Provisional chapter

3-Algebras in String Theory

Matsuo Sato

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1. Introduction

In this chapter, we review 3-algebras that appear as fundamental properties of string theory. 3-algebra is a generalization of Lie algebra; it is defined by a tri-linear bracket instead of by a bi-linear bracket, and satisfies fundamental identity, which is a generalization of Jacobi identity [1], [2], [3]. We consider 3-algebras equipped with invariant metrics in order to apply them to physics.

It has been expected that there exists M-theory, which unifies string theories. In M-theory, some structures of 3-algebras were found recently. First, it was found that by using $u(N) \oplus u(N)$ Hermitian 3-algebra, we can describe a low energy effective action of N coincident supermembranes [4], [5], [6], [7], [8], which are fundamental objects in M-theory.

With this as motivation, 3-algebras with invariant metrics were classified [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22]. Lie 3-algebras are defined in real vector spaces and tri-linear brackets of them are totally anti-symmetric in all the three entries. Lie 3-algebras with invariant metrics are classified into $_4$ algebra, and Lorentzian Lie 3-algebras, which have metrics with indefinite signatures. On the other hand, Hermitian 3-algebras are defined in Hermitian vector spaces and their tri-linear brackets are complex linear and anti-symmetric in the first two entries, whereas complex anti-linear in the third entry. Hermitian 3-algebras with invariant metrics are classified into $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras.

Moreover, recent studies have indicated that there also exist structures of 3-algebras in the Green-Schwartz supermembrane action, which defines full perturbative dynamics of a supermembrane. It had not been clear whether the total supermembrane action including fermions has structures of 3-algebras, whereas the bosonic part of the action can be described by using a tri-linear bracket, called Nambu bracket [23], [24], which is a generalization of Poisson bracket. If we fix to a light-cone gauge, the total action can be described by using

Poisson bracket, that is, only structures of Lie algebra are left in this gauge [25]. However, it was shown under an approximation that the total action can be described by Nambu bracket if we fix to a semi-light-cone gauge [26]. In this gauge, the eleven dimensional space-time of M-theory is manifest in the supermembrane action, whereas only ten dimensional part is manifest in the light-cone gauge.

The BFSS matrix theory is conjectured to describe an infinite momentum frame (IMF) limit of M-theory [27] and many evidences were found. The action of the BFSS matrix theory can be obtained by replacing Poisson bracket with a finite dimensional Lie algebra's bracket in the supermembrane action in the light-cone gauge. Because of this structure, only variables that represent the ten dimensional part of the eleven-dimensional space-time are manifest in the BFSS matrix theory. Recently, 3-algebra models of M-theory were proposed [26], [28], [29], by replacing Nambu bracket with finite dimensional 3-algebras' brackets in an action that is shown, by using an approximation, to be equivalent to the semi-light-cone supermembrane action. All the variables that represent the eleven dimensional space-time are manifest in these models. It was shown that if the DLCQ limit of the 3-algebra models of M-theory is taken, they reduce to the BFSS matrix theory [26], [28], as they should [30], [31], [32], [33], [34], [35].

2. Definition and classification of metric Hermitian 3-algebra

In this section, we will define and classify the Hermitian 3-algebras equipped with invariant metrics.

2.1. General structure of metric Hermitian 3-algebra

The metric Hermitian 3-algebra is a map $V \times V \times V \to V$ defined by $(x, y, z) \mapsto [x, y; z]$, where the 3-bracket is complex linear in the first two entries, whereas complex anti-linear in the last entry, equipped with a metric $< x, y >$, satisfying the following properties:

the fundamental identity

$$[[x, y; z], v; w] = [[x, v; w], y; z] + [x, [y, v; w]; z] - [x, y; [z, w; v]] \tag{id2}$$

the metric invariance

$$< [x, v; w], y > - < x, [y, w; v] > = 0 \tag{id3}$$

and the anti-symmetry

$$[x, y; z] = -[y, x; z] \tag{id4}$$

for

$$x,\ y,\ z,\ v,\ w \in V \tag{id5}$$

The Hermitian 3-algebra generates a symmetry, whose generators $D(x,\ y)$ are defined by

$$D(x,\ y)z := [z,\ x;\ y] \tag{id6}$$

From (□), one can show that $D(x,\ y)$ form a Lie algebra,

$$[D(x,\ y),\ D(v,\ w)] = D(D(x,\ y)v,\ w) - D(v,\ D(y,\ x)w) \tag{id7}$$

There is an one-to-one correspondence between the metric Hermitian 3-algebra and a class of metric complex super Lie algebras [19]. Such a class satisfies the following conditions among complex super Lie algebras $S = S_0 \oplus S_1$, where S_0 and S_1 are even and odd parts, respectively. S_1 is decomposed as $S_1 = V \oplus \overline{V}$, where V is an unitary representation of S_0: for $a \in S_0$, $u,\ v \in V$,

$$[a,\ u] \in V \tag{id8}$$

and

$$< [a,\ u],\ v > + < u,\ [a^*,\ v] > = 0 \tag{id9}$$

$\bar{v} \in \overline{V}$ is defined by

$$\bar{v} = < \ \ ,\ v > \tag{id10}$$

The super Lie bracket satisfies

$$[V,\ V] = 0, \quad [\overline{V},\ \overline{V}] = 0 \tag{id11}$$

From the metric Hermitian 3-algebra, we obtain the class of the metric complex super Lie algebra in the following way. The elements in S_0, V, and $\overline{V}$ are defined by (□), (□), and (□), respectively. The algebra is defined by (□) and

$$\begin{aligned}
&[D(x,\ y),\ z] := D(x,\ y)z = [z,\ x;\ y] \\
&[D(x,\ y),\ \bar{z}] := -\overline{D(y,\ x)z} = -\overline{[z,\ y;\ x]} \\
&[x,\ \bar{y}] := D(x,\ y) \\
&[x,\ y] := 0 \\
&[\bar{x},\ \bar{y}] := 0
\end{aligned} \tag{id12}$$

One can show that this algebra satisfies the super Jacobi identity and (▭)-(▭) as in [19].

Inversely, from the class of the metric complex super Lie algebra, we obtain the metric Hermitian 3-algebra by

$$[x, y; z] := \alpha[[y, \bar{z}], x] \tag{id13}$$

where α is an arbitrary constant. One can also show that this algebra satisfies (▭)-(▭) for (▭) as in [19].

2.2. Classification of metric Hermitian 3-algebra

The classical Lie super algebras satisfying (▭)-(▭) are $A(m-1, n-1)$ and $C(n+1)$. The even parts of $A(m-1, n-1)$ and $C(n+1)$ are $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$, respectively. Because the metric Hermitian 3-algebra one-to-one corresponds to this class of the super Lie algebra, the metric Hermitian 3-algebras are classified into $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebras.

First, we will construct the $u(m) \oplus u(n)$ Hermitian 3-algebra from $A(m-1, n-1)$, according to the relation in the previous subsection. $A(m-1, n-1)$ is simple and is obtained by dividing $sl(m, n)$ by its ideal. That is, $A(m-1, n-1) = sl(m, n)$ when $m \neq n$ and $A(n-1, n-1) = sl(n, n)/\lambda 1_{2n}$.

Real $sl(m, n)$ is defined by

$$\begin{pmatrix} h_1 & c \\ ic^{\dagger} & h_2 \end{pmatrix} \tag{id15}$$

where h_1 and h_2 are $m \times m$ and $n \times n$ anti-Hermite matrices and c is an $n \times m$ arbitrary complex matrix. Complex $sl(m, n)$ is a complexification of real $sl(m, n)$, given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{id16}$$

where α, β, γ, and δ are $m \times m$, $n \times m$, $m \times n$, and $n \times n$ complex matrices that satisfy

$$\mathrm{tr}\alpha = \mathrm{tr}\delta \tag{id17}$$

Complex $A(m-1, n-1)$ is decomposed as $A(m-1, n-1) = S_0 \oplus V \oplus \overline{V}$, where

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in S_0$$
$$\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \in V \qquad \text{(id18)}$$
$$\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \in \overline{V}$$

(▭) is rewritten as $V \to \overline{V}$ defined by

$$B = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \mapsto B^{\dagger} = \begin{pmatrix} 0 & 0 \\ \beta^{\dagger} & 0 \end{pmatrix} \qquad \text{(id19)}$$

where $B \in V$ and $B^{\dagger} \in \overline{V}$. (▭) is rewritten as

$$[X, Y; Z] = \alpha[[Y, Z^{\dagger}], X] = \alpha \begin{pmatrix} 0 & yz^{\dagger}x - xz^{\dagger}y \\ 0 & 0 \end{pmatrix} \qquad \text{(id20)}$$

for

$$X = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in V$$
$$Y = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in V \qquad \text{(id21)}$$
$$Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in V$$

As a result, we obtain the $u(m) \oplus u(n)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha\left(yz^{\dagger}x - xz^{\dagger}y\right) \qquad \text{(id22)}$$

where x, y, and z are arbitrary $n \times m$ complex matrices. This algebra was originally constructed in [8].

Inversely, from (), we can construct complex $A(m-1, n-1)$. (▭) is rewritten as

$$D(x, y) = \left(xy^{\dagger}, y^{\dagger}x\right) \in S_0 \qquad \text{(id23)}$$

(▭) and (▭) are rewritten as

$$\begin{aligned}
&[(xy^{\dagger},\ y^{\dagger}x),\ (x'y'^{\dagger},\ y'^{\dagger}x')] = ([xy^{\dagger},\ x'y'^{\dagger}],\ [y^{\dagger}x,\ y'^{\dagger}x'])\\
&[(xy^{\dagger},\ y^{\dagger}x),\ z] = xy^{\dagger}z - zy^{\dagger}x\\
&[(xy^{\dagger},\ y^{\dagger}x),\ w^{\dagger}] = y^{\dagger}xw^{\dagger} - w^{\dagger}xy^{\dagger}\\
&[x,\ y^{\dagger}] = (xy^{\dagger},\ y^{\dagger}x)\\
&[x,\ y] = 0\\
&[x^{\dagger},\ y^{\dagger}] = 0
\end{aligned} \tag{id24}$$

This algebra is summarized as

$$\left[\begin{pmatrix} xy^{\dagger} & z \\ w^{\dagger} & y^{\dagger}x \end{pmatrix}, \begin{pmatrix} x'y'^{\dagger} & z' \\ w'^{\dagger} & y'^{\dagger}x' \end{pmatrix}\right] \tag{id25}$$

which forms complex $A(m-1,\ n-1)$.

Next, we will construct the $sp(2n) \oplus u(1)$ Hermitian 3-algebra from $C(n+1)$. Complex $C(n+1)$ is decomposed as $C(n+1) = S_0 \oplus V \oplus \overline{V}$. The elements are given by

$$\begin{aligned}
&\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a^{T} \end{pmatrix} \in S_0\\
&\begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^{T} & 0 & 0 \\ 0 & -x_1^{T} & 0 & 0 \end{pmatrix} \in V\\
&\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_2 \\ y_2^{T} & 0 & 0 & 0 \\ -y_1^{T} & 0 & 0 & 0 \end{pmatrix} \in \overline{V}
\end{aligned} \tag{id26}$$

where α is a complex number, a is an arbitrary $n \times n$ complex matrix, b and c are $n \times n$ complex symmetric matrices, and x_1, x_2, y_1 and y_2 are $n \times 1$ complex matrices. (▭) is rewritten as $V \to \overline{V}$ defined by $B \mapsto \overline{B} = UB^{*}U^{-1}$, where $B \in V$, $\overline{B} \in \overline{V}$ and

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{id27}$$

Explicitly,

$$B = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \mapsto \bar{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_2^* & -x_1^* \\ -x_1^\dagger & 0 & 0 & 0 \\ -x_2^\dagger & 0 & 0 & 0 \end{pmatrix} \tag{id28}$$

(▭) is rewritten as

$$\begin{aligned} [X, Y; Z] &:= \alpha[[Y, \bar{Z}], X] \\ &= \alpha\left[\left[\begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z_2^* & -z_1^* \\ -z_1^\dagger & 0 & 0 & 0 \\ -z_2^\dagger & 0 & 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix}\right] \\ &= \alpha \begin{pmatrix} 0 & 0 & w_1 & w_2 \\ 0 & 0 & 0 & 0 \\ 0 & w_2^T & 0 & 0 \\ 0 & -w_1^T & 0 & 0 \end{pmatrix} \end{aligned} \tag{id29}$$

for

$$X = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \in V$$
$$Y = \begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix} \in V \qquad \text{(id30)}$$
$$Z = \begin{pmatrix} 0 & 0 & z_1 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & z_2^T & 0 & 0 \\ 0 & -z_1^T & 0 & 0 \end{pmatrix} \in V$$

where w_1 and w_2 are given by

$$(w_1, w_2) = -\left(y_1 z_1^\dagger + y_2 z_2^\dagger\right)(x_1, x_2) + \left(x_1 z_1^\dagger + x_2 z_2^\dagger\right)(y_1, y_2) + \left(x_2 y_1^T - x_1 y_2^T\right)\left(z_2^*, -z_1^*\right) \qquad \text{(id31)}$$

As a result, we obtain the $sp(2n) \oplus u(1)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha((y \odot \tilde{z})x + (\tilde{z} \odot x)y - (x \odot y)\tilde{z}) \qquad \text{(id32)}$$

for $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, where x_1, x_2, y_1, y_2, z_1, and z_2 are n-vectors and

$$\begin{aligned} \tilde{z} &= \left(z_2^*, -z_1^*\right) \\ a \odot b &= a_1 \cdot b_2 - a_2 \cdot b_1 \end{aligned} \qquad \text{(id33)}$$

3. 3-algebra model of M-theory

In this section, we review the fact that the supermembrane action in a semi-light-cone gauge can be described by Nambu bracket, where structures of 3-algebra are manifest. The 3-algebra Models of M-theory are defined based on the semi-light-cone supermembrane action. We also review that the models reduce to the BFSS matrix theory in the DLCQ limit.

3.1. Supermembrane and 3-algebra model of M-theory

The fundamental degrees of freedom in M-theory are supermembranes. The action of the covariant supermembrane action in M-theory [36] is given by

$$S_{M2} = \int d^3\sigma\Big(\sqrt{-G} + \frac{i}{4}\,{}^{\alpha\beta\gamma}\bar{\Psi}\Gamma_{MN}\partial_\alpha\Psi(\Pi_\beta{}^M\Pi_\gamma{}^N + \frac{i}{2}\Pi_\beta{}^M\bar{\Psi}\Gamma^N\partial_\gamma\Psi - \frac{1}{12}\bar{\Psi}\Gamma^M\partial_\beta\Psi\bar{\Psi}\Gamma^N\partial_\gamma\Psi)\Big) \quad \text{(id35)}$$

where $M, N = 0, \cdots, 10$, $\alpha, \beta, \gamma = 0, 1, 2$, $G_{\alpha\beta} = \Pi_\alpha{}^M\Pi_{\beta M}$ and $\Pi_\alpha{}^M = \partial_\alpha X^M - \frac{i}{2}\bar{\Psi}\Gamma^M\partial_\alpha\Psi$. Ψ is a $SO(1, 10)$ Majorana fermion.

This action is invariant under dynamical supertransformations,

$$\begin{aligned} \delta\Psi &= \\ \delta X^M &= -i\bar{\Psi}\Gamma^M \end{aligned} \quad \text{(id36)}$$

These transformations form the = 1 supersymmetry algebra in eleven dimensions,

$$[\delta_1, \delta_2]X^M = -2i_1\Gamma^M{}_2 \quad \text{(id37)}$$

$$[\delta_1, \delta_2]\Psi = 0 \quad \text{(id38)}$$

The action is also invariant under the κ-symmetry transformations,

$$\begin{aligned} \delta\Psi &= (1+\Gamma)\kappa(\sigma) \\ \delta X^M &= i\bar{\Psi}\Gamma^M(1+\Gamma)\kappa(\sigma) \end{aligned} \quad \text{(id39)}$$

where

$$\Gamma = \frac{1}{3!\sqrt{-G}}\,{}^{\alpha\beta\gamma}\Pi_\alpha^L\,\Pi_\beta^M\Pi_\gamma^N\Gamma_{LMN} \quad \text{(id40)}$$

If we fix the κ-symmetry (▭) of the action by taking a semi-light-cone gauge [26]Advantages of a semi-light-cone gauges against a light-cone gauge are shown in [37], [38], [39]

$$\Gamma^{012}\Psi = -\Psi \quad \text{(id42)}$$

we obtain a semi-light-cone supermembrane action,

$$S_{M2} = \int d^3\sigma\Big(\sqrt{-G} + \frac{i}{4}\,{}^{\alpha\beta\gamma}\Big(\bar{\Psi}\Gamma_{\mu\nu}\partial_\alpha\Psi\Big(\Pi_\beta{}^\mu\Pi_\gamma{}^\nu + \frac{i}{2}\Pi_\beta{}^\mu\bar{\Psi}\Gamma^\nu\partial_\gamma\Psi - \frac{1}{12}\bar{\Psi}\Gamma^\mu\partial_\beta\Psi\bar{\Psi}\Gamma^\nu\partial_\gamma\Psi\Big) + \bar{\Psi}\Gamma_{IJ}\partial_\alpha\Psi\partial_\beta X^I\partial_\gamma X^J\Big)\Big) \quad \text{(id43)}$$

where $G_{\alpha\beta} = h_{\alpha\beta} + \Pi_\alpha{}^\mu \Pi_{\beta\mu}$, $\Pi_\alpha{}^\mu = \partial_\alpha X^\mu - \frac{i}{2}\bar{\Psi}\Gamma^\mu\partial_\alpha\Psi$, and $h_{\alpha\beta} = \partial_\alpha X^I \partial_\beta X_I$.

In [26], it is shown under an approximation up to the quadratic order in $\partial_\alpha X^\mu$ and $\partial_\alpha\Psi$ but exactly in X^I, that this action is equivalent to the continuum action of the 3-algebra model of M-theory,

$$\begin{aligned} S_{cl} \;=\; & \int d^3\sigma\sqrt{-g}\Big(-\frac{1}{12}\{X^I, X^J, X^K\}^2 - \frac{1}{2}\big(A_{\mu ab}\{\varphi^a, \varphi^b, X^I\}\big)^2 \\ & -\frac{1}{3}E^{\mu\nu\lambda}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\{\varphi^a, \varphi^c, \varphi^d\}\{\varphi^b, \varphi^e, \varphi^f\} + \frac{1}{2}\Lambda \\ & -\frac{i}{2}\bar{\Psi}\Gamma^\mu A_{\mu ab}\{\varphi^a, \varphi^b, \Psi\} + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}\{X^I, X^J, \Psi\}\Big) \end{aligned} \qquad \text{(id44)}$$

where $I, J, K = 3, \cdots, 10$ and $\{\varphi^a, \varphi^b, \varphi^c\} = {}^{\alpha\beta\gamma}\partial_\alpha\varphi^a\partial_\beta\varphi^b\partial_\gamma\varphi^c$ is the Nambu-Poisson bracket. An invariant symmetric bilinear form is defined by $\int d^3\sigma\sqrt{-g}\varphi^a\varphi^b$ for complete basis φ^a in three dimensions. Thus, this action is manifestly VPD covariant even when the world-volume metric is flat. X^I is a scalar and Ψ is a $SO(1, 2) \times SO(8)$ Majorana-Weyl fermion satisfying (▭). $E^{\mu\nu\lambda}$ is a Levi-Civita symbol in three dimensions and Λ is a cosmological constant.

The continuum action of 3-algebra model of M-theory (▭) is invariant under 16 dynamical supersymmetry transformations,

$$\begin{aligned} & \delta X^I = i\Gamma^I\Psi \\ & \delta A_\mu(\sigma, \sigma') = \frac{i}{2}\Gamma_\mu\Gamma_I\big(X^I(\sigma)\Psi(\sigma') - X^I(\sigma')\Psi(\sigma)\big), \\ & \delta\Psi = -A_{\mu ab}\{\varphi^a, \varphi^b, X^I\}\Gamma^\mu\Gamma_I - \frac{1}{6}\{X^I, X^J, X^K\}\Gamma_{IJK} \end{aligned} \qquad \text{(id45)}$$

where $\Gamma_{012} = -$. These supersymmetries close into gauge transformations on-shell,

$$\begin{aligned} & [\delta_1, \delta_2]X^I = \Lambda_{cd}\{\varphi^c, \varphi^d, X^I\} \\ & [\delta_1, \delta_2]A_{\mu ab}\{\varphi^a, \varphi^b, \quad\} = \Lambda_{ab}\{\varphi^a, \varphi^b, A_{\mu cd}\{\varphi^c, \varphi^d, \quad\}\} \\ & \qquad - A_{\mu ab}\{\varphi^a, \varphi^b, \Lambda_{cd}\{\varphi^c, \varphi^d, \quad\}\} + 2i_2\Gamma^\nu{}_1 O^A_{\mu\nu} \\ & [\delta_1, \delta_2]\Psi = \Lambda_{cd}\{\varphi^c, \varphi^d, \Psi\} + \Big(i_2\Gamma^\mu{}_1\Gamma_\mu - \frac{i}{4}{}_2\Gamma^{KL}{}_1\Gamma_{KL}\Big)O^\Psi \end{aligned} \qquad \text{(id46)}$$

where gauge parameters are given by $\Lambda_{ab} = 2i_2\Gamma^\mu{}_1 A_{\mu ab} - i_2\Gamma_{JK1}X^J_a X^K_b$. $O^A_{\mu\nu} = 0$ and $O^\Psi = 0$ are equations of motions of $A_{\mu\nu}$ and Ψ, respectively, where

$$O^A_{\mu\nu} = A_{\mu ab}\{\varphi^a, \varphi^b, A_{\nu cd}\{\varphi^c, \varphi^d, \quad\}\} - A_{\nu ab}\{\varphi^a, \varphi^b, A_{\mu cd}\{\varphi^c, \varphi^d, \quad\}\}$$
$$+E_{\mu\nu\lambda}\left(-\{X^I, A^\lambda_{ab}\{\varphi^a, \varphi^b, X_I\}, \quad\} + \frac{i}{2}\{\bar{\Psi}, \Gamma^\lambda\Psi, \quad\}\right) \qquad \text{(id47)}$$
$$O^\Psi = -\Gamma^\mu A_{\mu ab}\{\varphi^a, \varphi^b, \Psi\} + \frac{1}{2}\Gamma_{IJ}\{X^I, X^J, \Psi\}$$

(▭) implies that a commutation relation between the dynamical supersymmetry transformations is

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \qquad \text{(id48)}$$

up to the equations of motions and the gauge transformations.

This action is invariant under a translation,

$$\delta X^I(\sigma) = \eta^I, \qquad \delta A^\mu(\sigma, \sigma') = \eta^\mu(\sigma) - \eta^\mu(\sigma') \qquad \text{(id49)}$$

where η^I are constants.

The action is also invariant under 16 kinematical supersymmetry transformations

$$\delta\Psi = \qquad \text{(id50)}$$

and the other fields are not transformed. is a constant and satisfy $\Gamma_{012} = .$ and should come from sixteen components of thirty-two $= 1$ supersymmetry parameters in eleven dimensions, corresponding to eigen values ± 1 of Γ_{012}, respectively. This $= 1$ supersymmetry consists of remaining 16 target-space supersymmetries and transmuted 16 κ-symmetries in the semi-light-cone gauge [26], [25], [40].

A commutation relation between the kinematical supersymmetry transformations is given by

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \qquad \text{(id51)}$$

A commutator of dynamical supersymmetry transformations and kinematical ones acts as

$$(\delta_2\delta_1 - \delta_1\delta_2)X^I(\sigma) = i_1\Gamma^I{}_2 \equiv \eta_0^I$$
$$(\delta_2\delta_1 - \delta_1\delta_2)A^\mu(\sigma, \sigma') = \frac{i}{2}{}_1\Gamma^\mu\Gamma_I(X^I(\sigma) - X^I(\sigma'))_2 \equiv \eta_0^\mu(\sigma) - \eta_0^\mu(\sigma') \qquad \text{(id52)}$$

where the commutator that acts on the other fields vanishes. Thus, the commutation relation is given by

$$\delta_2\delta_1 - \delta_1\delta_2 = \delta_\eta \qquad \text{(id53)}$$

where δ_η is a translation.

If we change a basis of the supersymmetry transformations as

$$\begin{aligned} \delta^{'} &= \delta + \delta \\ \delta^{'} &= i(\delta - \delta) \end{aligned} \qquad \text{(id54)}$$

we obtain

$$\begin{aligned} \delta_2^{'}\delta_1^{'} - \delta_1^{'}\delta_2^{'} &= \delta_\eta \\ \delta_2^{'}\delta_1^{'} - \delta_1^{'}\delta_2^{'} &= \delta_\eta \\ \delta_2^{'}\delta_1^{'} - \delta_1^{'}\delta_2^{'} &= 0 \end{aligned} \qquad \text{(id55)}$$

These thirty-two supersymmetry transformations are summarised as $\Delta = \left(\delta^{'}, \delta^{'}\right)$ and (□) implies the = 1 supersymmetry algebra in eleven dimensions,

$$\Delta_2\Delta_1 - \Delta_1\Delta_2 = \delta_\eta \qquad \text{(id56)}$$

3.2. Lie 3-algebra models of M-theory

In this and next subsection, we perform the second quantization on the continuum action of the 3-algebra model of M-theory: By replacing the Nambu-Poisson bracket in the action (□) with brackets of finite-dimensional 3-algebras, Lie and Hermitian 3-algebras, we obtain the Lie and Hermitian 3-algebra models of M-theory [26], [28], respectively. In this section, we review the Lie 3-algebra model.

If we replace the Nambu-Poisson bracket in the action (□) with a completely antisymmetric real 3-algebra's bracket [21], [22],

$$\begin{aligned} &\int d^3\sigma\sqrt{-g} \rightarrow \langle \quad \rangle \\ &\left\{\varphi^a, \varphi^b, \varphi^c\right\} \rightarrow [T^a, T^b, T^c] \end{aligned} \qquad \text{(id58)}$$

we obtain the Lie 3-algebra model of M-theory [26], [28],

$$\begin{aligned} S_0 \;=\; & < -\frac{1}{12}[X^I, X^J, X^K]^2 - \frac{1}{2}\left(A_{\mu ab}[T^a, T^b, X^I]\right)^2 \\ & -\frac{1}{3}E^{\mu\nu\lambda}A_{\mu ab}A_{\nu cd}A_{\lambda ef}[T^a, T^c, T^d][T^b, T^e, T^f] \\ & -\frac{i}{2}\bar{\Psi}\Gamma^\mu A_{\mu ab}[T^a, T^b, \Psi] + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}[X^I, X^J, \Psi] > \end{aligned} \tag{id59}$$

We have deleted the cosmological constant Λ, which corresponds to an operator ordering ambiguity, as usual as in the case of other matrix models [27], [41].

This model can be obtained formally by a dimensional reduction of the = 8 BLG model [4], [5], [6],

$$\begin{aligned} S_{=8BLG} \;=\; & \int d^3x < -\frac{1}{12}[X^I, X^J, X^K]^2 - \frac{1}{2}\left(D_\mu X^I\right)^2 - E^{\mu\nu\lambda}\Big(\frac{1}{2}A_{\mu ab}\partial_\nu A_{\lambda cd}T^a[T^b, T^c, T^d] \\ & + \frac{1}{3}A_{\mu ab}A_{\nu cd}A_{\lambda ef}[T^a, T^c, T^d][T^b, T^e, T^f]\Big) \\ & + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}[X^I, X^J, \Psi] > \end{aligned} \tag{id60}$$

The formal relations between the Lie (Hermitian) 3-algebra models of M-theory and the = 8 (= 6) BLG models are analogous to the relation among the = 4 super Yang-Mills in four dimensions, the BFSS matrix theory [27], and the IIB matrix model [41]. They are completely different theories although they are related to each others by dimensional reductions. In the same way, the 3-algebra models of M-theory and the BLG models are completely different theories.

The fields in the action (▭) are spanned by the Lie 3-algebra T^a as $X^I = X^I_a T^a$, $\Psi = \Psi_a T^a$ and $A^\mu = A^\mu_{ab} T^a \otimes T^b$, where $I = 3, \cdots, 10$ and $\mu = 0, 1, 2$. $< >$ represents a metric for the 3-algebra. Ψ is a Majorana spinor of SO(1,10) that satisfies $\Gamma_{012}\Psi = \Psi$. $E^{\mu\nu\lambda}$ is a Levi-Civita symbol in three-dimensions.

Finite dimensional Lie 3-algebras with an invariant metric is classified into four-dimensional Euclidean $_4$ algebra and the Lie 3-algebras with indefinite metrics in [9], [10], [11], [21], [22]. We do not choose $_4$ algebra because its degrees of freedom are just four. We need an algebra with arbitrary dimensions N, which is taken to infinity to define M-theory. Here we choose the most simple indefinite metric Lie 3-algebra, so called the Lorentzian Lie 3-algebra associated with $u(N)$ Lie algebra,

$$\begin{aligned} & [T^{-1}, T^a, T^b] = 0 \\ & [T^0, T^i, T^j] = [T^i, T^j] = f^{ij}{}_k T^k \\ & [T^i, T^j, T^k] = f^{ijk} T^{-1} \end{aligned} \tag{id61}$$

where $a = -1, 0, i$ $(i = 1, \cdots, N^2)$. T^i are generators of $u(N)$. A metric is defined by a symmetric bilinear form,

$$\begin{aligned} < T^{-1}, T^0 > &= -1 \\ < T^i, T^j > &= h^{ij} \end{aligned} \tag{id62}$$

and the other components are 0. The action is decomposed as

$$\begin{aligned} S = \mathrm{Tr}\Big(&-\frac{1}{4}\left(x_0^K\right)^2[x^I, x^J]^2 + \frac{1}{2}\left(x_0^I[x_I, x^J]\right)^2 - \frac{1}{2}\left(x_0^I b_\mu + [a_\mu, x^I]\right)^2 - \frac{1}{2}E^{\mu\nu\lambda}b_\mu[a_\nu, a_\lambda] \\ &+ i\bar{\Psi}_0\Gamma^\mu b_\mu\psi - \frac{i}{2}\bar{\Psi}\Gamma^\mu[a_\mu, \psi] + \frac{i}{2}x_0^I\bar{\Psi}\Gamma_{IJ}[x^J, \psi] - \frac{i}{2}\bar{\Psi}_0\Gamma_{IJ}[x^I, x^J]\psi\Big) \end{aligned} \tag{id63}$$

where we have renamed $X_0^I \to x_0^I$, $X_i^I T^i \to x^I$, $\Psi_0 \to \psi_0$, $\Psi_i T^i \to \psi$, $2A_{\mu 0i}T^i \to a_\mu$, and $A_{\mu ij}[T^i, T^j] \to b_\mu$. a_μ correspond to the target coordinate matrices X^μ, whereas b_μ are auxiliary fields.

In this action, T^{-1} mode; X_{-1}^I, Ψ_{-1} or A_{-1a}^μ does not appear, that is they are unphysical modes. Therefore, the indefinite part of the metric (▭) does not exist in the action and the Lie 3-algebra model of M-theory is ghost-free like a model in [42]. This action can be obtained by a dimensional reduction of the three-dimensional = 8 BLG model [4], [5], [6] with the same 3-algebra. The BLG model possesses a ghost mode because of its kinetic terms with indefinite signature. On the other hand, the Lie 3-algebra model of M-theory does not possess a kinetic term because it is defined as a zero-dimensional field theory like the IIB matrix model [41].

This action is invariant under the translation

$$\delta x^I = \eta^I, \qquad \delta a^\mu = \eta^\mu \tag{id64}$$

where η^I and η^μ belong to $u(1)$. This implies that eigen values of x^I and a^μ represent an eleven-dimensional space-time.

The action is also invariant under 16 kinematical supersymmetry transformations

$$\delta\psi = \hat{} \tag{id65}$$

and the other fields are not transformed. ˆ belong to $u(1)$ and satisfy Γ_{012}ˆ = ˆ. ˆ and should come from sixteen components of thirty-two = 1 supersymmetry parameters in eleven dimensions, corresponding to eigen values ± 1 of Γ_{012}, respectively, as in the previous subsection.

A commutation relation between the kinematical supersymmetry transformations is given by

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$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \qquad \text{(id66)}$$

The action is invariant under 16 dynamical supersymmetry transformations,

$$\begin{aligned}
&\delta X^I = i\Gamma^I\Psi \\
&\delta A_{\mu ab}[T^a, T^b, \quad] = i\Gamma_\mu\Gamma_I[X^I, \Psi, \quad] \\
&\delta\Psi = -A_{\mu ab}[T^a, T^b, X^I]\Gamma^\mu\Gamma_I - \frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK}
\end{aligned} \qquad \text{(id67)}$$

where $\Gamma_{012} = -$. These supersymmetries close into gauge transformations on-shell,

$$\begin{aligned}
&[\delta_1, \delta_2]X^I = \Lambda_{cd}[T^c, T^d, X^I] \\
&[\delta_1, \delta_2]A_{\mu ab}[T^a, T^b, \quad] = \Lambda_{ab}[T^a, T^b, A_{\mu cd}[T^c, T^d, \quad]] \\
&\qquad - A_{\mu ab}[T^a, T^b, \Lambda_{cd}[T^c, T^d, \quad]] + 2i_2\Gamma^\nu{}_1 O^A_{\mu\nu} \\
&[\delta_1, \delta_2]\Psi = \Lambda_{cd}[T^c, T^d, \Psi] + \left(i_2\Gamma^\mu{}_1\Gamma_\mu - \frac{i}{4}{}_2\Gamma^{KL}{}_1\Gamma_{KL}\right)O^\Psi
\end{aligned} \qquad \text{(id68)}$$

where gauge parameters are given by $\Lambda_{ab} = 2i_2\Gamma^\mu{}_1A_{\mu ab} - i_2\Gamma_{JK1}X^J_aX^K_b$. $O^A_{\mu\nu} = 0$ and $O^\Psi = 0$ are equations of motions of $A_{\mu\nu}$ and Ψ, respectively, where

$$\begin{aligned}
O^A_{\mu\nu} &= A_{\mu ab}[T^a, T^b, A_{\nu cd}[T^c, T^d, \quad]] - A_{\nu ab}[T^a, T^b, A_{\mu cd}[T^c, T^d, \quad]] \\
&\quad + E_{\mu\nu\lambda}\left(-[X^I, A^\lambda_{ab}[T^a, T^b, X_I], \quad] + \frac{i}{2}[\bar{\Psi}, \Gamma^\lambda\Psi, \quad]\right) \\
O^\Psi &= -\Gamma^\mu A_{\mu ab}[T^a, T^b, \Psi] + \frac{1}{2}\Gamma_{IJ}[X^I, X^J, \Psi]
\end{aligned} \qquad \text{(id69)}$$

(▭) implies that a commutation relation between the dynamical supersymmetry transformations is

$$\delta_2\delta_1 - \delta_1\delta_2 = 0 \qquad \text{(id70)}$$

up to the equations of motions and the gauge transformations.

The 16 dynamical supersymmetry transformations (▭) are decomposed as

$$
\begin{aligned}
\delta x^{I} &= i\Gamma^{I}\psi \\
\delta x_{0}^{I} &= i\Gamma^{I}\psi_{0} \\
\delta x_{-1}^{I} &= i\Gamma^{I}\psi_{-1} \\
\delta\psi &= -\left(b_{\mu}x_{0}^{I} + [a_{\mu}, x^{I}]\right)\Gamma^{\mu}\Gamma_{I} - \frac{1}{2}x_{0}^{I}[x^{J}, x^{K}]\Gamma_{IJK} \\
\delta\psi_{0} &= 0 \\
\delta\psi_{-1} &= -\mathrm{Tr}\left(b_{\mu}x^{I}\right)\Gamma^{\mu}\Gamma_{I} - \frac{1}{6}\mathrm{Tr}\left([x^{I}, x^{J}]x^{K}\right)\Gamma_{IJK} \\
\delta a_{\mu} &= i\Gamma_{\mu}\Gamma_{I}\left(x_{0}^{I}\psi - \psi_{0}x^{I}\right) \\
\delta b_{\mu} &= i\Gamma_{\mu}\Gamma_{I}[x^{I}, \psi] \\
\delta A_{\mu-1i} &= i\Gamma_{\mu}\Gamma_{I}\frac{1}{2}\left(x_{-1}^{I}\psi_{i} - \psi_{-1}x_{i}^{I}\right) \\
\delta A_{\mu-10} &= i\Gamma_{\mu}\Gamma_{I}\frac{1}{2}\left(x_{-1}^{I}\psi_{0} - \psi_{-1}x_{0}^{I}\right)
\end{aligned}
\tag{id71}
$$

and thus a commutator of dynamical supersymmetry transformations and kinematical ones acts as

$$
\begin{aligned}
(\delta_{2}\delta_{1} - \delta_{1}\delta_{2})x^{I} &= i_{1}\Gamma^{I}{}_{2} \equiv \eta^{I} \\
(\delta_{2}\delta_{1} - \delta_{1}\delta_{2})a^{\mu} &= i_{1}\Gamma^{\mu}\Gamma_{I}x_{0}^{I}{}_{2} \equiv \eta^{\mu} \\
(\delta_{2}\delta_{1} - \delta_{1}\delta_{2})A_{-1i}^{\mu}T^{i} &= \frac{1}{2}i_{1}\Gamma^{\mu}\Gamma_{I}x_{-1}^{I}{}_{2}
\end{aligned}
\tag{id72}
$$

where the commutator that acts on the other fields vanishes. Thus, the commutation relation for physical modes is given by

$$
\delta_{2}\delta_{1} - \delta_{1}\delta_{2} = \delta_{\eta} \tag{id73}
$$

where δ_{η} is a translation.

(▭), (▭), and (▭) imply the = 1 supersymmetry algebra in eleven dimensions as in the previous subsection.

3.3. Hermitian 3-algebra model of M-theory

In this subsection, we study the Hermitian 3-algebra models of M-theory [26]. Especially, we study mostly the model with the $u(N) \oplus u(N)$ Hermitian 3-algebra (▭).

The continuum action (▭) can be rewritten by using the triality of $SO(8)$ and the $SU(4) \times U(1)$ decomposition [8], [43], [44] as

$$\begin{aligned} S_{cl} = & \int d^3\sigma\sqrt{-g}\Big(-V - A_{\mu ba}\{Z^A, T^a, T^b\}A^{\mu}_{dc}\{Z_A, T^c, T^d\} \\ & + \frac{1}{3}E^{\mu\nu\lambda}A_{\mu ba}A_{\nu dc}A_{\lambda fe}\{T^a, T^c, T^d\}\{T^b, T^f, T^e\} \\ & + i\bar{\psi}^A\Gamma^{\mu}A_{\mu ba}\{\psi_A, T^a, T^b\} + \frac{i}{2}E_{ABCD}\bar{\psi}^A\{Z^C, Z^D, \psi^B\} - \frac{i}{2}E^{ABCD}Z_D\{\bar{\psi}_A, \psi_B, Z_C\} \\ & - i\bar{\psi}^A\{\psi_A, Z^B, Z_B\} + 2i\bar{\psi}^A\{\psi_B, Z^B, Z_A\}\Big) \end{aligned} \quad \text{(id75)}$$

where fields with a raised A index transform in the 4 of SU(4), whereas those with lowered one transform in the $\bar{4}$. $A_{\mu ba}$ ($\mu = 0, 1, 2$) is an anti-Hermitian gauge field, Z^A and Z_A are a complex scalar field and its complex conjugate, respectively. ψ_A is a fermion field that satisfies

$$\Gamma^{012}\psi_A = -\psi_A \quad \text{(id76)}$$

and ψ^A is its complex conjugate. $E^{\mu\nu\lambda}$ and E^{ABCD} are Levi-Civita symbols in three dimensions and four dimensions, respectively. The potential terms are given by

$$\begin{aligned} V = & \frac{2}{3}\Upsilon^{CD}_B\Upsilon^B_{CD} \\ \Upsilon^{CD}_B = & \{Z^C, Z^D, Z_B\} - \frac{1}{2}\delta^C_B\{Z^E, Z^D, Z_E\} + \frac{1}{2}\delta^D_B\{Z^E, Z^C, Z_E\} \end{aligned} \quad \text{(id77)}$$

If we replace the Nambu-Poisson bracket with a Hermitian 3-algebra's bracket [19], [20],

$$\begin{aligned} & \int d^3\sigma\sqrt{-g} \to \langle\ \ \rangle \\ & \{\varphi^a, \varphi^b, \varphi^c\} \to [T^a, T^b; \bar{T}^{\bar{c}}] \end{aligned} \quad \text{(id78)}$$

we obtain the Hermitian 3-algebra model of M-theory [26],

$$\begin{aligned} S = & < -V - A_{\mu\bar{b}a}[Z^A, T^a; \bar{T}^{\bar{b}}]\overline{A^{\mu}_{\bar{d}c}[Z_A, T^c; \bar{T}^{\bar{d}}]} + \frac{1}{3}E^{\mu\nu\lambda}A_{\mu\bar{b}a}A_{\nu\bar{d}c}A_{\lambda\bar{f}e}[T^a, T^c; \bar{T}^{\bar{d}}]\overline{[T^b, T^f; \bar{T}^{\bar{e}}]} \\ & + i\bar{\psi}^A\Gamma^{\mu}A_{\mu\bar{b}a}[\psi_A, T^a; \bar{T}^{\bar{b}}] + \frac{i}{2}E_{ABCD}\bar{\psi}^A[Z^C, Z^D; \bar{\psi}^B] - \frac{i}{2}E^{ABCD}\bar{Z}_D[\bar{\psi}_A, \psi_B; \bar{Z}_C] \\ & - i\bar{\psi}^A[\psi_A, Z^B; \bar{Z}_B] + 2i\bar{\psi}^A[\psi_B, Z^B; \bar{Z}_A] > \end{aligned} \quad \text{(id79)}$$

where the cosmological constant has been deleted for the same reason as before. The potential terms are given by

$$\begin{aligned} V &= \frac{2}{3}\Upsilon^{CD}_{B}\bar{\Upsilon}^{B}_{CD} \\ \Upsilon^{CD}_{B} &= [Z^{C}, Z^{D}; \bar{Z}_{B}] - \frac{1}{2}\delta^{C}_{B}[Z^{E}, Z^{D}; \bar{Z}_{E}] + \frac{1}{2}\delta^{D}_{B}[Z^{E}, Z^{C}; \bar{Z}_{E}] \end{aligned} \tag{id80}$$

This matrix model can be obtained formally by a dimensional reduction of the = 6 BLG action [8], which is equivalent to ABJ(M) action [7], [45]The authors of [46], [47], [48], [49] studied matrix models that can be obtained by a dimensional reduction of the ABJM and ABJ gauge theories on S^3. They showed that the models reproduce the original gauge theories on S^3 in planar limits.,

$$\begin{aligned} S_{=6BLG} &= \int d^3x < -V - D_{\mu}Z^{A}\overline{D^{\mu}Z_{A}} + E^{\mu\nu\lambda}\Big(\frac{1}{2}A_{\mu\bar{c}b}\partial_{\nu}A_{\lambda\bar{d}a}\bar{T}^{\bar{d}}[T^{a}, T^{b}; \bar{T}^{\bar{c}}] \\ &\quad + \frac{1}{3}A_{\mu\bar{b}a}A_{\nu\bar{d}c}A_{\lambda\bar{f}e}[T^{a}, T^{c}; \bar{T}^{\bar{d}}]\overline{[T^{b}, T^{f}; \bar{T}^{\bar{e}}]}\Big) \\ &\quad - i\bar{\Psi}^{A}\Gamma^{\mu}D_{\mu}\psi_{A} + \frac{i}{2}E_{ABCD}\bar{\Psi}^{A}[Z^{C}, Z^{D}; \psi^{B}] - \frac{i}{2}E^{ABCD}\bar{Z}_{D}[\bar{\Psi}_{A}, \psi_{B}; \bar{Z}_{C}] \\ &\quad - i\bar{\Psi}^{A}[\psi_{A}, Z^{B}; \bar{Z}_{B}] + 2i\bar{\Psi}^{A}[\psi_{B}, Z^{B}; \bar{Z}_{A}] > \end{aligned} \tag{id82}$$

The Hermitian 3-algebra models of M-theory are classified into the models with $u(m) \oplus u(n)$ Hermitian 3-algebra (▭) and $sp(2n) \oplus u(1)$ Hermitian 3-algebra (▭). In the following, we study the $u(N) \oplus u(N)$ Hermitian 3-algebra model. By substituting the $u(N) \oplus u(N)$ Hermitian 3-algebra (▭) to the action (▭), we obtain

$$\begin{aligned} S &= \mathrm{Tr}\Big(-\frac{(2\pi)^2}{k^2}V - (Z^{A}A^{R}_{\mu} - A^{L}_{\mu}Z^{A})(Z^{A}A^{R\mu} - A^{L\mu}Z^{A})^{\dagger} - \frac{k}{2\pi}\frac{i}{3}E^{\mu\nu\lambda}(A^{R}_{\mu}A^{R}_{\nu}A^{R}_{\lambda} - A^{L}_{\mu}A^{L}_{\nu}A^{L}_{\lambda}) \\ &\quad - \bar{\Psi}^{A}\Gamma^{\mu}(\psi_{A}A^{R}_{\mu} - A^{L}_{\mu}\psi_{A}) + \frac{2\pi}{k}(iE_{ABCD}\bar{\Psi}^{A}Z^{C}\psi^{\dagger B}Z^{D} - iE^{ABCD}Z^{\dagger}_{D}\overline{\psi^{\dagger}}_{A}Z^{\dagger}_{C}\psi_{B} \\ &\quad - i\bar{\Psi}^{A}\psi_{A}Z^{\dagger}_{B}Z^{B} + i\bar{\Psi}^{A}Z^{B}Z^{\dagger}_{B}\psi_{A} + 2i\bar{\Psi}^{A}\psi_{B}Z^{\dagger}_{A}Z^{B} - 2i\bar{\Psi}^{A}Z^{B}Z^{\dagger}_{A}\psi_{B})) \end{aligned} \tag{id83}$$

where $A^{R}_{\mu} \equiv -\frac{k}{2\pi}iA_{\mu\bar{b}a}T^{\dagger\bar{b}}T^{a}$ and $A^{L}_{\mu} \equiv -\frac{k}{2\pi}iA_{\mu\bar{b}a}T^{a}T^{\dagger\bar{b}}$ are $N \times N$ Hermitian matrices. In the algebra, we have set $\alpha = \frac{2\pi}{k}$, where k is an integer representing the Chern-Simons level. We choose $k = 1$ in order to obtain 16 dynamical supersymmetries. V is given by

$$\begin{aligned} V &= +\frac{1}{3}Z^{\dagger}_{A}Z^{A}Z^{\dagger}_{B}Z^{B}Z^{\dagger}_{C}Z^{C} + \frac{1}{3}Z^{A}Z^{\dagger}_{A}Z^{B}Z^{\dagger}_{B}Z^{C}Z^{\dagger}_{C} + \frac{4}{3}Z^{\dagger}_{A}Z^{B}Z^{\dagger}_{C}Z^{A}Z^{\dagger}_{B}Z^{C} \\ &\quad - Z^{\dagger}_{A}Z^{A}Z^{\dagger}_{B}Z^{C}Z^{\dagger}_{C}Z^{B} - Z^{A}Z^{\dagger}_{A}Z^{B}Z^{\dagger}_{C}Z^{C}Z^{\dagger}_{B} \end{aligned} \tag{id84}$$

By redefining fields as

$$\begin{aligned} Z^A &\to \left(\frac{k}{2\pi}\right)^{\frac{1}{3}} Z^A \\ A^\mu &\to \left(\frac{2\pi}{k}\right)^{\frac{1}{3}} A^\mu \\ \psi^A &\to \left(\frac{k}{2\pi}\right)^{\frac{1}{6}} \psi^A \end{aligned} \tag{id85}$$

we obtain an action that is independent of Chern-Simons level:

$$\begin{aligned} S = \mathrm{Tr}\Big(& -V - \left(Z^A A^R_\mu - A^L_\mu Z^A\right)\left(Z^A A^{R\mu} - A^{L\mu} Z^A\right)^\dagger - \frac{i}{3} E^{\mu\nu\lambda}\left(A^R_\mu A^R_\nu A^R_\lambda - A^L_\mu A^L_\nu A^L_\lambda\right) \\ & -\bar{\psi}^A \Gamma^\mu\left(\psi_A A^R_\mu - A^L_\mu \psi_A\right) + iE_{ABCD}\bar{\psi}^A Z^C \psi^{\dagger B} Z^D - iE^{ABCD} Z^\dagger_D \overline{\psi^\dagger}_A Z^\dagger_C \psi_B \\ & -i\bar{\psi}^A \psi_A Z^\dagger_B Z^B + i\bar{\psi}^A Z^B Z^\dagger_B \psi_A + 2i\bar{\psi}^A \psi_B Z^\dagger_A Z^B - 2i\bar{\psi}^A Z^B Z^\dagger_A \psi_B\Big) \end{aligned} \tag{id86}$$

as opposed to three-dimensional Chern-Simons actions.

If we rewrite the gauge fields in the action as $A^L_\mu = A_\mu + b_\mu$ and $A^R_\mu = A_\mu - b_\mu$, we obtain

$$\begin{aligned} S = \mathrm{Tr}\Big(& -V + \left([A_\mu, Z^A] + \{b_\mu, Z^A\}\right)\left([A^\mu, Z_A] - \{b^\mu, Z_A\}\right) + iE^{\mu\nu\lambda}\left(\frac{2}{3} b_\mu b_\nu b_\lambda + 2A_\mu A_\nu b_\lambda\right) \\ & +\bar{\psi}^A \Gamma^\mu\left([A_\mu, \psi_A] + \{b_\mu, \psi_A\}\right) + iE_{ABCD}\bar{\psi}^A Z^C \psi^{\dagger B} Z^D - iE^{ABCD} Z^\dagger_D \overline{\psi^\dagger}_A Z^\dagger_C \psi_B \\ & -i\bar{\psi}^A \psi_A Z^\dagger_B Z^B + i\bar{\psi}^A Z^B Z^\dagger_B \psi_A + 2i\bar{\psi}^A \psi_B Z^\dagger_A Z^B - 2i\bar{\psi}^A Z^B Z^\dagger_A \psi_B\Big) \end{aligned} \tag{id87}$$

where [,] and { , } are the ordinary commutator and anticommutator, respectively. The $u(1)$ parts of A^μ decouple because A^μ appear only in commutators in the action. b^μ can be regarded as auxiliary fields, and thus A^μ correspond to matrices X^μ that represents three space-time coordinates in M-theory. Among $N \times N$ arbitrary complex matrices Z^A, we need to identify matrices X^I $(I = 3, \cdots 10)$ representing the other space coordinates in M-theory, because the model possesses not $SO(8)$ but $SU(4) \times U(1)$ symmetry. Our identification is

$$\begin{aligned} Z^A &= iX^{A+2} - X^{A+6}, \\ X^I &= \hat{X}^I - ix^I 1 \end{aligned} \tag{id88}$$

where $\hat{X}^I$ and x^I are $su(N)$ Hermitian matrices and real scalars, respectively. This is analogous to the identification when we compactify ABJM action, which describes N M2 branes, and obtain the action of N D2 branes [50], [7], [51]. We will see that this identification works also in our case. We should note that while the $su(N)$ part is Hermitian, the $u(1)$ part is anti-Hermitian. That is, an eigen-value distribution of X^μ, Z^A, and not X^I determine the space-time in the Hermitian model. In order to define light-cone coordinates, we need to perform Wick rotation: $a^0 \to -ia^0$. After the Wick rotation, we obtain

$$A^0 = \hat{A}^0 - ia^0 1 \quad \text{(id89)}$$

where $\hat{A}^0$ is a $su(N)$ Hermitian matrix.

3.4. DLCQ Limit of 3-algebra model of M-theory

It was shown that M-theory in a DLCQ limit reduces to the BFSS matrix theory with matrices of finite size [30], [31], [32], [33], [34], [35]. This fact is a strong criterion for a model of M-theory. In [26], [28], it was shown that the Lie and Hermitian 3-algebra models of M-theory reduce to the BFSS matrix theory with matrices of finite size in the DLCQ limit. In this subsection, we show an outline of the mechanism.

DLCQ limit of M-theory consists of a light-cone compactification, $x^- \approx x^- + 2\pi R$, where $x^{\pm} = \frac{1}{\sqrt{2}}(x^{10} \pm x^0)$, and Lorentz boost in x^{10} direction with an infinite momentum. After appropriate scalings of fields [26], [28], we define light-cone coordinate matrices as

$$\begin{aligned} X^0 &= \frac{1}{\sqrt{2}}(X^+ - X^-) \\ X^{10} &= \frac{1}{\sqrt{2}}(X^+ + X^-) \end{aligned} \quad \text{(id91)}$$

We integrate out b^μ by using their equations of motion.

A matrix compactification [52] on a circle with a radius R imposes the following conditions on X^- and the other matrices Y:

$$\begin{aligned} X^- - (2\pi R)1 &= U^\dagger X^- U \\ Y &= U^\dagger Y U \end{aligned} \quad \text{(id92)}$$

where U is a unitary matrix. In order to obtain a solution to (▭), we need to take $N \to \infty$ and consider matrices of infinite size [52]. A solution to (▭) is given by $X^- = \overline{X}^- + \widetilde{X}^-$, $Y = \widetilde{Y}$ and

$$U = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & 0 & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & 0 & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \otimes 1_{n \times n} \in U(N) \quad \text{(id93)}$$

Backgrounds $\overline{X}^-$ are

$$\overline{X}^- = -T^3\bar{x}_0^- T^0 - (2\pi R)\mathrm{diag}(\cdots, s-1, s, s+1, \cdots) \otimes 1_{n\times n} \quad \text{(id94)}$$

in the Lie 3-algebra case, whereas

$$\overline{X}^- = -i\left(T^3\bar{x}^-\right)1 - i(2\pi R)\mathrm{diag}(\cdots, s-1, s, s+1, \cdots) \otimes 1_{n\times n} \quad \text{(id95)}$$

in the Hermitian 3-algebra case. A fluctuation $\tilde{x}$ that represents $u(N)$ parts of $\widetilde{X}^-$ and $\widetilde{Y}$ is

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & & \\ \ddots & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & & & \ddots & \\ \ddots & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & & & \\ & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & & \\ & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & \\ & & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \ddots \\ & \ddots & & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \ddots \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \quad \text{(id96)}$$

Each $\tilde{x}(s)$ is a $n\times n$ matrix, where s is an integer. That is, the (s, t)-th block is given by $\tilde{x}_{s,t} = \tilde{x}(s-t)$.

We make a Fourier transformation,

$$\tilde{x}(s) = \frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau\, x(\tau) e^{is\frac{\tau}{\widetilde{R}}} \quad \text{(id97)}$$

where $x(\tau)$ is a $n\times n$ matrix in one-dimension and $R\widetilde{R} = 2\pi$. From (▭)-(▭), the following identities hold:

$$\begin{aligned} \sum_t \tilde{x}_{s,t}\widetilde{x'}_{t,u} &= \frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau\; x(\tau)x'(\tau)e^{i(s-u)\frac{\tau}{\widetilde{R}}} \\ \mathrm{tr}\left(\sum_{s,t}\tilde{x}_{s,t}\widetilde{x'}_{t,s}\right) &= V\frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau\; \mathrm{tr}\left(x(\tau)x'(\tau)\right) \\ [\bar{x}^-, \tilde{x}]_{s,t} &= \frac{1}{2\pi\widetilde{R}}\int_0^{2\pi\widetilde{R}} d\tau\; \partial_\tau x(\tau)e^{i(s-t)\frac{\tau}{\widetilde{R}}} \end{aligned} \quad \text{(id98)}$$

where tr is a trace over $n\times n$ matrices and $V = \sum_s 1$.

Next, we boost the system in x^{10} direction:

$$\widetilde{X}'^{+} = \frac{1}{T}\widetilde{X}^{+}$$
$$\widetilde{X}'^{-} = T\widetilde{X}^{-} \quad \text{(id99)}$$

The DLCQ limit is achieved when $T \to \infty$, where the "novel Higgs mechanism" [50] is realized. In $T \to \infty$, the actions of the 3-algebra models of M-theory reduce to that of the BFSS matrix theory [27] with matrices of finite size,

$$S = \frac{1}{g^2}\int_{-\infty}^{\infty} d\tau \, \mathrm{tr}\left(\frac{1}{2}\left(D_0 x^P\right)^2 - \frac{1}{4}[x^P, x^Q]^2 + \frac{1}{2}\bar{\psi}\Gamma^0 D_0 \psi - \frac{i}{2}\bar{\psi}\Gamma^P[x_P, \psi]\right) \quad \text{(id100)}$$

where $P, Q = 1, 2, \cdots, 9$.

3.5. Supersymmetric deformation of Lie 3-algebra model of M-theory

A supersymmetric deformation of the Lie 3-algebra Model of M-theory was studied in [53] (see also [54], [55], [56]). If we add mass terms and a flux term,

$$S_m = \left\langle -\frac{1}{2}\mu^2\left(X^I\right)^2 - \frac{i}{2}\mu\bar{\Psi}\Gamma_{3456}\Psi + H_{IJKL}\,[X^I, X^J, X^K]X^L \right\rangle \quad \text{(id102)}$$

such that

$$H_{IJKL} = \begin{cases} -\frac{\mu}{6}{}_{IJKL} & (I, J, K, L = 3, 4, 5, 6 \text{ or } 7, 8, 9, 10) \\ 0 & (\text{otherwise}) \end{cases} \quad \text{(id103)}$$

to the action (▭), the total action $S_0 + S_m$ is invariant under dynamical 16 supersymmetries,

$$\delta X^I = i\Gamma^I\Psi$$
$$\delta A_{\mu ab}[T^a, T^b, \quad] = i\Gamma_\mu\Gamma_I[X^I, \Psi, \quad]$$
$$\delta\Psi = -\frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK} - A_{\mu ab}[T^a, T^b, X^I]\Gamma^\mu\Gamma_I + \mu\Gamma_{3456}X^I\Gamma_I \quad \text{(id104)}$$

From this action, we obtain various interesting solutions, including fuzzy sphere solutions [53].

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4. Conclusion

The metric Hermitian 3-algebra corresponds to a class of the super Lie algebra. By using this relation, the metric Hermitian 3-algebras are classified into $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebras.

The Lie and Hermitian 3-algebra models of M-theory are obtained by second quantizations of the supermembrane action in a semi-light-cone gauge. The Lie 3-algebra model possesses manifest = 1 supersymmetry in eleven dimensions. In the DLCQ limit, both the models reduce to the BFSS matrix theory with matrices of finite size as they should.

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Author details

Matsuo Sato[1]

1 Hirosaki University,, Japan

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