mathbf{A} \mathbf{A}^*|_\alpha^\alpha}. \quad (3)$$

Remark 2.1. *Note that for an arbitrary full-rank matrix, $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, a column-vector \mathbf{d}_j , and a row-vector \mathbf{d}_i , with appropriate sizes, respectively, we put*

$$\begin{aligned} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{d}_j)) &= \sum_{\beta \in I_{n,n}\{i\}} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{d}_j))_\beta^\beta, \quad \det(\mathbf{A}^* \mathbf{A}) = \sum_{\beta \in I_{n,n}} |\mathbf{A}^* \mathbf{A}|_\beta^\beta \text{ when } r = n, \\ \text{rdet}_j((\mathbf{A} \mathbf{A}^*)_{.j}(\mathbf{d}_i)) &= \sum_{\alpha \in I_{m,m}\{j\}} \text{rdet}_j((\mathbf{A} \mathbf{A}^*)_{.j}(\mathbf{d}_i))_\alpha^\alpha, \quad \det(\mathbf{A} \mathbf{A}^*) = \sum_{\alpha \in I_{m,m}} |\mathbf{A} \mathbf{A}^*|_\alpha^\alpha \text{ when } r = m. \end{aligned}$$

Furthermore, $\mathbf{P}_A = \mathbf{A}^\dagger \mathbf{A}$, $\mathbf{Q}_A = \mathbf{A} \mathbf{A}^\dagger$, $\mathbf{L}_A = \mathbf{I} - \mathbf{A}^\dagger \mathbf{A}$, and $\mathbf{R}_A = \mathbf{I} - \mathbf{A} \mathbf{A}^\dagger$ stand for some orthogonal projectors induced from \mathbf{A} .

Theorem 2.2. [30] *Let $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{B} \in \mathbb{H}^{r \times s}$, and $\mathbf{C} \in \mathbb{H}^{m \times s}$ be known and $\mathbf{X} \in \mathbb{H}^{n \times r}$ be unknown. Then, the matrix equation*

$$\mathbf{A} \mathbf{X} \mathbf{B} = \mathbf{C} \quad (4)$$

is consistent if and only if $\mathbf{A} \mathbf{A}^\dagger \mathbf{C} \mathbf{B} \mathbf{B}^\dagger = \mathbf{C}$. In this case, its general solution can be expressed as

$$\mathbf{X} = \mathbf{A}^\dagger \mathbf{C} \mathbf{B}^\dagger + \mathbf{L}_A \mathbf{V} + \mathbf{W} \mathbf{R}_B, \quad (5)$$

where \mathbf{V} and \mathbf{W} are arbitrary matrices over \mathbb{H} with appropriate dimensions.

The partial solution, $\mathbf{X}^0 = \mathbf{A}^\dagger \mathbf{C} \mathbf{B}^\dagger$, of (4) possesses the following determinantal representations.

Theorem 2.3. [20] Let $\mathbf{A} \in \mathbb{H}_{r_1}^{m \times n}$ and $\mathbf{B} \in \mathbb{H}_{r_2}^{r \times s}$. Then, $\mathbf{X}^0 = (x_{ij}^0) \in \mathbb{H}^{n \times r}$ has determinantal representations,

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{A}^* \mathbf{A})_{.i} \left(\mathbf{d}_{.j}^B \right) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_1, n}} |\mathbf{A}^* \mathbf{A}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_2, r}} |\mathbf{B} \mathbf{B}^*|_{\alpha}^{\alpha}},$$

or

$$x_{ij} = \frac{\sum_{\alpha \in I_{r_2, r} \setminus \{j\}} \text{rdet}_j \left((\mathbf{B} \mathbf{B}^*)_{.j} \left(\mathbf{d}_{.i}^A \right) \right)_{\alpha}^{\alpha}}{\sum_{\beta \in J_{r_1, n}} |\mathbf{A}^* \mathbf{A}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_2, r}} |\mathbf{B} \mathbf{B}^*|_{\alpha}^{\alpha}},$$

where

$$\mathbf{d}_{.j}^B = \left[\sum_{\alpha \in I_{r_2, r} \setminus \{j\}} \text{rdet}_j \left((\mathbf{B} \mathbf{B}^*)_{.j} (\tilde{\mathbf{c}}_{.k}) \right)_{\alpha}^{\alpha} \right] \in \mathbb{H}^{n \times 1}, \quad k = 1, \dots, n,$$

$$\mathbf{d}_{.i}^A = \left[\sum_{\beta \in J_{r_1, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{A}^* \mathbf{A})_{.i} (\tilde{\mathbf{C}}_{.l}) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1 \times r}, \quad l = 1, \dots, r,$$

are the column vector and the row vector, respectively. $\tilde{\mathbf{c}}_{.i}$ and $\tilde{\mathbf{c}}_{.j}$ are the i th row and the j th column of $\tilde{\mathbf{C}} = \mathbf{A}^* \mathbf{C} \mathbf{B}^*$.

3. Determinantal representations of a partial solution to the system (1)

Lemma 3.1. [7] Let $\mathbf{A}_1 \in \mathbb{H}^{m \times n}$, $\mathbf{B}_1 \in \mathbb{H}^{r \times s}$, $\mathbf{C}_1 \in \mathbb{H}^{m \times s}$, $\mathbf{A}_2 \in \mathbb{H}^{k \times n}$, $\mathbf{B}_2 \in \mathbb{H}^{r \times p}$, and $\mathbf{C}_2 \in \mathbb{H}^{k \times p}$ be given and $\mathbf{X} \in \mathbb{H}^{n \times r}$ is to be determined. Put $\mathbf{H} = \mathbf{A}_2 \mathbf{L}_{A_1}$, $\mathbf{N} = \mathbf{R}_{B_1} \mathbf{B}_2$, $\mathbf{T} = \mathbf{R}_H \mathbf{A}_2$, and $\mathbf{F} = \mathbf{B}_2 \mathbf{L}_N$. Then, the system (1) is consistent if and only if

$$\mathbf{A}_i \mathbf{A}_i^{\dagger} \mathbf{C}_i \mathbf{B}_i^{\dagger} \mathbf{B}_i = \mathbf{C}_i, \quad i = 1, 2; \tag{6}$$

$$\mathbf{T} [\mathbf{A}_2^{\dagger} \mathbf{X} \mathbf{B}_2^{\dagger} - \mathbf{A}_1^{\dagger} \mathbf{C}_1 \mathbf{B}_1^{\dagger}] \mathbf{F} = \mathbf{0}. \tag{7}$$

In that case, the general solution of (1) can be expressed as the following,

$$\begin{aligned} \mathbf{X} = & \mathbf{A}_1^{\dagger} \mathbf{C}_1 \mathbf{B}_1^{\dagger} + \mathbf{L}_{A_1} \mathbf{H}^{\dagger} \mathbf{A}_2 \mathbf{L}_T (\mathbf{A}_2^{\dagger} \mathbf{C}_2 \mathbf{B}_2^{\dagger} - \mathbf{A}_1^{\dagger} \mathbf{C}_1 \mathbf{B}_1^{\dagger}) \mathbf{B}_2 \mathbf{B}_2^{\dagger} + \mathbf{T}^{\dagger} \mathbf{T} (\mathbf{A}_2^{\dagger} \mathbf{C}_2 \mathbf{B}_2^{\dagger} - \mathbf{A}_1^{\dagger} \mathbf{C}_1 \mathbf{B}_1^{\dagger}) \mathbf{B}_2 \mathbf{N}^{\dagger} \mathbf{R}_{B_1} \\ & + \mathbf{L}_{A_1} (\mathbf{Z} - \mathbf{H}^{\dagger} \mathbf{H} \mathbf{Z} \mathbf{B}_2 \mathbf{B}_2^{\dagger}) - \mathbf{L}_{A_1} \mathbf{H}^{\dagger} \mathbf{A}_2 \mathbf{L}_T \mathbf{W} \mathbf{N} \mathbf{B}_2^{\dagger} + (\mathbf{W} - \mathbf{T}^{\dagger} \mathbf{T} \mathbf{W} \mathbf{N} \mathbf{N}^{\dagger}) \times \mathbf{R}_{B_1}, \end{aligned} \tag{8}$$

where \mathbf{Z} and \mathbf{W} are the arbitrary matrices over \mathbb{H} with compatible dimensions.

Some simplification of (8) can be derived due to the quaternionic analog of the following proposition.

Lemma 3.2. [32] *If $\mathbf{A} \in \mathbb{H}^{n \times n}$ is Hermitian and idempotent, then the following equation holds for any matrix $\mathbf{B} \in \mathbb{H}^{m \times n}$,*

$$\mathbf{A}(\mathbf{B}\mathbf{A})^\dagger = (\mathbf{B}\mathbf{A})^\dagger. \quad (9)$$

It is evident that if $\mathbf{A} \in \mathbb{H}^{n \times n}$ is Hermitian and idempotent, then the following equation is true as well,

$$(\mathbf{A}\mathbf{B})^\dagger \mathbf{A} = (\mathbf{A}\mathbf{B})^\dagger. \quad (10)$$

Since \mathbf{L}_{A_1} , \mathbf{R}_{B_1} , and \mathbf{R}_H are projectors, then using (9) and (10), we have, respectively,

$$\begin{aligned} \mathbf{L}_{A_1} \mathbf{H}^\dagger &= \mathbf{L}_{A_1} (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger = (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger = \mathbf{H}^\dagger, \\ \mathbf{N}^\dagger \mathbf{R}_{B_1} &= (\mathbf{R}_{B_1} \mathbf{B}_2)^\dagger \mathbf{R}_{B_1} = (\mathbf{R}_{B_1} \mathbf{B}_2)^\dagger = \mathbf{N}^\dagger, \\ \mathbf{T}^\dagger \mathbf{T} &= (\mathbf{R}_H \mathbf{A}_2)^\dagger \mathbf{R}_H \mathbf{A}_2 = (\mathbf{R}_H \mathbf{A}_2)^\dagger \mathbf{A}_2 = \mathbf{T}^\dagger \mathbf{A}_2, \\ \mathbf{L}_T &= \mathbf{I} - \mathbf{T}^\dagger \mathbf{T} = \mathbf{I} - \mathbf{T}^\dagger \mathbf{A}_2. \end{aligned} \quad (11)$$

Using (11) and (6), we obtain the following expression of (8),

$$\begin{aligned} \mathbf{X} &= \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger + \mathbf{H}^\dagger \mathbf{A}_2 (\mathbf{I} - \mathbf{T}^\dagger \mathbf{A}_2) (\mathbf{A}_2^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger - \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger) \mathbf{B}_2 \mathbf{B}_2^\dagger \\ &\quad + \mathbf{T}^\dagger \mathbf{A}_2 (\mathbf{A}_2^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger - \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger) \mathbf{B}_2 \mathbf{N}^\dagger + \mathbf{L}_{A_1} (\mathbf{Z} - \mathbf{H}^\dagger \mathbf{H} \mathbf{Z} \mathbf{B}_2 \mathbf{B}_2^\dagger) - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{L}_T \mathbf{W} \mathbf{N} \mathbf{B}_2^\dagger \\ &\quad + (\mathbf{W} - \mathbf{T}^\dagger \mathbf{T} \mathbf{W} \mathbf{N} \mathbf{N}^\dagger) \mathbf{R}_{B_1} = \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger + \mathbf{H}^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{H}^\dagger (\mathbf{A}_2 \mathbf{T}^\dagger - \mathbf{I}) \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger \mathbf{Q}_{B_2} \\ &\quad - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{T}^\dagger \mathbf{C}_2 \mathbf{N}^\dagger - \mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger \mathbf{B}_2 \mathbf{N}^\dagger + \mathbf{L}_{A_1} (\mathbf{Z} - \mathbf{H}^\dagger \mathbf{H} \mathbf{Z} \mathbf{B}_2 \mathbf{B}_2^\dagger) \\ &\quad - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{L}_T \mathbf{W} \mathbf{N} \mathbf{B}_2^\dagger + (\mathbf{W} - \mathbf{T}^\dagger \mathbf{T} \mathbf{W} \mathbf{N} \mathbf{N}^\dagger) \mathbf{R}_{B_1}. \end{aligned} \quad (12)$$

By putting $\mathbf{Z}_1 = \mathbf{W}_1 = \mathbf{0}$ in (12), the partial solution of (8) can be derived,

$$\begin{aligned} \mathbf{X}_0 &= \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger + \mathbf{H}^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{T}^\dagger \mathbf{C}_2 \mathbf{N}^\dagger + \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger \mathbf{Q}_{B_2} \\ &\quad - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger \mathbf{Q}_{B_2} - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{T}^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger - \mathbf{T}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_1^\dagger \mathbf{B}_2 \mathbf{N}^\dagger. \end{aligned} \quad (13)$$

Further we give determinantal representations of (13). Let $\mathbf{A}_1 = (a_{ij}^{(1)}) \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{B}_1 = (b_{ij}^{(1)}) \in \mathbb{H}_{r_2}^{r \times s}$, $\mathbf{A}_2 = (a_{ij}^{(2)}) \in \mathbb{H}_{r_3}^{k \times n}$, $\mathbf{B}_2 = (b_{ij}^{(2)}) \in \mathbb{H}_{r_4}^{r \times p}$, $\mathbf{C}_1 = (c_{ij}^{(1)}) \in \mathbb{H}^{m \times s}$, and $\mathbf{C}_2 = (c_{ij}^{(2)}) \in \mathbb{H}^{k \times p}$, and there exist $\mathbf{A}_1^\dagger = (a_{ij}^{(1), \dagger}) \in \mathbb{H}^{n \times m}$, $\mathbf{B}_2^\dagger = (b_{ij}^{(2), \dagger}) \in \mathbb{H}^{p \times r}$, $\mathbf{H}^\dagger = (h_{ij}^\dagger) \in \mathbb{H}^{n \times k}$, $\mathbf{N}^\dagger = (n_{ij}^\dagger) \in \mathbb{H}^{p \times r}$, and $\mathbf{T}^\dagger = (t_{ij}^\dagger) \in \mathbb{H}^{n \times k}$. Let $\text{rank } \mathbf{H} = \min\{\text{rank } \mathbf{A}_2, \text{rank } \mathbf{L}_{A_1}\} = r_5$, $\text{rank } \mathbf{N} = \min\{\text{rank } \mathbf{B}_2, \text{rank } \mathbf{R}_{B_1}\} = r_6$, and $\text{rank } \mathbf{T} = \min\{\text{rank } \mathbf{A}_2, \text{rank } \mathbf{R}_H\} = r_7$. Consider each term of (13) separately.

(i) By Theorem 2.3 for the first term, x_{ij}^{01} , of (13), we have

$$x_{ij}^{01} = \frac{\sum_{\beta \in I_{r_1, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\mathbf{d}_j^{B_1}) \right)_{\beta}^{\beta}}{\sum_{\beta \in I_{r_1, n}} |\mathbf{A}_1^* \mathbf{A}_1|_{\beta}^{\beta} \sum_{\alpha \in I_{r_2, p}} |\mathbf{B}_1 \mathbf{B}_1^*|_{\alpha}^{\alpha}}, \quad (14)$$

or

$$x_{ij}^{01} = \frac{\sum_{\alpha \in I_{r_2, q} \setminus \{j\}} \text{rdet}_j \left((\mathbf{B}_1 \mathbf{B}_1^*)_{.j} (\mathbf{d}_i^{A_1}) \right)_{\alpha}^{\alpha}}{\sum_{\beta \in I_{r_1, p}} |\mathbf{A}_1^* \mathbf{A}_1|_{\beta}^{\beta} \sum_{\alpha \in I_{r_2, q}} |\mathbf{B}_1 \mathbf{B}_1^*|_{\alpha}^{\alpha}}, \quad (15)$$

where

$$\mathbf{d}_j^{B_1} = \left[\sum_{\alpha \in I_{r_2, p} \setminus \{j\}} \text{rdet}_j \left((\mathbf{B}_1 \mathbf{B}_1^*)_{.j} (\tilde{c}_q^{(1)}) \right)_{\alpha}^{\alpha} \right] \in \mathbb{H}^{n \times 1}, \quad q = 1, \dots, n,$$

$$\mathbf{d}_i^{A_1} = \left[\sum_{\beta \in I_{r_1, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\tilde{c}_l^{(1)}) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1 \times r}, \quad l = 1, \dots, r,$$

are the column vector and the row vector, respectively. $\tilde{c}_q^{(1)}$ and $\tilde{c}_l^{(1)}$ are the q th row and the l th column of $\tilde{\mathbf{C}}_1 = \mathbf{A}_1^* \mathbf{C}_1 \mathbf{B}_1^*$.

(ii) Similarly, for the second term of (13), we have

$$x_{ij}^{02} = \frac{\sum_{\beta \in I_{r_5, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} (\mathbf{d}_j^{B_2}) \right)_{\beta}^{\beta}}{\sum_{\beta \in I_{r_5, n}} |\mathbf{H}^* \mathbf{H}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_4, r}} |\mathbf{B}_2 \mathbf{B}_2^*|_{\alpha}^{\alpha}}, \quad (16)$$

or

$$x_{ij}^{02} = \frac{\sum_{\alpha \in I_{r_4, r} \setminus \{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_{.j} (\mathbf{d}_i^H) \right)_{\alpha}^{\alpha}}{\sum_{\beta \in I_{r_5, n}} |\mathbf{H}^* \mathbf{H}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_4, r}} |\mathbf{B}_2 \mathbf{B}_2^*|_{\alpha}^{\alpha}}, \quad (17)$$

where

$$\mathbf{d}_j^{B_2} = \left[\sum_{\alpha \in I_{r_4, r} \setminus \{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_{.j} (\tilde{c}_q^{(2)}) \right)_{\alpha}^{\alpha} \right] \in \mathbb{H}^{n \times 1}, \quad q = 1, \dots, n,$$

$$\mathbf{d}_i^H = \left[\sum_{\beta \in I_{r_5, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} (\tilde{c}_l^{(2)}) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1 \times r}, \quad l = 1, \dots, r,$$

are the column vector and the row vector, respectively. $\tilde{\mathbf{c}}_q^{(2)}$ and $\tilde{\mathbf{c}}_l^{(2)}$ are the q th row and the l th column of $\tilde{\mathbf{C}}_2 = \mathbf{H}^* \mathbf{C}_2 \mathbf{B}_2^*$. Note that $\mathbf{H}^* \mathbf{H} = (\mathbf{A}_2 \mathbf{L}_{A_1})^* \mathbf{A}_2 \mathbf{L}_{A_1} = \mathbf{L}_{A_1} \mathbf{A}_2^* \mathbf{A}_2 \mathbf{L}_{A_1}$.

(iii) The third term of (13) can be obtained by Theorem 2.3 as well. Then

$$x_{ij}^{03} = \frac{\sum_{\beta \in J_{7,n}\{i\}} \text{cdet}_i \left((\mathbf{T}^* \mathbf{T})_{.i} \left(\mathbf{d}_j^N \right) \right)_\beta^\beta}{\sum_{\beta \in J_{7,n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta \sum_{\alpha \in I_{6,r}} |\mathbf{N} \mathbf{N}^*|_\alpha^\alpha}, \quad (18)$$

or

$$x_{ij}^{03} = \frac{\sum_{\alpha \in I_{6,r}\{j\}} \text{rdet}_j \left((\mathbf{N} \mathbf{N}^*)_{.j} \left(\mathbf{d}_i^T \right) \right)_\alpha^\alpha}{\sum_{\beta \in J_{7,n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta \sum_{\alpha \in I_{6,r}} |\mathbf{N} \mathbf{N}^*|_\alpha^\alpha}, \quad (19)$$

where

$$\mathbf{d}_j^N = \left[\sum_{\alpha \in I_{6,r}\{j\}} \text{rdet}_j \left((\mathbf{N} \mathbf{N}^*)_{.j} \left(\hat{\mathbf{c}}_q^{(2)} \right) \right)_\alpha^\alpha \right] \in \mathbb{H}^{m \times 1}, \quad q = 1, \dots, n,$$

$$\mathbf{d}_i^T = \left[\sum_{\beta \in J_{7,n}\{i\}} \text{cdet}_i \left((\mathbf{T}^* \mathbf{T})_{.i} \left(\hat{\mathbf{c}}_l^{(2)} \right) \right)_\beta^\beta \right] \in \mathbb{H}^{1 \times r}, \quad l = 1, \dots, r,$$

are the column vector and the row vector, respectively. $\hat{\mathbf{c}}_q^{(2)}$ is the q th row and $\hat{\mathbf{c}}_l^{(2)}$ is the l th column of $\hat{\mathbf{C}}_2 = \mathbf{T}^* \mathbf{C}_2 \mathbf{N}^*$. The following expression gives some simplify in computing. Since $\mathbf{T}^* \mathbf{T} = (\mathbf{R}_H \mathbf{A}_2)^* = \mathbf{A}_2^* \mathbf{R}_H^* \mathbf{R}_H \mathbf{A}_2 = \mathbf{A}_2^* \mathbf{R}_H \mathbf{A}_2$ and $\mathbf{R}_H = \mathbf{I} - \mathbf{H} \mathbf{H}^\dagger = \mathbf{I} - \mathbf{A}_2 \mathbf{L}_{A_1} (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger = \mathbf{I} - \mathbf{A}_2 (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger$, then $\mathbf{T}^* \mathbf{T} = \mathbf{A}_2^* \left(\mathbf{I} - \mathbf{A}_2 (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger \right) \mathbf{A}_2$.

(iv) Using (3) for determinantal representations of \mathbf{H}^\dagger and \mathbf{T}^\dagger in the fourth term of (13), we obtain

$$x_{ij}^{04} = \frac{\sum_{q=1}^n \sum_{z=1}^n \sum_{f=1}^r \sum_{\beta \in J_{7,n}\{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} \left(\mathbf{a}_q^{(2,H)} \right) \right)_\beta^\beta \sum_{\beta \in J_{7,n}\{q\}} \text{cdet}_q \left((\mathbf{T}^* \mathbf{T})_{.q} \left(\mathbf{a}_z^{(2,T)} \right) \right)_\beta^\beta x_{zf}^{01} q_{ff}}{\sum_{\beta \in J_{7,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta \sum_{\beta \in J_{7,n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta}, \quad (20)$$

where $\mathbf{a}_i^{(2,H)}$ and $\mathbf{a}_i^{(2,T)}$ are the i th columns of the matrices $\mathbf{H}^* \mathbf{A}_2$ and $\mathbf{T}^* \mathbf{A}_2$, respectively; q_{ff} is the (ff) th element of \mathbf{Q}_{B_2} with the determinantal representation,

$$q_{fj} = \frac{\sum_{\alpha \in I_{4,r}\{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_j \left(\ddot{\mathbf{b}}_f^{(2)} \right) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{4,r}} |\mathbf{B}_2 \mathbf{B}_2^*|_\alpha^\alpha},$$

and $\ddot{\mathbf{b}}_f^{(2)}$ is the f th row of $\mathbf{B}_2 \mathbf{B}_2^*$. Note that $\mathbf{H}^* \mathbf{A}_2 = \mathbf{L}_{A_1} \mathbf{A}_2^* \mathbf{A}_2$ and $\mathbf{T}^* \mathbf{A}_2 = \mathbf{A}_2^* \mathbf{R}_H \mathbf{A}_2 = \mathbf{A}_2^* (\mathbf{I} - \mathbf{A}_2 (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger) \mathbf{A}_2$.

(v) Similar to the previous case,

$$x_{ij}^{05} = \frac{\sum_{q=1}^n \sum_{f=1}^r \sum_{\beta \in J_{5,n}\{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} \left(\mathbf{a}_q^{(2,H)} \right) \right)_\beta^\beta x_{qf}^{01} q_{fj}}{\sum_{\beta \in J_{5,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta}, \quad (21)$$

(vi) Consider the sixth term by analogy to the fourth term. So,

$$x_{ij}^{06} = \frac{\sum_{q=1}^n \sum_{\beta \in J_{5,n}\{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} \left(\mathbf{a}_q^{(2,H)} \right) \right)_\beta^\beta \varphi_{qj}}{\sum_{\beta \in J_{5,n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta \sum_{\beta \in J_{7,n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta \sum_{\alpha \in I_{4,r}} |\mathbf{B}_2 \mathbf{B}_2^*|_\alpha^\alpha}, \quad (22)$$

where

$$\varphi_{qj} = \sum_{\beta \in J_{7,n}\{i\}} \text{cdet}_q \left((\mathbf{T}^* \mathbf{T})_{.q} \left(\psi_j^{B_2} \right) \right)_\beta^\beta, \quad (23)$$

or

$$\varphi_{qj} = \sum_{\alpha \in I_{4,r}\{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_j \left(\psi_q^T \right) \right)_\alpha^\alpha, \quad (24)$$

and

$$\psi_j^{B_2} = \left[\sum_{\alpha \in I_{4,r}\{f\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_j \left(\dot{\mathbf{c}}_q^{(2)} \right) \right)_\alpha^\alpha \right] \in \mathbb{H}^{1 \times n}, \quad q = 1, \dots, n,$$

$$\psi_q^T = \left[\sum_{\beta \in J_{7,n}\{q\}} \text{cdet}_q \left((\mathbf{T}^* \mathbf{T})_{.q} \left(\dot{\mathbf{c}}_{.l}^{(2)} \right) \right)_\beta^\beta \right] \in \mathbb{H}^{r \times 1}, \quad l = 1, \dots, r,$$

are the column vector and the row vector, respectively. $\check{c}_q(2)$ and $\check{c}_l(2)$ are the q th row and the l th column of $\check{\mathbf{C}}_2 = \mathbf{T}^* \mathbf{C}_2 \mathbf{B}_2^*$ for all $i = 1, \dots, n$ and $j = 1, \dots, p$.

(vii) Using (3) for determinantal representations of \mathbf{T}^\dagger and (2) for \mathbf{N}^\dagger in the seventh term of (13), we obtain

$$x_{ij}^{07} = \frac{\sum_{q=1}^n \sum_{f=1}^r \sum_{\beta \in J_{r_7, n} \{i\}} \text{cdet}_i \left((\mathbf{T}^* \mathbf{T})_{.i} \left(\mathbf{a}_{.q}^{(2,T)} \right) \right)_\beta^\beta x_{qf}^{01} \sum_{\alpha \in I_{r_6, r} \{j\}} \text{rdet}_j \left((\mathbf{N} \mathbf{N}^*)_{.j} \left(\mathbf{b}_{f.}^{(2,N)} \right) \right)_\alpha^\alpha}{\sum_{\beta \in J_{r_7, n}} |\mathbf{T}^* \mathbf{T}|_\beta^\beta \sum_{\alpha \in I_{r_6, r}} |\mathbf{N} \mathbf{N}^*|_\alpha^\alpha}, \quad (25)$$

where $\mathbf{a}_{.q}^{(2,T)}$ and $\mathbf{b}_{f.}^{(2,N)}$ are the q th column of $\mathbf{T}^* \mathbf{A}_2$ and the f th row of $\mathbf{B}_2 \mathbf{N}^* = \mathbf{B}_2 \mathbf{B}_2^* \mathbf{R}_{B_1}$, respectively.

Hence, we prove the following theorem.

Theorem 3.1. *Let $\mathbf{A}_1 \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{B}_1 \in \mathbb{H}_{r_2}^{r \times s}$, $\mathbf{A}_2 \in \mathbb{H}_{r_3}^{k \times n}$, $\mathbf{B}_2 \in \mathbb{H}_{r_4}^{r \times p}$, $\text{rank} \mathbf{H} = \text{rank}(\mathbf{A}_2 \mathbf{L}_{A_1}) = r_5$, $\text{rank} \mathbf{N} = (\mathbf{R}_{B_1} \mathbf{B}_2) = r_6$, and $\text{rank} \mathbf{T} = (\mathbf{R}_H \mathbf{A}_2) = r_7$. Then, for the partial solution (13), $\mathbf{X}_0 = (x_{ij}^0) \in \mathbb{H}^{n \times r}$, of the system (1), we have,*

$$x_{ij}^0 = \sum_{\delta} x_{ij}^{0\delta}, \quad (26)$$

where the term x_{ij}^{01} has the determinantal representations (14) and (15), x_{ij}^{02} —(16) and (17), x_{ij}^{03} —(18) and (19), x_{ij}^{04} —(20), x_{ij}^{05} —(21), x_{ij}^{06} —(23) and (24), and x_{ij}^{07} —(25).

4. Cramer's rules for special cases of (1)

In this section, we consider special cases of (1) when one or two equations are one-sided. Let in Eq.(1), the matrix \mathbf{B}_1 is vanished. Then, we have the system

$$\begin{cases} \mathbf{A}_1 \mathbf{X} = \mathbf{C}_1, \\ \mathbf{A}_2 \mathbf{X} \mathbf{B}_2 = \mathbf{C}_2. \end{cases} \quad (27)$$

The following lemma is extended to matrices with quaternion entries.

Lemma 4.1. [7] *Let $\mathbf{A}_1 \in \mathbb{H}^{m \times n}$, $\mathbf{C}_1 \in \mathbb{H}^{m \times r}$, $\mathbf{A}_2 \in \mathbb{H}^{k \times n}$, $\mathbf{B}_2 \in \mathbb{H}^{r \times p}$, and $\mathbf{C}_2 \in \mathbb{H}^{k \times p}$ be given and $\mathbf{X} \in \mathbb{H}^{n \times r}$ is to be determined. Put $\mathbf{H} = \mathbf{A}_2 \mathbf{L}_{A_1}$. Then, the following statements are equivalent:*

- i. System (27) is consistent.
- ii. $\mathbf{R}_{A_1} \mathbf{C}_1 = \mathbf{0}$, $\mathbf{R}_H (\mathbf{C}_2 - \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_2) = \mathbf{0}$, $\mathbf{C}_2 \mathbf{L}_{B_2} = \mathbf{0}$.
- iii. $\text{rank}[\mathbf{A}_1 \ \mathbf{C}_1] = \text{rank}[\mathbf{A}_1]$, $\text{rank} \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{B}_2 \end{bmatrix} = \text{rank}[\mathbf{B}_2]$, $\text{rank} \begin{bmatrix} \mathbf{A}_1 & \mathbf{C}_1 \mathbf{B}_2 \\ \mathbf{A}_2 & \mathbf{C}_2 \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$.

In this case, the general solution of (27) can be expressed as

$$\mathbf{X} = \mathbf{A}_1^\dagger \mathbf{C}_1 + \mathbf{L}_{A_1} \mathbf{H}^\dagger (\mathbf{C}_2 - \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_2) \mathbf{B}_2^\dagger + \mathbf{L}_{A_1} \mathbf{L}_H \mathbf{Z}_1 + \mathbf{L}_{A_1} \mathbf{W}_1 \mathbf{R}_{B_2}, \quad (28)$$

where \mathbf{Z}_1 and \mathbf{W}_1 are the arbitrary matrices over \mathbb{H} with appropriate sizes.

Since by (9), $\mathbf{L}_{A_1} \mathbf{H}^\dagger = \mathbf{L}_{A_1} (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger = (\mathbf{A}_2 \mathbf{L}_{A_1})^\dagger = \mathbf{H}^\dagger$, then we have some simplification of (28),

$$\mathbf{X} = \mathbf{A}_1^\dagger \mathbf{C}_1 + \mathbf{H}^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_2 \mathbf{B}_2^\dagger + \mathbf{L}_{A_1} \mathbf{L}_H \mathbf{Z}_1 + \mathbf{L}_{A_1} \mathbf{W}_1 \mathbf{R}_{B_2}.$$

By putting $\mathbf{Z}_1 = \mathbf{W}_1 = \mathbf{0}$, there is the following partial solution of (27),

$$\mathbf{X}_0 = \mathbf{A}_1^\dagger \mathbf{C}_1 + \mathbf{H}^\dagger \mathbf{C}_2 \mathbf{B}_2^\dagger - \mathbf{H}^\dagger \mathbf{A}_2 \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{B}_2 \mathbf{B}_2^\dagger. \quad (29)$$

Theorem 4.1. Let $\mathbf{A}_1 = (a_{ij}^{(1)}) \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{A}_2 = (a_{ij}^{(2)}) \in \mathbb{H}_{r_2}^{k \times n}$, $\mathbf{B}_2 = (b_{ij}^{(2)}) \in \mathbb{H}_{r_3}^{r \times p}$, $\mathbf{C}_1 = (c_{ij}^{(1)}) \in \mathbb{H}^{m \times r}$, and $\mathbf{C}_2 = (c_{ij}^{(2)}) \in \mathbb{H}^{k \times p}$, and there exist $\mathbf{A}_1^\dagger = (a_{ij}^{(1)\dagger}) \in \mathbb{H}^{n \times m}$, $\mathbf{B}_2^\dagger = (b_{ij}^{(2)\dagger}) \in \mathbb{H}^{p \times r}$, and $\mathbf{H}^\dagger = (h_{ij}^\dagger) \in \mathbb{H}^{n \times k}$. Let $\text{rank} \mathbf{H} = \min\{\text{rank} \mathbf{A}_2, \text{rank} \mathbf{L}_{A_1}\} = r_4$. Denote $\mathbf{A}_1^* \mathbf{C}_1 =: \widehat{\mathbf{C}}_1 = (\widehat{c}_{ij}^{(1)}) \in \mathbb{H}^{m \times r}$, $\mathbf{H}^* \mathbf{C}_2 \mathbf{B}_2^* =: \widehat{\mathbf{C}}_2 = (\widehat{c}_{ij}^{(2)}) \in \mathbb{H}^{n \times r}$, $\mathbf{H}^* \mathbf{A}_2 \mathbf{A}_1^* =: \widehat{\mathbf{A}}_2 = (\widehat{a}_{ij}^{(2)}) \in \mathbb{H}^{n \times m}$, and $\mathbf{C}_1 \mathbf{Q}_{B_2} =: \widehat{\mathbf{Q}} = (\widehat{q}_{ij}) \in \mathbb{H}^{m \times p}$. Then, the partial solution (29), $\mathbf{X}_0 = (x_{ij}^0) \in \mathbb{H}^{n \times r}$, possesses the following determinantal representations,

$$x_{ij}^0 = \frac{\sum_{\beta \in I_{r_1, n} \{i\}} \text{cdet}_i \left((\mathbf{A}_1^* \mathbf{A}_1)_{.i} (\widehat{\mathbf{C}}_j^{(1)}) \right)_\beta^\beta}{\sum_{\beta \in I_{r_1, n}} |\mathbf{A}_1^* \mathbf{A}_1|_\beta^\beta} + \frac{d_{ij}^{(\lambda)}}{\sum_{\beta \in I_{r_4, n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta \sum_{\alpha \in I_{r_3, r}} |\mathbf{B}_2 \mathbf{B}_2^*|_\alpha^\alpha} - \frac{\sum_{l=1}^m \delta_{il}^{(\mu)} \sum_{\alpha \in I_{r_3, r} \{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_{j.} (\widehat{\mathbf{Q}}_{l.}) \right)_\alpha^\alpha}{\sum_{\beta \in I_{r_4, n}} |\mathbf{H}^* \mathbf{H}|_\beta^\beta \sum_{\alpha \in I_{r_1, m}} |\mathbf{A}_1 \mathbf{A}_1^*|_\alpha^\alpha \sum_{\alpha \in I_{r_3, r}} |\mathbf{B}_2 \mathbf{B}_2^*|_\alpha^\alpha}$$

for all $\lambda = 1, 2$ and $\mu = 1, 2$. Here

$$d_{ij}^{(1)} := \sum_{\alpha \in I_{r_3, r} \{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_{j.} (\mathbf{v}_i^{(1)}) \right)_\alpha^\alpha, \quad \delta_{il}^{(1)} := \sum_{\alpha \in I_{r_1, m} \{l\}} \text{rdet}_l \left((\mathbf{A}_1 \mathbf{A}_1^*)_{l.} (\mathbf{u}_i^{(1)}) \right)_\alpha^\alpha,$$

and the row-vectors $\mathbf{v}_i^{(1)} = (v_{i1}^{(1)}, \dots, v_{ir}^{(1)})$ and $\mathbf{u}_i^{(1)} = (u_{i1}^{(1)}, \dots, u_{im}^{(1)})$ such that

$$v_{it}^{(1)} := \sum_{\beta \in J_{r_4, n}\{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} \left(\widehat{\mathbf{c}}_{.i}^{(2)} \right) \right)_{\beta}^{\beta}, \quad u_{iz}^{(1)} := \sum_{\beta \in J_{r_4, n}\{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} \left(\widehat{\mathbf{a}}_{.z}^{(2)} \right) \right)_{\beta}^{\beta}.$$

In another case,

$$d_{ij}^{(2)} := \sum_{\beta \in J_{r_4, n}\{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} \left(\mathbf{v}_{.j}^{(2)} \right) \right)_{\beta}^{\beta}, \quad g_{il}^{(2)} := \sum_{\beta \in J_{r_4, n}\{i\}} \text{cdet}_i \left((\mathbf{H}^* \mathbf{H})_{.i} \left(\mathbf{u}_{.l}^{(2)} \right) \right)_{\beta}^{\beta}.$$

and the column-vectors $\mathbf{v}_{.j}^{(2)} = (v_{1j}^{(2)}, \dots, v_{nj}^{(2)})$ and $\mathbf{u}_{.l}^{(2)} = (u_{1l}^{(2)}, \dots, u_{nl}^{(2)})$ such that

$$v_{qj}^{(2)} := \sum_{\alpha \in I_{r_3, r}\{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_{.j} \left(\widehat{\mathbf{c}}_{.q}^{(2)} \right) \right)_{\alpha}^{\alpha}, \quad u_{ql}^{(2)} := \sum_{\alpha \in I_{r_1, m}\{l\}} \text{rdet}_l \left((\mathbf{A}_1 \mathbf{A}_1^*)_{.l} \left(\mathbf{a}_{.q}^{(2)} \right) \right)_{\alpha}^{\alpha}$$

Proof. The proof is similar to the proof of Theorem 3.1.

Let in Eq.(1), the matrix \mathbf{A}_1 is vanished. Then, we have the system,

$$\begin{cases} \mathbf{X} \mathbf{B}_1 = \mathbf{C}_1, \\ \mathbf{A}_2 \mathbf{X} \mathbf{B}_2 = \mathbf{C}_2. \end{cases} \quad (30)$$

The following lemma is extended to matrices with quaternion entries as well.

Lemma 4.2. [7] *Let $\mathbf{B}_1 \in \mathbb{H}^{r \times s}$, $\mathbf{C}_1 \in \mathbb{H}^{n \times s}$, $\mathbf{A}_2 \in \mathbb{H}^{k \times n}$, $\mathbf{B}_2 \in \mathbb{H}^{r \times p}$, and $\mathbf{C}_2 \in \mathbb{H}^{k \times p}$ be given and $\mathbf{X} \in \mathbb{H}^{n \times r}$ is to be determined. Put $\mathbf{N} = \mathbf{R}_{\mathbf{B}_1} \mathbf{B}_2$. Then, the following statements are equivalent:*

- i. System (30) is consistent.
- ii. $\mathbf{R}_{\mathbf{A}_2} \mathbf{C}_2 = \mathbf{0}$, $(\mathbf{C}_2 - \mathbf{A}_2 \mathbf{C}_1 \mathbf{B}_1^{\dagger} \mathbf{B}_2) \mathbf{L}_{\mathbf{N}} = \mathbf{0}$, $\mathbf{C}_2 \mathbf{L}_{\mathbf{B}_2} = \mathbf{0}$.
- iii. $\text{rank}[\mathbf{A}_2 \quad \mathbf{C}_2] = \text{rank}[\mathbf{A}_2]$, $\text{rank} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{B}_1 \end{bmatrix} = \text{rank}[\mathbf{B}_1]$, $\text{rank} \begin{bmatrix} \mathbf{C}_2 & \mathbf{A}_2 \mathbf{C}_1 \\ \mathbf{B}_2 & \mathbf{B}_1 \end{bmatrix} = \text{rank}[\mathbf{B}_2 \quad \mathbf{B}_1]$.

In this case, the general solution of (30) can be expressed as

$$\mathbf{X} = \mathbf{C}_1 \mathbf{B}_1^{\dagger} + \mathbf{A}_2^{\dagger} (\mathbf{C}_2 - \mathbf{A}_2 \mathbf{C}_1 \mathbf{B}_1^{\dagger} \mathbf{B}_2) \mathbf{N}^{\dagger} \mathbf{R}_{\mathbf{B}_1} + \mathbf{L}_{\mathbf{A}_2} \mathbf{W}_2 \mathbf{R}_{\mathbf{B}_1} + \mathbf{Z}_2 \mathbf{R}_{\mathbf{N}} \mathbf{R}_{\mathbf{B}_1}, \quad (31)$$

where \mathbf{Z}_2 and \mathbf{W}_2 are the arbitrary matrices over \mathbb{H} with appropriate sizes.

Since by (10), $\mathbf{N}^{\dagger} \mathbf{R}_{\mathbf{B}_1} = (\mathbf{R}_{\mathbf{B}_1} \mathbf{B}_2)^{\dagger} \mathbf{R}_{\mathbf{B}_1} = \mathbf{N}^{\dagger}$, then some simplification of (31) can be derived,

$$\mathbf{X} = \mathbf{C}_1 \mathbf{B}_1^{\dagger} + \mathbf{A}_2^{\dagger} \mathbf{C}_2 \mathbf{N}^{\dagger} - \mathbf{A}_2 \mathbf{C}_1 \mathbf{B}_1^{\dagger} \mathbf{B}_2 \mathbf{N}^{\dagger} + \mathbf{L}_{\mathbf{A}_2} \mathbf{W}_2 \mathbf{R}_{\mathbf{B}_1} + \mathbf{Z}_2 \mathbf{R}_{\mathbf{N}} \mathbf{R}_{\mathbf{B}_1}.$$

By putting $\mathbf{Z}_2 = \mathbf{W}_2 = \mathbf{0}$, there is the following partial solution of (30),

$$\mathbf{X}_0 = \mathbf{C}_1 \mathbf{B}_1^{\dagger} + \mathbf{A}_2^{\dagger} \mathbf{C}_2 \mathbf{N}^{\dagger} - \mathbf{A}_2 \mathbf{C}_1 \mathbf{B}_1^{\dagger} \mathbf{B}_2 \mathbf{N}^{\dagger}. \quad (32)$$

The following theorem on determinantal representations of (29) can be proven similar to the proof of Theorem 3.1 as well.

Theorem 4.2. Let $\mathbf{B}_1 = (b_{ij}^{(1)}) \in \mathbb{H}_{r_1}^{r \times s}$, $\mathbf{A}_2 = (a_{ij}^{(2)}) \in \mathbb{H}_{r_2}^{k \times n}$, $\mathbf{B}_2 = (b_{ij}^{(2)}) \in \mathbb{H}_{r_3}^{r \times p}$, $\mathbf{C}_1 = (c_{ij}^{(1)}) \in \mathbb{H}^{n \times s}$, and $\mathbf{C}_2 = (c_{ij}^{(2)}) \in \mathbb{H}^{k \times p}$, and there exist $\mathbf{B}_1^\dagger = (b_{ij}^{(1), \dagger}) \in \mathbb{H}^{s \times r}$, $\mathbf{A}_2^\dagger = (a_{ij}^{(2), \dagger}) \in \mathbb{H}^{n \times k}$, $\mathbf{N}^\dagger = (n_{ij}^\dagger) \in \mathbb{H}^{p \times r}$. Let $\text{rank } \mathbf{N} = \min\{\text{rank } \mathbf{B}_2, \text{rank } \mathbf{R}_{\mathbf{B}_1}\} = r_4$. Denote $\mathbf{C}_1 \mathbf{B}_1^* =: \tilde{\mathbf{C}}_1 = (\tilde{c}_{ij}^{(1)}) \in \mathbb{H}^{n \times r}$, $\mathbf{A}_2^* \mathbf{C}_2 \mathbf{N}^* =: \tilde{\mathbf{C}}_2 = (\tilde{c}_{ij}^{(2)}) \in \mathbb{H}^{n \times r}$, $\mathbf{B}_1^* \mathbf{B}_2 \mathbf{N}^* =: \tilde{\mathbf{B}}_2 = (\tilde{b}_{ij}^{(2)}) \in \mathbb{H}^{s \times r}$, and $\mathbf{P}_{\mathbf{A}_2} \mathbf{C}_1 =: \tilde{\mathbf{P}} = (\tilde{p}_{ij}) \in \mathbb{H}^{n \times s}$. Then, the partial solution (32), $\mathbf{X}_0 = (x_{ij}^0) \in \mathbb{H}^{n \times r}$, possesses the following determinantal representations,

$$x_{ij}^0 = \frac{\sum_{\alpha \in I_{r_1, r}\{j\}} \text{rdet}_j \left((\mathbf{B}_1 \mathbf{B}_1^*)_j \left(\tilde{c}_{i \cdot}^{(1)} \right)_\alpha \right)}{\sum_{\alpha \in I_{r_1, r}} |\mathbf{B}_1 \mathbf{B}_1^*|_\alpha} + \frac{d_{ij}^{(\lambda)}}{\sum_{\beta \in I_{r_2, n}} |\mathbf{A}_2^* \mathbf{A}_2|_\beta \sum_{\alpha \in I_{r_4, r}} |\mathbf{N} \mathbf{N}^*|_\alpha} \\ - \frac{\sum_{z=1}^s \sum_{\beta \in I_{r_2, n}\{i\}} \text{cdet}_i \left((\mathbf{A}_2^* \mathbf{A}_2)_{\cdot i} (\tilde{\mathbf{p}}_{\cdot z})_\beta \right)_\beta^{(\mu)}}{\sum_{\beta \in I_{r_2, n}} |\mathbf{A}_2^* \mathbf{A}_2|_\beta \sum_{\beta \in I_{r_1, s}} |\mathbf{B}_1^* \mathbf{B}_1|_\beta \sum_{\alpha \in I_{r_4, r}} |\mathbf{N} \mathbf{N}^*|_\alpha}$$

for all $\lambda = 1, 2$ and $\mu = 1, 2$. Here

$$d_{ij}^{(1)} := \sum_{\alpha \in I_{r_3, r}\{j\}} \text{rdet}_j \left((\mathbf{N} \mathbf{N}^*)_j \left(\varphi_i^{(1)} \right)_\alpha \right), \quad g_{ii}^{(1)} := \sum_{\alpha \in I_{r_4, r}\{j\}} \text{rdet}_j \left((\mathbf{N} \mathbf{N}^*)_j \left(\psi_{z \cdot}^{(1)} \right)_\alpha \right),$$

and the row-vectors $\varphi_i^{(1)} = (\varphi_{i1}^{(1)}, \dots, \varphi_{ir}^{(1)})$ and $\psi_{i \cdot}^{(1)} = (\psi_{z1}^{(1)}, \dots, \psi_{zr}^{(1)})$ such that

$$\varphi_{it}^{(1)} = \sum_{\beta \in I_{r_2, n}\{i\}} \text{cdet}_i \left((\mathbf{A}_2^* \mathbf{A}_2)_{\cdot i} \left(c_{\cdot t}^{(2)} \right)_\beta \right), \quad \psi_{zv}^{(1)} = \sum_{\beta \in I_{r_1, n}\{z\}} \text{cdet}_z \left((\mathbf{B}_1^* \mathbf{B}_1)_{\cdot z} \left(b_{\cdot v}^{(2)} \right)_\beta \right).$$

In another case,

$$d_{ij}^{(2)} := \sum_{\beta \in I_{r_2, n}\{i\}} \text{cdet}_i \left((\mathbf{A}_2^* \mathbf{A}_2)_{\cdot i} \left(\varphi_j^{(2)} \right)_\beta \right), \quad g_{zj}^{(2)} := \sum_{\beta \in I_{r_1, n}\{z\}} \text{cdet}_z \left((\mathbf{B}_1^* \mathbf{B}_1)_{\cdot z} \left(\psi_j^{(2)} \right)_\beta \right),$$

and the column-vectors $\varphi_j^{(2)} = (\varphi_{1j}^{(2)}, \dots, \varphi_{nj}^{(2)})$ and $\psi_{\cdot j}^{(2)} = (\psi_{1j}^{(2)}, \dots, \psi_{sj}^{(2)})$ such that

$$\varphi_{qj}^{(2)} = \sum_{\alpha \in I_{r_4, r}\{j\}} \text{rdet}_j \left((\mathbf{N} \mathbf{N}^*)_j \left(c_{q \cdot}^{(2)} \right)_\alpha \right), \quad \psi_{uj}^{(2)} := \sum_{\alpha \in I_{r_4, r}\{j\}} \text{rdet}_j \left((\mathbf{N} \mathbf{N}^*)_j \left(b_{u \cdot}^{(2)} \right)_\alpha \right).$$

Now, suppose that the both equations of (1) are one-sided. Let in Eq.(1), the matrices \mathbf{B}_1 and \mathbf{A}_2 are vanished. Then, we have the system

$$\begin{cases} \mathbf{A}_1 \mathbf{X} = \mathbf{C}_1, \\ \mathbf{X} \mathbf{B}_2 = \mathbf{C}_2. \end{cases} \quad (33)$$

The following lemma is extended to matrices with quaternion entries.

Lemma 4.3. [31] Let $\mathbf{A}_1 \in \mathbb{H}^{m \times n}$, $\mathbf{B}_2 \in \mathbb{H}^{r \times p}$, $\mathbf{C}_1 \in \mathbb{H}^{m \times r}$, and $\mathbf{C}_2 \in \mathbb{H}^{n \times p}$ be given and $\mathbf{X} \in \mathbb{H}^{n \times r}$ is to be determined. Then, the system (33) is consistent if and only if $\mathbf{R}_{\mathbf{A}_1} \mathbf{C}_1 = \mathbf{0}$, $\mathbf{C}_2 \mathbf{L}_{\mathbf{B}_2} = \mathbf{0}$, and $\mathbf{A}_1 \mathbf{C}_2 = \mathbf{C}_1 \mathbf{B}_2$. Under these conditions, the general solution to (33) can be established as

$$\mathbf{X} = \mathbf{A}_1^\dagger \mathbf{C}_1 + \mathbf{L}_{\mathbf{A}_1} \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{L}_{\mathbf{A}_1} \mathbf{U} \mathbf{R}_{\mathbf{B}_2}, \quad (34)$$

where \mathbf{U} is a free matrix over \mathbb{H} with a suitable shape.

Due to the consistence conditions, Eq. (34) can be expressed as follows:

$$\begin{aligned} \mathbf{X} &= \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{A}_1^\dagger (\mathbf{C}_1 - \mathbf{A}_1 \mathbf{C}_2 \mathbf{B}_2^\dagger) + \mathbf{L}_{\mathbf{A}_1} \mathbf{U} \mathbf{R}_{\mathbf{B}_2} \\ &= \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{A}_1^\dagger (\mathbf{C}_1 - \mathbf{C}_1 \mathbf{B}_2 \mathbf{B}_2^\dagger) + \mathbf{L}_{\mathbf{A}_1} \mathbf{U} \mathbf{R}_{\mathbf{B}_2} = \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{R}_{\mathbf{B}_2} + \mathbf{L}_{\mathbf{A}_1} \mathbf{U} \mathbf{R}_{\mathbf{B}_2}, \end{aligned}$$

Consequently, the partial solution \mathbf{X}^0 to (33) is given by

$$\mathbf{X}^0 = \mathbf{A}_1^\dagger \mathbf{C}_1 + \mathbf{L}_{\mathbf{A}_1} \mathbf{C}_2 \mathbf{B}_2^\dagger, \quad (35)$$

or

$$\mathbf{X}^0 = \mathbf{C}_2 \mathbf{B}_2^\dagger + \mathbf{A}_1^\dagger \mathbf{C}_1 \mathbf{R}_{\mathbf{B}_2}. \quad (36)$$

Due to the expression (35), the following theorem can be proven similar to the proof of Theorem 3.1.

Theorem 4.3. Let $\mathbf{A}_1 = (a_{ij}^{(1)}) \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{B}_2 = (b_{ij}^{(2)}) \in \mathbb{H}_{r_2}^{r \times p}$, $\mathbf{C}_1 = (c_{ij}^{(1)}) \in \mathbb{H}^{m \times r}$, and $\mathbf{C}_2 = (c_{ij}^{(2)}) \in \mathbb{H}^{n \times r}$, and there exist $\mathbf{A}_1^\dagger = (a_{ij}^{(1), \dagger}) \in \mathbb{H}^{n \times m}$, $\mathbf{B}_2^\dagger = (b_{ij}^{(2), \dagger}) \in \mathbb{H}^{p \times r}$, and $\mathbf{L}_{\mathbf{A}_1} = \mathbf{I} - \mathbf{A}_1^\dagger \mathbf{A}_1 =: (l_{ij}) \in \mathbb{H}^{n \times n}$. Denote $\mathbf{A}_1^* \mathbf{C}_1 =: \widehat{\mathbf{C}}_1 = (\widehat{c}_{ij}^{(1)}) \in \mathbb{H}^{n \times r}$ and $\mathbf{L}_{\mathbf{A}_1} \mathbf{C}_2 \mathbf{B}_2^* =: \widehat{\mathbf{C}}_2 = (\widehat{c}_{ij}^{(2)}) \in \mathbb{H}^{n \times r}$. Then, the partial solution (35), $\mathbf{X}^0 = (x_{ij}^0) \in \mathbb{H}^{n \times s}$, possesses the following determinantal representation,

$$x_{ij}^0 = \frac{\sum_{\beta \in I_{r_1, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{A}_1^* \mathbf{A}_1)_{\cdot i} (\widehat{\mathbf{C}}_j^{(1)}) \right)_\beta^\beta}{\sum_{\beta \in I_{r_1, n}} |\mathbf{A}_1^* \mathbf{A}_1|_\beta^\beta} + \frac{\sum_{\alpha \in I_{r_2, r} \setminus \{j\}} \text{rdet}_j \left((\mathbf{B}_2 \mathbf{B}_2^*)_{\cdot j} (\widehat{\mathbf{C}}_i^{(2)}) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{r_2, r}} |\mathbf{B}_2 \mathbf{B}_2^*|_\alpha^\alpha}, \quad (37)$$

where $\widehat{c}_j^{(1)}$ is the j th column of $\widehat{\mathbf{C}}_1$ and $\widehat{c}_i^{(2)}$ is the i th row of $\widehat{\mathbf{C}}_2$.

Remark 4.1. In accordance to the expression (36), we obtain the same representations, but with the denotations, $\mathbf{C}_2 \mathbf{B}_2^* =: \widehat{\mathbf{C}}_2 = (\widehat{c}_{ij}^{(2)}) \in \mathbb{H}^{n \times r}$ and $\mathbf{A}_1^* \mathbf{C}_1 \mathbf{R}_{\mathbf{B}_2} =: \widehat{\mathbf{C}}_1 = (\widehat{c}_{ij}^{(1)}) \in \mathbb{H}^{n \times r}$.

Let in Eq.(1), the matrices \mathbf{B}_1 and \mathbf{B}_2 are vanished. Then, we have the system

$$\begin{cases} \mathbf{A}_1\mathbf{X} = \mathbf{C}_1, \\ \mathbf{A}_2\mathbf{X} = \mathbf{C}_2. \end{cases} \quad (38)$$

Lemma 4.4. [7] Suppose that $\mathbf{A}_1 \in \mathbb{H}^{m \times n}$, $\mathbf{C}_1 \in \mathbb{H}^{m \times r}$, $\mathbf{A}_2 \in \mathbb{H}^{k \times n}$, and $\mathbf{C}_2 \in \mathbb{H}^{k \times r}$ are known and $\mathbf{X} \in \mathbb{H}^{n \times r}$ is unknown, $\mathbf{H} = \mathbf{A}_2\mathbf{L}_{A_1}$, $\mathbf{T} = \mathbf{R}_H\mathbf{A}_2$. Then, the system (38) is consistent if and only if $\mathbf{A}_i\mathbf{A}_i^\dagger\mathbf{C}_i = \mathbf{C}_i$, for all $i = 1, 2$ and $\mathbf{T}(\mathbf{A}_2^\dagger\mathbf{C}_2 - \mathbf{A}_1^\dagger\mathbf{C}_1) = \mathbf{0}$. Under these conditions, the general solution to (38) can be established as

$$\mathbf{X} = \mathbf{A}_1^\dagger\mathbf{C}_1 + \mathbf{L}_{A_1}\mathbf{H}^\dagger\mathbf{A}_2(\mathbf{A}_2^\dagger\mathbf{C}_2 - \mathbf{A}_1^\dagger\mathbf{C}_1) + \mathbf{L}_{A_1}\mathbf{L}_H\mathbf{Y}, \quad (39)$$

where \mathbf{Y} is an arbitrary matrix over \mathbb{H} with an appropriate size.

Using (10) and the consistency conditions, we simplify (39) accordingly, $\mathbf{X}^0 = \mathbf{A}_1^\dagger\mathbf{C}_1 + \mathbf{H}^\dagger\mathbf{C}_2 - \mathbf{H}^\dagger\mathbf{A}_2\mathbf{A}_1^\dagger\mathbf{C}_1 + \mathbf{L}_{A_1}\mathbf{L}_H\mathbf{Y}$. Consequently, the following partial solution of (39) will be considered

$$\mathbf{X}^0 = \mathbf{A}_1^\dagger\mathbf{C}_1 + \mathbf{H}^\dagger\mathbf{C}_2 - \mathbf{H}^\dagger\mathbf{A}_2\mathbf{A}_1^\dagger\mathbf{C}_1. \quad (40)$$

In the following theorem, we give the determinantal representations of (40).

Theorem 4.4. Let $\mathbf{A}_1 = (a_{ij}^{(1)}) \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{A}_2 = (a_{ij}^{(2)}) \in \mathbb{H}_{r_2}^{k \times n}$, $\mathbf{C}_1 = (c_{ij}^{(1)}) \in \mathbb{H}^{m \times r}$, $\mathbf{C}_2 = (c_{ij}^{(2)}) \in \mathbb{H}^{k \times r}$, and there exist $\mathbf{A}_1^\dagger = (a_{ij}^{(1), \dagger}) \in \mathbb{H}^{n \times m}$, $\mathbf{H}_2^\dagger = (h_{ij}^\dagger) \in \mathbb{H}^{n \times s}$. Let $\text{rank } \mathbf{H} = \min\{\text{rank } \mathbf{A}_2, \text{rank } \mathbf{L}_{A_1}\} = r_3$. Denote $\mathbf{A}_1^*\mathbf{C}_1 =: \widehat{\mathbf{C}}_1 = (\widehat{c}_{ij}^{(1)}) \in \mathbb{H}^{n \times r}$, $\mathbf{H}^*\mathbf{C}_2 =: \widehat{\mathbf{C}}_2 = (\widehat{c}_{ij}^{(2)}) \in \mathbb{H}^{n \times r}$, and $\mathbf{H}^*\mathbf{A}_2 =: \widehat{\mathbf{A}}_2 = (\widehat{a}_{ij}^{(2)}) \in \mathbb{H}^{n \times n}$. Then, $\mathbf{X}^0 = (x_{ij}^0) \in \mathbb{H}^{n \times r}$ possesses the following determinantal representation,

$$\begin{aligned} x_{ij}^0 = & \frac{\sum_{\beta \in J_{r_1, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{A}_1^*\mathbf{A}_1)_{.i} (\widehat{\mathbf{c}}_j^{(1)}) \right)_\beta^\beta}{\sum_{\beta \in J_{r_1, n}} |\mathbf{A}_1^*\mathbf{A}_1|_\beta^\beta} + \frac{\sum_{\beta \in J_{r_3, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{H}^*\mathbf{H})_{.i} (\widehat{\mathbf{c}}_j^{(2)}) \right)_\beta^\beta}{\sum_{\beta \in J_{r_3, n}} |\mathbf{H}^*\mathbf{H}|_\beta^\beta} \\ & - \sum_{l=1}^n \frac{\sum_{\beta \in J_{r_3, n} \setminus \{i\}} \text{cdet}_i \left((\mathbf{H}^*\mathbf{H})_{.i} (\widehat{\mathbf{a}}_l^{(2)}) \right)_\beta^\beta}{\sum_{\beta \in J_{r_3, n}} |\mathbf{H}^*\mathbf{H}|_\beta^\beta} \cdot \frac{\sum_{\beta \in J_{r_1, n} \setminus \{l\}} \text{cdet}_l \left((\mathbf{A}_1^*\mathbf{A}_1)_{.l} (\widehat{\mathbf{c}}_j^{(1)}) \right)_\beta^\beta}{\sum_{\beta \in J_{r_1, n}} |\mathbf{A}_1^*\mathbf{A}_1|_\beta^\beta}, \end{aligned} \quad (41)$$

where $\widehat{\mathbf{c}}_j^{(1)}$, $\widehat{\mathbf{c}}_j^{(2)}$, and $\widehat{\mathbf{a}}_j^{(2)}$ are the j th columns of the matrices $\widehat{\mathbf{C}}_1$, $\widehat{\mathbf{C}}_2$, and $\widehat{\mathbf{A}}_2$, respectively.

Proof. The proof is similar to the proof of Theorem 3.1.

5. Conclusion

Within the framework of the theory of row-column determinants previously introduced by the author, we get determinantal representations (analogs of Cramer's rule) of partial solutions to the system of two-sided quaternion matrix equations $\mathbf{A}_1\mathbf{X}\mathbf{B}_1=\mathbf{C}_1$, $\mathbf{A}_2\mathbf{X}\mathbf{B}_2=\mathbf{C}_2$, and its special cases with 1 and 2 one-sided matrix equations. We use previously obtained by the author determinantal representations of the Moore-Penrose inverse. Note to give determinantal representations for all above matrix systems over the complex field, it is obviously needed to substitute all row and column determinants by usual determinants.

Conflict of interest

The author declares that there are no conflict interests.

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